Research Article

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On a comparison principle for Trudinger's equation

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Abstract: We study the comparison principle for non-negative solutions of the equation

$$\frac{\partial(|v|^{p-2}v)}{\partial t} = \operatorname{div}(|\nabla v|^{p-2}\nabla v), \quad 1 < p < \infty.$$

This equation is related to extremals of Poincaré inequalities in Sobolev spaces. We apply our result to obtain pointwise control of the large time behavior of solutions.

Keywords: Trudinger's equation, doubly non-linear equations, uniqueness, asymptotics

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1 Introduction

Among the so-called doubly non-linear evolutionary equations, Trudinger's equation

$$\frac{\partial(|v|^{p-2}v)}{\partial t} = \operatorname{div}(|\nabla v|^{p-2}\nabla v), \quad 1 < p < \infty, \quad (1.1)$$

is distinguished. We shall study its solutions in $\Omega_T = \Omega \times (0, T)$, where Ω is a domain in $\mathbb{R}^n$. The equation was originally considered by Trudinger in [22], where a Harnack inequality was studied for a wider class of evolutionary equations. He pointed out that no “intrinsic scaling” is needed for (1.1). The equation has two special features: it is *homogeneous* and it is *not translation invariant*, except for $p = 2$ when it reduces to the heat equation. The first property is, of course, an advantage.

While some progress has been made regarding continuity and regularity properties, the question about *uniqueness* for the Dirichlet boundary value problem seems to be unsettled (under natural assumptions). Sign-changing solutions certainly come with many challenges.

We shall prove a comparison principle in Theorem 1 for positive weak supersolutions/subsolutions belonging, by definition, to the Sobolev space $L^p(0, T; W^{1,p}(\Omega))$. One of the functions is required to be bounded away from zero. Nevertheless, we can allow that one of the functions has zero lateral boundary values (Corollary 1), which is relevant for a related eigenvalue problem. Furthermore, for Perron's method, a proper comparison principle is a *sine qua non*. Our result also implies that for $p \geq 2$, all non-negative continuous weak solutions are viscosity solutions in the sense of Crandall, Evans, and Lions in [3]. This is Theorem 3.

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Trudinger's equation has an interesting connection to extremals of Poincaré inequalities in the Sobolev space $W_0^{1,p}(\Omega)$. These are the minimizers of the Rayleigh quotient

$$\lambda_p = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx}. \quad (1.2)$$

(We refer the curious reader to [13] and [14].) Any extremal $u = u(x)$ solves the corresponding Euler–Lagrange equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) + \lambda_p |u|^{p-2} u = 0, \quad (1.3)$$

and

$$v(x, t) = e^{-\frac{\lambda_p}{p-1} t} u(x)$$

is a solution to Trudinger's equation. In [7], the following result is proved: if v is a weak solution of

$$\begin{cases} \frac{\partial(|v|^{p-2} v)}{\partial t} = \Delta_p v & \text{in } \Omega \times (0, \infty), \\ v = g & \text{on } \Omega \times \{0\}, \\ v = 0 & \text{on } \partial\Omega \times [0, \infty), \end{cases}$$

where $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2} \nabla v)$, then the limit

$$u = \lim_{t \rightarrow \infty} e^{+\frac{\lambda_p}{p-1} t} v(\cdot, t)$$

exists in $L^p(\Omega)$ and u is a solution of (1.3), possibly identically zero. It is also proved that solutions with extremals as initial data are separable. See [16] for some other related results. We address this question anew in Theorem 2, where we prove a uniqueness result in star-shaped domains. In Corollary 5, we treat the large time behavior in $C^{1,\alpha}$ -domains, not necessarily star-shaped.

Known related results. The existence of solutions has been addressed in for instance [1], [15, Paragraph 3.1, Chapter 4] and [18].

To the best of our knowledge, the comparison principles known so far are limited; in [1], a comparison principle is proved under the extra assumption

$$\frac{\partial(|v|^{p-2} v - |u|^{p-2} u)}{\partial t} \in L^1(\Omega_T),$$

where v is a supersolution and u a subsolution. In [8], a comparison principle is proved for a general class of doubly non-linear equations. The method there is right; however, for the parameter values yielding Trudinger's Equation, we are unable to verify the validity of the proof in that paper.

In [9, Theorem 2.1], a Harnack inequality is proved for strictly positive solutions, and in a subsequent comment it is stated that it is valid also for merely non-negative solutions. A similar result for positive solutions has also been obtained in [5]. Local regularity and regularity up to the boundary has been studied in [17]. The equation and its regularity have also been treated in [10, 11, 19–21, 23]. See also [2] for a viscosity approach to the equation.

Plan of the paper. In Section 2, we introduce some notation and define weak solutions. In Section 3, we prove the comparison principle. In Section 4, we establish uniqueness of solutions in star-shaped domains with zero lateral boundary condition. In Section 5, we show that one can compare with extremals of the Poincaré inequality in $C^{1,\alpha}$ -domains, and we use this to study the large time behavior. Finally, in Section 6, we prove that weak solutions are also viscosity solutions when $p \geq 2$. In the Appendix, we give a proof of the fact that the maximum of a subsolution and a constant is again a subsolution.

2 Preliminaries

We use standard notation. If Ω is a bounded and open set in $\mathbb{R}^n$, we use the notation

$$\Omega_T = \Omega \times (0, T),$$

and the parabolic boundary of Ω_T is

$$\partial_p \Omega_T = \Omega \times \{0\} \cup \partial\Omega \times [0, T].$$

We denote the time derivative of a function v by v_t . The positive part of a real quantity a is $a^+ = \max\{a, 0\}$. Let us define the weak supersolutions, subsolutions and solutions of the equation

$$(|v|^{p-2}v)_t = \Delta_p v, \quad 1 < p < \infty.$$

Notice that although the functions are non-negative, we shall often write $|v|^{p-2}v$ in place of v^{p-1} , the reason being that many auxiliary identities are valid also for sign-changing solutions.

Definition 1. We say that $v \in L^p_{\text{loc}}(0, T; W^{1,p}_{\text{loc}}(\Omega))$ is a *weak supersolution* in Ω_T if

$$\iint_{\Omega_T} (-|v|^{p-2}v \phi_t + |\nabla v|^{p-2} \nabla v \cdot \nabla \phi) dx dt \geq 0 \quad (2.1)$$

holds for any $\phi \in C_0^\infty(\Omega_T)$, $\phi \geq 0$. Similarly, $u \in L^p_{\text{loc}}(0, T; W^{1,p}_{\text{loc}}(\Omega))$ is a *weak subsolution* if

$$\iint_{\Omega_T} (-|u|^{p-2}u \phi_t + |\nabla u|^{p-2} \nabla u \cdot \nabla \phi) dx dt \leq 0. \quad (2.2)$$

A function is a *weak solution* if it is both a weak super- and subsolution.

By regularity theory, the weak solutions are locally Hölder continuous, see [9] and [23]. It is likely that the weak super- and subsolutions are semicontinuous (upon a change in a set of Lebesgue measure zero), but we do not know of any reference. For technical reasons, we shall assume that weak *subsolutions* are continuous, in the range $1 < p < 2$, when it is not sure that $\text{ess inf } u > 0$. An expedient tool is the weak Harnack inequality in [9] for weak supersolutions $v \geq 0$ in Ω_T . It implies that if

$$\text{ess inf}_{B \times (t_1, t_2)} v = 0$$

over a strict subdomain, then $v \equiv 0$ in $\Omega \times (0, t_1]$.

3 The comparison principle

The uniqueness of sufficiently *smooth* solutions that coincide on the parabolic boundary is evident. In particular, the time derivative is crucial. Indeed, for two such solutions u_1 and u_2 in $C(\overline{\Omega_T})$ with the same boundary and initial values on $\partial_p \Omega_T$, we *formally* use the test function $\phi = H_\delta((u_2 - u_1)^+)$, where

$$H_\delta(s) := \begin{cases} 1, & s \geq \delta, \\ \frac{s}{\delta}, & 0 < s < \delta, \\ 0, & s \leq 0, \end{cases} \quad (3.1)$$

approximates the Heaviside function, to obtain

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\Omega} \partial_t (|u_2|^{p-2}u_2 - |u_1|^{p-2}u_1) H_\delta((u_2 - u_1)^+) dx dt \\ &= -\frac{1}{\delta} \int_{t_1}^{t_2} \int_{\Omega \cap \{0 < u_2 - u_1 < \delta\}} \langle |\nabla u_2|^{p-2} \nabla u_2 - |\nabla u_1|^{p-2} \nabla u_1, \nabla(u_2 - u_1)^+ \rangle dx dt \leq 0. \end{aligned}$$

As $\delta \rightarrow 0$, it follows that

$$\int_{t_1}^{t_2} \int_{\Omega} \partial_t (|u_2|^{p-2}u_2 - |u_1|^{p-2}u_1)^+ dx dt \leq 0.$$

Integrating we see that

$$\int_{\Omega} [|u_2|^{p-2}u_2 - |u_1|^{p-2}u_1]^+ dx \Big|_{t_2} \leq \int_{\Omega} [|u_2|^{p-2}u_2 - |u_1|^{p-2}u_1]^+ dx \Big|_{t_1}.$$

The last integral becomes zero at $t_1 = 0$. We conclude that $u_2 \leq u_1$ and switching the functions we get the desired uniqueness $u_2 = u_1$. This simple proof was purely formal. It is not clear how to rescue this reasoning without access to the time derivatives. Therefore we must pay careful attention to the proper regularizations.¹

We have extracted some parts of our proof below of the Comparison Principle from [8]. Unfortunately, sign-changing solutions are not susceptible of our treatment. In the next theorem, the majorant can become infinite, if the parameter $\beta = 1$.

Theorem 1. *Let $v \geq 0$ be a weak supersolution and $u \geq 0$ a weak subsolution in $\Omega_T = \Omega \times (0, T)$, satisfying*

$$\liminf_{(y,s) \rightarrow (x,t)} v(y, s) \geq \limsup_{(y,s) \rightarrow (x,t)} u(y, s) \quad \text{when } (x, t) \in \partial\Omega \times (0, T) \quad (3.2)$$

on the lateral boundary. Then the inequality

$$\int_{\Omega} (|u|^{p-2}u - |\beta v|^{p-2}\beta v)^+ dx \Big|_{t=t_2} \leq \int_{\Omega} (|u|^{p-2}u - |\beta v|^{p-2}\beta v)^+ dx \Big|_{t=t_1} \quad (3.3)$$

holds for each constant $\beta > 1$ and for a.e. every time t_1 and t_2 , $0 < t_1 < t_2 < T$, under the following assumptions:

- *In the case $p > 2$,*

$$\operatorname{ess\,inf}_{\Omega_T} v > 0 \quad \text{or} \quad \operatorname{ess\,inf}_{\Omega_T} u > 0.$$

- *In the case $1 < p < 2$,*

$$\operatorname{ess\,inf}_{\Omega_T} u > 0,$$

or u is continuous and $\operatorname{ess\,inf}_{\Omega_T} v > 0$.

Proof. Let us first assume $v \geq c$ (and u continuous in the case $1 < p < 2$). Let $\gamma > 0$. Then $u < v + \gamma$ close enough to $\partial\Omega \times (0, T)$. Since $v \geq c > 0$, this implies that

$$u < v + \gamma = v \left(1 + \frac{\gamma}{v}\right) \leq v \left(1 + \frac{\gamma}{c}\right).$$

Define

$$\tilde{v}(x, t) = \left(1 + \frac{\gamma}{c}\right)v(x, t), \quad \beta = 1 + \frac{\gamma}{c}.$$

We will prove (3.3) with βv replaced by $\tilde{v}$. Adding up (2.1) for $\tilde{v}$ and (2.2) yields

$$\int_0^T \int_{\Omega} (|\tilde{v}|^{p-2}\tilde{v} - |u|^{p-2}u)\phi_t dx dt \leq \int_0^T \int_{\Omega} (|\nabla \tilde{v}|^{p-2}\nabla \tilde{v} - |\nabla u|^{p-2}\nabla u) \cdot \nabla \phi dx dt \quad (3.4)$$

for any non-negative $\phi \in C_0^\infty(\Omega_T)$.

We need a regularization. The Steklov average of a function $f(x, t)$ is

$$f_h(x, t) := \frac{1}{h} \int_{t-h}^t f(x, \tau) d\tau, \quad 0 < t-h, t < T, h > 0.$$

See [4, Lemma 3.2 in Chapter 3-(i)]. We note that if

$$\int_0^T \int_{\Omega} f(x, t)\phi_t(x, t) dx dt \leq 0, \quad \phi \in C_0^\infty(\Omega_T),$$

¹ A similar remark can be made for “proofs” of the inequality

$$\frac{d}{dt} \int_{\Omega} |\nabla u(x, t)|^p dx \leq 0, \quad u \in L^p(0, T, W_0^{1,p}(\Omega)).$$

then

$$\int_0^T \int_{\Omega} f(x, t + \tau) \phi_t(x, t) dx dt \leq 0, \quad -h < \tau < 0, \quad \phi \in C_0^\infty(\Omega \times (2h, T - h)).$$

Integrating τ from $-h$ to 0, we obtain

$$\int_0^T \int_{\Omega} f_h(x, t) \phi_t(x, t) dx dt \leq 0, \quad \phi \in C_0^\infty(\Omega \times (2h, T - h)).$$

Taking Steklov averages in (3.4), we deduce

$$\int_0^T \int_{\Omega} (|\tilde{v}|^{p-2} \tilde{v} - |u|^{p-2} u)_h \phi_t dx dt \leq \int_0^T \int_{\Omega} (|\nabla \tilde{v}|^{p-2} \nabla \tilde{v} - |\nabla u|^{p-2} \nabla u)_h \cdot \nabla \phi dx dt$$

for $\phi \geq 0$ belonging to $C_0^\infty(\Omega \times (2h, T - h))$. Since the Steklov average is differentiable with respect to t , we may integrate by parts to obtain

$$-\int_0^T \int_{\Omega} \partial_t (|\tilde{v}|^{p-2} \tilde{v} - |u|^{p-2} u)_h \phi dx dt \leq \int_0^T \int_{\Omega} (|\nabla \tilde{v}|^{p-2} \nabla \tilde{v} - |\nabla u|^{p-2} \nabla u)_h \cdot \nabla \phi dx dt. \quad (3.5)$$

Recall the function H_δ from equation (3.1) and define the function

$$G_\delta(s) = \begin{cases} s - \frac{\delta}{2}, & s \geq \delta, \\ \frac{s^2}{2\delta}, & 0 < s < \delta, \\ 0, & s \leq 0. \end{cases}$$

Note that $G'_\delta(s) = H_\delta(s)$ and that $H_\delta(s)$ is an approximation of the Heaviside function $H(s)$. For t_1 and t_2 with $2h < t_1 < t_2 < T - h$, we also define the cut-off function

$$\eta_\varepsilon(t) = H_\varepsilon(t - t_1) - H_\varepsilon(t - t_2),$$

where $\varepsilon > 0$ is small.

Near the boundary $\partial\Omega$ we have $|u|^{p-2}u < |\tilde{v}|^{p-2}\tilde{v}$, so that the Steklov average $[|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}]_h < 0$ there. We now choose the test function²

$$\phi(x, t) = \eta_\varepsilon(t) H_\delta([|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}]_h)$$

and observe that

$$\partial_t G_\delta([|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}]_h) = H_\delta([|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}]_h) \partial_t ([|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}]_h).$$

With this particular choice of ϕ , inequality (3.5) becomes

$$\begin{aligned} & \int_0^T \int_{\Omega} \eta_\varepsilon(t) \partial_t G_\delta([|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}]_h) dx dt \\ & \leq \int_0^T \int_{\Omega} \eta_\varepsilon(t) (|\nabla \tilde{v}|^{p-2} \nabla \tilde{v} - |\nabla u|^{p-2} \nabla u)_h \cdot \nabla H_\delta([|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}]_h) dx dt. \end{aligned} \quad (3.6)$$

It is straightforward to send $\varepsilon \rightarrow 0$.

² By estimating $\nabla \phi$ in the same way as B_h on the following two pages, one can argue that ϕ is regular enough to be approximated by smooth functions.

We now focus on the left-hand side of (3.5). As $\varepsilon \rightarrow 0$, it becomes

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\Omega} \partial_t G_{\delta}([|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}]_h) dx dt \\ &= \int_{\Omega} G_{\delta}([|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}]_h) dx \Big|_{t=t_2} - \int_{\Omega} G_{\delta}([|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}]_h) dx \Big|_{t=t_1}. \end{aligned}$$

Notice that $G_{\delta}(s) \leq s$. We may now let $h \rightarrow 0$ to obtain³

$$\int_{\Omega} G_{\delta}(|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}) dx \Big|_{t=t_2} - \int_{\Omega} G_{\delta}(|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}) dx \Big|_{t=t_1},$$

for a.e. t_1 and t_2 . As $h \rightarrow 0$, we shall justify that (3.6) implies

$$\begin{aligned} & \int_{\Omega} G_{\delta}(|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}) dx \Big|_{t=t_2} - \int_{\Omega} G_{\delta}(|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}) dx \Big|_{t=t_1} \\ & \leq \int_{t_1}^{t_2} \int_{\Omega} (|\nabla \tilde{v}|^{p-2}\nabla \tilde{v} - |\nabla u|^{p-2}\nabla u) \cdot \nabla H_{\delta}(|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}) dx dt \\ & = -\frac{1}{\delta} \iint_{\Omega_{\delta}} (|\nabla u|^{p-2}\nabla u - |\nabla \tilde{v}|^{p-2}\nabla \tilde{v}) \cdot \nabla (|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}) dx dt, \end{aligned} \quad (3.7)$$

where

$$\Omega_{\delta} = \Omega \times (t_1, t_2) \cap \{0 < |u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v} < \delta\}.$$

The right-hand member in (3.6) is (after sending $\varepsilon \rightarrow 0$)

$$\iint_{\Omega_{\delta,h}} \overbrace{(|\nabla \tilde{v}|^{p-2}\nabla \tilde{v} - |\nabla u|^{p-2}\nabla u)_h}^{=A_h(x,t)} \cdot \overbrace{\nabla H_{\delta}([|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}]_h)}^{=B_h(x,t)} dx dt,$$

where the integration is over the compact subset $\Omega_{\delta,h}$ defined by

$$\Omega_{\delta,h} = \Omega \times (t_1, t_2) \cap \{0 < [|u|^{p-2}u]_h - [|\tilde{v}|^{p-2}\tilde{v}]_h < \delta\}.$$

For all small h , $\Omega_{\delta,h} \subset K \times (t_1, t_2)$, for some compact subset $K \subset \subset \Omega$. We have that if

- $A_h \rightarrow A_0$ strongly in $L_{\text{loc}}^{p/(p-1)}(\Omega_T)$,
- B_h is bounded in $L_{\text{loc}}^p(\Omega_T)$,

then, upon extracting a subsequence, the integrals converge as they should:

$$\iint_{\Omega_{\delta,h}} A_h(x, t) B_h(x, t) dx dt \rightarrow \iint_{\Omega_{\delta}} A_0(x, t) B_0(x, t) dx dt.$$

The convergence of A_h follows from the properties of the Steklov averages. Concerning the factor B_h , which is not a Steklov average, we have that

$$B_h = \frac{1}{\delta} \nabla ([|u|^{p-2}u]_h - [|\tilde{v}|^{p-2}\tilde{v}]_h)$$

in $\Omega_{\delta,h}$ and $B_h = 0$ otherwise. We have two cases.

³ The limit holds at the Lebesgue points of

$$t \rightarrow \int_{\Omega} G_{\delta}(|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}) dx.$$

If $p > 2$, we may assume that $0 \leq u \leq L$, by Corollary 6 in the Appendix. Using Hölder's inequality and that $[\tilde{v}^{p-1}]_h \leq [u^{p-1}]_h$, we have

$$|[\tilde{v}^{p-2} \nabla \tilde{v}]_h| \leq [\tilde{v}^{p-1}]_h^{\frac{p-2}{p-1}} [|\nabla \tilde{v}|^{p-1}]_h^{\frac{1}{p-1}} \leq [u^{p-1}]_h^{\frac{p-2}{p-1}} [|\nabla \tilde{v}|^{p-1}]_h^{\frac{1}{p-1}} \leq L^{\frac{p-2}{p-1}} [|\nabla \tilde{v}|^{p-1}]_h^{\frac{1}{p-1}}.$$

Integration yields the bound

$$\iint_{\Omega_{\delta,h}} |[\tilde{v}^{p-2} \nabla \tilde{v}]_h|^p dx dt \leq L^{\frac{p(p-2)}{p-1}} \iint_{\Omega_{\delta,h}} [|\nabla \tilde{v}|^{p-1}]_h^{\frac{p}{p-1}} dx dt \leq C \iint_{K \times (t_1, t_2)} |\nabla \tilde{v}|^p dx dt$$

when h is small. This bound is uniform in h . A similar estimate holds for the part of B_h that involves u . Thus the L^p -norm of B_h is uniformly bounded. This was the case $p > 2$.

For $p < 2$ we proceed as follows. Since $\tilde{v} > c$ near the lateral boundary, we can replace u by the sub-solution $\max\{u, \sigma\}$, where $0 < \sigma < c$. See Proposition 2 in the Appendix. Then the factors u^{p-2} and $\tilde{v}^{p-2}$ are bounded from above so that

$$|[\tilde{v}^{p-2} \nabla \tilde{v}]_h| \leq c^{p-2} [|\nabla \tilde{v}|]_h, \quad |[u^{p-2} \nabla u]_h| \leq \sigma^{p-2} [|\nabla u|]_h.$$

Thus (3.6) implies (3.7), by a standard procedure.

We recall that for fixed $\gamma > 0$ (defining $\tilde{v}$), Ω_δ is compactly contained in $\Omega \times (0, T)$. Using the identity

$$\nabla(|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}) = (p-1)|u|^{p-2}(\nabla u - \nabla \tilde{v}) + (p-1)\nabla \tilde{v}(|u|^{p-2} - |\tilde{v}|^{p-2}),$$

and rewriting the right-hand side of (3.7), we obtain

$$\begin{aligned} & \int_{\Omega} G_\delta(|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}) dx \Big|_{t=t_2} - \int_{\Omega} G_\delta(|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}) dx \Big|_{t=t_1} \\ & \leq -\frac{(p-1)}{\delta} \iint_{\Omega_\delta} |u|^{p-2} \overbrace{(|\nabla u|^{p-2}\nabla u - |\nabla \tilde{v}|^{p-2}\nabla \tilde{v}) \cdot (\nabla u - \nabla \tilde{v})}^{\geq 0} dx dt \\ & \quad - \frac{(p-1)}{\delta} \iint_{\Omega_\delta} (|u|^{p-2} - |\tilde{v}|^{p-2})(|\nabla u|^{p-2}\nabla u - |\nabla \tilde{v}|^{p-2}\nabla \tilde{v}) \cdot \nabla \tilde{v} dx dt \\ & \leq -\frac{(p-1)}{\delta} \iint_{\Omega_\delta} (|u|^{p-2} - |\tilde{v}|^{p-2})(|\nabla u|^{p-2}\nabla u - |\nabla \tilde{v}|^{p-2}\nabla \tilde{v}) \cdot \nabla \tilde{v} dx dt. \end{aligned} \quad (3.8)$$

It is straightforward to see that for a.e. time t_1 and t_2 we get

$$\lim_{\delta \rightarrow 0} \int_{\Omega} G_\delta(|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v}) dx \Big|_{t=t_i} = \int_{\Omega} (|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v})^+ dx \Big|_{t=t_i},$$

where $i = 1, 2$.

It now remains to verify that the right-hand side tends to zero as $\delta \rightarrow 0$. The difficulty is that the available inequality $0 < u^{p-1} - \tilde{v}^{p-1} < \delta$, with a very small δ does not necessarily imply that also the quantity $u^{p-2} - \tilde{v}^{p-2}$ is of the order $O(\delta)$, if it so happens that both u and $\tilde{v}$ are very small. It is at this point that the assumption of a lower bound c is crucial.

The elementary inequality

$$(1+x)^{\frac{p-2}{p-1}} \leq 1 + \frac{p-2}{p-1}x, \quad p > 2, x > 0,$$

implies that

$$u^{p-2} = (\tilde{v}^{p-1} + [u^{p-1} - \tilde{v}^{p-1}])^{\frac{p-2}{p-1}} < (\tilde{v}^{p-1} + \delta)^{\frac{p-2}{p-1}} \leq \tilde{v}^{p-2} + \frac{p-2}{p-1} \frac{\delta}{\tilde{v}}.$$

Thus we have arrived at

$$0 < u^{p-2} - \tilde{v}^{p-2} \leq \frac{p-2}{p-1} \frac{\delta}{\tilde{v}} \leq \frac{p-2}{p-1} \frac{\delta}{c} = O(\delta), \quad p > 2,$$

since $\tilde{v} > v \geq c$. In the case $1 < p < 2$, the above elementary inequality is reversed and a similar reasoning yields

$$0 < \tilde{v}^{p-2} - u^{p-2} \leq \frac{2-p}{p-1} \frac{\delta}{\tilde{v}} \leq \frac{2-p}{p-1} \frac{\delta}{c} = O(\delta), \quad p < 2.$$

Hence we can kill the denominator δ below:

$$\left| \frac{(p-1)}{\delta} \iint_{\Omega_\delta} (|u|^{p-2} - |\tilde{v}|^{p-2})(|\nabla u|^{p-2} \nabla u - |\nabla \tilde{v}|^{p-2} \nabla \tilde{v}) \cdot \nabla \tilde{v} \, dx \, dt \right| \leq C \iint_{\Omega_\delta} (|\nabla u|^p + |\nabla \tilde{v}|^p) \, dx \, dt,$$

where we have also used Young's inequality for the terms involving ∇u and $\nabla \tilde{v}$. Since the integral

$$\int_{t_1}^{t_2} \int_{\{|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v} > 0\}} (|\nabla u|^p + |\nabla \tilde{v}|^p) \, dx \, dt$$

is convergent and $\text{meas}(\Omega_\delta) \rightarrow 0$, it is clear that

$$\iint_{\Omega_\delta} (|\nabla u|^p + |\nabla \tilde{v}|^p) \, dx \, dt \rightarrow 0$$

as $\delta \rightarrow 0$. Therefore, letting $\delta \rightarrow 0$ in (3.8), we arrive at

$$\int_{\Omega} (|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v})^+ \, dx \Big|_{t=t_2} \leq \int_{\Omega} (|u|^{p-2}u - |\tilde{v}|^{p-2}\tilde{v})^+ \, dx \Big|_{t=t_1}.$$

This is valid for a.e. t_1 and t_2 . This concludes the case $v \geq c$.

Finally, we note that if we instead assume $u \geq c$ (but not necessarily $v \geq c$), then (3.2) implies that also v is bounded away from zero near $\partial\Omega \times [t_1, t_2]$. By the weak Harnack inequality in [9], also $\text{ess inf } v > 0$, the infimum being taken over $\Omega \times [t_1, t_2]$. The same argument goes through again. $\square$

Remark 1. The conclusion in (3.3) is still valid, if assumption (3.2) is replaced by the requirement

$$(u - v)^+ \in L^p_{\text{loc}}(0, T; W^{1,p}_0(\Omega)).$$

Now the boundary values are taken in Sobolev's sense.

Corollary 1 (Comparison Principle). *If inequality (3.2) is valid on the whole parabolic boundary $\partial_p\Omega$, then $v \geq u$ in Ω_T .*

Proof. Let $\beta > 1$ be arbitrary. As $t_1 \rightarrow 0$, the right-hand side of inequality (3.3) approaches zero so that

$$\int_{\Omega} (|u|^{p-2}u - |\beta v|^{p-2}\beta v)^+ \, dx \Big|_{t_2} \leq 0$$

for a.e. t_2 . We conclude that

$$(|u|^{p-2}u - |\beta v|^{p-2}\beta v)^+ = 0.$$

The result follows as $\beta \rightarrow 1$. $\square$

4 Comparison in star-shaped domains

The restriction $\text{ess inf } v > 0$ or $\text{ess inf } u > 0$ is *not* assumed in this section. In star-shaped domains, we may prove uniqueness for non-negative solutions, provided that the lateral boundary values are zero. Convex domains are of this type. The initial values, say $g(x)$, have to be attained for a supersolution (or subsolution) v at least in the sense that

$$\lim_{t \rightarrow 0^+} \int_{\Omega} |v(x, t)^{p-1} - g(x)^{p-1}| \, dx = 0. \quad (4.1)$$

In the comparison principle below, at least the “smaller function” has zero lateral boundary values. (We aim at the eigenvalue problem (1.3).)

Theorem 2. Suppose Ω is star-shaped. Let $v \geq 0$ be a weak supersolution with initial values $v(x, 0) = g(x)$. Assume that $u \geq 0$ is a weak subsolution with the same initial values $u(x, 0) = g(x)$ and with zero lateral values

$$u \in L^p_{\text{loc}}(0, T; W^{1,p}_0(\Omega)).$$

In the range $1 < p < 2$ we assume, in addition, that u is continuous. Then $v \geq u$ a.e. in Ω_T .

Remark 2. If the lateral boundary values are taken in the classical sense

$$\lim_{(y,\tau) \rightarrow (x,t)} u(y, \tau) = 0 \quad \text{when } (x, t) \in \partial\Omega \times (0, T),$$

the conclusion is still valid.

Proof. We may exclude the case $v \equiv 0$, since then also $g \equiv 0$. Thus, $v > 0$ in Ω_T by the weak Harnack inequality ([9, Theorem 7.1]). We may assume that Ω is star-shaped with respect to the origin. Consider the weak supersolution

$$w_\alpha(x, t) = v(\alpha x, \alpha^p t), \quad 0 < \alpha < 1.$$

Let $0 < t_1 < t_2 < T$. Now, again by the same weak Harnack inequality,

$$w_\alpha(x, t) \geq c_\alpha > 0 \quad \text{when } (x, t) \in \bar{\Omega} \times [t_1, t_2],$$

where c_α is a positive constant. (For the original v this inequality may fail.) Hence, the comparison principle in Theorem 1 applies for w_α and u in the subdomain $\Omega \times [t_1, t_2]$ so that

$$\int_{\Omega} [u^{p-1} - (\beta w_\alpha)^{p-1}]^+ dx \Big|_{t_2} \leq \int_{\Omega} [u^{p-1} - (\beta w_\alpha)^{p-1}]^+ dx \Big|_{t_1},$$

when $\beta > 1$. Now we can send t_1 to 0. It follows that

$$\int_{\Omega} [u^{p-1} - (\beta w_\alpha)^{p-1}]^+ dx \Big|_{t_2} \leq \int_{\Omega} [g(x)^{p-1} - (\beta g(\alpha x))^{p-1}]^+ dx.$$

We can now safely send $\alpha \rightarrow 1-$ and $\beta \rightarrow 1+$, whence

$$\int_{\Omega} [u(x, t_2)^{p-1} - v(x, t_2)^{p-1}]^+ dx \leq 0.$$

We conclude that $u(x, t_2) \leq v(x, t_2)$ when $x \in \Omega$. It follows that $u \leq v$ a.e. in Ω_T . □

In the next corollary we again assume that the lateral boundary values are taken in the sense that

$$v \in L^p_{\text{loc}}(0, T; W^{1,p}_0(\Omega))$$

and the initial values are as in (4.1).

Corollary 2. Assume $g \in L^p(\Omega)$ and Ω star-shaped. Then the solution v of the problem

$$\begin{cases} \partial_t(|v|^{p-2}v) = \Delta_p v, & (x, t) \in \Omega \times (0, T), \\ v(x, 0) = g(x) \geq 0, & x \in \Omega, \\ v(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \end{cases}$$

is unique.

In particular, we obtain that solutions with extremals of (1.2) as initial data are separable.

Corollary 3. Assume that Ω is a star-shaped domain. Suppose v is a weak solution of

$$\begin{cases} \partial_t(|v|^{p-2}v) = \Delta_p v, & (x, t) \in \Omega \times (0, T), \\ v(x, 0) = u_p(x), & x \in \Omega, \\ v(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \end{cases}$$

where u_p is an extremal of (1.2). Then

$$v = e^{-\frac{\lambda_p t}{p-1}} u_p.$$

5 Extremals and large time behavior

When the initial data are comparable to an extremal of (1.2), we are able to extend the comparison principle to more general domains than star-shaped ones. If the boundary is $C^{1,\alpha}$ -regular, it is known that the extremals are $C^{1,\alpha}$ up to the boundary and that the estimates are uniform if the $C^{1,\alpha}$ -norm of the boundary is uniformly controlled. See [12, Theorem 1]. As a consequence we have the following lemma for the extremals of (1.2) under an exhaustion

$$\Omega = \bigcup_{j=1}^{\infty} \Omega_j, \quad \Omega_j \subset\subset \Omega_{j+1}$$

of the domain.

Lemma 1. *Suppose Ω is a $C^{1,\alpha}$ -domain. Then there are a sequence of uniformly $C^{1,\alpha}$ -regular sets $\Omega_j \subset\subset \Omega$ exhausting Ω and a sequence of numbers $c_j \rightarrow 1$ such that*

$$c_j u_p^j \leq u_p \quad \text{in } \Omega_j.$$

Here u_p^j and u_p are the extremals of (1.2) in Ω_j and Ω , respectively.

Proof. We argue by contradiction. Take $\varepsilon > 0$. If the result is not true, then for any choice of Ω_j and for each $j > 0$, there is a point $x_j \in \Omega_j$ such that

$$\frac{u_p^j(x_j)}{u_p(x_j)} \geq 1 + \varepsilon. \quad (5.1)$$

It is clear that x_j must converge to a point in $\partial\Omega$ since the quotient is uniformly convergent to 1 in every compact subdomain of Ω .

Suppose for simplicity that $x_j \rightarrow 0 \in \partial\Omega$ and that near the origin the boundary of Ω is the hyperplane $x_1 = 0$.⁴ We choose the boundary of Ω_j to be the hyperplane $x_1 = \frac{1}{j}$. By [12, Theorem 1], the functions u_p^j are uniformly $C^{1,\alpha}(U \cap \{x_1 \geq \frac{1}{j}\})$ for some neighborhood U of the origin. By inspecting the functions

$$v^j(x) = u_p^j\left(x + \frac{e_1}{j}\right), \quad x \in U \cap \{x_1 \geq 0\},$$

it is easy to conclude from Ascoli's Theorem that up to a subsequence, $\nabla v^j(0) = \nabla u_p^j(\frac{e_1}{j})$ converges to $\nabla u_p(0)$ when $j \rightarrow \infty$.

We may also assume that $x_j = \frac{e_1}{j} + z_j e_1$, where $z_j > 0$ and $z_j \rightarrow 0$ (otherwise we may just translate the origin along the hyperplane). We note that $\nabla u_p(0)$ and $\nabla u_p^j(x_j)$ both point in the e_1 direction. Taylor expansion gives

$$u_p(x_j) = \left(\frac{1}{j} + z_j\right) |\nabla u_p(0)| + \mathcal{O}(x_j^{1+\alpha}). \quad (5.2)$$

Since $\nabla u_p^j(\frac{e_1}{j})$ converges to $\nabla u_p(0)$, we may write

$$\nabla u_p^j\left(\frac{e_1}{j}\right) = |\nabla u_p(0)| e_1 + \delta_j e_1, \quad \delta_j \rightarrow 0.$$

Again, using Taylor expansion

$$\begin{aligned} u_p^j(x_j) &= \nabla u_p^j\left(\frac{e_1}{j}\right) \cdot e_1 z_j + \mathcal{O}(z_j^{1+\alpha}) \\ &= (|\nabla u_p(0)| e_1 + \delta_j e_1) e_1 z_j + \mathcal{O}(z_j^{1+\alpha}) \\ &= |\nabla u_p(0)| z_j + \delta_j z_j + \mathcal{O}(z_j^{1+\alpha}) \\ &= |\nabla u_p(0)| \left(\frac{1}{j} + z_j\right) + \mathcal{O}(x_j^{1+\alpha}) - \frac{|\nabla u_p(0)|}{j} - \mathcal{O}(x_j^{1+\alpha}) + \delta_j z_j + \mathcal{O}(z_j^{1+\alpha}), \end{aligned}$$

⁴ The geometry can be transformed into this upon making a local coordinate transformation with a $C^{1,\alpha}$ -function.

where $\delta_j \rightarrow 0$. Hence

$$u_p^j(x_j) = u_p(x_j) - \frac{|\nabla u_p(0)|}{j} - \mathcal{O}(x_j^{1+\alpha}) + \delta_j z_j + \mathcal{O}(z_j^{1+\alpha}) \leq u_p(x_j) + \mathcal{O}(x_j^{1+\alpha}) + \delta_j x_j e_1,$$

since $|z_j| \leq |x_j|$. Therefore, using (5.2) again,

$$\begin{aligned} \frac{u_p^j(x_j)}{u_p(x_j)} &\leq \frac{u_p(x_j) + \mathcal{O}(x_j^{1+\alpha}) + \delta_j x_j e_1}{u_p(x_j)} \\ &= 1 + \frac{\mathcal{O}(x_j^{1+\alpha}) + \delta_j x_j e_1}{u_p(x_j)} \\ &= 1 + \frac{\mathcal{O}(x_j^{1+\alpha}) + \delta_j x_j e_1}{x_j e_1 |\nabla u_p(0)| + \mathcal{O}(x_j^{1+\alpha})} \rightarrow 1. \end{aligned}$$

This contradicts the antithesis (5.1). $\square$

In the next proposition the assumption that $g \geq u_p$ admits multiplication of u_p by arbitrarily small constants. Thus the restriction is crucial only near the boundary $\partial\Omega$.

Proposition 1. Assume that Ω is a $C^{1,\alpha}$ -domain. Suppose v is a non-negative weak solution of

$$\begin{cases} \partial_t(|v|^{p-2}v) = \Delta_p v, & (x, t) \in \Omega \times (0, T), \\ v(x, 0) = g(x) \geq 0, & x \in \Omega, \\ v(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T). \end{cases}$$

Assume in addition that $g \geq u_p$, where u_p is an extremal of (1.2). Then

$$v \geq e^{-\frac{\lambda_p}{p-1}t} u_p$$

in $\Omega \times (0, T)$. Similarly, if $g \leq u_p$, then the reverse inequality holds.

Proof. By Harnack's inequality ([9, Theorem 2.1]) we can again conclude that $v > 0$ in Ω_T . Indeed, if v vanishes at some point, then g vanishes identically, which is excluded by the hypothesis.

Let Ω_j be a sequence of smooth subdomains exhausting Ω . Let u_p^j be the extremal in Ω_j with eigenvalue λ_p^j and the same L^p -norm as u_p . In [14, p. 189], it is argued for that $\lambda_p^j \rightarrow \lambda_p$. By Lemma 1, we may choose Ω_j for which there are constants $c_j \rightarrow 1$ such that $c_j u_p^j \leq u_p$ in Ω_j for j large enough. Since $\min\{v\} > 0$ in $\overline{\Omega_j} \times (0, T)$ and $v(x, 0) = g(x) \geq u_p(x) \geq c_j u_p^j(x)$ for all $x \in \Omega_j$, comparison (Corollary 1) implies

$$v \geq c_j e^{-\frac{\lambda_p^j}{p-1}t} u_p^j.$$

We may pass to the limit in the above inequality and conclude that $v \geq e^{-\frac{\lambda_p}{p-1}t} u_p$. The reversed inequality can be proved similarly. $\square$

As a corollary we obtain that solutions with extremals of (1.2) as initial data are separable.

Corollary 4. Assume that Ω is a $C^{1,\alpha}$ -domain. Suppose v is a weak solution of

$$\begin{cases} \partial_t(|v|^{p-2}v) = \Delta_p v, & (x, t) \in \Omega \times (0, T), \\ v(x, 0) = u_p(x), & x \in \Omega, \\ v(x, t) = 0, & x \in \partial\Omega, \end{cases}$$

where u_p is an extremal of (1.2). Then

$$v = e^{-\frac{\lambda_p}{p-1}t} u_p.$$

We can now obtain pointwise control of the large time behavior assuming that the initial data g satisfy $0 \leq g \leq u_p$, where u_p is an extremal of (1.2).

Corollary 5. Assume that Ω is a $C^{1,\alpha}$ -domain and that $g \in C(\overline{\Omega})$ satisfies $0 \leq g \leq u_p$, where u_p is an extremal of (1.2). Then the convergence

$$u(x) = \lim_{t \rightarrow \infty} e^{\frac{\lambda_p}{p-1}t} v(x, t)$$

is uniform in $\overline{\Omega}$. If in addition $g \geq w_p$, where w_p is another extremal, then the limit function $u(x)$ is non-zero and therefore an extremal.

Proof. Let τ_k be an increasing sequence of positive numbers such that $\tau_k \rightarrow \infty$ as $k \rightarrow \infty$. In [7, Theorem 1] it is proved that $\lim_{k \rightarrow \infty} e^{\frac{\lambda_p \tau_k}{p-1}} v(\cdot, \tau_k)$ exists in $L^p(\Omega)$. We now argue that this convergence is uniform in $\overline{\Omega}$. Let

$$v^k(x, t) = e^{\frac{\lambda_p \tau_k}{p-1}} v(x, t + \tau_k).$$

We remark that $e^{-\frac{\lambda_p t}{p-1}} u_p$ is a solution of Trudinger's equation. By the comparison with extremals (Proposition 1) and the fact that v is non-negative,

$$0 \leq v^k(x, t) \leq e^{-\frac{\lambda_p t}{p-1}} u_p(x) \leq u_p(x),$$

for $(x, t) \in \Omega \times [-1, 1]$ for all $k \in \mathbb{N}$ large enough. These bounds together with the local Hölder continuity ([11, Theorem 2.8] and [10, Theorem 2.5]) give that v^k is uniformly bounded in $C^\alpha(B \times [0, 1])$ for any ball $B \subset \subset \Omega$. Using a covering argument and that u_p is continuous up to the boundary, it is standard to conclude that v^k is equicontinuous in $\overline{\Omega} \times [0, 1]$. From this the result follows as in [6, proof of Theorem 1.3].

The last part of the statement follows from the observation that in this case we also have the bound

$$v^k(x, t) \geq e^{-\frac{\lambda_p t}{p-1}} w_p(x),$$

which forces the limit to be non-zero. The result then follows from [7, Theorem 1]. $\square$

Remark 3. We note that Corollary 5 can be proved using Theorem 2 if Ω is a star-shaped Lipschitz domain and not necessarily a $C^{1,\alpha}$ -domain.

6 Weak and viscosity solutions

We shall prove that weak solutions are also viscosity solutions when $p \geq 2$; see the definition in [2]. For the general theory we refer to [3]. As always, this requires at least a comparison principle for classical solutions.

Theorem 3. Let $p \geq 2$. A weak solution $v \geq 0$ of

$$\partial_t(v^{p-1}) = \Delta_p v$$

in Ω_T is also a viscosity solution.

Proof. We treat supersolutions and subsolutions separately. The result follows by combining the parts.

Part 1: Supersolution. By [9, Theorem 7.1], either $v > 0$ in $\Omega \times (0, T)$ or there is a time t^* such that v is identically zero in $\Omega \times (0, t^*)$ and $v > 0$ in $\Omega \times (t^*, T)$. Therefore, we may as well assume that $v > 0$. Suppose that there is a test function⁵ ϕ such that $\phi(x_0, t_0) = v(x_0, t_0)$ and $\phi < v$ near (x_0, t_0) .⁶ We assume, towards a contradiction, that the viscosity inequality for ϕ fails:

$$\partial_t(\phi(x_0, t_0)^{p-1}) < \Delta_p \phi(x_0, t_0).$$

By continuity, the inequalities

$$\partial_t(\phi^{p-1}) < \Delta_p \phi, \quad \phi > 0$$

hold in a neighborhood of (x_0, t_0) . Again, by continuity, we can find a $\lambda > 1$ and a possibly smaller cylindrical neighborhood $B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta)$ such that $\lambda \phi < v$ on the parabolic boundary of $B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta)$.

⁵ C^2 in x and C^1 in t .

⁶ The definition in [2] does not require strict inequality away from (x_0, t_0) . This however can be accomplished by subtracting $(x - x_0)^4 + (t - t_0)^4$ from ϕ .

Note also that the inequality

$$\partial_t((\lambda\phi)^{p-1}) < \Delta_p(\lambda\phi)$$

remains true in $B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta)$ by homogeneity. Therefore, Theorem 1 implies

$$\lambda\phi \leq v$$

in $B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta)$. This contradicts that $\phi(x_0, t_0) = v(x_0, t_0)$ since $\phi(x_0, t_0) = v(x_0, t_0) > 0$.

Part 2: Subsolution. Now we prove that v is a viscosity subsolution. Suppose $\phi(x_0, t_0) = v(x_0, t_0)$ and $\phi > v$ otherwise. If $v(x_0, t_0) > 0$, we may argue as in the case of a supersolution. Suppose instead that $v(x_0, t_0) = 0$. Then ϕ is a non-negative function attaining a minimum at (x_0, t_0) . Hence, $\phi_t(x_0, t_0) = 0$, $\nabla\phi(x_0, t_0) = 0$ and $D^2\phi(x_0, t_0) \geq 0$. Hence,

$$\partial_t(\phi^{p-1}) - \Delta_p\phi \leq 0,$$

as required. $\square$

A Appendix

The pointwise maximum of a subsolution and a positive constant is again a subsolution. To be on the safe side, we present a proof that does *not* use the comparison principle.

Proposition 2. *Let $u \geq 0$ be a weak subsolution in $\Omega_T = \Omega \times (0, T)$ and $c > 0$. If $p < 2$, assume in addition that u is continuous. Then the function $\max\{u, c\}$ is also a weak subsolution in Ω_T .*

Proof. Since u is a subsolution, we have as in (3.5)

$$-\int_0^T \int_\Omega \partial_t([u^{p-1} - c^{p-1}]_h) \phi \, dx \, dt \geq \int_0^T \int_\Omega [|\nabla u|^{p-2} \nabla u]_h \cdot \nabla \phi \, dx \, dt, \quad (\text{A.1})$$

for $\phi \geq 0$ in $C_0^\infty(\Omega_T)$, provided that the parameter h in the Steklov average is small enough. Let $\psi \geq 0$ be a test function in $C_0^\infty(\Omega_T)$. Now

$$\phi(x, t) = \psi(x, t) H_\delta([u^{p-1} - c^{p-1}]_h)$$

is a valid test function in (A.1) for $0 < h < h_\psi$ say; here H_δ is as in the proof of Theorem 1. Using the rule

$$\partial_t G_\delta([u^{p-1} - c^{p-1}]_h) = H_\delta([u^{p-1} - c^{p-1}]_h) \partial_t([u^{p-1} - c^{p-1}]_h)$$

and integrating by parts, we obtain

$$\begin{aligned} \int_0^T \int_\Omega G_\delta([u^{p-1} - c^{p-1}]_h) \psi_t \, dx \, dt &\geq \int_0^T \int_\Omega H_\delta([u^{p-1} - c^{p-1}]_h) [|\nabla u|^{p-2} \nabla u]_h \cdot \nabla \psi \, dx \, dt \\ &\quad + \frac{1}{\delta} \iint_{\{0 < [u^{p-1} - c^{p-1}]_h < \delta\}} [|\nabla u|^{p-2} \nabla u]_h \cdot \nabla [u^{p-1}]_h \psi \, dx \, dt. \end{aligned}$$

Now we wish to pass $h \rightarrow 0$. By standard reasoning with Steklov averages we obtain

$$\begin{aligned} \int_0^T \int_\Omega G_\delta(u^{p-1} - c^{p-1}) \psi_t \, dx \, dt &\geq \int_0^T \int_\Omega H_\delta(u^{p-1} - c^{p-1}) |\nabla u|^{p-2} \nabla u \cdot \nabla \psi \, dx \, dt \\ &\quad + \limsup_{h \rightarrow 0} \frac{1}{\delta} \iint_{\{0 < [u^{p-1} - c^{p-1}]_h < \delta\}} \psi [|\nabla u|^{p-2} \nabla u]_h \cdot \nabla [u^{p-1}]_h \, dx \, dt. \end{aligned} \quad (\text{A.2})$$

We claim that, upon extracting a subsequence,

$$\lim_{h \rightarrow 0} \iint_{\{0 < [u^{p-1} - c^{p-1}]_h < \delta\}} \psi [|\nabla u|^{p-2} \nabla u]_h \cdot \nabla [u^{p-1}]_h \, dx \, dt = (p-1) \iint_{\{0 < u^{p-1} - c^{p-1} < \delta\}} \psi u^{p-2} |\nabla u|^p \, dx \, dt \geq 0,$$

so that the last integral in (A.2) can be thrown away (before we send δ to zero).

This requires some estimates. First, we note the convergence

$$[|\nabla u|^{p-2} \nabla u]_h \rightarrow |\nabla u|^{p-2} \nabla u \quad \text{in } L_{\text{loc}}^{\frac{p}{p-1}}(\Omega_T)$$

for the first factor. The second factor is treated separately depending on the sign of $p - 2$. In the case $p > 2$ we have by Hölder's inequality

$$\begin{aligned} |[u^{p-2} \nabla u]_h| &= \left| \frac{1}{h} \int_{t-h}^h u^{p-2} |\nabla u| \, d\tau \right| \\ &\leq \left(\frac{1}{h} \int_{t-h}^h u^{p-1} \, d\tau \right)^{\frac{p-2}{p-1}} \left(\frac{1}{h} \int_{t-h}^h |\nabla u|^{p-1} \, d\tau \right)^{\frac{1}{p-1}} \\ &\leq ([u^{p-1}]_h)^{\frac{p-2}{p-1}} \left(\frac{1}{h} \int_{t-h}^h |\nabla u|^{p-1} \, d\tau \right)^{\frac{1}{p-1}} \\ &\leq (c^{p-1} + \delta)^{\frac{p-2}{p-1}} [|\nabla u|^p]_h^{\frac{1}{p}}, \end{aligned}$$

in the set $\{0 < [u^{p-1} - c^{p-1}]_h < \delta\}$. Hence,

$$\iint_{\{0 < [u^{p-1} - c^{p-1}]_h < \delta\}} \psi |[u^{p-2} \nabla u]_h|^p \, dx \, dt \leq C \iint_{\Omega_T} \psi [|\nabla u|^p]_h \, dx \, dt,$$

where $C = (c^{p-1} + \delta)^{\frac{p(p-2)}{p-1}}$. The last integral is uniformly bounded in h .

In the case $p < 2$ we instead argue as follows. The assumed continuity implies that $u > \frac{c}{2}$ in the support of ψ , when $[u^{p-1} - c^{p-1}]_h > 0$ and for $h < h(c)$ small enough.

As a consequence, when $[u^{p-1} - c^{p-1}]_h > 0$, we have the estimate

$$|[u^{p-2} \nabla u]_h| = \left| \frac{1}{h} \int_{t-h}^h u^{p-2} \nabla u \, d\tau \right| \leq \left| \frac{c}{2} \right|^{p-2} \frac{1}{h} \int_{t-h}^h |\nabla u| \, d\tau \leq \left| \frac{c}{2} \right|^{p-2} [|\nabla u|]_h.$$

Therefore,

$$\iint_{\{0 < [u^{p-1} - c^{p-1}]_h < \delta\}} \psi |[u^{p-2} \nabla u]_h|^p \, dx \, dt \leq \left| \frac{c}{2} \right|^{p(p-2)} \iint_{\Omega_T} \psi [|\nabla u|]_h^p \, dx \, dt.$$

In both cases we can conclude that the integrand in the problematic term is a product of a function converging in $L_{\text{loc}}^{p/(p-1)}$ and a function with compact support which is bounded in L_{loc}^p . This is enough to pass $h \rightarrow 0$, after having extracted a subsequence.

Throwing away the positive term and letting $\delta \rightarrow 0$, we obtain

$$\iint_{\Omega_T \cap \{u > c\}} (u^{p-1} - c^{p-1}) \psi_t \, dx \, dt \geq \iint_{\Omega_T \cap \{u > c\}} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi \, dx \, dt. \quad (\text{A.3})$$

Here we have used that $H_\delta(u^{p-1} - c^{p-1}) \rightarrow \chi_{\{u > c\}}$ in L^q for any $q > 0$ and in particular in L^p . When multiplied with the $L^{\frac{p}{p-1}}$ -function $|\nabla u|^{p-2} \nabla u$, the product converges. It now remains to note that

$$\begin{aligned} \iint_{\Omega_T \cap \{u > c\}} (u^{p-1} - c^{p-1}) \psi_t \, dx \, dt &= \int_0^T \int_{\Omega} (\max\{u, c\})^{p-1} \psi_t \, dx \, dt - \int_0^T \int_{\Omega} c^{p-1} \psi_t \, dx \, dt \\ &= \int_0^T \int_{\Omega} (\max\{u, c\})^{p-1} \psi_t \, dx \, dt. \end{aligned}$$

Therefore (A.3) is equivalent to

$$\int_0^T \int_{\Omega} (\max\{u, c\})^{p-1} \psi_t \, dx \, dt \geq \int_0^T \int_{\Omega} |\nabla \max\{u, c\}|^{p-2} \nabla \max\{u, c\} \cdot \nabla \psi \, dx \, dt.$$

This is the desired inequality for a subsolution. $\square$

Corollary 6. *Under the same assumptions as in Proposition 2, $u \in L_{\text{loc}}^{\infty}(\Omega_T)$.*

Proof. By [9, Lemma 5.1], the weak subsolution $\max\{u, 1\}$ is locally bounded. So is, of course, then u . $\square$

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References

- [1] H. W. Alt and S. Luckhaus, Quasilinear elliptic-parabolic differential equations, *Math. Z.* **183** (1983), no. 3, 311–341.
- [2] T. Bhattacharya and L. Marazzi, On the viscosity solutions to Trudinger's equation, *NoDEA Nonlinear Differential Equations Appl.* **22** (2015), no. 5, 1089–1114.
- [3] M. G. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), no. 1, 1–67.
- [4] E. DiBenedetto, *Degenerate Parabolic Equations*, Universitext, Springer, New York, 1993.
- [5] U. Gianazza and V. Vespi, A Harnack inequality for solutions of doubly nonlinear parabolic equations, *J. Appl. Funct. Anal.* **1** (2006), no. 3, 271–284.
- [6] R. Hynd and E. Lindgren, Hölder estimates and large time behavior for a nonlocal doubly nonlinear evolution, *Anal. PDE* **9** (2016), no. 6, 1447–1482.
- [7] R. Hynd and E. Lindgren, Large time behavior of solutions of Trudinger's equation, preprint (2017), <https://arxiv.org/abs/1702.01630>.
- [8] A. V. Ivanov, P. Z. Mkrtchyan and V. Jäger, Existence and uniqueness of a regular solution of the Cauchy–Dirichlet problem for a class of doubly nonlinear parabolic equations, *J. Math. Sci.* **84** (1997), 845–855.
- [9] J. Kinnunen and T. Kuusi, Local behaviour of solutions to doubly nonlinear parabolic equations, *Math. Ann.* **337** (2007), no. 3, 705–728.
- [10] T. Kuusi, R. Laleoglu, J. Siljander and J. M. Urbano, Hölder continuity for Trudinger's equation in measure spaces, *Calc. Var. Partial Differential Equations* **45** (2012), no. 1–2, 193–229.
- [11] T. Kuusi, J. Siljander and J. M. Urbano, Local Hölder continuity for doubly nonlinear parabolic equations, *Indiana Univ. Math. J.* **61** (2012), no. 1, 399–430.
- [12] G. M. Lieberman, The Dirichlet problem for quasilinear elliptic equations with continuously differentiable boundary data, *Comm. Partial Differential Equations* **11** (1986), no. 2, 167–229.
- [13] P. Lindqvist, On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, *Proc. Amer. Math. Soc.* **109** (1990), no. 1, 157–164.
- [14] P. Lindqvist, On a nonlinear eigenvalue problem, in: *Fall School in Analysis* (Jyväskylä 1994), Report 68, Univ. Jyväskylä, Jyväskylä (1995), 33–54.
- [15] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod; Gauthier-Villars, Paris, 1969.
- [16] J. J. Manfredi and V. Vespi, Large time behavior of solutions to a class of doubly nonlinear parabolic equations, *Electron. J. Differential Equations* **1994** (1994), Paper No. 2.
- [17] M. M. Porzio and V. Vespi, Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations, *J. Differential Equations* **103** (1993), no. 1, 146–178.
- [18] P. A. Raviart, Sur la résolution de certaines équations paraboliques non linéaires, *J. Funct. Anal.* **5** (1970), 299–328.
- [19] J. Siljander, Boundedness of the gradient for a doubly nonlinear parabolic equation, *J. Math. Anal. Appl.* **371** (2010), no. 1, 158–167.
- [20] S. Sturm, Existence of weak solutions of doubly nonlinear parabolic equations, *J. Math. Anal. Appl.* **455** (2017), no. 1, 842–863.
- [21] S. Sturm, Pointwise estimates via parabolic potentials for a class of doubly nonlinear parabolic equations with measure data, *Manuscripta Math.* **157** (2018), no. 3–4, 295–322.
- [22] N. S. Trudinger, Pointwise estimates and quasilinear parabolic equations, *Comm. Pure Appl. Math.* **21** (1968), 205–226.
- [23] V. Vespi, On the local behaviour of solutions of a certain class of doubly nonlinear parabolic equations, *Manuscripta Math.* **75** (1992), no. 1, 65–80.