Non-linear Integer Programming Model and Algorithms for Connected p-facility Location Problem

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Abstract In this paper, a new location analysis method is presented. Given a connected graph $G = (V, E)$ with nonnegative edge cost c_e for each edge $e \in E$, d_{ij} is the cost of the shortest path between vertices i and j in the graph. The *Connected p-facility Location Problem* (CpLP) is to choose p vertices from V so as to minimize the total cost of shortest path of pair-wise of these p vertices. This problem is proved to be NP-hard and non-linear integer programming is formulated. Then, two algorithms are designed for solving the CpLP. One is a heuristic algorithm based on classical maximum clique approach, while the second one is genetic algorithm. Finally, computational results show the comparison between these two algorithms.

Keywords connected location; maximum clique; heuristic algorithm; genetic algorithm

1 Introduction

Given a connected graph $G = (V, E)$ with nonnegative edge cost c_e for each edge $e \in E$, d_{ij} is the cost of the shortest path between vertices i and j in the graph. The *Connected p-facility Location Problem* (CpLP) is to choose p vertices from V such that the total shortest path cost of each pair of these p vertices is minimized.

Facility Location Problem (FL) is a classical issue which has been studied since Weber Location Problem in 1909. The problem looks for the best locations for a set of facilities that must satisfy requests of service coming from a given set of customers. We use the term “facility” here in its broadest sense. That is, it is meant to include entities such as factories, warehouse, retail outlets, hospitals, electronic switching centers and emergency warning sirens. The ubiquity of location decision-making has led to a strong interest in location analysis and modeling within the operations research and management science community. There are many facility location models, such as covering^[1, 2], p-center^[3, 4], p-dispersion^[5] and p-median^[3].

Cooper^[6] formulated the *Facility Location-Allocation* (FLA) that provides a valuable method in deciding where to place facilities coupled with determining how to assign demand to the located facilities in order to utilize resources effectively. Another interesting location analysis method is to combine the vehicle routing problem with facility location, which is formulated to *Location-Routing Problem* (LRP). Many variations of LRPs exist, such as deterministic LRP^[7], dynamic LRP^[8] and planar LRP^[9]. Much more work about LRP can be obtained from^[10].

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To optimize transportation between facilities, Gupta^[11] introduce *Connected Facility Location Problem* (ConFL) which corresponds to the above mentioned facilities. The facilities need to be installed and connected with each other and customer nodes need to be assigned to them. The ConFL problem consists of finding an assignment of each customer to exactly one facility and connecting these facilities via a Steiner tree. Furthermore, the ConFL is transformed into the minimum Steiner arborescence problem and solved by an exact branch-and-cut method. Ten different integer programming formulations for ConFL have been presented by [12]. Several relevant special ConFLs are considered in some recent works, such as capacity constrained ConFL^[13–15], hop constrained ConFL^[16, 17].

In this paper, a new location analysis method is presented by paying more attention to the sum distance/cost of pair-wise of these selected facilities. This problem comes from the real situation when dealing with natural disasters. In order to relieve the damage of large-scale disasters, many countries pay attention to emergency management issues. Mitigation and preparation are carefully considered in the one hand, while quick and efficient response is done on the other hand. Relief materials are very important in the whole process of disaster response. Huge demands are supplied from all over the country, sometimes outside the country from international support. Distribution center plays an important role in reserving, distributing, and transporting relief goods in order to enhance the ability of material organization. Huge demand for materials, especially for large-scale disasters, need to be transported from all other centers. The objective is to minimize the total transportation cost or time between each pair of the selected distribution centers.

The remainder of this paper is organized as follows. In Section 2, we will formulate a mathematic model for CpLP in detail and present the NP-hardness proof for this problem. Two efficient algorithms are designed for solving this problem in Section 3. The first one is based on procedure for maximum clique problem, while the other one is genetic algorithm. Computational results are shown in Section 4. In Section 5, we draw conclusions with further researches discussion.

2 Model formulation and complexity analysis

In this section, the mathematical model for CpLP will be formulated. And complexity of this problem will be analyzed.

2.1 Basic model

Let $G = (V, E)$ be a connected graph with nonnegative edge cost c_e for each edge $e \in E$. Parameters and decision variables are defined below.

Parameters

p : the number of vertices need to be selected.

d_{ij} : the cost of the shortest path between vertices i and j in the graph $G = (V, E)$.

Decision variables

$$y_i = \begin{cases} 1, & \text{vertex } i \text{ is selected,} \\ 0, & \text{otherwise,} \end{cases}$$

Now the non-linear integer programming formulation is presented below.

$$\min z = \frac{1}{2} \sum_{i \in V} \sum_{j \in V} d_{ij} y_i y_j \quad (1)$$

subject to

$$\sum_{i \in V} y_i = p \quad (2)$$

$$y_i = \{0, 1\}, \quad \forall i \in V \quad (3)$$

2.2 NP-hardness proof

By reduction from maximum clique problem, we will prove the problem is NP-hard. Given an undirected graph G , a *clique* is a subgraph of G in which all the pairs of vertices are adjacent. Two vertices are said to be *adjacent* if they are connected by an edge. The *maximum clique problem*, which is one of the most important NP-hard problems^[18] in discrete mathematics and theoretical computer science, is to find a clique with maximum number of vertices.

The above definition of maximum clique problem is called the *optimization version*. In order to perform the proof, let's introduce the *recognition version* of the maximum clique problem.

Given an undirected graph G and integer p . Whether G has a clique which has at least p vertices?

Then, we define recognition version of *Connected p -facility Location Problem* as follows.

Given a connected graph $G = (V, E)$ with nonnegative edge cost c_e for each edge $e \in E$, d_{ij} is the cost of the shortest path between vertices i and j in the graph. Also, given an integer p and nonnegative number w . Whether G has a subset with p vertices whose sum cost of each pair of these p vertices is at most w ?

Since the recognition version of a combinatorial problem has the same complexity as optimization version, we only need to prove the recognition version of CpLP to be NP-hard.

Theorem 1 *The recognition version of CpLP is NP-hard even when all edge with unit cost.*

Proof For any instance of recognition version of maximum clique problem, we will construct an instance of CpLP.

For an instance I_{MCP} of maximum clique problem with undirected graph $G = (V, E)$ and integer p , let's construct an instance I_{CpLP} . Define a weighted graph $G = (V, E)$ with $c_e = 1$ for any $e \in E$, integer p and $w = \frac{p(p-1)}{2}$. We will show I_{CpLP} has a subset with p vertices whose sum shortest path cost of each pair of these p vertices is at most $\frac{p(p-1)}{2}$ if and only if I_{MCP} has a clique which has at least p vertices. At first, if I_{MCP} has a clique C which has at least p vertices, the cost of shortest path between each pair of vertices in C equals 1, then the sum cost equals $\frac{p(p-1)}{2}$. On the other hand, if I_{CpLP} has a subset with p vertices whose sum shortest path cost of each pair of these p vertices is at most $\frac{p(p-1)}{2}$, any pair of vertices in this subset is adjacent in G , then this subset is a clique of G with p vertices. This is the end of the proof. $\square$

Since the CpLP is NP-hard, it is hard to solve this problem exactly especially for large-scale

instances. A lower bound for CpLP is presented as follows for comparison of our algorithms.

$$lowerbound = \text{the sum } d_{ij} \text{ of minimum } \frac{p(p-1)}{2} \text{ pairs of vertices.}$$

3 Algorithms

In this section, two algorithms will be presented. The first one is a heuristic algorithm based on classical maximum clique approach, while the second one is genetic algorithm.

3.1 Algorithm based on maximum clique procedure

Since the maximum clique problem is also NP-hard, we design a heuristic polynomial algorithm for the maximum clique problem at the beginning. Given an undirected graph $G = (V, E)$, let $G(U)$ be the induced graph for any subset $U \subseteq V$ in $G = (V, E)$. $d_{G(U)}(v)$ is the degree of vertex $v \in U$ in graph $G(U)$ and $N_{G(U)}(v)$ is the neighbor of vertex $v \in U$ in graph $G(U)$. Now, we present the maximum clique procedure (MCP).

Procedure MCP($G = (V, E)$)

Input a graph $G = (V, E)$

Output a clique C and its size $NumC$

1. $C = \emptyset, U = V;$
 2. **while** $U \neq \emptyset$
 3. $u = \max\{v | d_{G(U)}(v), v \in U\}$
 4. $C = C \cup \{u\}$
 5. $U = N_{G(U)}(u) \setminus \{u\}$
 6. **endwhile**
 7. $NumC = \text{size}(C)$
 8. **return** C and $NumC$
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There are much more research on maximum clique problem. One standard approach is based on the branch-and-bound method, such as [19–22]. Our procedure MCP is a simple greedy method which can find a clique within a very short time. Next, we will present our algorithm for solving the CpLP based on procedure MCP. The algorithm begins with an empty graph, then executes the procedure MCP while adding the edges.

Algorithm based on maximum clique procedure

Input a connected graph $G = (V, E)$ with nonnegative edge cost c_e for each edge $e \in E$, and an integer p

Output a subset C of V with p vertices

1. Calculate the shortest path d_{ij} for every pair of vertices i and j using Floyd-Warshall algorithm;

2. $E' = \emptyset, NumC = 0$;
 3. $visited(ij) = 0, \forall i, j \in V$;
 4. **while** $NumC < p$
 5. $e = \min\{(i, j) | d_{ij} \text{ and } visited(ij) = 0, i, j \in V\}$;
 6. $E' = E' \cup \{e\}$;
 7. $[C, NumC] = \text{procedure MCP}(G = (V, E'))$;
 8. $visited(e) = 1$;
 9. **endwhile**
 10. **while** $NumC > p$
 11. $v = \max\{i | \sum_{j \in C} d_{ij}, i \in C\}$;
 12. $C = C \setminus \{v\}$;
 13. $NumC = NumC - 1$;
 14. **endwhile**
 15. **return** C
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3.2 Genetic algorithm for CpLP

The solution is coded as a vector of V dimensions, and V is vertex set. Each vector is a chromosome, and each gene is binary corresponding to variable y_i . The fitness function of a chromosome is defined as follows.

$$fitness(y) = M / \left(\frac{1}{2} \sum_{i \in V} \sum_{j \in V} d_{ij} y_i y_j + M' \times \left| p - \sum_{i \in V} y_i \right| + 1 \right) \quad (*)$$

M : constant.

M' : penalty factor for the number of uncovered nodes.

P_c : the probability of crossover operation.

P_m : the probability of mutation operation.

NP : size of the population.

NG : the generation of GA.

$|p - \sum_{i \in V} y_i|$ is absolute value function. The crossover and mutation are simple binary valued and roulette wheel selection is applied. Then, the genetic algorithm is shown below.

Genetic algorithm for CpLP

Step 1. Generate the initial population. And calculate the fitness of each chromosome according to (*).

Step 2. **(Selection)** If generation is more than NG , stop and output the best chromosome; else, applying roulette wheel selection method to generate the next population.

Step 3. **(Crossover)** For two selected chromosomes in the population, applying simple binary valued crossover method to generate two chromosomes according to P_c .

Step 4. **(Mutation)** For any chromosome, randomly choose a gene to mutate according to P_m . Go to Step 2.

4 Experimental results

In this section, random method is introduced to generate the basic connected graphs. Vertices are placed randomly on a 400×400 square. Two vertices are adjacent when their Euclidean distance is no more than a predetermined radius. Two such graphs are shown as Figure 1.

Figure 1 Two randomly generated graph with 100 vertices and radius is 80 and 150

We have implemented the algorithm based on maximum clique procedure and genetic algorithm in Matlab 7.1 and carried out computational experiments to evaluate them. The computer used has a Inter(R) Core(TM) i5 2.30GHz CPU and Windows 7 operating system.

Parameters for genetic algorithm are set as follows. $NP = 100$, $NG = 500$, $P_c = 0.6$, $P_m = 0.1$, $M = 1000$, $M' = 1000000$.

In the first group of instances, we will analyse the performance of our presented algorithms when the radius is fixed and number of vertices increases. $p = 10$ facilities need to be chosen from the vertex set. Radius is 100, and the number of vertices is increasing from 40 to 1000. In this group of instances, sparse or dense graphs are also considered. Experimental results are shown in Table 1, from which we can draw the following conclusions.

- 1) The objective value obtained from algorithm based on Procedure MCP is better than that of genetic algorithm. According to further analysis by comparing with the lower bound of CpLP, the produced value of algorithm based on Procedure MCP is about 3 to 7 times of lower bound while the objective value of genetic algorithm is about 5 to 2.1×10^5 times of lower bound. Figure 2 shows the comparison among these values.
- 2) The solution output from genetic algorithm is not stability especially for large scale instances. When the number of vertices goes to 1000, there are 445 vertices been chosen. At

the same time, the objective value becomes higher 2.0×10^7 , which is a very bad solution.

- 3) When the number of vertices is lower than 100, both of these two algorithms can find better solution.
- 4) The computational CPU time of algorithm based on Procedure MCP is much faster than that of genetic algorithm under the current parameters.

Table 1 Computational results for fixed connectivity radius

Number of vertices	Number of edges	Genetic algorithm			Algorithm based on procedure MCP			Lowerbound
		Number of facilities	Cost	Cputime (seconds)	Number of facilities	Cost	Cputime (seconds)	
40	127	10	6.38E+03	5.632	10	4.39E+03	0.374	1.29E+03
50	198	10	7.44E+03	7.317	10	3.49E+03	0.531	1.10E+03
60	284	10	7.74E+03	9.095	10	3.67E+03	1.264	959.2391
70	422	10	1.04E+04	11.201	10	2.36E+03	1.185	655.6718
80	509	10	8.32E+03	13.447	10	2.37E+03	1.373	561.9503
90	644	10	6.42E+03	16.552	10	2.34E+03	1.887	545.4202
100	808	10	7.78E+03	18.861	10	2.14E+03	2.496	545.8485
110	983	10	7.19E+03	21.84	10	1.96E+03	3.307	492.4224
120	1206	10	8.37E+03	25.07	10	2.20E+03	4.821	434.2451
130	1455	10	9.93E+03	28.813	10	1.42E+03	3.292	443.1823
140	1609	11	9.28E+03	32.604	10	1.71E+03	5.428	329.0939
150	1855	12	1.07E+04	36.707	10	1.63E+03	4.072	311.7418
200	3097	45	1.98E+05	61.605	10	1.30E+03	7.987	328.996
300	6987	93	9.46E+05	132.257	10	1.00E+03	14.773	152.1773
500	19392	216	4.88E+06	389.096	10	648.5159	21.763	114.6109
1000	80342	445	2.00E+07	1837.231	10	618.774	96.424	95.4558

Figure 2 Comparison of objective value among lower bound, algorithm based on Procedure MCP and genetic algorithm

In the second group of instances, we will analyse the performance of our presented algorithms when the number of vertices is fixed and the radius is different. Also, $p = 10$ facilities need to be chosen from the vertex set. There are 100 vertices randomly placed on the square. The radius changes from 60 to 570. Obviously, the graph will be complete when radius is larger than $400\sqrt{2}$. Experimental results are shown in Table 2, from which we can draw the following conclusions.

Table 2 Computational results for fixed number of vertices

Radius	Number	Genetic algorithm			Algorithm based on procedure MCP			Lowerbound
	of edges	Number of facilities	Cost	Cputime (seconds)	Number of facilities	Cost	Cputime (seconds)	
60	368	10	1.16E+04	18.767	10	2.15E+03	2.886	5.36E+02
70	422	10	8.31E+03	18.798	10	2.37E+03	2.543	5.07E+02
80	567	10	1.01E+04	18.798	10	2.19E+03	2.309	511.5218
90	645	10	966E+03	18.798	10	2.26E+03	2.013	497.084
100	861	10	9.09E+03	18.908	10	1.16E+03	1.123	428.4923
110	980	10	8.37E+03	18.907	10	2.17E+03	2.73	429.5922
120	1147	10	7.28E+03	18.814	10	2.06E+03	2.512	536.5452
130	1205	10	9.13E+03	18.829	10	1.87E+03	1.826	378.5094
140	1611	10	7.41E+03	18.938	10	1.51E+03	1.981	485.7258
150	1574	10	8.23E+03	18.829	10	2.09E+03	2.574	479.5754
200	2387	10	7.73E+03	18.783	10	2.01E+03	2.355	459.9384
250	3135	10	8.41E+03	18.783	10	1.73E+03	1.763	488.6672
350	4562	10	9.46E+03	18.907	10	1.69E+03	1.95	541.5531
570	4950	10	8.40E+03	18.845	10	2.32E+03	2.761	574.7385

- 1) The performance of algorithm based on Procedure MCP is better than that of genetic algorithm not only for the objective value but also for computational CPU time.
- 2) Both of the two algorithms are stability.
- 3) The objective values and computational CPU times of the presented algorithms do not vary too much when radius grows larger.
- 4) The CpLP is not sensitive for the radius when the number of vertices is fixed.

In the third group of instances, we will analyse the performance of our presented algorithms when the number of vertices and radius are all fixed while number of facilities changes. There are 100 vertices randomly placed on the square and radius is 100. The number of facilities increases from 5 to 50. Table 3 shows the computational results.

- 1) The algorithm based on Procedure MCP can produce better solution no matter number of facilities is small or large and the algorithm is more stability. The objective value is

about 2 to 5 times of the lower bound. The genetic algorithm can also produce better solution only when the number of facilities is large, while it is not a better algorithm for small number of facilities.

- 2) The computational CPU time of algorithm based on Procedure MCP becomes much longer when the number of facilities increases. While the genetic algorithm is not sensitive for the facility number too much, it is because the coding method is up to total number of vertices.

Table 3 Computational results for fixed connectivity radius and number of vertices

p	Number of edges	Genetic algorithm			Algorithm based on procedure MCP			Lowerbound
		Cost	Cputime (seconds)	Cost/low erbound	Cost	Cputime (seconds)	Cost/low erbound	
5	617	2.11E+03	18.845	37.79	2.43E+02	0.515	4.35	5.59E+01
10	593	1.91E+04	18.892	11.62	4.71E+03	4.352	2.87	1.64E+03
15	563	1.91E+04	18.892	11.62	4.71E+03	4.352	2.87	1.64E+03
20	537	4.19E+04	18.845	9.16	1.81E+04	14.289	3.95	4.58E+03
25	538	6.55E+04	18.829	7.89	3.01E+04	18.33	3.62	8.30E+03
30	558	1.01E+05	18.798	7.39	4.58E+04	28.533	3.36	1.36E+04
35	613	1.27E+05	18.923	5.75	7.10E+04	37.893	3.20	2.22E+04
40	592	1.50E+05	18.845	4.36	8.63E+04	42.666	2.51	3.43E+04
45	556	1.99E+05	18.829	3.89	1.24E+05	53.321	2.43	5.12E+04
50	539	2.56E+05	18.798	3.55	1.87E+05	67.346	2.58	7.23E+04

From the above groups of instances, the algorithm based on Procedure MCP always produces better solution. Furthermore, other better procedure for maximum clique problem can improve the performance of our presented algorithm. On the other hand, genetic algorithm can also produce better solution if increase the size of the population and the generation of GA.

5 Conclusion

In this paper, we present a new location analysis method by considering the distance between each pair of selected vertices, which is called connected p -facilities location problem(CpLP). This problem is proved to be NP-hard and mathematical model is formulated. For solving CpLP, we present two algorithms. One is based on maximum clique procedure, while the other one is genetic method. From the computational results of three groups of instances, these two algorithms can produce better solutions by comparing with the lower bound of CpLP.

For further researches, new coding methods can improve the performance of genetic algorithm. Another direction is to design new algorithm based on relaxing the mathematical model of CpLP.

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