Uncertain Comprehensive Evaluation Method Based on Expected Value

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Abstract The author presents a new comprehensive evaluation method based on uncertainty theory in this paper. According to this kind of method, the evaluation quality of each evaluated index given by every expert is regarded as an uncertain variable. Then a comparison rule of uncertain comprehensive evaluation is proposed by means of expected value of uncertain variable. Some properties about the comparison rule are discussed. At last, an example about the comparison of urban environment quality is given to illustrate the uncertain comprehensive evaluation method.

Keywords uncertain comprehensive evaluation; uncertainty theory; uncertain variable; expected value; variance

1 Introduction

Comprehensive evaluation is an objective and reasonable overall assessment of the evaluation object whose theory and methods are widely used in society science and natural science. Its basic idea is to translate the multi-index (related multiple factors) into an index which can reflect comprehensive situation.

There always exists indeterminacy when we conduct a quantitative evaluation. Different ways to deal with the indeterminacy produce different methods of comprehensive evaluation.

One method of comprehensive evaluation is multivariate statistical analysis method based on probability theory. Some indeterminacy are stochastic phenomenon and researchers describe them with random variables, and then some researchers use mathematical statistics methods as well as multivariate techniques to study and solve the problem with multi-index, namely multivariate statistical analysis method which has been widely used in meteorology, medicine, agriculture, economic and many other fields. [1] has introduced that multivariate statistical analysis method contains PCA (principal component analysis) method, factor analysis method,

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cluster analysis method and some other methods. [2] and [3] have applied the methods into practical problems. We can find that multivariate techniques can statistically estimate relationships between different variables, and correlate how important each one is to the final outcome and where dependencies exist between them, which are methodologies for estimating population based upon sample data. But [4] and [5] have pointed out that all the methods of multivariate statistical analysis have some common shortcomings. One is that they need enough sample data and high-quality samples to give meaningful results, so the reliance on sample data limits its application. Another one is the relativity of the evaluation results, because we can't see the actual level of evaluation objects from the results of assessment directly.

Another method is called fuzzy comprehensive evaluation method. It is proposed by [6] based on fuzzy sets theory that was originally introduced by [7]. It refers to do overall evaluation of some thing or phenomena affected by a variety of factors, which involve the fuzzy factors that are distinctive to stochastic factors. It has been studied by many researchers and widely used in many fields which implies its advantages. However, it has some insufficiencies. One is indicated by [8] that the result of fuzzy comprehensive evaluation based on the principle of maximum membership degree is a kind of vague, and this method may also cause loss of information, even abnormal conclusions. [9–11] point out some other deficiencies. Although [12–14] have proposed some solutions, the problems can't be fundamentally solved by fuzzy theory.

When we are lack of observed data about the unknown state of nature, the estimated probability distribution may be far from the cumulative frequency. [15] asserted that probability theory may lead to counterintuitive results in this case. We have to invite some domain experts to evaluate the belief degree that each event will occur. It is not suitable to employ random variable or fuzzy variable to describe the uncertain factor. So we need a new consistent mathematical system to deal with the belief degree, such as empirical estimation. Fortunately, Liu^[16] founded uncertainty theory in 2007 to deal with the indeterminacy of some information and knowledge which are neither subjective probability nor fuzziness, it has become a branch of axiomatic mathematics based on normality, self-duality, countable subadditivity, and product measure axioms. [17] made many surveys on practical problems to support the fact that uncertainty theory is a legitimate mathematical system to model human uncertainty. Note that probability is essentially a limit of frequencies, but uncertainty is the legitimate approach to deal with belief degree.

So far uncertainty theory has been widely used in science and engineering. For example, [18] treated risks as uncertain variables with uncertain functions and proposed some comparison rules to compare uncertain variables. [19] proposed a comprehensive evaluation method based on uncertainty theory. In that method, the corresponding weight value and remark to every index in evaluated system are characterized as uncertain variables and the evaluated result is the remark whose expected value is the biggest. But this method also has its defects such as we can't make a comparison between two evaluated objects which belongs to the same remark.

This paper proposed a new uncertain comprehensive evaluation method and a comparison rule based on uncertainty theory which can evaluate, compare and sequence the evaluated objects. The remainder of this paper is organized as follows. In Section 2, some basic concepts of uncertainty theory are reviewed. In Section 3, the uncertain evaluated method is introduced.

In Section 4, an numerical example is given to illustrate this method. Section 5 concludes this paper.

2 Preliminaries

In this section, we shall briefly introduce some basic concepts and results selected from uncertainty theory.

Let Γ be a nonempty set and $\mathcal{L}$ a σ -algebra over Γ . Each element Λ in the σ -algebra $\mathcal{L}$ is assigned a number $\mathcal{M}\{\Lambda\}$ which indicates the belief degree that Λ will occur. The set function $\mathcal{M}\{\cdot\}$ is called an uncertain measure if it satisfies the following axioms:

- (1) (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ ;
- (2) (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
- (3) (Subadditivity Axiom) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leq \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. In order to obtain an uncertain measure of compound event, [16] defined the fourth axiom called product axiom.

(4) (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \dots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \dots$, respectively.

Definition 1 ([16]) *An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

In order to describe an uncertain variable in practice, Liu^[16] introduces the concept of uncertainty distribution $\Phi : \mathbb{R} \rightarrow [0, 1]$ of an uncertain variable ξ as follows,

Definition 2 ([16]) *The uncertainty distribution Φ of an uncertain variable ξ is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}, \quad x \in \mathbb{R}.$$

for any real number x .

The inverse function Φ^{-1} is called the inverse uncertainty distribution of ξ .

Definition 3 ([16]) *An uncertain variable ξ is called linear if it has a linear uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ \frac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } x \geq b \end{cases}$$

denoted by $\mathcal{L}(a, b)$, where a and b are real numbers with $a < b$.

The inverse uncertainty distribution of $\mathcal{L}(a, b)$ is

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b.$$

Definition 4 ([16]) *An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ \frac{x-a}{2(b-a)}, & \text{if } a \leq x \leq b \\ \frac{x+c-2b}{2(c-b)}, & \text{if } b \leq x \leq c \\ 1, & \text{if } x \geq c \end{cases}$$

denoted by $\mathcal{Z}(a, b, c)$, where a, b and c are real numbers with $a < b < c$.

The inverse uncertainty distribution of $\mathcal{Z}(a, b, c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5, \\ (2 - 2\alpha)b + (2\alpha - 1)c, & \text{if } \alpha \geq 0.5. \end{cases}$$

Definition 5 ([16]) *The uncertain variables $\xi_1, \xi_2, \dots, \xi_n$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^n (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^n \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \dots, B_n$ of real numbers.

Theorem 6 ([16]) *Assume that ξ_1 and ξ_2 are independent linear uncertain variables $\mathcal{L}(a_1, b_1)$ and $\mathcal{L}(a_2, b_2)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a linear uncertain variable $\mathcal{L}(a_1 + a_2, b_1 + b_2)$, i.e.,*

$$\mathcal{L}(a_1, b_1) + \mathcal{L}(a_2, b_2) = \mathcal{L}(a_1 + a_2, b_1 + b_2).$$

The product of a linear uncertain variable $\mathcal{L}(a, b)$ and a scalar number $k > 0$ is also a linear uncertain variable $\mathcal{L}(ka, kb)$, i.e.,

$$k \cdot \mathcal{L}(a, b) = \mathcal{L}(ka, kb).$$

Definition 7 ([16]) *Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\} dr - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\} dr,$$

provided that at least one of the two integrals is finite.

Theorem 8 ([16]) *Let ξ be an uncertain variable with regular uncertainty distribution Φ . If its expected value exists, then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha.$$

Especially, if $\xi \sim \mathcal{L}(a, b)$, then $E[\xi] = \frac{a+b}{2}$. If $\xi \sim \mathcal{Z}(a, b, c)$, then $E[\xi] = \frac{a+2b+c}{4}$.

Theorem 9 ([16]) Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers a and b , we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

Definition 10 ([16]) Let ξ be an uncertain variable with finite expected value e . Then the variance of ξ is defined by $V[\xi] = E[(\xi - e)^2]$.

The variance of linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ is

$$V[\xi] = \frac{(b - a)^2}{12}.$$

The variance of uncertain variable provides a degree of the spread of the distribution around its expected value. A small value of variance indicates that the uncertain variable is tightly concentrated around its expected value. And a large value of variance indicates that the uncertain variance has a wide spread around its expected value.

3 Method of uncertain comprehensive evaluation

When we carry on an evaluation, there always exists indeterminacy due to the complexity of the system itself and the subjectivity of domain experts. In many situations, the determination of evaluated factors and their weights is unable to be perfect and the scores of indices given by experts who have their own opinions and preferences hardly reflect the real situation. Here we suppose that the evaluated indices, the corresponding weight values and the weights of experts are determined. In addition, the score of every index is regarded as uncertain variable and an uncertain evaluation model can be set up. Note that the evaluation indices are independent to each other in this method.

Let us first introduce the following indices and parameters:

i : indices, $i = 1, 2, \dots, m$;

j : experts, $j = 1, 2, \dots, n$;

ω_i : weight of the i -th index, $\sum_{i=1}^m \omega_i = 1, \omega_i \geq 0, i = 1, 2, \dots, m$,

$$\mathbf{W} = \begin{pmatrix} \omega_1, & \omega_2, & \dots, & \omega_m \end{pmatrix};$$

ν_j : weight of the j -th expert, $\sum_{j=1}^n \nu_j = 1, \nu_j \geq 0, j = 1, 2, \dots, n$,

$$\mathbf{V} = \begin{pmatrix} \nu_1, & \nu_2, & \dots, & \nu_n \end{pmatrix}^T;$$

ξ_{ij} : uncertain evaluation quality of the i -th index given by the j -th expert, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$;

Φ_{ij} : uncertainty distribution of ξ_{ij} , $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$;

Φ_{ij}^{-1} : inverse uncertainty distribution of ξ_{ij} , $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$;

ξ_j : uncertain total evaluation quality given by the j -th expert, $j = 1, 2, \dots, n$;

Φ_j : uncertainty distribution of ξ_j , $j = 1, 2, \dots, n$;

ξ : uncertain comprehensive evaluation quality of the evaluated object;

Φ : uncertainty distribution of ξ .

The uncertain evaluation matrix R is displayed as follows:

$$R = \begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \xi_{m1} & \xi_{m2} & \cdots & \xi_{mn} \end{pmatrix}.$$

Then the above matrix R is multiplied by weight vector $\mathbf{W}$ to obtain another vector $\mathbf{B}$, whose elements are respectively denoted by $\xi_j (j = 1, 2, \dots, n)$. The formula of uncertain evaluated vector $\mathbf{B}$ is

$$\begin{aligned} \mathbf{B} = \mathbf{W} \cdot \mathbf{R} &= \begin{pmatrix} \omega_1 & \omega_2 & \cdots & \omega_m \end{pmatrix} \cdot \begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \xi_{m1} & \xi_{m2} & \cdots & \xi_{mn} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^m \omega_i \xi_{i1}, & \sum_{i=1}^m \omega_i \xi_{i2}, & \cdots, & \sum_{i=1}^m \omega_i \xi_{in} \end{pmatrix} \\ &\triangleq \begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \end{pmatrix}, \end{aligned}$$

where $\xi_j = \sum_{i=1}^m \omega_i \xi_{ij}$ represents the weighted total scores given by the j -th expert, ($j = 1, 2, \dots, n$).

Next, we multiply vector $\mathbf{B}$ by vector $\mathbf{V}$ to get the formula of uncertain comprehensive score ξ . Then we have

$$\begin{aligned} \xi &= \mathbf{B} \cdot \mathbf{V} = \begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \end{pmatrix} \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{pmatrix} \\ &= \nu_1 \xi_1 + \nu_2 \xi_2 + \cdots + \nu_n \xi_n \\ &= \sum_{j=1}^n \nu_j \xi_j = \sum_{j=1}^n \sum_{i=1}^m \nu_j \omega_i \xi_{ij}. \end{aligned}$$

According to the above method, every evaluated object can be assigned an uncertain variable which indicates the total score. For the sake of obtaining the comparison result of comprehensive evaluation, a comparison rule should be well defined in order to overcome the unable comparison between uncertain variables.

Definition 11 (*Comparison Rule of Comprehensive Evaluation*) Let ξ and η be two independent uncertain variables with finite expected values. Then ξ is said to be better than η if $E[\xi] > E[\eta]$.

Theorem 12 Let ξ_{ij} ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$) be independent variables with uncertainty distributions Φ_{ij} and inverse uncertainty distributions $\Phi_{ij}^{-1}(\alpha)$, respectively. Assume that $\xi = \sum_{j=1}^n \sum_{i=1}^m \nu_j \omega_i \xi_{ij}$, where ν_j and ω_i are all nonnegative weights. Then we have

$$E[\xi] = \sum_{j=1}^n \sum_{i=1}^m \nu_j \omega_i E[\xi_{ij}] = \sum_{j=1}^n \sum_{i=1}^m \nu_j \omega_i \int_0^1 \Phi_{ij}^{-1}(\alpha) d\alpha.$$

Proof Because uncertain variables ξ_{ij} ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$) are independent and $\xi = \sum_{j=1}^n \sum_{i=1}^m \nu_j \omega_i \xi_{ij}$, it follows from Theorem 9 that

$$E[\xi] = \sum_{j=1}^n \sum_{i=1}^m \nu_j \omega_i E[\xi_{ij}].$$

According to Theorem 8, we have

$$E[\xi_{ij}] = \int_0^1 \Phi_{ij}^{-1}(\alpha) d\alpha.$$

Then we obtain

$$E[\xi] = \sum_{j=1}^n \sum_{i=1}^m \nu_j \omega_i E[\xi_{ij}] = \sum_{j=1}^n \sum_{i=1}^m \nu_j \omega_i \int_0^1 \Phi_{ij}^{-1}(\alpha) d\alpha.$$

The theorem is proved.

The property of the comparison rule we defined above are stated as follows.

Property Let ξ_{ij} ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$) be independent uncertain variables with finite expected values, and also let η_{ij} ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$) be independent uncertain variables with finite expected values. Denote

$$\xi = \sum_{j=1}^n \sum_{i=1}^m \nu_j \omega_i \xi_{ij}, \quad \eta = \sum_{j=1}^n \sum_{i=1}^m \nu_j \omega_i \eta_{ij}.$$

Then we have the following assertions:

- (1) $E[\xi] > E[\eta]$ if $E[\xi_{ij}] > E[\eta_{ij}]$ for each pair of (i, j) , ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$)
- (2) The reverse of assertion (1) is not true.

Proof (1) For simplicity, we only prove the case of $i = 1, j = 1, 2$. It follows from Theorem 12 that

$$\begin{aligned} E[\xi] &= \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} \cdot \begin{pmatrix} E[\xi_{11}] & E[\xi_{12}] \\ E[\xi_{21}] & E[\xi_{22}] \end{pmatrix} \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \\ &= \begin{pmatrix} \omega_1 E[\xi_{11}] + \omega_2 E[\xi_{21}] & \omega_1 E[\xi_{12}] + \omega_2 E[\xi_{22}] \end{pmatrix} \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \\ &= \nu_1 \omega_1 E[\xi_{11}] + \nu_1 \omega_2 E[\xi_{21}] + \nu_2 \omega_1 E[\xi_{12}] + \nu_2 \omega_2 E[\xi_{22}], \end{aligned}$$

and

$$\begin{aligned} E[\eta] &= \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} \cdot \begin{pmatrix} E[\eta_{11}] & E[\eta_{12}] \\ E[\eta_{21}] & E[\eta_{22}] \end{pmatrix} \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \\ &= \begin{pmatrix} \omega_1 E[\eta_{11}] + \omega_2 E[\eta_{21}] & \omega_1 E[\eta_{12}] + \omega_2 E[\eta_{22}] \end{pmatrix} \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \end{aligned}$$

$$= \nu_1\omega_1 E[\eta_{11}] + \nu_1\omega_2 E[\eta_{21}] + \nu_2\omega_1 E[\eta_{12}] + \nu_2\omega_2 E[\eta_{22}].$$

Since $E[\xi_{ij}] > E[\eta_{ij}]$, $i = 1, 2, j = 1, 2$, it's obvious that $E[\xi] > E[\eta]$.

(2) We can't know directly whether $E[\xi]$ is greater than $E[\eta]$ if $E[\xi_{ij}] > E[\eta_{ij}]$ does not hold for each pair of (i, j) . We should compute them according to Theorem 12 and compare them. There will be two counter-examples to illustrate the property in Section 4 (see Example 2, 3).

If the evaluation results of two objects are equal in the sense of mathematical expectation, we can say they are without much difference. But strictly speaking, they are not the same completely. In order to make difference between them, we can compare their variances further if necessary.

Remark 13 If $E[\xi] = E[\eta]$, then two uncertain variables ξ and η can be compared by comparing their variances $V[\xi]$ and $V[\eta]$, in order to make difference between them.

If $V[\xi] < V[\eta]$, then ξ is more concentrated around the expected value than η . That is, ξ is better than η .

According to the above theorem and definition, the evaluated objects can be sorted by the ranking of expected values of the corresponding uncertain variables.

4 Example

The quality of urban environment is related to human beings closely. The evaluation and comparison about the quality of urban environment is always attracted much attention. In this section, we will apply this new uncertain comprehensive evaluation method into urban environment quality evaluation. Due to that the urban environment is related to many aspects, we elect six independent representative factors, which includes air quality, water quality, noise control situation, waste disposal situation, clean energy utilization and greening condition as evaluation indices according to [20]. Suppose that there don't exist observed data about these evaluation indicators, or we can't get enough observed data because of the indeterminacy of the indicators. So we invite some domain experts to give their empirical estimation.

Linear uncertain variable is a popular type of uncertain variable. Now, we take some examples in which the uncertain scores of evaluated indices can be described by linear uncertain variables.

Example 1 Suppose that two cities A and B are elected as evaluated objects. The evaluated aspects are consisted of the above six indices. The number of domain experts is five and their weights are the same. So it's clear that $i=1, 2, 3, 4, 5, 6, j=1, 2, 3, 4, 5$. We use ξ and η to represent the comprehensive uncertain scores of city A and city B, respectively. The weight value vectors are

$$W = \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_5 & \omega_6 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.25 & 0.1 & 0.15 & 0.1 & 0.15 \end{pmatrix},$$

and

$$V = \begin{pmatrix} \nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 \end{pmatrix}^T = \begin{pmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{pmatrix}^T.$$

The uncertain evaluation matrix R_1 of city A, in which the element ξ_{ij} has the uncertainty

distributions Φ_{ij} , is displayed as follows:

$$R_1 = \begin{pmatrix} \mathcal{L}(81, 87) & \mathcal{L}(77, 79) & \mathcal{L}(81, 83) & \mathcal{L}(83, 89) & \mathcal{L}(81, 85) \\ \mathcal{L}(83, 93) & \mathcal{L}(80, 88) & \mathcal{L}(83, 87) & \mathcal{L}(76, 80) & \mathcal{L}(81, 83) \\ \mathcal{L}(72, 80) & \mathcal{L}(72, 78) & \mathcal{L}(76, 80) & \mathcal{L}(81, 83) & \mathcal{L}(70, 76) \\ \mathcal{L}(82, 88) & \mathcal{L}(82, 90) & \mathcal{L}(84, 92) & \mathcal{L}(81, 83) & \mathcal{L}(82, 86) \\ \mathcal{L}(71, 77) & \mathcal{L}(72, 80) & \mathcal{L}(76, 80) & \mathcal{L}(72, 80) & \mathcal{L}(81, 83) \\ \mathcal{L}(81, 85) & \mathcal{L}(81, 89) & \mathcal{L}(84, 88) & \mathcal{L}(84, 92) & \mathcal{L}(83, 85) \end{pmatrix}.$$

The uncertain evaluation matrix R_2 of city B, in which the element η_{ij} has the uncertainty distributions Ψ_{ij} , is presented as follows:

$$R_2 = \begin{pmatrix} \mathcal{L}(81, 85) & \mathcal{L}(72, 80) & \mathcal{L}(74, 80) & \mathcal{L}(81, 83) & \mathcal{L}(76, 80) \\ \mathcal{L}(82, 90) & \mathcal{L}(77, 79) & \mathcal{L}(81, 87) & \mathcal{L}(70, 78) & \mathcal{L}(77, 79) \\ \mathcal{L}(61, 71) & \mathcal{L}(61, 69) & \mathcal{L}(71, 79) & \mathcal{L}(76, 80) & \mathcal{L}(64, 72) \\ \mathcal{L}(77, 79) & \mathcal{L}(81, 83) & \mathcal{L}(81, 85) & \mathcal{L}(74, 80) & \mathcal{L}(65, 75) \\ \mathcal{L}(63, 73) & \mathcal{L}(68, 80) & \mathcal{L}(68, 78) & \mathcal{L}(62, 74) & \mathcal{L}(74, 80) \\ \mathcal{L}(67, 79) & \mathcal{L}(75, 81) & \mathcal{L}(81, 85) & \mathcal{L}(81, 85) & \mathcal{L}(67, 79) \end{pmatrix}.$$

Since uncertain variables ξ_{ij} and η_{ij} are all linear, we can easily compute $E[\xi_{ij}]$ and $E[\eta_{ij}]$ according to Theorem 8. The expectation evaluation of city A and city B are displayed as follows:

$$\left(E[\xi_{ij}] \right)_{6 \times 5} = \begin{pmatrix} 84 & 78 & 82 & 86 & 83 \\ 88 & 84 & 85 & 78 & 82 \\ 76 & 75 & 78 & 82 & 73 \\ 85 & 86 & 88 & 82 & 84 \\ 74 & 76 & 78 & 76 & 82 \\ 83 & 85 & 86 & 88 & 84 \end{pmatrix}, \quad \left(E[\eta_{ij}] \right)_{6 \times 5} = \begin{pmatrix} 83 & 76 & 77 & 82 & 78 \\ 86 & 78 & 84 & 74 & 78 \\ 66 & 65 & 75 & 78 & 68 \\ 78 & 82 & 83 & 77 & 78 \\ 68 & 74 & 73 & 68 & 70 \\ 73 & 78 & 83 & 83 & 73 \end{pmatrix}.$$

According to Theorem 12, we obtain

$$E[\xi] = \sum_{j=1}^5 \sum_{i=1}^6 \nu_j \omega_i E[\xi_{ij}] = 82.43, \quad E[\eta] = \sum_{j=1}^5 \sum_{i=1}^6 \nu_j \omega_i E[\eta_{ij}] = 77.54.$$

Obviously, $E[\xi] > E[\eta]$, which means the environment quality of city A is better than city B.

In fact, we can find that $E[\xi_{ij}] > E[\eta_{ij}]$ for each pair of (i, j) . According to the property of Theorem 12, we know that $E[\xi] > E[\eta]$.

Example 2 Let us exam Example 1 in another case.

Assume that the uncertain evaluation matrix R_1 of city A is

$$R_1 = \begin{pmatrix} \mathcal{L}(81, 87) & \mathcal{L}(77, 79) & \mathcal{L}(81, 83) & \mathcal{L}(83, 89) & \mathcal{L}(81, 85) \\ \mathcal{L}(82, 90) & \mathcal{L}(77, 79) & \mathcal{L}(81, 87) & \mathcal{L}(70, 78) & \mathcal{L}(77, 79) \\ \mathcal{L}(72, 80) & \mathcal{L}(72, 78) & \mathcal{L}(76, 80) & \mathcal{L}(81, 83) & \mathcal{L}(70, 76) \\ \mathcal{L}(77, 79) & \mathcal{L}(81, 83) & \mathcal{L}(81, 85) & \mathcal{L}(74, 80) & \mathcal{L}(65, 75) \\ \mathcal{L}(71, 77) & \mathcal{L}(72, 80) & \mathcal{L}(76, 80) & \mathcal{L}(72, 80) & \mathcal{L}(81, 83) \\ \mathcal{L}(67, 79) & \mathcal{L}(75, 81) & \mathcal{L}(81, 85) & \mathcal{L}(81, 85) & \mathcal{L}(67, 79) \end{pmatrix}.$$

The uncertain evaluation matrix R_2 of city B is

$$R_2 = \begin{pmatrix} \mathcal{L}(81, 85) & \mathcal{L}(72, 80) & \mathcal{L}(74, 80) & \mathcal{L}(81, 83) & \mathcal{L}(76, 80) \\ \mathcal{L}(83, 93) & \mathcal{L}(80, 88) & \mathcal{L}(83, 87) & \mathcal{L}(76, 80) & \mathcal{L}(81, 83) \\ \mathcal{L}(61, 71) & \mathcal{L}(61, 69) & \mathcal{L}(71, 79) & \mathcal{L}(76, 80) & \mathcal{L}(64, 72) \\ \mathcal{L}(82, 88) & \mathcal{L}(82, 90) & \mathcal{L}(84, 92) & \mathcal{L}(81, 83) & \mathcal{L}(82, 86) \\ \mathcal{L}(63, 73) & \mathcal{L}(68, 80) & \mathcal{L}(68, 78) & \mathcal{L}(62, 74) & \mathcal{L}(74, 80) \\ \mathcal{L}(81, 85) & \mathcal{L}(81, 89) & \mathcal{L}(84, 88) & \mathcal{L}(84, 92) & \mathcal{L}(83, 85) \end{pmatrix}.$$

Since uncertain variables ξ_{ij} and η_{ij} are all linear, we can easily calculate $E[\xi_{ij}]$ and $E[\eta_{ij}]$ according to Theorem 8. The expectation evaluation of city A and city B are respectively displayed as follows:

$$\left(E[\xi_{ij}] \right)_{6 \times 5} = \begin{pmatrix} 84 & 78 & 82 & 86 & 83 \\ 86 & 78 & 84 & 74 & 78 \\ 76 & 75 & 78 & 82 & 73 \\ 78 & 82 & 83 & 77 & 78 \\ 74 & 76 & 78 & 76 & 82 \\ 73 & 78 & 83 & 83 & 73 \end{pmatrix}, \quad \left(E[\eta_{ij}] \right)_{6 \times 5} = \begin{pmatrix} 83 & 76 & 77 & 82 & 78 \\ 88 & 84 & 85 & 78 & 82 \\ 66 & 65 & 75 & 78 & 68 \\ 85 & 86 & 88 & 82 & 84 \\ 68 & 74 & 73 & 68 & 70 \\ 83 & 85 & 86 & 88 & 84 \end{pmatrix}.$$

We compute $E[\xi]$ and $E[\eta]$ by Theorem 12, and obtain

$$E[\xi] = \sum_{j=1}^5 \sum_{i=1}^6 \nu_j \omega_i E[\xi_{ij}] = 79.69, \quad E[\eta] = \sum_{j=1}^5 \sum_{i=1}^6 \nu_j \omega_i E[\eta_{ij}] = 80.28.$$

Since $E[\xi] < E[\eta]$, the urban environment quality of city A is a little worse than that of city B.

Example 3 Let us consider a more complicated example.

Assume that the uncertain evaluation matrix R_1 of city A, where the element ξ_{ij} has uncertainty distributions $\Phi_{ij} = \mathcal{L}(a_{ij}, b_{ij})$ respectively, is displayed as

$$R_1 = \begin{pmatrix} \mathcal{L}(82, 90) & \mathcal{L}(88, 94) & \mathcal{L}(86, 90) & \mathcal{L}(83, 87) & \mathcal{L}(90, 96) \\ \mathcal{L}(66, 76) & \mathcal{L}(62, 70) & \mathcal{L}(62, 68) & \mathcal{L}(72, 76) & \mathcal{L}(61, 67) \\ \mathcal{L}(86, 94) & \mathcal{L}(82, 88) & \mathcal{L}(85, 89) & \mathcal{L}(89, 95) & \mathcal{L}(92, 96) \\ \mathcal{L}(70, 76) & \mathcal{L}(64, 70) & \mathcal{L}(63, 69) & \mathcal{L}(63, 67) & \mathcal{L}(68, 72) \\ \mathcal{L}(46, 54) & \mathcal{L}(44, 52) & \mathcal{L}(50, 54) & \mathcal{L}(50, 60) & \mathcal{L}(45, 49) \\ \mathcal{L}(65, 71) & \mathcal{L}(67, 73) & \mathcal{L}(69, 75) & \mathcal{L}(62, 68) & \mathcal{L}(70, 76) \end{pmatrix}.$$

The uncertain evaluation matrix R_2 of city B, where the element η_{ij} has uncertain distributions $\Psi_{ij} = \mathcal{L}(a'_{ij}, b'_{ij})$, is displayed as

$$R_2 = \begin{pmatrix} \mathcal{L}(65, 75) & \mathcal{L}(62, 72) & \mathcal{L}(61, 73) & \mathcal{L}(81, 85) & \mathcal{L}(86, 96) \\ \mathcal{L}(84, 90) & \mathcal{L}(85, 95) & \mathcal{L}(80, 92) & \mathcal{L}(70, 82) & \mathcal{L}(60, 72) \\ \mathcal{L}(88, 96) & \mathcal{L}(80, 86) & \mathcal{L}(80, 90) & \mathcal{L}(85, 95) & \mathcal{L}(88, 98) \\ \mathcal{L}(66, 74) & \mathcal{L}(60, 70) & \mathcal{L}(60, 66) & \mathcal{L}(58, 64) & \mathcal{L}(66, 78) \\ \mathcal{L}(43, 53) & \mathcal{L}(45, 55) & \mathcal{L}(50, 58) & \mathcal{L}(52, 62) & \mathcal{L}(42, 54) \\ \mathcal{L}(65, 77) & \mathcal{L}(66, 78) & \mathcal{L}(70, 80) & \mathcal{L}(65, 73) & \mathcal{L}(65, 77) \end{pmatrix}.$$

Since uncertain variables ξ_{ij} and η_{ij} are all linear, we can easily compute $E[\xi]$ and $E[\eta]$ according to Theorem 8. The expectation evaluation matrices of city A and city B are respectively displayed as follows:

$$\left(E[\xi_{ij}] \right)_{6 \times 5} = \begin{pmatrix} 86 & 91 & 88 & 85 & 93 \\ 71 & 66 & 65 & 74 & 64 \\ 90 & 85 & 87 & 92 & 94 \\ 73 & 67 & 66 & 65 & 70 \\ 50 & 48 & 52 & 55 & 47 \\ 68 & 70 & 72 & 65 & 73 \end{pmatrix}, \quad \left(E[\eta_{ij}] \right)_{6 \times 5} = \begin{pmatrix} 70 & 67 & 67 & 83 & 91 \\ 87 & 90 & 86 & 76 & 66 \\ 92 & 83 & 85 & 90 & 93 \\ 70 & 65 & 63 & 61 & 72 \\ 48 & 50 & 54 & 57 & 48 \\ 71 & 72 & 75 & 69 & 71 \end{pmatrix}.$$

We compute $E[\xi]$ and $E[\eta]$ by Theorem 12 and obtain

$$E[\xi] = \sum_{j=1}^5 \sum_{i=1}^6 \nu_j \omega_i E[\xi_{ij}] = 73.82, \quad E[\eta] = \sum_{j=1}^5 \sum_{i=1}^6 \nu_j \omega_i E[\eta_{ij}] = 73.82.$$

Now we find by chance that $E[\xi] = E[\eta]$, which means the urban environment quality of city A is the same as that of city B. But it may miss some information and result in an unreasonable conclusion. In order to be more fair, we should compare $V[\xi]$ and $V[\eta]$.

It follows from Theorem 6 and Definition 10 that we can firstly compute the uncertainty distributions of ξ and η , which are stated as follows:

$$\begin{aligned} \xi &\sim \mathcal{L} \left(\sum_{j=1}^5 \sum_{i=1}^6 \nu_j \omega_i a_{ij}, \sum_{j=1}^5 \sum_{i=1}^6 \nu_j \omega_i b_{ij} \right) = \mathcal{L}(70.81, 76.83), \\ \eta &\sim \mathcal{L} \left(\sum_{j=1}^5 \sum_{i=1}^6 \nu_j \omega_i a'_{ij}, \sum_{j=1}^5 \sum_{i=1}^6 \nu_j \omega_i b'_{ij} \right) = \mathcal{L}(68.99, 78.65). \end{aligned}$$

Then we can compute the variances of both ξ and η and obtain

$$\begin{aligned} V[\xi] &= \frac{(b-a)^2}{12} = \frac{(76.83-70.81)^2}{12} \approx 3.02, \\ V[\eta] &= \frac{(b'-a')^2}{12} = \frac{(78.65-68.99)^2}{12} \approx 5.15. \end{aligned}$$

Obviously, $V[\xi] < V[\eta]$, which implies the environment quality of city A is more stable than city B. Even though $E[\xi] = E[\eta]$, we can say the environment quality of city A is better than that of city B.

5 Conclusion

A new method for uncertain comprehensive evaluation was proposed in this paper based on uncertainty theory. We regarded the human uncertainty appears in evaluation as uncertain variable and set up an evaluation model. A comparison rule based on expected value was proposed to compare uncertain variables, and the property of the rule was discussed. Finally, three numerical examples were given to illustrate the method.

This method combines qualitative evaluation with quantitative evaluation, and it is easy to calculate. The comparison rule based on expected value is equivalent to compare the average

scores of the evaluation objects. When the number of domain experts is large, the result is very reliable.

In addition to expected value criterion, we can propose some other comparison rules based on optimistic value, pessimistic value and so on, which will become our further work.

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