Research Article

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On the regularity of solutions to the plastoelasticity problem

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Abstract: We prove the local regularity of solutions to the problem of minimizing

$$\int_{\Omega} [L_{\infty}(\nabla v(x)) + f(x)v(x)] dx \quad \text{on } u^0 + W_0^{1,\infty}(\Omega),$$

where L_{∞} is either $\frac{1}{2}|\xi|^2$ for $|\xi| \leq 1$ and $+\infty$ for $|\xi| > 1$, or a more general convex, extended-valued function.

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1 Introduction

We shall consider the following problem P_{∞} : given $\Omega \subset \mathbb{R}^N$ open and bounded, $f \in L^2(\Omega)$ and $u^0 \in W^{1,\infty}(\Omega)$, minimize

$$I_{\infty}(v) = \int_{\Omega} [L_{\infty}(\nabla v(x)) + f(x)v(x)] dx \quad \text{on } u^0 + W_0^{1,\infty}(\Omega), \quad (1.1)$$

where

$$L_{\infty}(\xi) = l_{\infty}(|\xi|) = \begin{cases} \frac{1}{2}|\xi|^2 & \text{for } |\xi| \leq 1, \\ +\infty & \text{for } |\xi| > 1, \end{cases} \quad (1.2)$$

where $|\cdot|$ is the Euclidean norm. In the special case $f \equiv \mu > 0$, $\Omega \subset \mathbb{R}^2$, the solution u_{∞} represents the stress function in a torsion problem, with parameter μ , of an elastic bar with cross section Ω .

For the case of $\Omega \subset \mathbb{R}^2$ a disk of radius R about the origin, $f \equiv 1$ and $u^0 \equiv 0$, a solution for small R is $u_C(x) = \frac{1}{4}|x|^2 - \frac{1}{4}R^2$, while, as $R > 2$, it becomes

$$u_C(x) = \begin{cases} \frac{1}{4}|x|^2 + 1 - R & \text{for } |x| \leq 2, \\ |x| - R & \text{for } |x| \geq 2, \end{cases}$$

whose gradient is

$$\nabla u_C(x) = \begin{cases} \frac{1}{2}x & \text{for } |x| \leq 2, \\ \frac{x}{|x|} & \text{for } |x| \geq 2. \end{cases}$$

The sets $E = \{x \in \Omega : |\nabla u_{\infty}(x)| < 1\}$ and $P = \{x \in \Omega : |\nabla u_{\infty}(x)| = 1\}$ are called the elastic and plastic subsets of Ω , and in the example one can see the occurrence of P as R becomes larger.

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The investigation of the regularity properties of the solution, both for the original problem and for related problems expressed in the form of variational inequalities, has been the subject of several papers, [1–4, 6–8, 12] to name a few (the more complex problem of the regularity of the stress tensor has been investigated by Seregin; see [11] and the references therein). In the example presented above, we can notice that the gradient of the solution is continuous on Ω , but that the Hessian has a jump discontinuity at $|x| = 2$: in particular, this implies that the solution cannot possibly be more regular than being in $W^{2,2}(\Omega)$.

In [3], Brézis and Stampacchia proved that, for $u^0 \equiv 0$ and Ω either convex with $\partial\Omega$ Lipschitzian, or Ω arbitrary with $\partial\Omega \in C^2$, the solution u belongs to $W^{2,2}(\Omega)$. Let $d_\partial(x)$ be the distance of a point $x \in \Omega$ from $\partial\Omega$. In [2], Brézis and Sibony, for the case $u^0 \equiv 0$ and $f \equiv \mu$, succeeded in proving that the minimization problem (P_∞) is equivalent to the following problem: find $u \in \mathbb{K}_2 \doteq \{v \in W^{1,\infty} : |v(x)| \leq d_\partial(x) \text{ a.e. on } \Omega\}$ such that, for every $v \in \mathbb{K}_2$, we have

$$\int_{\Omega} \langle \nabla u(x), \nabla(u(x) - v(x)) \rangle dx + \int_{\Omega} f(x)(v(x) - u(x)) dx \geq 0,$$

i.e., they proved that the solution to (P_∞) solves a variational inequality *with an obstacle*. Hence, as one can see in the example above, the solution to the minimization problem for $f \equiv \mu$ is such that the plastic subset P consists of those $x \in \Omega$ where the solution agrees with the obstacle. Since this result, the papers on the regularity of the solutions focused on the regularity of the obstacle, be it the distance function d_∂ (whose regularity depends on the regularity assumptions on $\partial\Omega$) or a more general obstacle function ψ . Brézis and Kinderlehrer [1], and, independently, Frehse [6], did show that, under the condition that the obstacle ψ is $C^2(\overline{\Omega})$, the solution to a variational inequality with obstacle ψ is in $C^{1,1}(\Omega)$. Always for a variational inequality, the condition on the obstacle was relaxed to $\psi \in W^{2,\infty}(\overline{\Omega})$ in [7]. However, the regularity conditions imposed on the obstacle and, in particular, the regularity conditions on $\partial\Omega$, prevent the application of the regularity results in the case where Ω is less regular. In fact, in the simple case where $\Omega \subset \mathbb{R}^2$ is a square centered at the origin, the function distance to the boundary has a gradient that is discontinuous along the diagonal lines. A careful analysis of the regularity of the solution to problem (P_∞) , with $u^0 \equiv 0$ and $f \equiv \mu$, hence in a special case of our problem (P_∞) , and under the condition that $\Omega \subset \mathbb{R}^2$ is such that $\partial\Omega$ consists of finitely many C^3 curves (hence, covering the case of Ω being a square), was carried out by Caffarelli and Friedman in [4]. In particular, they did show that the set of points of discontinuity of ∇d_∂ is contained in the elastic part E of Ω .

Notice, however, that the equivalence of the problem of solving a variational inequality with an obstacle and the problem of minimizing (1.1) does not hold in the general case treated here, i.e., where $f \in L^2$ depends on x , not even in the case $u^0 \equiv 0$. In fact, consider the problem of minimizing (1.1) on $\Omega = (-\alpha, \alpha) \subset \mathbb{R}^1$, with $u^0 \equiv 0$ and $f(x) = \chi_{(-\alpha,0)}(x) - \chi_{(0,\alpha)}(x)$, i.e., the problem of minimizing

$$\int_{-\alpha}^{\alpha} \left[\frac{1}{2} (u'(x))^2 + f(x) \right] dx, \quad x(-\alpha) = 0, \quad x(\alpha) = 0,$$

subject to the condition $|x'| \leq 1$. One can see that, for α sufficiently large, on the interval $(-\alpha, 0)$ the solution to the minimization of (1.1) is

$$u_1(x) = \begin{cases} -(x + \alpha) & \text{for } x \in \left(-\alpha, -\frac{\alpha}{2} - 1\right), \\ \frac{1}{2} \left(x + \frac{\alpha}{2}\right)^2 - \frac{\alpha}{2} + \frac{1}{2} & \text{for } x \in \left(-\frac{\alpha}{2} - 1, -\frac{\alpha}{2} + 1\right), \\ x & \text{for } x \in \left(-\frac{\alpha}{2} + 1, 0\right). \end{cases}$$

Consider instead the problem of minimizing the same integral, with the same boundary conditions, but subject to the obstacle $|u(x)| \leq d_\partial(x)$, i.e., subject to

$$u(x) \leq \inf\{x + \alpha, x - \alpha\} \quad \text{and} \quad u(x) \geq \sup\{-(x + \alpha), -(x - \alpha)\}. \quad (1.3)$$

On the interval $(-\alpha, 0)$ the solution is not anymore symmetric with respect to the point $x = -\frac{\alpha}{2}$ and it is given by

$$u_2(x) = \begin{cases} -(x + \alpha) & \text{for } x \in (-\alpha, -\sqrt{2\alpha}), \\ \frac{1}{2}(x - (1 - \sqrt{2\alpha}))^2 - \alpha - \frac{1}{2} + \sqrt{2\alpha} & \text{for } x \in (-\sqrt{2\alpha}, 0). \end{cases}$$

We have that v_2 is a solution to the variational inequality problem with obstacle (1.3), so that the two problems are not equivalent.

In Section 3 of the present paper, for $\Omega \subset \mathbb{R}^N$ open and bounded, $f \in L^2(\Omega)$ and u^0 Lipschitzian, we show that the solution u_∞ to problem (P_∞) is in $W_{\text{loc}}^{2,2}(\Omega)$. We stress the point that in the conditions for our result there is neither a mention of $\partial\Omega$ whatsoever nor of any other condition on Ω , besides it being open and bounded: the regularity to u_∞ we prove comes entirely from it being a solution to a variational problem.

Notice that our result, when applied to the case of elastoplasticity, i.e., when $f \equiv \mu$ and $u^0 \equiv 0$, but without any regularity imposed on $\partial\Omega$, contains the main property mentioned before, i.e., that the points of discontinuity of ∇d_δ are contained in the elastic part E of Ω : in fact, whenever ∇d_δ is discontinuous, d_δ is not in $W_{\text{loc}}^{2,2}(\Omega)$ and hence, by the present result, it cannot coincide (locally) with u_∞ , so that the points of discontinuity of ∇d_δ must be in E .

The main technical difficulty to our approach comes from the fact that we are minimizing a Lagrangian that is an extended-valued convex function. In fact, although some regularity results for variational problems without upper bounds on the growth of l are known, [9, 10], we believe that there are no previous examples of a regularity result of this kind, proved for a variational problem having an extended-valued Lagrangian, a case that demands a different approach.

Our main purpose is to prove the following theorem, where we use the notation

$$\|v\|_{1,\infty} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|}.$$

Theorem 1.1. *Let Ω be open and bounded, let L_∞ as in (1.2), let $f \in L^2(\Omega)$, let u^0 be such that $\|u^0\|_{1,\infty} = 1 - d_\infty$ with $d_\infty > 0$, and let $u_\infty \in W^{1,\infty}(\Omega)$ be the solution to problem (P_∞) . Then $u_\infty \in W_{\text{loc}}^{2,2}(\Omega)$.*

In the case that we exclude, i.e., $d_\infty = 0$, hence when u^0 satisfies the condition $\|\nabla u^0\|_\infty \leq 1$, we might have that u^0 is the only function in $u^0 + W_0^{1,\infty}(\Omega)$ satisfying the condition $|\nabla u| \leq 1$. If this is the case, there are no non-trivial variations, a fact that prevents any regularity proof.

The Lagrangian whose restriction to the unit ball we are minimizing needs not be $\frac{1}{2}|\xi|^2$: a general case is presented in Section 4.

2 Notations and preliminary results

For a smooth rotationally symmetric Lagrangian L , i.e., of the form $L(\xi) = l(|\xi|)$ with l differentiable, we have

$$\nabla L(\xi) = l'(|\xi|) \frac{\xi}{|\xi|}.$$

Whenever the assumptions on l imply that $l''(0)$ exists, we shall consider the map $g : t \rightarrow l'(t)/t$ to be extended at 0 as $l''(0)$.

The Hessian matrix of a twice differentiable function $\xi \rightarrow l(|\xi|)$ is

$$H_l(\xi) = l''(|\xi|) \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|} + \frac{l'(|\xi|)}{|\xi|} \left[I - \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|} \right].$$

We set $\mathbb{K} = \{v \in u^0 + W_0^{1,\infty}(\Omega) : \|v\|_{1,\infty} \leq 1\}$. For a function $f : \Omega \rightarrow \mathbb{R}$ and for $\omega \Subset \Omega$, we set

$$\delta_i^h f(x) = \frac{f(x + he_i) - f(x)}{h},$$

defined for $x \in \omega$ and for a suitably small h .

Proposition 2.1. *There exists K_∞ such that, for any $v \in \mathbb{K}$, we have $\|v\|_{L^2} \leq K_\infty$.*

Proposition 2.2. *The function $u_\infty \in \mathbb{K}$ is the solution to problem (P_∞) if and only if for every $v \in \mathbb{K}$ we have*

$$\int_{\Omega} [\langle \nabla u_\infty(x), \nabla v(x) - \nabla u_\infty(x) \rangle + f(x)(v(x) - u_\infty(x))] dx \geq 0. \quad (2.1)$$

Proof. For the (extended-valued) convex function L_∞ , we have that $\xi \in \partial L_\infty(\xi)$ for every ξ such that $|\xi| \leq 1$ and hence, for $v \in \mathbb{K}$,

$$\int_{\Omega} [L_\infty(\nabla v) - L_\infty(\nabla u_\infty) + f(v - u_\infty)] dx \geq \int_{\Omega} [\langle \nabla u_\infty, \nabla v - \nabla u_\infty \rangle + f(v - u_\infty)] dx \geq 0.$$

Conversely, let $u_\infty \in \mathbb{K}$ be the solution to (P_∞) and fix $v \in \mathbb{K}$. Then we have

$$\frac{1}{\varepsilon} \int_{\Omega} [L_\infty(\nabla u_\infty + \varepsilon \nabla(v - u_\infty)) - L_\infty(\nabla u_\infty)] dx + \int_{\Omega} f(v - u_\infty) dx \geq 0. \quad (2.2)$$

Since

$$\varepsilon \rightarrow \frac{1}{\varepsilon} (L_\infty(\nabla u_\infty + \varepsilon \nabla(v - u_\infty)) - L_\infty(\nabla u_\infty))$$

converges pointwise to $\langle \nabla u_\infty(x), \nabla(v - u_\infty)(x) \rangle$ and in absolute value is bounded above by 2, we obtain

$$\int_{\Omega} [\langle \nabla L_\infty(\nabla u_\infty(x)), \nabla(v - u_\infty)(x) \rangle + f(v - u_\infty)] dx \geq 0, \quad (2.3)$$

as desired. $\square$

We shall approximate the function l_∞ by convex functions that are C^2 except at a point different from zero.

Lemma 2.3. *Let $l : \mathbb{R} \rightarrow \mathbb{R}^+$ be convex, symmetric, $C^1(\mathbb{R})$, and C^2 with the exception of finitely many points t_j , with $t_j \neq 0$, and assume that*

$$\limsup_{z \rightarrow 0} \left| \frac{l'(z)}{z} - l''(z) \right| < +\infty.$$

Moreover, let there exist K such that, whenever l'' exists, we have $l'' \leq K$. Let $u \in W_{\text{loc}}^{2,2}(\Omega)$. Then, for $s = 1, \dots, N$,

$$\frac{l'(|\nabla u|)}{|\nabla u|} u_{x_s} \in W_{\text{loc}}^{1,2}(\Omega),$$

and we have

$$\frac{d}{dx_i} \frac{l'(|\nabla u|)}{|\nabla u|} u_{x_s} = \left(l''(|\nabla u|) - \frac{l'(|\nabla u|)}{|\nabla u|} \right) \left\langle \frac{\nabla u}{|\nabla u|}, \nabla u_{x_i} \right\rangle \frac{u_{x_s}}{|\nabla u|} + \frac{l'(|\nabla u|)}{|\nabla u|} u_{x_s x_i}.$$

Proof. Under the assumptions, the function $g(t) = l'(t)/t$ (extended as $l''(0)$ at 0) is globally Lipschitzian and differentiable for $t \notin \{0, t_j : 1 \leq j \leq r\}$, with derivative $g'(t) = \frac{1}{t}(l''(t) - l'(t)/t)$. The map $x \rightarrow |\nabla u(x)|$ is in $W_{\text{loc}}^{1,2}(\Omega)$ and

$$\frac{d}{dx_i} |\nabla u| = \left\langle \frac{\nabla u}{|\nabla u|}, \nabla u_{x_i} \right\rangle.$$

Moreover, $\frac{d}{dx_i} |\nabla u| = 0$ a.e. on $\{x : |\nabla u(x)| \in \{0, t_i : 1 \leq j \leq r\}\}$, so that $\frac{d}{dx_i} g(|\nabla u(x)|) = 0$ a.e. on the same set. Then the map $x \rightarrow g(|\nabla u(x)|)$ is in $W_{\text{loc}}^{1,2}(\Omega)$ and

$$\frac{d}{dx_i} g(|\nabla u|) = \frac{1}{|\nabla u|} \left(l''(|\nabla u|) - \frac{l'(|\nabla u|)}{|\nabla u|} \right) \left\langle \frac{\nabla u}{|\nabla u|}, \nabla u_{x_i} \right\rangle \chi_{\{x : |\nabla u(x)| \notin \{0; t_j\}\}} + 0 \chi_{\{x : |\nabla u(x)| \in \{0; t_j\}\}}.$$

Then we have

$$\begin{aligned} \frac{d}{dx_i} \frac{l'(|\nabla u|)}{|\nabla u|} u_{x_s} &= \left(\frac{d}{dx_i} g(|\nabla u|) \right) u_{x_s} + g(|\nabla u|) u_{x_s x_i} \\ &= \frac{u_{x_s}}{|\nabla u|} \left(l''(|\nabla u|) - \frac{l'(|\nabla u|)}{|\nabla u|} \right) \left\langle \frac{\nabla u}{|\nabla u|}, \nabla u_{x_i} \right\rangle \chi_{\{x : |\nabla u(x)| \notin \{0; t_j\}\}} \\ &\quad + 0 \chi_{\{x : |\nabla u(x)| \in \{0; t_j\}\}} + \frac{l'(|\nabla u|)}{|\nabla u|} u_{x_s x_i}, \end{aligned} \quad (2.4)$$

as desired. $\square$

Besides (P_∞) , the problem of minimizing (1.1), we shall also consider solutions to the problem (P_l) of minimizing

$$\int_{\Omega} [l(|\nabla v(x)|) + f(x)u(x)] dx \quad \text{on } u^0 + W_0^{1,\infty}(\Omega)$$

for a convex function l , satisfying the assumptions of Lemma 2.3, and set $L(\xi) = l(|\xi|)$.

Lemma 2.4. *Let l be as in Lemma 2.3, let Ω and f be as in Theorem 1.1, let u be a solution to problem P_l , and let $\phi \in W^{1,2}(\Omega)$ with support compactly contained in Ω . Then, for $s = 1, \dots, N$, we have*

$$\int_{\Omega} \left\langle \frac{d}{dx_s} \nabla L(\nabla u), \nabla \phi \right\rangle = \int_{\Omega} f \left(\frac{d}{dx_s} \phi \right).$$

Proof. Under the conditions of the lemma, a solution u is in $W_{\text{loc}}^{2,2}(\Omega)$ (see, e.g., [5]) and satisfies the Euler–Lagrange condition: for every $\phi \in W^{1,2}(\Omega)$ with support compactly contained in Ω , we have

$$\int_{\Omega} [\langle \nabla L(\nabla u), \nabla \phi \rangle + f\phi] dx = 0.$$

Fix ϕ ; then $\delta_s^h \phi$ is in $W^{1,2}(\Omega)$ with support compactly contained in Ω and we obtain

$$-\int_{\Omega} f \delta_s^h \phi = \int_{\Omega} \langle \nabla L(\nabla u), \nabla \delta_s^h \phi \rangle = -\int_{\Omega} \langle \delta_s^h \nabla L(\nabla u), \nabla \phi \rangle.$$

From Lemma 2.3 we infer that $\nabla L(\nabla u) \in W_{\text{loc}}^{1,2}(\Omega)$; then, $\delta_s^h \nabla L(\nabla u) \rightarrow \frac{d}{dx_s} \nabla L(\nabla u)$ in $L^2(\omega)$. □

3 Proof of Theorem 1.1

Proof. (a) Let l_α be the inf-convolution of l_∞ and of $t \rightarrow \frac{1}{2}at^2$, i.e.,

$$l_\alpha(t) = \inf_{x \in \mathbb{R}} \left\{ \frac{1}{2} \alpha (t - x)^2 + l_\infty(x) \right\}.$$

A computation shows that, for $t \geq 0$,

$$l_\alpha(t) = \frac{1}{2} \frac{\alpha}{\alpha + 1} t^2 \chi_{\{t \leq \frac{\alpha+1}{\alpha}\}} + \frac{1}{2} (\alpha(t - 1)^2 + 1) \chi_{\{t > \frac{\alpha+1}{\alpha}\}},$$

so that

$$l'_\alpha(t) = \frac{\alpha}{\alpha + 1} t \chi_{\{t \leq \frac{\alpha+1}{\alpha}\}} + \alpha(t - 1) \chi_{\{t > \frac{\alpha+1}{\alpha}\}} \quad \text{and} \quad l''_\alpha(t) = \frac{\alpha}{\alpha + 1} \chi_{\{t < \frac{\alpha+1}{\alpha}\}} + \alpha \chi_{\{t > \frac{\alpha+1}{\alpha}\}}.$$

We have that l is C^2 except at the point $t = \frac{\alpha+1}{\alpha}$. Moreover, we have that

$$\frac{l'_\alpha(t)}{t} = \frac{\alpha}{\alpha + 1} \quad \text{for } t \leq \frac{\alpha + 1}{\alpha} \quad \text{and} \quad \frac{l'_\alpha(t)}{t} \geq \frac{\alpha}{\alpha + 1} \quad \text{for } t \geq \frac{\alpha + 1}{\alpha}.$$

Taking α large, we shall assume that $l'_\alpha(t)/t \geq \frac{1}{2}$ and that $l_\alpha(t) \geq \frac{1}{4}t^2$. We shall set $L_\alpha(\xi) = l_\alpha(|\xi|)$. In particular, Lemma 2.3 applies to l_α .

(b) Let u_α be the solution to the problem of minimizing

$$I_\alpha(v) = \int_{\Omega} [L_\alpha(\nabla v) + fv] dx$$

on $u^0 + W_0^{1,2}(\Omega)$. From the properties of l_α we obtain that $u_\alpha \in W_{\text{loc}}^{2,2}(\Omega)$ (see [5]). Moreover, since $l_\alpha \leq l_\infty$, recalling that u_∞ is the solution to problem (P_∞) and that $l_\infty \geq l_\alpha$, we have

$$I_\infty(u_\infty) \geq I_\alpha(u_\infty) \geq I_\alpha(u_\alpha) \geq \int_{\Omega} \left[\frac{1}{4} |\nabla u_\alpha|^2 + f u_\alpha \right] dx. \quad (3.1)$$

To estimate $\int_{\Omega} |\nabla u_{\alpha}|^2 dx$ call P the Poincaré constant in $W_0^{1,2}(\Omega)$ (we assume $P \geq 1$); from (3.1) we have

$$\begin{aligned} \int_{\Omega} |\nabla u_{\alpha}|^2 dx &\leq 4\{I_{\infty}(u_{\infty}) + \|f\|_{L^2}(P(\|\nabla u_{\alpha}\|_{L^2} + \|\nabla u_{\infty}\|_{L^2}) + \|u_{\infty}\|_{L^2})\} \\ &\leq 4\{I_{\infty}(u_{\infty}) + \|f\|_{L^2}P(\|\nabla u_{\alpha}\|_{L^2} + \|u_{\infty}\|_{W^{1,2}})\}, \end{aligned}$$

and hence

$$\int_{\Omega} |\nabla u_{\alpha}|^2 dx \leq 4\{I_{\infty}(u_{\infty}) + 2P^2\|f\|_{L^2}^2 + P\|f\|_{L^2}\|u_{\infty}\|_{W^{1,2}}\}. \quad (3.2)$$

We also obtain

$$\int_{\Omega} |u_{\alpha}|^2 \leq \int_{\Omega} |u_{\infty}|^2 + P^2 \left[2 \int_{\Omega} |\nabla u_{\alpha}|^2 + 2 \int_{\Omega} |\nabla u_{\infty}|^2 \right].$$

Hence, there exists a constant K , not depending on α , such that

$$\|\nabla u_{\alpha}\|_{L^2(\Omega)} \leq K, \quad \|u_{\alpha}\|_{L^2(\Omega)} \leq K. \quad (3.3)$$

Then we can assume the existence of a function $u \in u^0 + W_0^{1,2}(\Omega)$ and of a sequence α_n such that $u_{\alpha_n} \rightarrow u$ in $L^2(\Omega)$ and $\nabla u_{\alpha_n} \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$.

In addition, from (3.1) we also have

$$\int_{\Omega} l_{\alpha}(|\nabla u_{\alpha}|) dx \leq \|f\|K + I_{\infty}(u_{\infty}) = K_1. \quad (3.4)$$

(c) Fix x^0 and let δ^0 be such that $B(x^0, 2\delta^0) \subset \Omega$. Let $\eta \in C_0^{\infty}(B(x^0, 2\delta^0))$ be such that $0 \leq \eta \leq 1$ and that $\eta(x) = 1$ for $x \in B(x^0, \delta^0)$. Set $\gamma^s = (u_{\alpha})_{x_s}$; then the map $\phi = \eta^2 \cdot \gamma^s$ is in $W^{1,2}(B(x^0, 2\delta^0))$ and we obtain $\nabla \phi = 2\eta \nabla \eta \gamma^s + \eta^2 \nabla \gamma^s$.

From Lemma 2.4 we infer that, for every s ,

$$\int_{B(x^0, 2\delta^0)} \sum_i \left(\frac{d}{dx_s} \left(\frac{l'_{\alpha}(|\nabla(u_{\alpha})|)}{|\nabla u|} (u_{\alpha})_{x_i} \right) \right) \left(\eta^2 \frac{d}{dx_i} \gamma^s + 2\eta \eta_{x_i} \gamma^s \right) dx = G_s^{\alpha}, \quad (3.5)$$

where $G_s^{\alpha} = \int_{B(x^0, 2\delta^0)} f(2\eta \eta_{x_s} (u_{\alpha})_{x_s} + \eta^2 (u_{\alpha})_{x_s x_s}) dx$. Summing the previous equations over s , we obtain

$$\begin{aligned} \int_{B(x^0, 2\delta^0)} \sum_{i,s} \left(\frac{d}{dx_s} \left(\frac{l'_{\alpha}(|\nabla(u_{\alpha})|)}{|\nabla(u_{\alpha})|} (u_{\alpha})_{x_i} \right) \right) \eta^2 \frac{d}{dx_i} \gamma^s \\ = - \int_{B(x^0, 2\delta^0)} \sum_{i,s} \left(\frac{d}{dx_s} \left(\frac{l'_{\alpha}(|\nabla(u_{\alpha})|)}{|\nabla(u_{\alpha})|} (u_{\alpha})_{x_i} \right) \right) 2\eta \eta_{x_i} \gamma^s + \sum_s G_s^{\alpha}. \end{aligned} \quad (3.6)$$

(d) Set H_{α} to be the Hessian matrix of (u_{α}) and notice that $\nabla(|\nabla(u_{\alpha})|) = H \frac{\nabla u_{\alpha}}{|\nabla(u_{\alpha})|} = 0$ for $x \in \{|\nabla u_{\alpha}(x)| = 0\}$. Then, from (2.4), the left-hand side of (3.6) can be written as

$$\int_{B(x^0, 2\delta^0)} \eta^2 \left\{ \left(l''_{\alpha}(|\nabla(u_{\alpha})|) - \frac{l'_{\alpha}(|\nabla(u_{\alpha})|)}{|\nabla u_{\alpha}|} \right) \left| H_{\alpha} \frac{\nabla u_{\alpha}}{|\nabla(u_{\alpha})|} \right|^2 + \frac{l'_{\alpha}(|\nabla u_{\alpha}|)}{|\nabla u_{\alpha}|} |H_{\alpha}|^2 \right\}.$$

To discuss the right-hand side of (3.6), notice that, because of (2.4), the identity

$$\frac{d}{dx_s} \left(\frac{l'_{\alpha}(|\nabla(u_{\alpha})|)}{|\nabla(u_{\alpha})|} (u_{\alpha})_{x_i} u_{x_s} \right) = (u_{\alpha})_{x_s} \frac{d}{dx_s} \left(\frac{l'_{\alpha}(|\nabla u_{\alpha}|)}{|\nabla u_{\alpha}|} (u_{\alpha})_{x_i} \right) + \frac{l'_{\alpha}(|\nabla(u_{\alpha})|)}{|\nabla u_{\alpha}|} (u_{\alpha})_{x_i} (u_{\alpha})_{x_s x_s}$$

holds. Hence, the right-hand side of (3.6) becomes

$$- \sum_{i,s} \int_{B(x^0, 2\delta^0)} 2\eta \eta_{x_i} \left[\frac{d}{dx_s} \left(\frac{l'_{\alpha}(|\nabla(u_{\alpha})|)}{|\nabla u_{\alpha}|} (u_{\alpha})_{x_i} (u_{\alpha})_{x_s} \right) - \frac{l'_{\alpha}(|\nabla u_{\alpha}|)}{|\nabla u_{\alpha}|} (u_{\alpha})_{x_i} (u_{\alpha})_{x_s x_s} \right] + \sum_s G_s^{\alpha}, \quad (3.7)$$

and (3.6) becomes, by setting $(\Lambda)_{i,s} = \frac{d}{dx_s} \frac{d}{dx_i} \eta^2$,

$$\begin{aligned} & \int_{B(x^0, 2\delta^0)} \eta^2 \left\{ \left(l''_\alpha(|\nabla(u_\alpha)|) - \frac{l'_\alpha(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} \right) \left| H_\alpha \frac{\nabla(u_\alpha)}{|\nabla(u_\alpha)|} \right|^2 + \frac{l'_\alpha(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} |H_\alpha|^2 \right\} \\ &= \int_{B(x^0, 2\delta^0)} \frac{l'_\alpha(|\nabla u_\alpha|)}{|\nabla u_\alpha|} \nabla u_\alpha \Lambda \nabla u_\alpha + 2\eta \left\langle \nabla \eta, \frac{\nabla u_\alpha}{|\nabla u_\alpha|} \right\rangle \frac{l'_\alpha(|\nabla u_\alpha|)}{|\nabla u_\alpha|} \Delta u_\alpha + \sum_s G_s^\alpha. \end{aligned} \quad (3.8)$$

In particular, set $\tilde{\lambda} = |\Lambda|$, $D = \sup |\nabla \eta|$ and write

$$\sum_s G_s^\alpha \leq 2 \|f\|_{L^2} D \|\nabla u_\alpha\|_{L^2} + \int_\Omega \left[\frac{1}{8} \eta^2 |H_\alpha|^2 + 4\eta^2 |f|^2 \right]$$

to obtain

$$\begin{aligned} \int_{B(x^0, 2\delta^0)} \eta^2 \frac{l'_\alpha(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} |H_\alpha|^2 dx &\leq \int_{B(x^0, 2\delta^0)} \left[\tilde{\lambda} l'_\alpha(|\nabla(u_\alpha)|) |\nabla(u_\alpha)| + \eta^2 \frac{1}{2} \frac{l'_\alpha(|\nabla u_\alpha|)}{|\nabla u_\alpha|} |H_\alpha|^2 + 2D^2 \frac{l'_\alpha(|\nabla u_\alpha|)}{|\nabla u_\alpha|} \right] dx \\ &\quad + 2D \|f\|_{L^2} \|\nabla u_\alpha\|_{L^2} + \int_{B(x^0, 2\delta^0)} \left[\frac{1}{8} \eta^2 |H_\alpha|^2 + 4\eta^2 |f|^2 \right] dx. \end{aligned} \quad (3.9)$$

For every $t \in \mathbb{R}$, we have that

$$\frac{l'_\alpha(|t|)}{|t|} \leq \frac{\alpha}{\alpha+1} + l'_\alpha(|t|)|t|.$$

We obtain

$$\begin{aligned} \frac{1}{2} \int_{B(x^0, 2\delta^0)} \eta^2 \frac{l'_\alpha(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} |H_\alpha|^2 dx &\leq \int_{B(x^0, 2\delta^0)} \left[\frac{2D^2 \alpha}{\alpha+1} + (l'_\alpha(|\nabla(u_\alpha)|) |\nabla(u_\alpha)|) (\tilde{\lambda} + 2D^2) \right] dx + 2D \|f\|_{L^2} \|\nabla u_\alpha\|_{L^2} \\ &\quad + \int_{B(x^0, 2\delta^0)} \left[\frac{1}{8} \eta^2 |H_\alpha|^2 + 4\eta^2 |f|^2 \right] dx, \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{8} \int_{B(x^0, 2\delta^0)} \eta^2 |H_\alpha|^2 dx &\leq \int_{B(x^0, 2\delta^0)} \left[\frac{2D^2 \alpha}{\alpha+1} + (l'_\alpha(|\nabla(u_\alpha)|) |\nabla(u_\alpha)|) (\tilde{\lambda} + 2D^2) \right] dx + 2D \|f\|_{L^2} \|\nabla u_\alpha\|_{L^2} \\ &\quad + \int_{B(x^0, 2\delta^0)} 4\eta^2 |f|^2 dx. \end{aligned}$$

Since $\eta = 1$ on $B(x^0, \delta^0)$, recalling (3.2), we obtain that, for constants k_1 and k_2 not depending on α ,

$$\int_{B(x^0, \delta^0)} |H_\alpha|^2 dx \leq k_1 \int_{B(x^0, 2\delta^0)} l'_\alpha(|\nabla u_\alpha|) |\nabla u_\alpha| dx + k_2. \quad (3.10)$$

(e) Claim: Given $x^* \in \Omega$ such that $B(x^*, \delta^*) \subset \Omega$, there exists K^* , depending on u^0 and x^* , but not on α , such that

$$\int_{B(x^*, d_\infty \delta^*)} l'(|\nabla u_\alpha|) |\nabla u_\alpha| dx \leq K^*.$$

To prove this claim, set

$$v(x) = \begin{cases} u^0(x^*) & \text{if } |x - x^*| \leq d_\infty \delta^*, \\ u^0 \left(x^* + \frac{x - x^*}{|x - x^*|(1 - d_\infty)} (|x - x^*| - d_\infty \delta^*) \right) & \text{if } d_\infty \delta^* \leq |x - x^*| \leq \delta^*, \\ u^0(x) & \text{if } x \in \{\Omega \setminus \overline{B}(x^*, \delta^*)\}. \end{cases} \quad (3.11)$$

We have that $v \in u^0 + W_0^{1,\infty}(\Omega)$ and that $\nabla v \equiv 0$ on $B(x^*, d_\infty \delta^*)$. Set

$$\xi(x) = x^* + \frac{x - x^*}{|x - x^*|(1 - d_\infty)}(|x - x^*| - d_\infty \delta^*).$$

Then, for $d_\infty \delta^* \leq |x - x^*| \leq \delta^*$, we have

$$\nabla v(x) = \frac{1}{1 - d_\infty} \left[\left(1 - \frac{d_\infty \delta^*}{|x - x^*|}\right) \nabla u^0(\xi(x)) + \left\langle \nabla u^0(\xi(x)), \frac{x - x^*}{|x - x^*|} \right\rangle \frac{d_\infty \delta^*}{|x - x^*|} \frac{x - x^*}{|x - x^*|} \right],$$

so that

$$|\nabla v(x)| = \left| \nabla u^0 \left(x^* + \frac{x - x^*}{|x - x^*|(1 - d_\infty)} (|x - x^*| - d_\infty \delta^*) \right) \frac{1}{1 - d_\infty} \right| \leq 1.$$

Hence, $\|v\|_{1,\infty} \leq 1$ and $\|v\|_{L^2} \leq K_\infty$.

Since $\varepsilon \rightarrow \frac{1}{\varepsilon} (L_\alpha(\nabla u_\alpha(x) + \varepsilon \nabla v(x) - \nabla u_\alpha(x)) - L_\alpha(\nabla u_\alpha(x)))^+$ is non-decreasing, we have

$$\frac{1}{\varepsilon} (L_\alpha(\nabla u_\alpha(x) + \varepsilon \nabla v(x) - \nabla u_\alpha(x)) - L_\alpha(\nabla u_\alpha(x)))^+ \leq (L_\alpha(\nabla v(x)) - L_\alpha(\nabla u_\alpha(x)))^+,$$

and hence

$$\int_{\Omega} \frac{1}{\varepsilon} (L_\alpha(\nabla u_\alpha(x) + \varepsilon \nabla v(x) - \nabla u_\alpha(x)) - L_\alpha(\nabla u_\alpha(x)))^+ \leq \frac{1}{2} |\Omega| + K_1.$$

Since u_α is a minimum, we have

$$\frac{1}{\varepsilon} \int_{\Omega} [L_\alpha(\nabla u_\alpha + \varepsilon \nabla(v - u_\alpha)) - L_\alpha(\nabla u_\alpha)] dx + \int_{\Omega} f(v - u_\alpha) dx \geq 0, \quad (3.12)$$

so that

$$\begin{aligned} \int_{\Omega} \frac{(L_\alpha(\nabla u_\alpha + \varepsilon \nabla(v - u_\alpha)) - L_\alpha(\nabla u_\alpha))^-}{\varepsilon} dx &\leq \int_{\Omega} \left[\frac{(L_\alpha(\nabla u_\alpha + \varepsilon \nabla(v - u_\alpha)) - L_\alpha(\nabla u_\alpha))^+}{\varepsilon} + f(v - u_\alpha) \right] dx \\ &\leq \frac{1}{2} |\Omega| + K_1 + \int_{\Omega} f(v - u_\alpha) dx. \end{aligned}$$

By dominated convergence,

$$\begin{aligned} &\int_{\Omega} (\langle \nabla L_\alpha(\nabla u_\alpha(x)), \nabla(v - u_\alpha)(x) \rangle)^+ dx \\ &= \lim_{\varepsilon_n \rightarrow 0^+} \int_{\Omega} \frac{1}{\varepsilon_n} (L_\alpha(\nabla u_\alpha(x) + \varepsilon_n \nabla v(x) - \nabla u_\alpha(x)) - L_\alpha(\nabla u_\alpha(x)))^+ dx \end{aligned} \quad (3.13)$$

while, by Fatou's Lemma,

$$\int_{\Omega} (\langle \nabla L_\alpha(\nabla u_\alpha), \nabla(v - u_\alpha) \rangle)^- \leq \liminf \int_{\Omega} \frac{1}{\varepsilon_n} (L_\alpha(\nabla u_\alpha(x) + \varepsilon_n \nabla v(x) - \nabla u_\alpha(x)) - L_\alpha(\nabla u_\alpha(x)))^- dx.$$

Hence,

$$\int_{\Omega} (\langle \nabla L_\alpha(\nabla u_\alpha), \nabla(v - u_\alpha) \rangle)^- \leq \frac{1}{2} |\Omega| + K_1 + \int_{\Omega} f(v - u_\alpha).$$

Then

$$\begin{aligned} \int_{\Omega} |\langle \nabla L_\alpha(\nabla u_\alpha(x)), \nabla(v - u_\alpha)(x) \rangle| dx &= \int_{\Omega} (\langle \nabla L_\alpha(\nabla u_\alpha), \nabla(v - u_\alpha) \rangle)^- dx + \int_{\Omega} (\langle \nabla L_\alpha(\nabla u_\alpha(x)), \nabla(v - u_\alpha)(x) \rangle)^+ dx \\ &\leq |\Omega| + 2K_1 + \int_{\Omega} f(v - u_\alpha) \\ &\leq |\Omega| + 2K_1 + \|f\|_{L^2} (K_\infty + K) \doteq K^*, \end{aligned}$$

and, since $\nabla v = 0$ on $B(x^0, d_\infty \delta^*)$, in particular we obtain

$$\int_{B(x^0, d_\infty \delta^*)} l'_\alpha(|\nabla u_\alpha|)|\nabla u_\alpha| \leq K^*,$$

thus proving the claim.

(f) By a covering argument, given $\omega \in \Omega$, there exists a constant, independent of α , which bounds $\int_\omega l'_\alpha(|\nabla u_\alpha|)|\nabla u_\alpha|$. In particular, there exists K^0 such that

$$\int_{B(x^0, 2\delta^0)} l'_\alpha(|\nabla u_\alpha|)|\nabla u_\alpha| dx \leq K^0.$$

Then from (3.10) we obtain $\int_{B(x^0, \delta^0)} |H_\alpha|^2 dx \leq k_1 K^0 + k_2$. Hence, again by a covering argument, given $\omega \in \Omega$, there exists a constant K_ω , independent of α , such that

$$\int_\omega |H_\alpha|^2 dx \leq K_\omega \quad \text{and} \quad \int_\omega l'_\alpha(|\nabla u_\alpha|)|\nabla u_\alpha| dx \leq K_\omega. \quad (3.14)$$

In particular, the previous estimate implies that the family of the restrictions to ω of the maps ∇u_{α_n} converges to ∇u in $L^2(\omega)$ and that u is in $W^{2,2}(\omega)$. With ω being arbitrary, we obtain that $u \in W_{\text{loc}}^{2,2}(\Omega)$. The remainder of the proof hence consists in showing that $u = u_\infty$.

(g) First, we claim that $|\{x \in \Omega : |\nabla u(x)| > 1\}| = 0$. Otherwise, there exists a subset of Ω of positive measure such that $|\nabla u| > 1$ on this subset. By taking a sequence of compact sets ω_ν invading Ω , we infer that there exists an $\omega \in \Omega$, an $\varepsilon > 0$ and a set $\omega_\varepsilon \subset \omega$ with $|\omega_\varepsilon| \geq \varepsilon$, such that $|\nabla u| \geq 1 + \varepsilon$ on ω_ε . Then the restrictions to ω_ε of ∇u_{α_n} converge to ∇u in $L^2(\omega_\varepsilon)$. Hence, there exists a set $\omega'_\varepsilon \subset \omega$, with $|\omega'_\varepsilon| \geq \frac{1}{2}\varepsilon$, such that, for n sufficiently large, we have $|\nabla u_{\alpha_n}(x)| \geq 1 + \frac{1}{2}\varepsilon$ for $x \in \omega'_\varepsilon$. Hence, for $x \in \omega'_\varepsilon$ and n large, we have

$$l'_{\alpha_n}(|\nabla u_{\alpha_n}(x)|)|\nabla u_{\alpha_n}(x)| = \alpha_n(|\nabla u_{\alpha_n}(x)| - 1)|\nabla u_{\alpha_n}(x)| \geq \alpha_n \frac{1}{2} \varepsilon \left(1 + \frac{1}{2} \varepsilon\right),$$

and hence

$$\int_{\omega_{\alpha_n}} l'_{\alpha_n}(|\nabla u_{\alpha_n}|)|\nabla u_{\alpha_n}| dx \geq \left(1 + \frac{1}{2} \varepsilon\right) \frac{1}{4} \varepsilon^2 \alpha_n,$$

a contradiction to (3.14). This proves that $|\{x \in \Omega : |\nabla u(x)| > 1\}| = 0$.

(h) Take again a sequence of compact sets ω_ν invading Ω to show the existence of a subsequence $\alpha_{n'}$ such that $\nabla u_{\alpha_{n'}}$ converges to ∇u pointwise a.e. in Ω . Fix x , let (α''_ν) be any subsequence of $(\alpha_{n'})$ and assume that $l_{\alpha''_\nu}(|\nabla u_{\alpha''_\nu}(x)|) \rightarrow z(x)$. When $z(x) > \frac{1}{2}$, since

$$\sup_{|t| \leq \frac{\alpha+1}{\alpha}} \{l_\alpha(t)\} = \frac{1}{2} \frac{\alpha+1}{\alpha} \rightarrow \frac{1}{2},$$

we must have that, definitely, $|\nabla u_{\alpha''_\nu}(x)| > 1$, so that $|\nabla u(x)| = 1$ and $z(x) > \frac{1}{2} = l_\infty(|\nabla u(x)|)$. When $z(x) < \frac{1}{2}$, we must have that, definitely, $|\nabla u_{\alpha''_\nu}(x)| < 1$, so that $|\nabla u_{\alpha''_\nu}(x)| \rightarrow |\nabla u(x)| \leq 1$ and the uniform convergence of $\frac{1}{2} \frac{\alpha}{\alpha+1} (\cdot)^2$ to $\frac{1}{2} (\cdot)^2$ on $[-1, 1]$ implies that

$$l_{\alpha''_\nu}(|\nabla u_{\alpha''_\nu}(x)|) \rightarrow \frac{1}{2} (|\nabla u|)^2 = l_\infty(|\nabla u|).$$

Consider the case $z(x) = \frac{1}{2}$. For ε arbitrary, we must have that, definitely, $|\nabla u_{\alpha''_\nu}(x)| \geq 1 - \varepsilon$, hence $|\nabla u(x)| = 1$, so that $s(x) = \frac{1}{2} = l_\infty(|\nabla u(x)|)$. Hence, in any case, we have that $z(x) \geq l_\infty(|\nabla u(x)|)$. The arbitrariness of (α''_ν) then implies that

$$\liminf l_{\alpha_{\nu'}}(|\nabla u_{\alpha_{\nu'}}(x)|) \geq l_\infty(|\nabla u(x)|).$$

On the other hand, we have $\int_{\Omega} f(x)u(x) dx = \lim \int_{\Omega} f(x)u_{\alpha_n}(x) dx$. Hence, by Fatou's Lemma,

$$\begin{aligned} \int_{\Omega} [l_{\infty}(|\nabla u(x)|) + f(x)u(x)] dx &\leq \int_{\Omega} \liminf [l_{\alpha_{\nu'}}(|\nabla u_{\alpha_{\nu'}}(x)|) + f(x)u_{\alpha_{\nu'}}(x)] dx \\ &\leq \liminf \int_{\Omega} [l_{\alpha_{\nu'}}(|\nabla u_{\alpha_{\nu'}}(x)|) + f(x)u_{\alpha_{\nu'}}(x)] dx \\ &= \liminf I_{\alpha_{\nu'}}(u_{\alpha_{\nu'}}) \leq I_{\infty}(u_{\infty}). \end{aligned}$$

Since $u \in u^0 + W_0^{1,2}(\Omega)$, the uniqueness of solutions to problem (P_{∞}) implies that $u = u_{\infty}$, thus proving the theorem. $\square$

4 A more general case

The purpose of this section is to extend the results from the function L_{∞} of Section 1 to a more general convex function, still satisfying some suitable assumptions.

Consider the problem of minimizing

$$I_{\infty}^g(v) = \int_{\Omega} [g_{\infty}(|\nabla v(x)|) + f(x)v(x)] dx \quad \text{on } u^0 + W_0^{1,\infty}(\Omega), \quad (4.1)$$

where

$$G_{\infty}(\xi) = g_{\infty}(|\xi|) = \begin{cases} g(|\xi|) & \text{for } |\xi| \leq 1, \\ +\infty & \text{for } |\xi| > 1. \end{cases} \quad (4.2)$$

The function g in (4.2) satisfies the following assumption.

Assumption 4.1. g is a symmetric, convex, $C^2(\mathbb{R})$ function such that $g(0) = 0$ and, for a positive c_g and for $|t| \leq 1$, we have

$$\frac{g'(t)}{t} \geq c_g \quad \text{and} \quad g''(t) \geq c_g \frac{g'(t)}{t}.$$

From the assumption it follows that $g''(t) \geq c_g^2$ on $[-1, 1]$, and we shall assume $c_g \leq 1$. Our model case is the map

$$g_{\infty}(|\xi|) = \begin{cases} \sqrt{1 + |\xi|^2} & \text{for } |\xi| \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

For $g(t) = \sqrt{1 + t^2}$, a suitable constant c_g is $c_g = \frac{1}{2}$.

Theorem 4.2. Let Ω be open and bounded, let g satisfy Assumption 4.1, let $f \in L^2(\Omega)$, let u^0 be such that $\|u^0\|_{1,\infty} = 1 - d_{\infty}$ with $d_{\infty} > 0$, and let $u_{\infty}^g \in W^{1,\infty}(\Omega)$ be the solution to the problem of minimizing (4.1). Then $u_{\infty}^g \in W_{\text{loc}}^{2,2}(\Omega)$.

Proof. We will follow the steps of the proof of Theorem 1.1. Let g_{α} be the inf-convolution of g_{∞} , i.e.,

$$g_{\alpha}(t) = \inf_{x \in \mathbb{R}} \frac{1}{2} \alpha(t - x)^2 + g_{\infty}(x).$$

For $t \in \mathbb{R}$, we have $g_{\alpha}(t) \leq g_{\infty}(t)$ and $g_{\alpha}(t) \leq \frac{1}{2} \alpha t^2 + g_{\infty}(0) = \frac{1}{2} \alpha t^2$. Since $g(t) \geq \frac{1}{2} c_g t^2$, we obtain that

$$g_{\alpha}(t) \geq \frac{1}{2} \frac{c_g}{(1 + \frac{c_g}{\alpha})} t^2.$$

Again, we can assume that α is so large that

$$g_{\alpha}(t) \geq \frac{c_g}{4} t^2. \quad (4.3)$$

Introduce the maximal monotone, set-valued map

$$\partial g_\infty(t) = \begin{cases} (-\infty, g'(-1)] & \text{for } t = -1, \\ g'(t) & \text{for } |t| \leq 1, \\ [g'(1), +\infty) & \text{for } t = 1, \end{cases} \quad (4.4)$$

and call, as usual, $J_\alpha(t) = (\text{id} + \frac{1}{\alpha} \partial g_\infty)^{-1}(t)$ the resolvent and $(g_\alpha)' = \alpha(\text{id} - J_\alpha)$ the Yosida approximation to ∂g_∞ , so that $(g_\alpha)'(t) \in \partial g_\infty(J_\alpha(t))$. We obtain that

$$g'_\alpha(t) = \begin{cases} \alpha(t+1) & \text{for } t \leq \left(-1 + \frac{1}{\alpha} g'(-1)\right), \\ g'(J_\alpha(t)) = \alpha(t - J_\alpha(t)) & \text{for } -1 + \frac{1}{\alpha} g'(-1) \leq t \leq \left(1 + \frac{1}{\alpha} g'(1)\right), \\ \alpha(t-1) & \text{for } t \geq \left(1 + \frac{1}{\alpha} g'(1)\right), \end{cases}$$

so that g_α is C^2 except at the points $t = -1 + \frac{1}{\alpha} g'(-1)$ and $t = 1 + \frac{1}{\alpha} g'(1)$.

We claim that, for α sufficiently large, for $t \in \mathbb{R}$ we have

$$\frac{g'_\alpha(t)}{t} \geq \frac{1}{2} c_g, \quad (4.5)$$

$$g''_\alpha(t) \geq \frac{1}{2} c_g \frac{g'_\alpha(t)}{t}. \quad (4.6)$$

Consider first $0 \leq t < 1 + \frac{1}{\alpha} g'(1)$, so that $0 \leq J_\alpha(t) \leq 1$. We have

$$\frac{g'_\alpha(t)}{t} = \frac{g(J_\alpha(t))}{t} = \frac{g(J_\alpha(t))}{J_\alpha(t)} \frac{J_\alpha(t)}{t} \geq c_g \frac{J_\alpha(t)}{t}$$

and

$$g''_\alpha(t) = g''(J_\alpha(t)) J'_\alpha(t) \geq c_g \frac{g'(J_\alpha(t))}{J_\alpha(t)} J'_\alpha(t) = c_g \frac{g'(J_\alpha(t))}{t} \frac{t}{J_\alpha(t)} J'_\alpha(t),$$

so that, to prove the claim, it is enough to prove that, for α large, we have $\frac{J_\alpha(t)}{t} \geq \frac{1}{2}$ and $2J'_\alpha(t) \frac{t}{J_\alpha(t)} \geq 1$. In fact, we have

$$J'_\alpha(t) = \frac{1}{1 + \frac{1}{\alpha} g''(J_\alpha(t))} \geq \frac{1}{1 + 1/(\alpha \sup_{|\tau| \leq 1} \{g''(\tau)\})} \geq \frac{1}{2}$$

for α large. Then writing

$$J_\alpha(t) = \int_0^t J'_\alpha(s) ds \geq \frac{1}{2} t$$

proves (4.5). Inequality (4.6) follows by remarking that $|t| \geq |J_\alpha(t)|$. Consider the case $t > 1 + \frac{1}{\alpha} g'(1)$. Then $J_\alpha(t) = 1$, $g'_\alpha(t) = \alpha(t-1)$ and $g''_\alpha(t) = \alpha$, so that

$$\frac{g'_\alpha(t)}{t} = \alpha \frac{t-1}{t} \geq \frac{g'(1)}{1 + \frac{1}{\alpha} g'(1)} \geq \frac{1}{2} g'(1) \geq \frac{1}{2} c_g.$$

Moreover,

$$g''_\alpha(t) = \alpha \geq \alpha \frac{t-1}{t} = \frac{g'_\alpha(t)}{t} \geq \frac{1}{2} c_g \frac{g'_\alpha(t)}{t}.$$

The symmetry of g proves the claim on $\mathbb{R}$.

Let u_α be the solution to the problem of minimizing

$$I_\alpha^g(v) = \int_{\Omega} [G_\alpha(\nabla v) + f v] dx$$

on $u^0 + W_0^{1,2}(\Omega)$. From the properties of G_α we have

$$I_\infty^g(u_\infty^g) \geq I_\alpha^g(u_\infty^g) \geq I_\alpha^g(u_\alpha) \geq \int_{\Omega} \left[\frac{c_g}{4} |\nabla u_\alpha|^2 + f u_\alpha \right] dx. \quad (4.7)$$

Then from the estimates in (b) of the proof of Theorem 1.1 we obtain that there exist constants K and K_1 such that

$$\|\nabla u_\alpha\|_{L^2(\Omega)} \leq K, \quad \|u_\alpha\|_{L^2(\Omega)} \leq K, \quad \int_{\Omega} g_\alpha(|\nabla u_\alpha|) dx \leq K_1, \quad (4.8)$$

and we can assume the existence of a function $u^g \in u^0 + W_0^{1,2}(\Omega)$ and of a sequence α_n such that $u_{\alpha_n} \rightarrow u^g$ in $L^2(\Omega)$ and $\nabla u_{\alpha_n} \rightharpoonup \nabla u^g$ weakly in $L^2(\Omega)$.

Moreover, from the properties of g_α , we still have that $u_\alpha \in W_{\text{loc}}^{2,2}(\Omega)$ and that Lemmas 2.3 and 2.4 apply to u_α and g_α . Hence, for x^0 , δ^0 , η , and γ^s as in (c) of the proof of Theorem 1.1, we have again that the map $\phi = \eta^2 \cdot \gamma^s$ is in $W^{1,2}(B(x^0, 2\delta^0))$ and $\nabla \phi = 2\eta \nabla \eta \gamma^s + \eta^2 \nabla \gamma^s$. Now, for the left-hand side of the Euler–Lagrange equation (3.6) we obtain the estimate

$$\begin{aligned} & \int_{B(x^0, 2\delta^0)} \eta^2 \left\{ \left(g_\alpha''(|\nabla(u_\alpha)|) - \frac{g_\alpha'(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} \right) \left| H_\alpha \frac{\nabla u_\alpha}{|\nabla(u_\alpha)|} \right|^2 + \frac{g_\alpha'(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} |H_\alpha|^2 \right\} \\ & \geq \int_{B(x^0, 2\delta^0)} \eta^2 \left\{ \left(\frac{1}{2} c_g - 1 \right) \frac{g_\alpha'(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} \left| H_\alpha \frac{\nabla u_\alpha}{|\nabla(u_\alpha)|} \right|^2 + \frac{g_\alpha'(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} |H_\alpha|^2 \right\} \\ & \geq \int_{B(x^0, 2\delta^0)} \eta^2 \frac{1}{2} c_g \frac{g_\alpha'(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} |H_\alpha|^2. \end{aligned}$$

Then (3.8) still holds. Keeping the notations of (3.7), we now obtain

$$\begin{aligned} & \frac{1}{2} c_g \int_{B(x^0, 2\delta^0)} \eta^2 \frac{g_\alpha'(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} |H_\alpha|^2 dx \\ & \leq \int_{B(x^0, 2\delta^0)} \left[\tilde{\lambda} g_\alpha'(|\nabla(u_\alpha)|) |\nabla(u_\alpha)| + \eta^2 \frac{1}{4} c_g \frac{g_\alpha'(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} |H_\alpha|^2 + 4 \frac{1}{c_g} D^2 \frac{g_\alpha'(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} \right] dx \\ & \quad + 2D \|f\|_{L^2} \|\nabla u_\alpha\|_{L^2} + \int_{B(x^0, 2\delta^0)} \left[\frac{1}{8} (c_g)^2 \eta^2 |H_\alpha|^2 + \eta^2 |f|^2 \right] dx. \end{aligned}$$

Hence,

$$\begin{aligned} & \frac{1}{4} (c_g)^2 \int_{B(x^0, 2\delta^0)} \eta^2 |H_\alpha|^2 dx \leq \frac{1}{4} c_g \int_{B(x^0, 2\delta^0)} \eta^2 \frac{g_\alpha'(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} |H_\alpha|^2 dx \\ & \leq \int_{B(x^0, 2\delta^0)} \left[\tilde{\lambda} g_\alpha'(|\nabla(u_\alpha)|) |\nabla(u_\alpha)| + 4 \frac{1}{c_g} D^2 \frac{g_\alpha'(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} \right] dx + 2D \|f\|_{L^2} \|\nabla u_\alpha\|_{L^2} \\ & \quad + \int_{B(x^0, 2\delta^0)} \left[\frac{1}{8} (c_g)^2 \eta^2 |H_\alpha|^2 + \eta^2 |f|^2 \right] dx, \end{aligned}$$

so that

$$\begin{aligned} & \frac{(c_g)^2}{8} \int_{B(x^0, 2\delta^0)} \eta^2 |H_\alpha|^2 dx \leq \int_{B(x^0, 2\delta^0)} \left[\tilde{\lambda} g_\alpha'(|\nabla(u_\alpha)|) |\nabla(u_\alpha)| + \frac{4D^2}{c_g} \frac{g_\alpha'(|\nabla(u_\alpha)|)}{|\nabla(u_\alpha)|} \right] dx \\ & \quad + 2D \|f\|_{L^2} \|\nabla u_\alpha\|_{L^2} + \int_{B(x^0, 2\delta^0)} \eta^2 |f|^2 dx, \end{aligned}$$

and we obtain the analog of (3.10), i.e., that there exist constants k_1^g and k_2^g such that

$$\int_{B(x^0, \delta^0)} |H_\alpha|^2 dx \leq k_1^g \int_{B(x^0, 2\delta^0)} g_\alpha'(|\nabla u_\alpha|) |\nabla u_\alpha| dx + k_2^g. \quad (4.9)$$

We next have the analog of the claim in (e), i.e., that given $x^* \in \Omega$ such that $B(x^*, \delta^*) \subset \Omega$, there exists K^* , depending on u^0 and x^* , but not on α , such that

$$\int_{B(x^*, d_{\infty}\delta^*)} g'(|\nabla u_\alpha|)|\nabla u_\alpha| dx \leq K^*. \quad (4.10)$$

To prove the claim define v as in (3.11); we have that $g_\alpha(|\nabla v|) \leq g(|\nabla v|) \leq g(1)$. Since g_α is convex, we obtain that

$$\int_{\Omega} \frac{G_\alpha(\nabla u_\alpha + \varepsilon \nabla(v - u_\alpha)) - G_\alpha(\nabla u_\alpha)}{\varepsilon} dx \leq \int_{\Omega} (G_\alpha(\nabla v(x)) - G_\alpha(\nabla u_\alpha(x)))^+ dx \leq g(1)|\Omega| + K_1. \quad (4.11)$$

Since u_α is a minimum, we have

$$\frac{1}{\varepsilon} \int_{\Omega} [G_\alpha(\nabla u_\alpha + \varepsilon \nabla(v - u_\alpha)) - G_\alpha(\nabla u_\alpha)] dx + \int_{\Omega} f(v - u_\alpha) dx \geq 0. \quad (4.12)$$

This gives

$$\frac{1}{\varepsilon} \int_{\Omega} (G_\alpha(\nabla u_\alpha + \varepsilon \nabla(v - u_\alpha)) - G_\alpha(\nabla u_\alpha))^- dx \leq g(1)|\Omega| + K_1 + \int_{\Omega} f(v - u_\alpha) dx.$$

Then, by dominated convergence and Fatou's Lemma, we have

$$\begin{aligned} \int_{\Omega} |\langle \nabla L_\alpha(\nabla u_\alpha), \nabla(v - u_\alpha) \rangle| dx &= \int_{\Omega} (\langle \nabla L_\alpha(\nabla u_\alpha), \nabla(v - u_\alpha) \rangle)^- dx + \int_{\Omega} (\langle \nabla L_\alpha(\nabla u_\alpha), \nabla(v - u_\alpha) \rangle)^+ dx \\ &\leq 2g(1)|\Omega| + 2K_1 + \|f\|_{L^2}(K_\infty + K) \doteq K^*, \end{aligned}$$

and we obtain

$$\int_{B(x^0, d_{\infty}\delta^*)} g'_\alpha(|\nabla u_\alpha|)|\nabla u_\alpha| \leq K^*.$$

In turn, this implies that there exists K^0 such that $\int_{B(x^0, 2\delta^0)} g'_\alpha(|\nabla u_\alpha|)|\nabla u_\alpha| \leq K^0$. From (4.9) and a covering argument, to $\omega \in \Omega$ we associate a constant K_ω , independent of α , such that

$$\int_{\omega} |H_\alpha|^2 dx \leq K_\omega \quad \text{and} \quad \int_{\omega} g'_\alpha(|\nabla u_\alpha|)|\nabla u_\alpha| dx \leq K_\omega. \quad (4.13)$$

Then the family of the restrictions to ω of the maps ∇u_{α_n} converges to ∇u^g in $L^2(\omega)$ and $u^g \in u^0 + W^{2,2}(\omega)$. We can assume the existence of a subsequence, still denoted by α_n , such that $\nabla u_{\alpha_n} \rightarrow \nabla u^g$ pointwise a.e. in Ω .

It is left to show that $u^g = u_\infty^g$. First, we infer that $|\nabla u^g(x)| \leq 1$ for a.e. $x \in \Omega$. In fact, otherwise, there exist ε and $\omega'_\varepsilon \subset \omega$, with $|\omega'_\varepsilon| \geq \frac{1}{2}\varepsilon$, such that, for n sufficiently large, we have $|\nabla u_{\alpha_n}(x)| \geq 1 + \frac{1}{2}\varepsilon$ for $x \in \omega'_\varepsilon$. Hence, for $x \in \omega'_\varepsilon$ and n large, we have

$$\begin{aligned} g'_{\alpha_n}(|\nabla u_{\alpha_n}(x)|)|\nabla u_{\alpha_n}(x)| &= \alpha_n(|\nabla u_{\alpha_n}(x)| - J_{\alpha_n}(|\nabla u_{\alpha_n}(x)|))|\nabla u_{\alpha_n}(x)| \\ &= \alpha_n(|\nabla u_{\alpha_n}(x)| - 1)|\nabla u_{\alpha_n}(x)| \\ &\geq \alpha_n \frac{1}{2} \varepsilon \left(1 + \frac{1}{2} \varepsilon\right), \end{aligned}$$

and hence

$$\int_{\omega_{\alpha_n}} g'_{\alpha_n}(|\nabla u_{\alpha_n}|)|\nabla u_{\alpha_n}| dx \geq \left(1 + \frac{1}{2} \varepsilon\right) \frac{1}{4} \varepsilon^2 \alpha'_n,$$

contradicting (4.13). Hence, $|\nabla u^g| \leq 1$ a.e. in Ω .

Then, as in (h) of the proof of Theorem 1.1, let α'_n be such that $\nabla u'_{\alpha'_n}$ converges to ∇u^g pointwise a.e.; fix x and let α''_n be a subsequence such that $g_{\alpha''_n}(|\nabla u_{\alpha''_n}(x)|) \rightarrow z(x)$. When $z(x) > g(1)$, from

$$\sup_{|t| \leq 1 + \frac{1}{\alpha} g'(1)} g(|t|) = g(1)$$

we have that $|\nabla u^g(x)| = 1$ and that $\lim g_{\alpha_n''}(|\nabla u_{\alpha_n''}(x)|) > g(|\nabla u^g(x)|)$. When $z(x) < g(1)$, we must have, definitely, $|\nabla u_{\alpha_n''}(x)| < 1$ and, by the uniform convergence of $g_{\alpha_n''}$ to g , we obtain that

$$\lim g_{\alpha_n''}(|\nabla u_{\alpha_n''}(x)|) = g(|\nabla u^g(x)|).$$

In the case $z(x) = g(1)$, for ε arbitrary we must have that, definitely, $|\nabla u_{\alpha_n''}(x)| \geq 1 - \varepsilon$, hence $|\nabla u(x)| = 1$, so that $g(|\nabla u^g(x)|) = \lim g_{\alpha_n''}(|\nabla u_{\alpha_n''}(x)|)$. Then we obtain that $g(|\nabla u^g(x)|) \leq \liminf g_{\alpha_n'}(|\nabla u_{\alpha_n'}(x)|)$ for $x \in \Omega$. Finally, Fatou's Lemma implies

$$\begin{aligned} \int_{\Omega} [g_{\infty}(|\nabla u^g(x)|) + f(x)u^g(x)] dx &\leq \int_{\Omega} \liminf [g_{\alpha_n'}(|\nabla u_{\alpha_n'}(x)|) + f(x)u_{\alpha_n'}(x)] dx \\ &\leq \liminf \int_{\Omega} [l_{\alpha_{v'}}(|\nabla u_{\alpha_{v'}}(x)|) + f(x)u_{\alpha_{v'}}(x)] dx \\ &= \liminf I_{\alpha_{v'}}(u_{\alpha_{v'}}) \leq I_{\infty}^g(u_{\infty}^g), \end{aligned}$$

which implies $u^g = u_{\infty}^g$. □

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