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# Inverse Limit Spaces Satisfying a Poincaré Inequality

**Abstract:** We give conditions on Gromov-Hausdorff convergent inverse systems of metric measure graphs which imply that the measured Gromov-Hausdorff limit (equivalently, the inverse limit) is a PI space i.e., it satisfies a doubling condition and a Poincaré inequality in the sense of Heinonen-Koskela [12]. The Poincaré inequality is actually of type $(1, 1)$. We also give a systematic construction of examples for which our conditions are satisfied. Included are known examples of PI spaces, such as Laakso spaces, and a large class of new examples. As follows easily from [4], generically our examples have the property that they do not bilipschitz embed in any Banach space with Radon-Nikodym property. For Laakso spaces, this was noted in [4]. However according to [7] these spaces admit a bilipschitz embedding in $L_1$. For Laakso spaces, this was announced in [5].

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## 1 Introduction

This paper is part of a series concerning bilipschitz embeddability and PI spaces, i.e. metric measure spaces which satisfy a doubling condition and a Poincaré inequality; [2], [3], [4], [8], [5], [6], [9], [7]. In this paper we give a systematic construction of PI spaces as inverse limits, or equivalently Gromov-Hausdorff limits, of certain inverse systems of metric measure graphs which we term “admissible” (see Section 2 for the definition). Included are known examples of PI spaces, such as Laakso spaces ([15]) and a large class of new examples.

Our main result is:

**Theorem 1.1.** *The measured Gromov-Hausdorff limit of an admissible inverse system is a PI space satisfying a* $(1, 1)$*-Poincaré inequality. Moreover, the doubling constant* $\beta$ *and the constants* $\tau$, $\Lambda$ *in the Poincaré inequality depend only on the constants* $2 \le m \in \mathbb{N}$, $\Delta$, $\theta$, $C \in (0, \infty)$ *in conditions (1)–(6) for admissible inverse systems.*

For these limit spaces the dimension of the cotangent bundle is 1, the topological dimension is 1 and except in certain “degenerate” cases, the Hausdorff dimension is $> 1$. It follows from [7] that the spaces we construct admit bilipschitz embeddings in $L_1$. For Laakso spaces, this was announced in [5]. However, except in the degenerate cases, they do not bilipschitz embed in any Banach space with the Radon-Nikodym Property. For Laakso spaces, this was noted in [4].

One of the novelties in this paper is a new approach to proving the Poincaré inequality that exploits the fact that the metric measure space is the limit of an inverse system

$$X_0 \xleftarrow{\pi_0} \cdots \xleftarrow{\pi_{i-1}} X_i \xleftarrow{\pi_i} \cdots .$$

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The argument, which is by induction, involves averaging a function on $X_{i+1}$ over the fibers of the projection map $\pi_i : X_{i+1} \to X_i$, to produce a function on $X_i$. The averaging operator is defined by specifying, for each $x \in X_i$, a probability measure $\mathcal{D}_i(x)$ supported on the fiber $\pi_i^{-1}(x) \subset X_{i+1}$; for a generic point $x \in X_i$, the choice of $\mathcal{D}_i(x)$ is canonical. The key point is that under a certain condition (see Axiom (6) from Definition 2.10) this canonical assignment extends to one that is continuous with respect to the weak topology on Radon measures, and that is compatible with the operation of taking upper gradients. This new proof of the Poincaré inequality is robust and applies verbatim to certain higher dimensional inverse systems.

We mention that in [16], A. Schioppa has shown that the examples constructed in the present paper provide (many) examples of metric spaces $(X, d)$ which admit a continuum of mutually singular measures $\mu$ such that $(X, d, \mu)$ is doubling and admits a Poincaré inequality. He also observes that by way of contrast, from a recent announcement of P. Jones and M. Csornyei, it follows that if $\underline{d}$ is the standard metric on $\mathbb{R}^n$ and $\mu$ is a measure on $\mathbb{R}^n$ such that $(\mathbb{R}^n, \underline{d}, \mu)$ is doubling and admits a Poincaré inequality, then $\mu$ is absolutely continuous with respect to Lebesgue measure.

### Organization of the paper

In Section 2, after we recall some standard material, we state the six axioms which define admissible inverse systems, discuss the role of the axioms, and draw some simple consequences. Among a number of other things, we show in Corollary 2.16 that the topological dimension of the inverse limit is 1.

In Section 3, for each $X_i$, we verify, with uniform constants, the Poincaré inequality locally at the scale associated with $X_i$, as well as the (global) doubling condition.

In Section 4, the last three axioms are reformulated in terms of what we call “continuous fuzzy sections” of the maps $\pi_i : X_{i+1} \to X_i$ of our inverse system. This reformulation plays a role in several places in the paper.

In Section 5, using the continuous fuzzy sections, we prove that the $X_i$'s satisfy a uniform Poincaré inequality; this implies that the Gromov-Hausdorff limit $X_\infty$ has a Poincaré inequality ([1, 13]) thereby proving Theorem 1.1.

In Section 6 we construct a natural probability measure on the family of paths in $X_k$ which are lifts of some fixed path in $X_j$ $(j < k)$.

In Section 7, we give a second, essentially different, proof of the Poincaré inequality for $X_\infty$ using the probability measure on path families.

In Section 8 we show how to construct large families of examples of admissible inverse systems. The construction produces a sequence of partial inverse systems

$$X_0 \xleftarrow{\pi_0} \cdots \xleftarrow{\pi_{i-1}} X_i$$

by induction on $i$; in the inductive step, roughly speaking, one makes independent choices locally in $X_i$ to produce $X_{i+1}$. Both fuzzy sections and the measure on path families play a role in the discussion.

In Section 9 we show that for an admissible inverse system, the dimension of the fibre of the cotangent bundle of the limit is 1.

In Section 10, we show that except in degenerate cases, limits of admissible systems do not bilipschitz embed in any Banach space with the Radon-Nikodym Property.

In Section 11 we briefly indicate how our previous discussion can be extended to certain higher dimensional inverse systems. In this case, depending on which building blocks one uses, for example the Heisenberg group with its Carnot-Caratheodory metric, the resulting inverse limit spaces need not bilipschitz embed in $L_1$.

## 2 Preliminaries

In this section we begin by collecting some standard definitions. Then we give the axioms for an admissible inverse system, briefly indicate the role of each of the axioms and observe some elementary consequences.

## 2.1 The doubling condition and the Poincaré inequality

We now recall some relevant definitions. Let $(X, d, \mu)$ denote a metric measure space, with $\mu$ a Borel measure on $X$, which is finite and nonzero on metric balls $B_r(x)$ if $0 < r < \infty$.

For $U$ measurable, with $0 < \mu(U) < \infty$, we set

$$f_U = \frac{1}{\mu(U)} \int_U f\, d\mu\,. \tag{2.1}$$

The measure $\mu$ is said to satisfy a *doubling condition* if there exists $\beta = \beta(R)$ such that for all $x \in X$

$$\mu(B_{2r}(x)) \le \beta \cdot \mu(B_r(x))\,, \qquad (r \le R)\,. \tag{2.2}$$

If $(X, d)$ is a metric space, $f : X \to \mathbb{R}$ and a nonnegative Borel function $g : X \to \mathbb{R}_+$, we say that $g$ is an *upper gradient* for $f$ if for all rectifiable curves $c : [0, L] \to X$ parameterized by arclength,

$$|f(c(L)) - f(c(0))| \le \int_0^L g(c(s))\, ds\,. \tag{2.3}$$

We say that $(X, d, \mu)$ *satisfies a* $(1, p)$*-Poincaré inequality* if for some $\Lambda$ and $\tau = \tau(R)$, we have for every bounded continuous function $f$ and every upper gradient $g$,

$$\int_{B_r(x)} |f - f_{B_r(x)}|\, d\mu \le \tau r \cdot \left( \int_{B_{\Lambda r}(x)} g^p\, d\mu \right)^{\frac{1}{p}} \qquad (r \le R)\,. \tag{2.4}$$

This definition and the definition of upper gradient are due to Heinonen-Koskela [12]; for additional information on the Poincaré inequality, see [11], [14].

It was shown in [13, Theorem 2] that $(X, d, \mu)$ satisfies a $(1, p)$-Poincaré inequality if and only if there exists $\underline{\tau} = \underline{\tau}(R)$ such that for every Lipschitz function $f$,

$$\int_{B_r(x)} |f - f_{B_r(x)}|\, d\mu \le \tau r \cdot \left( \int_{B_{\Lambda r}(x)} (\mathrm{Lip}\, f(x))^p\, d\mu \right)^{\frac{1}{p}} \qquad (r \le R)\,, \tag{2.5}$$

where $\mathrm{Lip}\, f$ denotes the pointwise Lipschitz constant of $f$:

$$\mathrm{Lip}\, f(x) := \limsup_{d(x',x)\to 0} \frac{|f(x') - f(x)|}{d(x', x)} \qquad (x' \neq x)\,.$$

**Definition 2.6.** If (2.2) and (2.4) hold, we say that $(X, d, \mu)$ is a PI space.

*Remark* 2.7. The examples constructed in this paper will satisfy (2.5) with $p = 1$, which is the strongest version of the Poincaré inequality.

## 2.2 Axioms for admissible inverse systems

We will consider inverse systems of connected metric measure graphs,

$$X_0 \xleftarrow{\pi_0} \cdots \xleftarrow{\pi_{i-1}} X_i \xleftarrow{\pi_i} \cdots\,. \tag{2.8}$$

Let $\mathrm{St}(x, G)$ denote the star of a vertex $x$ in a graph $G$, i.e. the union of the edges containing $x$.

We assume that each $X_i$ is connected and is equipped with a path metric $d_i$ and a measure $\mu_i$, such that the following conditions hold, for some constants $2 \le m \in \mathbb{Z}$, $\Delta$, $\theta$, $C \in (0, \infty)$ and every $i \in \mathbb{Z}$:

1. (Bounded local metric geometry) $(X_i, d_i)$ is a nonempty connected graph with all vertices of valence $\leq \Delta$, and such that every edge of $X_i$ is isometric to an interval of length $m^{-i}$ with respect to the path metric $d_i$.
2. (Simplicial projections are open) If $X_i'$ denotes the graph obtained by subdividing each edge of $X_i$ into $m$ edges of length $m^{-(i+1)}$, then $\pi_i$ induces a map $\pi_i : (X_{i+1}, d_{i+1}) \to (X_i', d_i)$ which is open, simplicial, and an isometry on every edge.
3. (Controlled fiber diameter) For every $x_i \in X_i'$, the inverse image $\pi_i^{-1}(x_i) \subset X_{i+1}$ has $d_{i+1}$-diameter at most $\theta \cdot m^{-(i+1)}$.
4. (Bounded local metric measure geometry.) The measure $\mu_i$ restricts to a constant multiple of arclength on each edge $e_i \subset X_i$, and $\frac{\mu_i(e_{i,1})}{\mu_i(e_{i,2})} \in [C^{-1}, C]$ for any two adjacent edges $e_{i,1}, e_{i,2} \subset X_i$.
5. (Compatibility with projections)
$$(\pi_i)_\star(\mu_{i+1}) = \mu_i\,,$$
where $(\pi_i)_\star(\mu_{i+1})$ denotes the pushforward of $\mu_{i+1}$ under $\pi_i$.
6. (Continuity) For all vertices $v_i' \in X_i'$, and $v_{i+1} \in \pi_i^{-1}(v_i')$, the quantity
$$\frac{\mu_{i+1}(\pi_i^{-1}(e_i') \cap \mathrm{St}(v_{i+1}, X_{i+1}))}{\mu_i(e_i')} \tag{2.9}$$
is the same for all edges $e_i' \in \mathrm{St}(v_i', X_i')$.

**Definition 2.10.** An inverse system of metric measure graphs as in (2.8) is called *admissible* if it satisfies (1)–(6).

## 2.3 Discussion of the axioms and some elementary consequences

Let us give a brief indication of the relevant consequences of each of our axioms. Note that the first three axioms deal only with the metric and not the measure. Indeed, taken together, Axioms (1) and (2) have the following purely combinatorial content which is worth noting at the outset, since it helps to picture the restricted class of inverse systems that we are considering.

**Proposition 2.11.** *Let $\{v_i\}$ denote a compatible sequence of vertices, i.e. $v_i$ is a vertex of $X_i$ and $\pi_i(v_{i+1}) = v_i$, for all $i \geq 0$. Then for all but at most $\Delta$ values of $i$, the restriction of the locally surjective map $\pi_i$ to the open star of $v_{i+1}$ is actually* 1-1.

*Proof.* From the local surjectivity of $\pi_i$ it follows that the number of edges emanating from $v_i$ is a nondecreasing function of $i$. Therefore, from the uniform bound $\Delta$ on the degree of a vertex, of $X_i$, for all $i$, the proposition follows. □

Axiom (1) includes the statement that $\pi_i : X_{i+1} \to X_i'$ is a finite-to-one simplicial map. This implies that the vertices of $X_{i+1}$ are precisely the inverse images of vertices of $X_i'$. The second part of Axiom (1) states that the restriction of $\pi_i$ to every edge is an isometry. In particular, $\pi_i : (X_{i+1}, d_{i+1}) \to (X_i, d_i)$ is 1-Lipschitz, i.e. distance nonincreasing. Axiom (1) also implies that for all $K > 0$, if the ball in $X_i$ of radius $\leq K \cdot m^{-i}$ is rescaled to unit size, then the metric geometry has a uniform bound depending on $K$ but independent of $i$.

Axiom (2), stating that $\pi_i$ is open, implies that if $c$ is a rectifiable path parameterized by arc length and $\pi_i(x_{i+1}) = c(0)$, then there exists a lift $\widetilde{c}$ parameterized by arc length, with $\widetilde{c}(0) = x_{i+1}$. In general, $\widetilde{c}$ is not unique. By Axiom (1), the paths $c$ and $\widetilde{c}$ have equal lengths and in addition, for all $i \geq 0$, $x_{i+1} \in X_{i+1}$ and $r > 0$, we have
$$\pi_i(B_r(x_{i+1})) = B_r(\pi_i(x_{i+1}))\,, \qquad B_r(x_{i+1}) \subset \pi_i^{-1}(B_r(\pi_i(x_{i+1})))\,. \tag{2.12}$$
Axiom (2) is actually a consequence of Axioms (4), (5).

Axiom (3), together with (2.12), gives

$$B_r(x_{i+1}) \subset \pi_i^{-1}(B_r(\pi_i(x_{i+1}))) \subset B_{r+\theta m^{-(i+1)}}(x_{i+1}) . \tag{2.13}$$

This statement, which can be iterated, says that inverse images of balls are themselves comparable to balls. It is used in the inductive arguments which control the constants in the doubling and Poincaré inequalities.

Axioms (1)–(3) imply that for all $x_{i+1,1}, x_{i+1,2} \in X_{i+1}$, we have

$$d_i(\pi_i(x_{i+1,1}), \pi_i(x_{i+1,2})) \le d_{i+1}(x_{i+1,1}, x_{i+1,2}) \le d_i(\pi_i(x_{i+1,1}), \pi_i(x_{i+1,2})) + 2\theta \cdot m^{-(i+1)} ; \tag{2.14}$$

compare (2.12), (2.13).

Note also that Axioms (1) and (3) together imply that for all $i$ and all $x_i \in X_i$ the cardinality $\text{card}(\pi_i^{-1}(x_i))$ satisfies

$$\text{card}(\pi_i^{-1}(x_i)) \le \Delta^{\theta+1} , \tag{2.15}$$

since any two points of $\pi_i^{-1}(x_i)$ are connected by an edge path of length $\le \theta \cdot m^{-(i+1)}$ and there are at most $\Delta^{\theta+1}$ such paths which start at a given point of $\pi_i^{-1}(x_i)$.

Axiom (4) implies that on scale $m^{-i}$ the metric measure geometry of $X_i$ is bounded. As a consequence, for balls $B_{cm^{-i}}(x_i) \subset X_i$ there is a doubling condition and Poincaré inequality with constants which depend only on $c$ and are independent of $i$; see for example Lemma 3.1.

Axiom (5) is used in showing that the sequence $(X_i, d_i, \mu_i)$ converges in the measured Gromov-Hausdorff sense. It also plays a role in the inductive arguments verifying the doubling condition and the Poincaré inequality.

Axiom (6) is the least obvious of our axioms. However, it enters crucially in both of the proofs that we give of the bound on the constant in the Poincaré inequality for $(X_\infty, d_\infty, \mu_\infty)$; see Sections 5–7. Here is a very brief indication of the role of Axiom (6). Given Axioms (1)–(5), the disintegration $x \mapsto \mathcal{D}_i(x)$ of the measure $\mu_{i+1}$ with respect to the mapping $\pi_i : X_{i+1} \to X_i$, can be used to push a function $f_{i+1} : X_{i+1} \to \mathbb{R}$ down to a function $f_i : X_i \to \mathbb{R}$. If $f_{i+1}$ is Lipschitz, then Axiom (4) implies that away from the vertices of $X_i'$, the pointwise Lipschitz constant of $f_i$ is controlled by that of $f_{i+1}$. It follows from Axiom (6) that $f_i$ is continuous at vertices, and hence the Lipschitz control holds at the vertices of $X_i'$ as well. This construction is a key part of the induction step in our first proof of the Poincaré inequality. (Absent Axiom (6), even if $f_{i+1}$ is Lipschitz, the function $f_i$ need not be continuous at the vertices of $X_i$.)

Dually, given Axioms (1)–(5), there is a natural probability measure $\Omega$ on the collection $\Gamma$ of lifts to $X_{i+1}$ of an edge path $\gamma_i' \subset X_i'$. If Axiom (6) holds, this measure has the additional property of being independent of the orientation of $\gamma_i'$. This turns out to be required for the proof of the Poincaré inequality based on path families.

## 2.4 The inverse limit

We recall that the *inverse limit* of the inverse system $\{X_i\}$ is the collection $X_\infty$ of compatible sequences, i.e.

$$X_\infty = \{(v_i) \in \prod_i X_i \mid \pi_i(v_{i+1}) = v_i \text{ for all } i \ge 0\} .$$

For all $i \ge 0$, one has a projection map $\pi_i^\infty : X_\infty \to X_i$ that sends $(v_j) \in X_\infty$ to $v_i$.

For any $(v_i), (w_i) \in X_\infty$, the sequence $\{d_j(v_j, w_j)\}$ is nondecreasing since the projection maps $\{\pi_j\}$ are 1-Lipschitz, and bounded above by (2.14); therefore we have a well-defined metric on the inverse limit given by

$$d_\infty((v_i), (w_i)) = \lim_{j \to \infty} d_j(v_j, w_j) .$$

The projection map $\pi_i^\infty : (X_\infty, d_\infty) \to (X_i, d_i)$ is 1-Lipschitz.

We now record a consequence of the above discussion:

**Corollary 2.16.** *The inverse limit $X_\infty$ has topological dimension* 1.

*Proof.* By the path lifting argument in the discussion of Axiom (2), one may take an edge $\gamma_0 \subset X_0$, and lift it isometrically to a compatible family $\{\gamma_j \subset X_j\}_{j\geq 0}$ which produces a geodesic segment in $X_\infty$. Therefore $X_\infty$ has topological dimension at least 1.

If $\mathcal{U}_i$ is the cover of $X_i$ by open stars of vertices, and $\hat{\mathcal{U}}_i$ is the inverse image of $\mathcal{U}_i$ under the projection map $X_\infty \to X_i$, then $\hat{\mathcal{U}}_i$ has 1-dimensional nerve, and the diameter of each open set $U \in \hat{\mathcal{U}}_i$ is $\lesssim m^{-i}$, see (2.13). For any compact subset $K \subset X_\infty$, and any open cover $\mathcal{U}$ of $K$, some $\hat{\mathcal{U}}_i$ will provide a refinement of $\mathcal{U}$; this shows that $K$ has topological dimension $\leq 1$. As $X_\infty$ is locally compact, it follows that $X_\infty$ has topological dimension $\leq 1$. □

We now discuss the measure on $X_\infty$. For every $i$, one obtains a subalgebra $\Sigma_i$ of the Borel $\sigma$-algebra on $X_\infty$ by taking the inverse image of the Borel $\sigma$-algebra on $X_i$. One readily checks using (2.13) that the $\sigma$-algebra generated by the countable union $\cup_i \Sigma_i$ is the full Borel $\sigma$-algebra on $X_\infty$. The $\sigma$-algebra $\Sigma_i$ has a measure $\hat{\mu}_i$ induced from $\mu_i$ by the projection $\pi_i^\infty$. Axiom (5) implies that the measures $\hat{\mu}_i$ on the increasing family $\{\Sigma_i\}$ are compatible under restriction, and by applying the Caratheodory extension theorem, one gets that the $\hat{\mu}_i$'s extend uniquely to a Borel measure $\mu_\infty$ on $X_\infty$.

## 2.5 Measured Gromov-Hausdorff convergence

In view of (2.14), and since $\pi_i^\infty$ is also surjective, it follows easily that the sequence of mappings $\{\pi_i^\infty : (X_\infty, d_\infty) \to (X_i, d_i)\}$ is Gromov-Hausdorff convergent; in particular the Gromov-Hausdorff limit is isometric to $(X_\infty, d_\infty)$. By bringing in Axiom (5), we get that the sequence $\{\pi_i^\infty : (X_\infty, d_\infty, \mu_\infty) \to (X_i, d_i, \mu_i)\}$ is convergent in the pointed measured Gromov-Hausdorff sense; for the definition, see [10]. Hence, we obtain:

**Proposition 2.17.** *The sequence $(X_i, d_i, \mu_i)$ converges in the pointed measured Gromov-Hausdorff sense to $(X_\infty, d_\infty, \mu_\infty)$.*

# 3 Bounded local geometry and verification of doubling

Consider an admissible inverse system as in (2.8), with constants, $2 \leq m \in \mathbb{N}$, $\Delta, \theta, C \in (0, \infty)$ as in (1)–(6). The following lemma asserts the existence of a local doubling condition, and a local Poincaré inequality.

**Lemma 3.1.** *For all $K > 0$, there exists $\beta' = \beta'(m, \Delta, \theta, C, K)$, $\tau = \tau(m, \Delta, \theta, C, K)$, $\Lambda(m, \Delta, \theta, C, K)$, such that for balls $B_r(x_i) \subset X_i$, with*

$$r \leq K \cdot m^{-i}\,,$$

*a doubling condition and (1,1)-Poincaré inequality hold, with constants $\beta' = \beta'(m, \Delta, \theta, C, K)$, $\tau = \tau(m, \Delta, \theta, C, K)$, $\Lambda = 1$.*

*Proof.* Since for $r$ as in the lemma, we are concerned with a connected 1-dimensional complex with bounded geometry, the doubling condition is apparent and the Poincaré inequality follows easily from the Poincaré inequality on the real line. □

Next we verify the doubling condition for balls of arbitrary radius.

**Lemma 3.2.** *There is a constant $\beta = \beta(\Delta, \theta, C, R)$ such that for all $i$ and all $r \leq R$, the doubling condition holds for $X_i$ with constant $\beta$.*

*Proof.* First, observe that since for all $k$, from (2.13) and by Axiom (5), $(\pi_k)_\star(\mu_{k+1}) = \mu_k$, we get for $x_{k+1} \in X_{k+1}$,

$$\mu_k(B_r(\pi_k(x_{k+1}))) \leq \mu_{k+1}(B_{r+\theta m^{-(k+1)}}(x_{k+1})) . \tag{3.3}$$

$$\mu_{k+1}(B_r(x_{k+1})) \leq \mu_k(B_r(\pi_k(x_{k+1}))) . \tag{3.4}$$

First assume that $R = 1$. Let $j$ be such that $m^{-(j+1)} < \frac{r}{1+2\theta} \leq m^{-j}$. Let $x_i \in X_i$ and consider $B_r(x_i)$. If $j \geq i$, the conclusion follows from from Lemma 3.1. Otherwise, for $j+1 \leq k \leq i$ inductively define $x_{k-1} = \pi_{k-1}(x_k)$. Since, $m^{-(j+1)} \leq \frac{r}{1+2\theta}$ by (3.3), (3.4) and induction we get

$$\mu_j(B_{\frac{r}{1+2\theta}}(x_j)) \leq \mu_i(B_r(x_i))) \leq \mu_i(B_{2r}(x_i)) \leq \mu_j(B_{2r}(x_j)) . \tag{3.5}$$

Since $x_j \in X_j$ and $\frac{r}{1+2\theta} \leq m^{-j}$, the conclusion follows from (3.5) and Lemma 3.1.

Now if $R > 1$, the doubling inequality for $X_0$, with $\beta = \beta(R)$ follows from the fact that $X_0$ has controlled degree. If we take $j = 0$, the conclusion follows as above. □

# 4 Continuous fuzzy sections

Let $\mathcal{P}(Z)$ denote the space of Borel probability measures on $Z$ with the weak topology.

**Definition 4.1.** Given a map of metric spaces $\pi : Y \to X$, a *fuzzy section of* $\pi$ is a Borel measurable map $\mathcal{D} : X \to \mathcal{P}(Y)$ such that $\mathcal{D}(x)$ is supported on $\pi^{-1}(x)$, for all $x \in X$. $\mathcal{D}$ is called a *continuous fuzzy section* if it is continuous with respect to the metric topology on $X$ and the weak topology of $\mathcal{P}(Y)$. The fuzzy sections in this paper are all atomic, i.e. $\mathcal{D}(x)$ is a finite convex combination of Dirac masses.

Here, we will observe that given an admissible inverse system $\{(X_i, d_i, \mu_i, \pi_i)\}$ as in (2.8), each of the maps $\pi_i : X_{i+1} \to X_i$ has a naturally associated continuous fuzzy section $\mathcal{D}_i$ defined via the measures $\mu_i, \mu_{i+1}$, which satisfies for some $c_0 > 0$,

$$\mathcal{D}_i(x_i)(x_{i+1}) \geq c_0 \qquad (\text{for all } i,\ x_i \in X_i,\ x_{i+1} \in \pi_i^{-1}(x_i)) , \tag{4.2}$$

and has the additional property that if $e_{i+1} \subset X_{i+1}$ is an edge mapped isomorphically onto an edge $e_i' \subset X_i'$, then $x_i \mapsto \mathcal{D}_i(x_i)(x_{i+1})$ is constant as $x_i$ varies in the interior of $e_i'$ and $x_{i+1}$ varies in the interior of $e_{i+1}$; see (4.4). This is used in Section 5 in the proof of the Poincaré inquality. We also observe that conversely, given an inverse system of metric graphs $(X_i, d_i)$, as in (2.8) which satisfies (1)–(3), and a sequence of continuous fuzzy sections $\mathcal{D}_i$ satisfying (4.2), there is a naturally associated sequence of measures $\mu_i$ such that $\mu_0$ is normalized to be 1-dimensional Lebesgue measure and $(X_i, d_i, \mu_i)$ satisfies Axioms (1)–(6). This reformulation is used in Section 8, in which of examples of admissible systems are constructed.

Consider an admissible inverse system as in (2.8). Let $\mathrm{int}(e_i')$ denote an open edge of $X_i'$, and $\mathrm{int}(e_{i+1})$ an open edge of $X_{i+1}$, which is a component of $\pi_i^{-1}(\mathrm{int}(e_i'))$. For $x_i \in \mathrm{int}(e_i')$, $x_{i+1} \in \pi_i^{-1}(x_i)$ we define

$$\mathcal{D}_i(x_i)(x_{i+1}) = \frac{\mu_{i+1}(e_{i+1})}{\mu_i(e_i')} . \tag{4.3}$$

Thus, $\mathcal{D}_i$ is continuous on $\mathrm{int}(e_i')$, and in fact, constant in the sense that for $x_{i,1}, x_{i,2} \in \mathrm{int}(e_i')$, $x_{i+1,1} \in e_{i+1} \cap \pi_i^{-1}(x_{i,1})$,

$$\mathcal{D}_i(x_{i,1})(x_{i+1,1}) = \mathcal{D}_i(x_{i,2})(x_{i+1,2}) . \tag{4.4}$$

Next, suppose $v_i'$ is a vertex of $X_i'$ and $e_i'$ is an edge of $X_i'$ with $v_i'$ as one of its endpoints. If $v_{i+1} \in \pi_i^{-1}(v_i')$ then $v_{i+1}$ is a vertex of $X_{i+1}$ and we define

$$\mathcal{D}_i(v_i')(v_{i+1}) = \frac{\mu_{i+1}(\pi_i^{-1}(e_i') \cap \mathrm{St}(v_{i+1}, X_{i+1}))}{\mu_i(e_i')} = \sum_{e_{i+1} \in \mathrm{St}(v_{i+1})} \frac{\mu_{i+1}(e_{i+1})}{\mu_i(e_i')} . \tag{4.5}$$

By (2.9) of Axiom (6) (the continuity condition) $\mathcal{D}_i(x_i')(x_{i+1})$ is well defined independent of the choice of $e_i'$ with endpoint $v_i'$.

**Lemma 4.6.** *$\mathcal{D}_i$ is a continuous fuzzy section satisfying (4.2).*

*Proof.* This follows immediately from (4.3), (4.5) that $\mathcal{D}_i$ is continuous

*Remark* 4.7. Note that $\mathcal{D}_i$ is simply the disintegration of $\mu_{i+1}$ with respect to the map $\pi_i : X_{i+1} \to X_i$.

From (2.15), together with Axioms (3) and (4), it follows that $\mathcal{D}_i$ satisfies the lower bound (4.2). □

The next proposition provides a sort of converse to the previous lemma.

**Proposition 4.8.** *Suppose the inverse system in (2.8) satisfies (1)–(3). Let $\mathcal{D}_i$ denote a continuous fuzzy section of $\pi_i$, $i = 0, 1, \ldots$, satisfying (4.2) and (4.4). Let $\mu_0$ denote 1-dimensional Lebesgue measure and define $\mu_i$ inductively by (4.3). Then $\mu_i$ satisfies (4)–(6) for all i.*

*Proof.* Axiom (5) follows directly from the definition of $\mu_i$ via (4.3) and the fact that $\mathcal{D}_i(x_i)$ is a probability measure for all $x_i$. Axiom (6) follows directly from the assumption that the fuzzy section $\mathcal{D}_i$ is continuous.

To verify Axiom (4), let $e_{i,1}$, $e_{i,2}$ denote edges of $X_i$ with a common vertex $v_i$ of $X_i$. Define $v_k$ by downward induction, by setting $v_{k-1} = \pi_{k-1}(v_k)$. Let $j \geq 0$ be either the largest value of $k$ such that $v_k$ is a vertex of $X'_k$ which is not a vertex of $X_k$, or if there is no such $k$, put $j = 0$. In either case, it is clear that $\mu_j(\pi_j \circ \cdots \pi_{i-1}(e_{i,1})) = \mu_j(\pi_j \circ \cdots \pi_{i-1}(e_{i,2}))$.

From Proposition 2.11 we get:

(*) *For all but at most $\Delta$ values of k, the (locally surjective) map $\pi_{k-1}$ is 1-1 in a neighborhood of $v_k$.*

Suppose, as in (*), the (locally surjective) map $\pi_k$ is 1-1 in a neighborhood of $v_{k+1}$, and $e_{k+1,1}$, $e_{k+1,2}$, are edges with common vertex $v_{k+1}$. Since $\mathcal{D}_k$ is continuous, by (4.3), we have

$$\frac{\mu_{k+1}(e_{k+1,1})}{\mu_{k+1}(e_{k+1,2})} = \frac{\mu_k(\pi_k(e_{k+1,1})}{\mu_k(\pi_k(e_{k+1,2}))} \,. \tag{4.9}$$

For the remaining values of $k$, by (4.2),

$$c_0 \leq \frac{\mu_{k+1}(e_{k+1,1})}{\mu_{k+1}(e_{k+1,2})} \leq c_0^{-1} \,. \tag{4.10}$$

It follows that (4) holds with $C = (c_0)^{\Delta}$. □

# 5 Proof of the Poincaré inequality and of Theorem 1.1

In this section $i \geq 0$ will be fixed.

Given $f_{i+1} : X_{i+1} \to \mathbb{R}$, we can perform integration of $f_{i+1}$ over the fibers $\{\pi_i^{-1}(x_i)\}_{x_i \in X_i}$ of $\pi_i : X_{i+1} \to X_i$ with respect to the family of measures $\{\mathcal{D}_i(x_i)\}_{x_i \in X_i}$, to produce a function on $X_i$ which we denote by $\mathcal{I}_{\mathcal{D}_i} f_{i+1}$. Thus,

$$\mathcal{I}_{\mathcal{D}_i} f_{i+1}(x_i) := \sum_{x_{i+1} \in \pi_i^{-1}(x_i)} \mathcal{D}_i(x_i)(x_{i+1}) f_{i+1}(x_{i+1}) \,. \tag{5.1}$$

By (4.3), (5.1), for all $A_i \subset X_i$, we have

$$\int_{A_i} \mathcal{I}_{\mathcal{D}_i} f_{i+1} \, d\mu_i = \int_{\pi_i^{-1}(A_i)} f_{i+1} \, d\mu_{i+1} \,; \tag{5.2}$$

this also expresses the fact that $\mathcal{D}_i$ is the disintegration of $\mu_{i+1}$ with respect to $\pi_i$ and $\mu_i$ is the pushforward of $\mu_{i+1}$ by $\pi_i$.

Now suppose $f_{i+1}$ is Lipschitz and let Lip $f_{i+1}(x_{i+1})$ denote the pointwise Lipschitz constant at $x_{i+1} \in X_{i+1}$. Let $e'_i$ denote an edge of $X'_i$ and $e_{i+1} \subset \pi_i^{-1}(e'_i)$ an edge of $X_{i+1}$. Since by (4.4), the function $\mathcal{D}_i(x_i)(x_{i+1})$ is constant as $x_i$ varies in $\mathrm{int}(e'_i)$ and $x_{i+1}$ varies in $\pi_i^{-1}(x_i) \cap \mathrm{int}(e_{i+1})$, and since the restriction of $\pi_i$ to $e_{i+1}$ is an isometry, it follows that that the restriction of $\mathcal{I}_{\mathcal{D}_i} f_{i+1}$ to $\mathrm{int}(e'_i)$ is Lipschitz, and

$$\mathrm{Lip}(\mathcal{I}_{\mathcal{D}_i} f_{i+1})(x_i) \leq \sum_{x_{i+1} \in \pi_i^{-1}(x_i)} \mathcal{D}_i(x_i)(x_{i+1}) \,\mathrm{Lip}\, f_{i+1}(x_{i+1}) = \mathcal{I}_{\mathcal{D}_i}(\mathrm{Lip}\, f_{i+1})(x_i)\,. \tag{5.3}$$

The following lemma depends crucially on the continuity assumption, Axiom (6) (as well as on Axiom (4)); see also (4.5).

**Lemma 5.4.** *If $f_{i+1} : X_{i+1} \to \mathbb{R}$ is Lipschitz then so is $\mathcal{I}_{\mathcal{D}_i} f_{i+1}$ and for all $x_i \in X_i$ (including $x_i = v'_i$, a vertex of $X'_i$), we have*

$$\mathrm{Lip}(\mathcal{I}_{\mathcal{D}_i} f_{i+1})(x_i) \leq \mathcal{I}_{\mathcal{D}_i}(\mathrm{Lip}\, f_{i+1})(x_i)\,. \tag{5.5}$$

*Proof.* Clearly, it suffices to check that (5.5) holds for $x_i = v'_i$ a vertex of $X'_i$. Let $v'_i$ a vertex of $e'_i$, $y_i \in \mathrm{int}(e'_i)$ and $v_{i+1} \in \pi_i^{-1}(v'_i)$. Then,

$$\mathcal{I}_{\mathcal{D}_i} f_{i+1}(y_i) = \sum_{v_{i+1} \in \pi_i^{-1}(v'_i)} \;\sum_{y_{i+1} \in \pi_i^{-1}(y_i) \cap \mathrm{St}(v_{i+1}, X_{i+1})} \mathcal{D}_i(y_i)(y_{i+1}) f_{i+1}(y_{i+1})\,. \tag{5.6}$$

and since the fuzzy section $\mathcal{D}_i$ is *continuous*,

$$\mathcal{I}_{\mathcal{D}_i} f_{i+1}(v'_i) = \sum_{v_{i+1} \in \pi_i^{-1}(v'_i)} \mathcal{D}_i(v_i)(v_{i+1}) f_{i+1}(v_{i+1}) = \sum_{v_{i+1} \in \pi_i^{-1}(v'_i)} \;\sum_{y_{i+1} \in \pi_i^{-1}(y_i) \cap \mathrm{St}(v_{i+1}, X_{i+1})} \mathcal{D}_i(y_i)(y_{i+1}) f_{i+1}(v_{i+1})\,. \tag{5.7}$$

By subtracting (5.7) from (5.6), dividing through by $d_i(y_i, v'_i) = d_{i+1}(y_{i+1}, v_{i+1})$ and letting $y_i \to v'_i$, we easily obtain (5.5). □

*Remark* 5.8. We could as well have worked throughout with upper gradients. If $g_{i+1}$ is an upper gradient for $f_{i+1} : X_{i+1} \to \mathbb{R}$, then a similar argument based on the continuity of $\mathcal{D}_i$ shows that $\mathcal{I}_{\mathcal{D}_i} g_{i+1}$ is an upper gradient for $f_i = \mathcal{I}_{\mathcal{D}_i} f_{i+1}$.

**Proposition 5.9.** *Given an admissible inverse system as in (2.8), for all $i$ and $R$, a $(1,1)$-Poincaré inequality holds for balls $B_r(x_i) \subset X_i$, with $r \leq R$, $\tau = \tau(R, \delta, \theta, C)$ and $\Lambda = 2(1+\theta)$.*

*Proof.* Without essential loss of generality, it suffices to assume $R = 1$. Given $0 < r \leq 1$, let $j$ be such that

$$m^{-(j+1)} < r \leq m^{-j}\,.$$

Let $B_r(x_i) \subset X_i$. If $r \leq m^{-i}$ then Lemma 3.1 applies. Thus, we can assume $m^{-i} < r$.

For $j+1 \leq k \leq i$, inductively define

$$x_{k-1} = \pi_{k-1} \circ \cdots \circ \pi_{i-1}(x_i)\,, \tag{5.10}$$

$$\begin{aligned} U_j &= B_r(x_j)\,, \\ U_k &= \pi_{k-1}^{-1}(U_{k-1}) \qquad j+1 \leq k \leq i\,. \end{aligned} \tag{5.11}$$

By (2.13) and induction, we have

$$B_r(x_i) \subset U_i \subset B_{(1+\theta)r}(x_i)\,. \tag{5.12}$$

Given a Lipschitz function $f_i : X_i \to \mathbb{R}$, set

$$f_{k-1} = \mathcal{I}_{\mathcal{D}_{k-1}} f_k \qquad j+1 \leq k \leq i\,, \tag{5.13}$$

$$\hat{f}_k = f_{k-1} \circ \pi_{k-1}^{-1}\,. \tag{5.14}$$

Then for all $A_{k-1} \subset X_{k-1}$ and $A_k := \pi_{k-1}^{-1}(A_{k-1})$, we have

$$(f_k)_{A_k} = (f_{k-1})_{A_{k-1}} = (\hat{f}_k)_{A_k}\,. \tag{5.15}$$

In particular, since $(\hat{f}_i)_{U_i} = (f_{i-1})_{U_{i-1}}$, we get

$$\begin{aligned}\int_{U_i} |f_i - (f_i)_{U_i}|\, d\mu_i &\le \int_{U_i} |f_i - \hat{f}_i|\, d\mu_i + \int_{U_i} |\hat{f}_i - (\hat{f}_i)_{U_i}|\, d\mu_i \\ &= \int_{U_i} |f_i - \hat{f}_i|\, d\mu_i + \int_{U_{i-1}} |f_{i-1} - (f_{i-1})_{U_{i-1}}|\, d\mu_{i-1}\,,\end{aligned}$$

and by induction,

$$\int_{U_i} |f_i - (f_i)_{U_i}|\, d\mu_i \le \sum_{k=j+1}^{i} \int_{U_k} |f_k - \hat{f}_k|\, d\mu_k + \int_{B_r(x_j)} |f_j - (f_j)_{B_r(x_j)}|\, d\mu_j\,. \tag{5.16}$$

Using Lemma 3.1, Lemma 5.4, (5.12) and induction, for $\tau = \tau(\Delta, \theta, C)$, the Poincaré inequality on $B_r(x_j)$ gives following estimate for the second term on the r.h.s of (5.16).

$$\begin{aligned}\int_{B_r(x_j)} |f_j - (f_j)_{B_r(x_j)}|\, d\mu_j &\le \tau r \cdot \int_{B_r(x_j)} \operatorname{Lip} f_j\, d\mu_j \\ &\le \tau r \cdot \int_{U_i} \operatorname{Lip} f_i\, d\mu_i \\ &\le \tau r \cdot \int_{B_{(1+\theta)r}(x_i)} \operatorname{Lip} f_i\, d\mu_i\,.\end{aligned} \tag{5.17}$$

Next we estimate the remaining terms on the r.h.s. of (5.16). For all $j+1 \le k \le i$, let $\{x_{k,t}\}$ denote a maximal $m^{-k}$-separated subset of $U_k$. Set $V_{k,t} = \pi_{k-1}^{-1}(B_{m^{-k}}(\pi_{k-1}(x_{k,t})))$. Then by (2.13) we have $V_{k,t} \subset B_{(1+\theta)m^{-k}}(x_{k,t})$. Morever, $\{V_{k,t}\}$ covers $U_k$ and it follows from the doubling condition that the collection of balls, $\{B_{(1+\theta)m^{-k}}(x_{k,t})\}$ has multiplicity bounded by a constant $M(\beta, \theta)$, with $\beta$ the local doubling constant in Lemma 3.1.

Set $U_{i,k,t} = (\pi_k \circ \cdots \circ \pi_{i-1})^{-1}(B_{(1+\theta)m^{-k}}(x_{k,t}))$. By (5.15), we have

$$(f_k - \hat{f}_k)_{V_{k,t}} = 0\,.$$

Thus, we get

$$\begin{aligned}&\int_{V_{k,t}} |(f_k - \hat{f}_k)|\, d\mu_k \\ &\quad = \int_{V_{k,t}} |(f_k - \hat{f}_k) - (f_k - \hat{f}_k)_{V_{k,t}}|\, d\mu_k \\ &\quad = \int_{V_{k,t}} |(f_k - \hat{f}_k) - (f_k - \hat{f}_k)_{B_{(1+\theta)m^{-k}}(x_{k,t})}|\, d\mu_k \\ &\qquad + |(f - \hat{f})_{B_{(1+\theta)m^{-k}}(x_{k,t})} - (f - \hat{f})_{V_{k,t}}| \cdot \mu_k(V_{k,t}) \\ &\quad \le 2 \int_{B_{(1+\theta)m^{-k}}(x_{k,t})} |(f_k - \hat{f}_k) - (f_k - \hat{f}_k)_{B_{(1+\theta)m^{-k}}(x_{k,t})}|\, d\mu_k \\ &\quad \le 4\tau(1+\theta)m^{-k} \cdot \int_{B_{(1+\theta)m^{-k}}(x_{k,t})} \operatorname{Lip} f_k\, d\mu_k \\ &\quad \le 4\tau(1+\theta)m^{-k} \cdot \int_{U_{i,k,t}} \operatorname{Lip} f_i\, d\mu_i\,,\end{aligned}$$

where the penultimate inquality comes from using $\operatorname{Lip}(f_k - \hat{f}_k) \le 2 \operatorname{Lip} f_k$ and applying the Poincaré inquality on $B_{(1+\theta)m^{-k}}(x_{k,t})$. By summing this estimate over $t$, $k$, and using $\bigcup_t U_{i,k,t} \subset B_{2(1+\theta)r}(x_i)$, together with the bound $M(\beta, \theta)$ on the multiplicity of the collection of balls, $\{B_{(1+\theta)m^{-k}}(x_{k,t})\}$, the proof is completed. □

*Proof of Theorem 1.1.* We have observed in Proposition 2.17 that $\{(X_n, d_n, \mu_n)\}$ converges to $(X_\infty, d_\infty, \mu_\infty)$ in the measured Gromov-Hausdorff sense. Since [1], [13], the doubling condition and Poincaré inequality with uniform constants pass to measured Gromov-Hausdorff limits, [1], [13], the theorem follows from Propositions 3.2, 5.9. □

## 6 A probability measure on the lifts of a path

In this section we define a probability measure $\Omega$ on the set of lifts to $X_i$ $(i > k)$ of a path $\gamma_k$ in $X_k$ and establish a particular property which is a consequence of Axiom (6); see Proposition 6.13. This property plays a role in Section 7, in which we give an alternative proof of the Poincaré inequality. The measure $\Omega$ has an interpretation in terms of Markov chains which is explained in Remark 6.15 at the end of the section; it also enters in Section 8, in which we construct examples of admissible inverse systems. We begin with the case $i = k + 1$ from which the general case follows easily.

A *vertex path* in $X'_k$ is a sequence of vertices $v'_{0,k}, \dots, v'_{N+1,k}$ such that each pair of consecutive vertices are the vertices of an edge of $X'_k$. Associated to a vertex path is the *path* $\gamma'_k = e'_{0,k} \cup \dots \cup e'_{N,k}$, which we will always assume is parameterized by arclength. Similarly, we define a *path* $\gamma_{k+1} = e_{0,k+1} \cup \dots \cup e_{N,k+1}$ in $X_{k+1}$ associated to $v_{0,k+1}, \dots, v_{N+1,k+1}$. We denote by $\Gamma$, the (finite) collection of all $\gamma_{k+1}$ that are lifts of $\gamma'_k$.

Below, given $e'_k$ and a lift $e_{k+1}$, by slight abuse of notation (compare (4.3)) we write

$$\mathcal{D}_k(e'_k)(e_{k+1}) := \frac{\mu_{k+1}(e_{k+1})}{\mu_k(e'_k)}\,. \tag{6.1}$$

Define a measure $\Omega$ on $\Gamma$ by setting

$$\Omega(\gamma_{k+1}) := \mathcal{D}_k(e'_{0,k})(e_{0,k+1}) \times \left(\frac{\mathcal{D}_k(e'_{1,k})(e_{1,k+1})}{\mathcal{D}_k(v'_{1,k})(v_{1,k+1})}\right) \times \dots \times \left(\frac{\mathcal{D}_k(e'_{N,k})(e_{n,k+1})}{\mathcal{D}_k(v'_{N,k})(v_{N,k+1})}\right)\,, \tag{6.2}$$

where by (4.5), we can write

$$\mathcal{D}_k(v'_{j,k})(v_{j,k+1}) = \sum_{\underline{e}_{j,k+1} \in \pi_k^{-1}(e'_{j,k}) \cap \mathrm{St}(v_{j,k+1})} \frac{\mu_{k+1}(\underline{e}_{j,k+1})}{\mu_k(e'_{j,k})}\,. \tag{6.3}$$

For a path, $\gamma'_k = e'_{0,k}$, consisting of a single edge, and a lift, $\gamma_{k+1} = e_{0,k+1}$, we just have

$$\Omega(e_{0,k+1}) = \mathcal{D}_k(e'_{0,k})(e_{0,k+1})\,. \tag{6.4}$$

Since $\mathcal{D}_k(x'_{0,,k})(\cdot)$ is a probability measure, it follows directly from the definitions that $\Omega$ is a probability measure in this case.

We now check an important property of $\Omega$ which in particular, implies that $\Omega$ is a probability measure for arbitrary $\gamma'_k$; see (6.5). Let $\psi'_k$ denote a path consisting of $N + 1$ edges obtained from $\gamma'_k$ by adjoining a single edge $e'_{N+1,k}$. Let $\Psi$ denote the collection of all lifts of $\psi'_k$ and let $\Omega_{\psi'_{k+1}}$ denote the measure on $\Psi$ (defined as in (6.2)). Let $\overline{\Psi}$ denote the collection of lifts of $\psi'_k$ containing the *fixed* lift $\gamma_{k+1}$ of $\gamma'_k$. Then it follows from (6.1) and (6.2), together with (6.3) applied to the vertices $v'_{N+1,k}$, $v_{N+1,k+1}$, that

$$\Omega_{\psi'_{k+1}}(\overline{\Psi}) = \Omega(\gamma_{k+1})\,. \tag{6.5}$$

It now follows by induction that $\Omega$ is a probability measure for arbitrary $\gamma'_k$; compare Remark 6.15.

*Remark* 6.6. Note that if we understand (6.3) to be the definition of $\mathcal{D}_k(v'_{j,k})(v_{j,k+1})$, then the discussion to this point has not made use of Axiom (6).

Recall that Axiom (6) implies that $\mathcal{D}_k(v'_{j,k})(v_{j,k+1})$ depends only on $v'_{j,k}$, $v_{j,k+1}$, and in particular (compare (6.3)) we also have

$$\mathcal{D}_k(v'_{j,k})(v_{j,k+1}) = \sum_{\underline{e}_{j-1,k+1} \in \pi_k^{-1}(e'_{j-1,k}) \cap \mathrm{St}(v_{j,k+1})} \frac{\mu_{k+1}(\underline{e}_{j-1,k+1})}{\mu_k(e'_{j-1,k})}\,. \tag{6.7}$$

If we rewrite the expression in (6.2) for $\Omega$ as

$$\Omega(\gamma_{k+1}) = \frac{\mathcal{D}_k(e'_{0,k})(e_{0,k+1}) \times \cdots \times \mathcal{D}_k(e'_{N,k})(e_{N,k+1})}{\mathcal{D}_k(v'_{1,k})(v_{1,k+1}) \times \cdots \times \mathcal{D}_k(v'_{N,k})(v_{N,k+1})}, \tag{6.8}$$

we easily obtain:

**Proposition 6.9.** *For an admissible inverse system, the measure $\Omega$ is invariant under the operation of reversing the orientations of $\gamma'_k$, $\gamma_{k+1}$.*

It follows immediately from Proposition 6.9, that (6.5) also holds if the additional edge is adjoined at the begining of $\gamma'_k$ rather than at the end. From this and an argument by induction, we get the following: For arbitrary $\gamma'_k$, if $\psi'_k$ is any path containing $\gamma'_k$, $\gamma_{k+1}$ is any fixed lift of $\gamma'_k$ and $\overline{\Psi}$ denotes the collection of all lifts of $\psi'_k$ containing $\gamma_{k+1}$ then (6.5) holds. This gives:

**Corollary 6.10.** *If $e'_{j,k}$ is any edge contained in $\gamma'_k$, $e_{j,k+1} \in \pi^{-1}(e'_{j,k})$ and $\overline{\Gamma}$ denotes the collection of lifts of $\gamma'_k$ which contain $e_{j,k+1}$, then*

$$\Omega(\overline{\Gamma}) = \mathcal{D}_k(e'_{j,k})(e_{j,k+1}) = \frac{\mu_{k+1}(e_{j,k+1})}{\mu_k(e'_{j,k})}. \tag{6.11}$$

Next, we give a consequence of (6.11) which is used in the alternate proof of the Poincaré inequality given in Section 7.

Suppose that $\gamma'_k$ is the subdivision of a path in $X_k$ consisting of the union of $L$ edges $e_{0,k} \cup \cdots \cup e_{L,k}$ of $X_k$. (Thus, $\gamma'_k$ has $L \cdot m$ edges $e'_{j,k}$.) Assume that $\gamma'_k$ is parameterized by arclength. Define $\Phi : \Gamma \times [0, L \cdot m^{-k}] \to X_k$ by

$$\Phi(\gamma_{k+1}, t) = \gamma_{k+1}(t)$$

Let $\mathcal{L}$ denote Lebesgue measure on $X_{k+1}$.

We claim that on any fixed $e_{\ell,k}$ in the domain of $\gamma_k$, we have

$$\Phi_\star(\Omega \times \mathcal{L}) = \frac{m^{-k}}{\mu_{k+1}(\pi_k^{-1}(e_{\ell,k}))} \cdot \mu_{k+1},$$

where $\Phi_\star$ denotes push forward under the map $\Phi$. To see it, note that for any $e_{j,k+1}$ we have

$$\mathcal{L} = \mu_{k+1} \cdot \frac{m^{-(k+1)}}{\mu_{k+1}(e_{j,k+1})},$$

If $e'_{j,k} \subset e_{\ell,k}$ and $e_{j,k+1} \subset \pi_k^{-1}(e'_{j,k})$, then on $e_{j,k+1}$ we have by (6.11)

$$\Phi_\star(\Omega \times \mathcal{L}) = \frac{\mu_{k+1}(e_{j,k+1})}{\mu_k(e'_{j,k})} \cdot \mathcal{L}.$$

Combining the previous two relations gives

$$\Phi_\star(\Omega \times \mathcal{L}) = \frac{m^{-(k+1)}}{\mu_k(e'_{j,k})} \cdot \mu_{k+1} = \frac{m^{-k}}{\mu_{k+1}(\pi_k^{-1}(e_{\ell,k}))} \cdot \mu_{k+1}, \tag{6.12}$$

where the last equality follows by because $\mu_k$ is a constant multiple of Lebesgue measure on $e_{\ell,k}$ and $(\pi_k)_\star(\mu_{k+1}) = \mu_k$.

Finally, we give a generalization of the above. Put $\pi_k^i = \pi_k \circ \cdots \circ \pi_{i-1}$. Write $X_k^i$ for $X_k$ with each of its edges subdivided into edges of length $m^{-(i-1)}$. Then $\pi_k^i$ is maps edges of $X^i$ to edges of $(X_k^i)'$. It is easy to see that after rescaling of the metric and measure on both $X_k^i$ and $X_i$ by a factor $m^{i-1}$, Axioms (1)–(6) are satisfied (where the verification of Axiom (6) is by induction). In addition, the $X_k^i$ with rescaled metric has the property that the rescaled $\mu_i$ is a constant multiple of $\mathcal{L}$ on the edges of the rescaled $X_k$ (which have length $m^{i-k-1}$ in the rescaled metric). As a consequence, by the same argument which led to (6.12), we get:

**Proposition 6.13.** *Let $\gamma_k$ denote a path in $X_k$ which is the union of edges $e_k$ of $X_k$ and let $\gamma_k^i$ denote its subdivision in $X_k^i$. If $\Gamma$ denotes collection of lifts of $\gamma_k^i \subset X_k^i$ to $X_i$, then there is a probability measure $\Omega$ on $\Gamma$ such that*

$$\Phi_\star(\Omega \times \mathcal{L}) = \frac{m^{-k}}{\mu_i((\pi_k^i)^{-1}(e_{\ell,k}))} \cdot \mu_i \qquad (\text{on } e_{\ell,k}). \tag{6.14}$$

*Remark* 6.15. The definition of $\Omega$ in (6.8) can be understood in terms of Markov chains. This gives a more general perspective on why it is a probability measure. Associated to $\gamma'_{k+1}$ is a discrete time Markov chain whose collection of states is $\bigcup_{j=0}^{N}(\pi_k^{-1}(e'_{k,j}), j)$. The probability of being in a state $(e_{j,k+1}, j)$ at time 0 is 0 unless $j = 0$, in which case the probability is $\mathcal{D}(e_{0,k})(e_{0,k+1})$. The probability of transition from a state $(e_{j_1,k+1}, j_1)$ at time $j$ to a state $(e_{j_2,k+1}, j_2)$ at time $j+1$ is 0 unless $j_1 = j$, $j_2 = j+1$ and there exists $\gamma_{k+1} \in \Gamma$ such that $e_{j,k+1}$, $e_{j+1,k+1}$ are consecutive edges of $\gamma_{k+1}$ with common vertex $v_{j+1,k+1}$, and such that $e_{j_1,k+1} = e_{j,k+1}$ and $e_{j_2,k+1} = e_{j+1,k+1}$. In this case the transition probability is

$$\frac{\mathcal{D}(e'_{j+1,k})(e_{j+1,k+1})}{\mathcal{D}(v'_{j+1,k})(e_{j+1,k+1})} := \frac{\mu_{k+1}(e_{j,k+1})}{\sum_{\underline{e}_{j,k+1} \in \pi_k^{-1}(e'_{j,k}) \cap \mathrm{St}(v_{j,k+1})} \mu_{k+1}(\underline{e}_{j,k+1})};$$

For this Markov chain, the probability of observing a sequence of states $(e_{j_0,k+1}, 0), (e_{j_1,k+1}, 1), \ldots, (e_{j_N,k+1}, N)$ is zero unless there exists $\gamma_{k+1} = e_{0,k+1} \cup \cdots \cup e_{N,k+1} \in \Gamma$, with $e_{j_0,k+1} = e_{0,k+1}, \ldots, e_{j_N,k+1} = e_{N,k+1}$, in which case this probability is $\Omega(\gamma_{k+1})$.

Note that the in above discussion we need not assume that Axiom (6) holds. However, this assumption is required for Proposition 6.9 whose consequence, Proposition 6.13, is crucial for the alternate proof of the Poincaré inequality given in the next section.

# 7 A proof of the Poincaré inequality using measured path families

In this section we give an second proof based on measured path families that the Poincaré inequality holds for $(X_\infty, d_\infty, \mu_\infty)$.[1] This is closer in spirit to other proofs of the Poincaré inequality [17].

Suppose $k \le i$, $v_k$ is a vertex of $X_k$, $e_{0,k}, e_{1,k}$ are edges belonging to the star of $v_k$ in $X_k$, and $Z_\ell = (\pi_k^i)^{-1}(e_{\ell,k}) \subset X_i$ for $\ell \in \{0, 1\}$. Let $\gamma_k : [0, 2m^{-k}] \to X_k^i$ denote a unit speed parametrization of the path $e_{0,k} \cup e_{1,k}$ and $\gamma_k^i$ its subdivision in $X_k^i$. Let $\Gamma$ denote the space of lifts $\gamma_i : [0, 2m^{-k}] \to X_i$ of $\gamma_k^i$ and let $\Omega$ denote the probability measure on $\Gamma$ constructed in Section 6. Let $\Phi : \Gamma \times [0, 2m^{-k}] \to Z_0 \cup Z_1 \subset X_i$ denote the tautological map $(s, \gamma_i) \mapsto \gamma_i(s)$.

Recall from (2.3) the definition of an upper gradient $g$ of a function $f$ on a metric space.

**Lemma 7.1.** *Let $k < i$, $Z_0, Z_1$ are as above. Let $u : X_i \to \mathbb{R}$ denote a Lipschitz function and $g : X_i \to \mathbb{R}$ an upper gradient for $u$. Then*

$$\left| \fint_{Z_0} u\, d\mu_i - \fint_{Z_1} u\, d\mu_i \right| \le \hat{C} m^{-k} \fint_{Z_0 \cup Z_1} g\, d\mu_i\,.$$

*Proof.* With Axiom (4) and (6.14) of Proposition 6.13 (which is used twice below) we get:

$$\begin{aligned} \left| \fint_{Z_0} u\, d\mu_i - \fint_{Z_1} u\, d\mu_i \right| &= \left| \fint_{[0,m^{-k}]\times\Gamma} (u \circ \Phi)\, d(\mathcal{L} \times \Omega) - \fint_{[m^{-k},2m^{-k}]\times\Gamma} (u \circ \Phi)\, d(\mathcal{L} \times \Omega) \right| \\ &\le \fint_{[0,m^{-k}]\times\Gamma} \left| u(\gamma_i(t)) - u(\gamma_i(t + m^{-k})) \right| 2\, d\mathcal{L}(t)\, d\Omega(\eta) \end{aligned}$$

[1] As a matter of convenience, some of the notational conventions of this section are somewhat at variance with those of other sections and (given that this is our second proof of the Poincaré inequality) the style of presentation is slightly more informal.

$$\leq \fint_{[0,m^{-k}]\times\Gamma} \int_{[0,m^{-k}]} g\circ\gamma_i(t+s)\, d\mathcal{L}(s)\, d\mathcal{L}(t)\, d\Omega(\hat\eta)$$

$$= \int_{[0,m^{-k}]} \left( \fint_{[0,m^{-k}]\times\Gamma} g\circ\gamma_i(t+s)\, d\mathcal{L}(t)\, d\Omega(\gamma_i)\right) d\mathcal{L}(s)$$

$$\leq \hat C \int_{[0,m^{-k}]} \left( \fint_{Z_0\cup Z_1} g\, d\mu_i \right) d\mathcal{L}(s) = \hat C\, m^{-k} \fint_{Z_0\cup Z_1} g\, d\mu_i\,.$$

□

**Theorem 7.2.** *$(X_\infty, d_\infty, \mu_\infty)$ satisfies a Poincaré inequality.*

*Proof.* It suffices to prove that $(X_i, d_i, \mu_i)$ satisfies a Poincaré inequality for every $i \in \mathbb{Z}$, with constant independent of $i$; see [1], [13]. We fix $i \in \mathbb{Z}$, and let $u : X_i \to \mathbb{R}$ denote a Lipschitz function with upper gradient $g : X_i \to \mathbb{R}$. For every $k \leq i$, let $\mathfrak{U}^i_k$ denote the collection of subsets of $X_i$ of the form $U^i_k = (\pi^i_k)^{-1}(e_k)$, where $e_k$ is an edge of $X_k$. Let $u_{i,k} : X_i \to \mathbb{R}$ denote a step function such that for every $U^i_k \in \mathfrak{U}^i_k$,

$$u_{i,k}(x_i) = \fint_{U^i_k} u\, d\mu_i\,,$$

for $\mu_i$-a.e. $x_i \in U^i_k$. In particular, $u_{i,i}$ satisfies

$$u_{i,i}(x_i) = \fint_{e_i} u\, d\mu_i\,,$$

for all edges $e_i$ of $X_i$ and $\mu_i$-a.e. $x_i \in e_i$.

Let $k < i$, and $U^i_k = (\pi^i_k)^{-1}(e_k) \in \mathfrak{U}^i_k$. If two elements $U^i_{0,k+1} = (\pi^i_{k+1})^{-1}(e_{0,k+1})$, $U^i_{1,k+1} = (\pi^i_{k+1})^{-1}(e_{1,k+1}) \in \mathfrak{U}^i_{k+1}$ are contained in some $U_k$, then by Axiom (3) (the diameter bound on fibres) $e_{0,k+1}, e_{1,k+1}$ are at distance $\leq C = C(\theta)m^{-k}$ in $X_{k+1}$, and so by Lemma 7.1 and induction, we have

$$\left| \fint_{U^i_{0,k+1}} u\, d\mu_i - \fint_{U^i_{1,k+1}} u\, d\mu_i \right| \leq \hat C \cdot m^{-k} \fint_{CU^i_k} g\, d\mu_i\,,$$

where $CU^i_k$ denotes of a tubular neighborhood of radius $C(\theta)m^{-k}$ around $e_k$; see (2.13).

Since at most a definite number of elements of $\mathfrak{U}^i_{k+1}$ are contained in a fixed $U^i_k$ (see (2.15)) this gives for all $k \leq i-1$,

$$\int_{U^i_j} |u_{i,k} - u_{i,k+1}|\, d\mu_i \leq C_1 m^{-k} \int_{CU^i_{k+1}} g\, d\mu_k\,. \tag{7.3}$$

where $C_1 = C_1(m, \Delta, \theta)$.

Now suppose $j \leq i$, $v_j$ is a vertex of $X_j$, and let $Z = (\pi^i_j)^{-1}(\mathrm{St}(v_j, X_j)) \subset X_i$. By (7.3) (with notation as above) we have

$$\int_Z |u_{i,i} - u_{i,j}|\, d\mu_i \leq \sum_{k=j}^{i-1} \int_Z |u_{k,j+1} - u_{k,j}|\, d\mu_k \leq \sum_{k=j}^{i-1} C_1 m^{-j} \int_{CZ} g\, d\mu_i \leq C_1 m^{-j} \int_{CZ} g\, d\mu_i\,. \tag{7.4}$$

Applying the Poincaré inequality for each edge $e_i$ of $Z$ gives

$$\int_Z |u - u_{i,i}|\, d\mu_i \leq m^{-i} \int_Z g\, d\mu_i\,. \tag{7.5}$$

Since $X_j$ has a valence bound independent of $j$, it follows from Lemma 7.1 that

$$\int_Z |u_{i,j} - u_Z|\, d\mu_i \le \hat{C} m^{-j} \int_Z g\, d\mu_i\,. \tag{7.6}$$

Combining (7.4), (7.5), and (7.6) we obtain

$$\int_Z |u - u_Z|\, d\mu_i \le \int_Z (|u - u_{i,i}| + |u_{i,i} - u_{i,j}| + |u_{i,j} - u_Z|)\, d\mu_i \le C m^{-j} \cdot \int_{CZ} g\, d\mu_i\,. \tag{7.7}$$

Since $X_i$ has valence bounded independent of $i$ and edges of length $m^{-i}$, it suffices to prove the Poincaré inequality for balls $B_r(x_i)$ where $r$ is at least comparable to $m^{-i}$, since otherwise $B_r(x_i)$ lies in the star of some vertex $v_i \in X_i$, and the result is trivial; see Lemma 3.1. Thus, we may assume that there is a $j \le k$ with $m^{-j}$ comparable to $r$ and a vertex $v_j \in X_j$ such that $\pi^i_j(B_r(x_i)) \subset \mathrm{St}(v_i, X_i)$. Letting $Z = (\pi^i_j)^{-1}(\mathrm{St}(v_j, X_j))$, we have $B_r(x_i) \subset Z$ and $\mu_i(Z)/\mu_k(B_r(x_i))$ has a definite bound; see Axiom (4). Then

$$\fint_{B_r(x_i)} |u - u_{B_r(x_j)}|\, d\mu_i \le C \fint_Z |u - u_Z|\, d\mu_i \le C\, m^{-j} \fint_{CZ} g\, d\mu_i \le C\, m^{-j} \fint_{B_{Cr}(x_i)} g\, d\mu_i\,.$$

This suffices to complete the proof. □

# 8 Construction of admissible inverse systems

In view of Theorem 1.1, it is natural to ask for explicit examples of admissible inverse systems and whether (and in what sense) it is possible to classify them. In this section we will content ourselves with giving an inductive procedure for constructing admissible inverse systems, which makes it clear that combinatorially distinct admissible inverse systems exist in great abundance. We will also give a simple example of an inverse system of metric graphs satisfying Axioms (1)–(3) which cannot be given the structure of an admissible inverse system, i.e. for this inverse system, a sequence of measures $\mu_k$, satisfying Axioms (4)–(6) does not exist; see Example 8.15.

## 8.1 Admissible edge inverses; the simplest special case

Given an admissible inverse system $\{X_i\}_{i \in \mathbb{Z}_+}$, one may think of $X_{k+1}$ as the union the subgraphs $\pi_k^{-1}(e_k)$, where $e_k \subset X_k$ ranges over all edges of $X_k$. The following definition axiomatizes the properties of these subgraphs, up to rescaling of the metric and the measure.

**Definition 8.1.** An *admissible edge inverse* is a map $(Y_1, d_1, \nu_1) \xrightarrow{\pi} (Y_0, d_0, \nu_0)$ of finite metric measure graphs, satisfying the following conditions for some integer $m \ge 2$:

1. $(Y_0, d_0, \nu_0)$ is a copy of the unit interval $[0, 1]$ with the usual metric and measure. $Y_1$ is a nonempty, finite, possibly disconnected graph, such that every edge $e_1 \subset Y_1$ is isometric to an interval of length $\frac{1}{m}$. The restriction of $d_1$ to every component of $Y_1$ is the associated path metric. The restriction of the measure $\nu_1$ to $e_1$ is a nonzero multiple of the arclength.
2. If $Y_0'$ denotes the result of subdividing $Y_0 \simeq [0, 1]$ into $m$ edges of length $\frac{1}{m}$, then $\pi : Y_1 \to Y_0'$ is open, and its restriction to any edge $e_1 \subset Y_1$ maps $e$ isometrically onto an edge of $Y_0'$.
3. (Compatibility with projections) The pushforward $\pi_*(\nu_1)$ is $\nu_0$.
4. (Continuity) For every vertex $v \in Y_0'$, and every $w \in \pi^{-1}(v) \subset Y_1$, the quantity

$$\frac{\nu_1(\pi^{-1}(e_0) \cap \mathrm{St}(w, Y_1))}{\nu_0(e_0)}$$

is the same for all edges $e \subset \mathrm{St}(v, Y_0')$.

Note that if $\{X_i\}_{i\ge 0}$ is an admissible inverse system with subdivision parameter $m$, then for any $i$ and any edge $e \subset X_i$, the restriction of $\pi_i$ to $\pi_i^{-1}(e)$ yields an admissible edge inverse $\pi_i : \pi_i^{-1}(e) \to e$, modulo rescaling the metric and normalizing the measure.

Fix $m, n \ge 2$, and an admissible edge inverse $\pi : (Y_1, \nu_1) \to (Y_0, \nu_0)$ with subdivision parameter $m$. We now assume further that if $v \in \{0, 1\}$ is an endpoint of $Y_0 \simeq [0, 1]$ then $\pi^{-1}(v)$ has cardinality $n$. For each such endpoint, choose and identification of the set of inverse images with the set $\{1, \dots, n\}$. Moreover, assume that

$$Y_1 \text{ is connected and } d_1 \text{ is a length metric on } Y_1. \tag{8.2}$$

$$\text{If } v \in \{0,1\} \text{ is an endpoint of } Y_0 \simeq [0,1] \text{ and } w \in \pi^{-1}(v) \text{, then } w \text{ has degree 1, and the unique edge containing } w \text{ has } \nu_1 \text{ measure } \frac{1}{mn}. \tag{8.3}$$

## 8.2 Inductive construction of admissible inverse systems

Fix $m$ and $N < \infty$ and assume that for each integer $n$ with $1 \le n \le N$ we have a finite nonempty family $\mathcal{G}(n)$ of edge inverses as above such that for $v$ an endpoint of $Y_0$, the cardinality of $\pi^{-1}(v)$ is $n$. The existence of such families will be shown in a subsequent subsection. In fact, with suitable choice of parameters, we will show that it is possible to choose finite families $\mathcal{G}(n)$ with arbitrarily large cardinality.

Choose a sequence, $\{n(k)\}$, with $n(k) \le N$ for all $k$. Using elements of the family $\mathcal{G}^{n(k)}$ as building blocks, we can construct inverse systems of metric measure graphs, using the procedure described below.

We begin with a connected metric measure graph $(X_0, d_0, \mu_0)$, with $d_0$ the length metric, for which the degree is bounded and such that the restriction of $(d_0, \mu_0)$ to every edge of $X_0$ is a copy of $[0, 1]$ with the usual Lebesgue measure $\mathcal{L}$.

Then we iterate the following procedure to construct $X_{k+1}$ and a map $\pi_k : X_{k+1} \to X_k$, for every $k$:

- We choose $n = n(k) \le N$ and corresponding family $\mathcal{G}(n(k))$ as above.
- We construct the inverse image $\pi_k^{-1}(V_k)$ of the vertex set $V_k \subset X_k$. This is defined to be $V_k \times \{1, \dots, n\}$, and the projection map is the projection on the first factor, $\pi_k : V_k \times \{1, \dots, n\} \to V_k \subset X_k$.
- For each edge $e_k \subset X_k$, we choose a copy of some admissible edge inverse $(Y_0, Y_1, \pi) \in \mathcal{G}(n(k))$, with the metrics rescaled by $m^{-k}$, the measures rescaled by $\mu_k(e_k)$. Then we identify $Y_0$ with $e_k$ and identify the inverse images of the endpoints $\{0, 1\} = Y_0$ with the inverse images of the endpoints of $e_k$ using the identifications of these sets with $\{1, \dots, n\}$. Finally, modulo the above identifications, we define the projection map $\pi_k : \pi_k^{-1}(e_k) \to e_k \subset X_k$ to be the projection map $\pi : Y_1 \to Y_0$,
- We define $d_{k+1}$ to be the path metric on $X_{k+1}$ which agrees with the given metric on edges.

**Lemma 8.4.** *Any inverse system constructed as above is admissible, where the parameters $\Delta, \theta, C$ depend only on $\{\mathcal{G}(n)\}$ $(n \le N)$ and the degree bound for $X_0$.*

*Proof.* Note that $X_0$ is assumed to have bounded degree and $n(k) \le N$ for all $k$. Also, for fixed $k$, $\{\mathcal{G}(n))\}$ is a finite collection, and each $Y_1 \in \mathcal{G}$ is a finite graph, so that in particular, there is a uniform bound on the degree for at vertices of elements of $\mathcal{G}(n)$ for all $n$. It then follows from (8.3) that there is a uniform bound on the degree of vertices of $X_k$ which is independent of $k$. It now clear that Axioms (1) and (2) hold.

Axiom (3) the bound on fibre diameters follows directly from the connectedness assumption (8.2).

Axiom (4), local bounded metric measure geometry, follows from the finiteness discussion above, together with (8.3). Namely, by (8.3), for $v_k \in V_k$ and $w_{k+1} \in \pi_k^{-1}(v_k)$ up to scaling of the metric and the measure, the local geometry at $w_{k+1}$ is the same as the local geometry at $v_k$.

Axiom (5) is immediate from (C), while Axiom (6) follows from (D) and (8.3). ☐

## 8.3 Relaxing some of the conditions

Next point out some generalizations of the construction above, in which some of the conditions are relaxed.

We can relax (8.3), requiring instead that $\mathcal{G}$ contains nonempty subsets of edge inverses satisfying (8.3), and that the rest have the weaker property that for each vertex $v \in Y_1$ projecting to one of the endpoints 0, 1, of $Y_0$, the $\nu_1$ measure of the edges leaving $v$ is exactly $\frac{1}{mn}$. For subsequent purposes note that in terms of the continuous fuzzy section defined as (4.5), this can be written equivalently as follows. Let 0, 1 denote the vertices of $Y_0 = [0, 1]$, $\ell \in \{0, 1\}$, and let $w \in \pi^{-1}(\ell)$. Then $\ell \in \{0, 1\}$,

$$\mathcal{D}(\ell)(w) = \frac{1}{n} \,. \tag{8.5}$$

The remainder of the discussion of this subsection applies equally well to the general case (discussed subsequent subsections) in which (8.5) is replaced by the assumption that for either endpoint $\ell \in \{0, 1\}$, of $Y_0 = [0, 1]$, $\mathcal{D}(\ell)(\cdot)$ is an arbitrary probability measure taking positive values on every point of $\pi^{-1}(\ell)$; compare (8.6).

We may drop the requirement (8.2), and instead ask that $\mathcal{G}$ contain a nonempty subset $\mathcal{G}_c$ for which the corresponding $Y_1$ is connected. Then to ensure the point inverses $\pi_k^{-1}(v)$ have controlled diameter, it suffices to ensure that the set of edges $e \subset X_k$ for which the inverse image $\pi_k^{-1}(e)$ is chosen from $\mathcal{G}_c$ forms a $\tilde{C}m^{-k}$ net in $X_k$, where $\tilde{C}$ is independent of $k$.

Let $\ell \in \{0, 1\}$ denote the endpoints of $Y_0 = [0, 1]$. Denote by $\mathcal{G}_\ell$, $\mathcal{G}_1$, the subset of $\mathcal{G}$ for which every vertex of $\pi^{-1}(\ell)$ has degree 1. Put $\mathcal{G}_0 \cap \mathcal{G}_1 = \mathcal{G}_{0,1}$. To ensure the existence of the valence bound $\Delta$ as in Axiom (1), we can fix a number $K$, and whenever an edge $e \subset X_k$ has a vertex whose degree exceeds $K$ and choose the edge inverse from $\mathcal{G}_\ell$, the vertex has degree exceeding $K$ (or from $\mathcal{G}_{0,1}$ if both vertices have degree exceeding $K$).

Thus, if $\mathcal{G}$ contains nonempty subsets $\mathcal{G}_c$, $\mathcal{G}_c \cap G_0$ $\mathcal{G}_c \cap \mathcal{G}_1$ $\mathcal{G}_c \cap \mathcal{G}_{0,1}$ we can start by making choices from these subsets at sufficiently many edges to form a $\tilde{C}m^{-k}$ net, and then, for the remaining edges make arbitrary choices from $\mathcal{G}$.

## 8.4 Admissible edge inverses; the general case

Next, we give the definition of admissible edge inverses in the general case.

We will retain (A)–(D). However, we are going to use the reformulation of (C) in terms of continuous fuzzy sections.

As discussed in the special case which we have already treated, the connectedness assumption (8.2) is dropped. (As before, in the inductive construction, for each $k$, we will assume as before that the edges with connected $Y_1$ form a $\tilde{C}m^{-k}$-net where $\tilde{C}$ is independent of $k$.)

For some $N_1$, the inverse images of the endpoints $\ell \in \{0, 1\}$ of $Y_0 = [0, 1]$ are assumed to have cardinalities, $n_0, n_1 \leq N_1$, where possibly $n_0 \neq n_1$. We choose identifications of $\pi^{-1}(\ell)$ with $1, \ldots, n_\ell$. Let the continuous fuzzy section $\mathcal{D}$ be defined in terms of $\nu_0$, $\nu_1$ as in (4.3)–(4.5); see also Proposition 4.8.. In place of (8.5), we simply assume that $\mathcal{D}(\ell)$ is an arbitrary probability measure on $\pi^{-1}(\ell)$ such that

$$\mathcal{D}(\ell)(w) > c_0' > 0 \,, \tag{8.6}$$

for all $w \in \pi^{-1}(\ell)$.

Suppose we choose to regard $\mathcal{D}(0)(\cdot)$ and $\mathcal{D}(1)(\cdot)$ as having been specified. Then as (4.4), (4.5), the measure $\nu_1$ provides an extension of $\mathcal{D}$ as a continuous fuzzy section to all of $Y_1$. Conversely, any such extension provides a measure $\nu_1$ satisfying (C) i.e. the pushforward of $\nu_1$ under $\pi$ is $\nu_0$; see (4.3) and Proposition 4.8 . With this much understood, it will be convenient to formulate the rest of the discussion of this section in terms of $\mathcal{D}$ (rather than $\nu_1$).

We let $\mathcal{G}_c \cap \mathcal{G}_0$, $\mathcal{G}_c \cap \mathcal{G}_1$ and $\mathcal{G}_c \cap \mathcal{G}_{0,1}$ retain their previous meanings. Similarly, (8.3) is dropped with the proviso that as before, we will only consider collections $\mathcal{G}$ such that $\mathcal{G}_c \cap \mathcal{G}_0$, $\mathcal{G}_c \cap \mathcal{G}_1$ and $\mathcal{G}_c \cap \mathcal{G}_{0,1}$ are nonempty, so that in the inductive construction, we are at liberty make choices from these subsets when the degree of vertices exceeds a preselected $K$ and/or to ensure that edges with connected edge inverses form $\tilde{C}m^{-k}$-dense subset of $X_k$. The existence of such $\mathcal{G}$ is guaranteed by the following Proposition 8.7.

**Proposition 8.7.** *Assume that the cardinalities $n_0, n_1$ of $\pi^{-1}(\ell)$ satisfy $n_\ell \le N_1$, $\ell \in \{0, 1\}$. Let $\mathcal{D}$ be specified arbitrarily on $\pi^{-1}(0) \cup \pi^{-1}(1)$ subject to the condition that (8.6) holds for some $c_0' > 0$. Let $\mathcal{G}$ denote the collection of edge inverses for which $\mathcal{D}$ has the specified restriction to $\pi^{-1}(0) \cup \pi^{-1}(1)$ and such that in addition, $Y_1$ has $\le m \cdot N_1$ edges and for all $i/m \in Y_0'$ and $w \in \pi^{-1}(i/m)$,*

$$\mathcal{D}(i/m)(w) \ge c_0' . \tag{8.8}$$

*Then $\mathcal{G}_c \cap \mathcal{G}_{0,1}$ has cardinality $\ge m - 1$.*

*Proof.* Fix some vertex $\underline{i}/m$ of $Y_0'$ which is not an endpoint. (Each such choice will determine a different $Y_1$ as in the proposition.) The combinatorial structure of $Y_1$ is specified by stipulating that:

1) $\pi^{-1}(\underline{i}/m)$ consists of a single vertex $\underline{w}$.

2) For every $w_{0,s} \in \pi^{-1}(0)$ the segment $[0, \underline{i}/m] \subset Y_0'$ from $v_0$ to $y'$ has a unique lift $\gamma_s$ with initial point $w_{0,s}$ (and final point $w$).

3) For every $w_{1,t} \in \pi^{-1}(v_1)$ , the segment $[\underline{i}/m, 1] \subset Y_0'$ has a unique lift $\gamma_t$ with final point $w_{1,t}$ (and initial point $w$).

$\mathcal{D}$ is given as follows. $\mathcal{D}(\underline{i}/m)(\underline{w}) = 1$. If $w \in \gamma_s$, $w \ne \underline{w}$ then $\mathcal{D}(\pi(w))(w) = \mathcal{D}(0)(w_{0,s})$. If $w \in \gamma_t$, $w \ne \underline{w}$ then $\mathcal{D}(\pi(w))(w) = \mathcal{D}(1)(w_{1,t})$. □

*Remark* 8.9. Although Proposition 8.7 shows the existence of $\mathcal{G}$ with $\mathcal{G}_c \cap \mathcal{G}_{0,1} \ne \emptyset$, it has the drawback that the combinatorial and metric structure of $Y_1$ depends only on $n_0, n_1$. However, as we will see below, in the general case, we actually do obtain many more examples of admissible inverse systems that in the simplest special case.

*Remark* 8.10. Fix $\ell \in \{0, 1\}$, say $\ell = 0$. There is an obvious 1-1 correspondence between arbitrary admissible edge inverses $(Y_1, d_1, v_1) \xrightarrow{\pi} (Y_0, d_0, v_0)$ with subdivision parameter $m$ and admissible edge inverses $(\hat{Y}_1, \hat{d}_1, \hat{v}_1) \xrightarrow{\hat{\pi}} (\hat{Y}_0, \hat{d}_0, \hat{v}_0)$ with subdivision parameter $m + 1$, such that all vertices in $\pi^{-1}(0)$ have degree 1. Here, after suitable rescaling of the metric and the measure, we regard $(Y_0, d_0, v_0)$ as $\hat{\pi}^{-1}([1/(m+1), \dots, 1])$. Also, each vertex in $\hat{\pi}^{-1}(0)$ is connected to the corresponding vertex in $\hat{\pi}^{-1}(1/(m+1))$ by a unique edge which projects under $\hat{\pi}$ to $[0, 1/(m+1)]$. Note that with the obvious identifications, $\mathcal{D}(\ell) \mid \pi^{-1}(\ell)$ remains unchanged, for $\ell$ both $\ell = 0$ and $\ell = 1$. If the edge inverse with subdivision parameter $m$ is connected, then so is the new one with subdivision parameter $m + 1$. Of course, the construction can also be done with the endpoint $\ell = 1$, of with both endpoints (in which case one obtains an edge inverse with subdivision parameter $m + 2$, for which the inverse images of both endpoints have degree 1).

## 8.5 General inductive construction

Choose constants, $c_0' > 0$, $0 < c_0 < \; < c_0'$, $N_1$, $N_2 \ge m \cdot N_1$, $\tilde{C}$ and $K$. It will be clear that the constants in Axioms (1)–(6), and hence, the constants in the doubling condition and Poincaré inquality, can be estimated in terms of these parameters.

For each vertex $v_k$ of $X_k$, we specify arbitrarily the cardinality $n(v_k)$ of $\pi_k^{-1}(v_k)$ subject only to $n(v_k) \le N_1$. We also choose an ordering of $\pi_k^{-1}(v_k)$. Finally, we choose an ordering of the vertices of $X_k$.

For each $v_k$ we choose a probability measure $\mathcal{D}_k$ on $\pi_k^{-1}(v_k)$ such that

$$\mathcal{D}_k(v_k)(v_{k+1}) \ge c_0' , \tag{8.11}$$

for all $v_k$, $v_{k+1} \in \pi_k^{-1}(v_k)$.

For each edge $e_k$, the ordering of its vertices induces an identification of $e_k$ with $Y_0 = [0, 1]$ and the specified $\mathcal{D}_k$ on the boundary of $e_k$ induces a probability measure $\mathcal{D}$ on $\pi^{-1}(0) \cup \pi^{-1}(1)$.

Denote by $\mathcal{G}$ the collection of admissible edge inverses with at most $N_2$ edges, such that $\mathcal{D}$ on $Y_1$, extends $\mathcal{D}$ on $\pi^{-1}(0) \cup \pi^{-1}(1)$ and such that in addition

$$\mathcal{D}(y)(w) \geq c_0 \,, \tag{8.12}$$

for all $y \in Y_0 = [0, 1]$ and $w \in \pi^{-1}(y)$. By Proposition 8.7, $\mathcal{G}_c \cap \mathcal{G}_{0,1}$ has cardinality $\geq m - 1$; compare however Remark 8.13.

Now we proceed mutadis mutandis as we did earlier. Namely, for each edge of $X_k$, we select an admissible edge inverse from the corresponding $\mathcal{G}$, subject to the stipulation that where necessary, we select from $\mathcal{G}_c$, $\mathcal{G}_c \cap G_0$, etc. In this way the construction of $(X_{k+1}, d_{k+1}, \mu_{k+1})$ is completed.

*Remark* 8.13. It will be clear from the discussion of subsequent subsections that the cardinality of $\mathcal{G}$ with $\mathcal{G}_c \cap \mathcal{G}_{0,1}$ will tend to infinity as any of $N_1$, $N_2$ $1/c_0'$, $\tilde{C}$ or $\mathcal{K}$ goes to infinity.

*Remark* 8.14. It will be seen below that if we assume that the values of $\mathcal{D}$ on $\pi^{-1}(0) \cup \pi^{-1}(1)$ can all be expressed as fractions (possibly not in lowest terms) with denominator $d$, then $c_0$ can be estimated from below in terms of $c_0'$, $N_2$, $d$; see Proposition 8.20.

**Example 8.15.** It is easy to construct examples of $\pi_k : X_{k+1} \to X_k$, such that for *no* choice of $\mathcal{D}_k$ on the inverse images of the vertices, is there an extension of $\mathcal{D}_k$ to a continuous fuzzy section to $X_{k+1}$. For instance, let $m \geq 2$ and let $X_k$ consist of 2 oriented edges $e, f$ with a common intitial point $x$ and a common final points $y$. Let $\pi_k^{-1}(x) = \{p, q\}$ and $\pi_k^{-1}(y) = \{r, s\}$. Let $\pi^{-1}(e)$ consist of two paths with disjoint interiors, one of which joins $p$ to $r$ and one of which joins $q$ to $s$. Let $\pi^{-1}(f)$ consist of a path joining $p$ to $r$, a path joining $q$ to $r$ and and a path joining $q$ to $s$, such that all 3 of these paths have disjoint interiors.

Suppose there exists a continuous fuzzy section $\mathcal{D}_k$. Using Axiom (6) (the continuity condition) and the structure of $\pi_k^{-1}(e)$ it follows that $\mathcal{D}(x)(p) = \mathcal{D}(y)(r)$, while from the structure of $\pi_k^{-1}(f)$, it follows that $D_k(p) > \mathcal{D}_k(r)$.

Having described the inductive construction in the general case, we devote the remainder of this section to the construction of large families of admissible edge inverses.

## 8.6 Quotients of edge inverses

Let $(Y_0, \hat{Y}_1, \hat{\pi})$ be an admissible edge inverse as in the previous subsection and assume $Y_0' \neq Y_1$. Form a quotient space $Y_1$ of $\hat{Y}_1$, by choosing some edge $e_j'$ in the interior of $Y_0'$ and identifying a pair of distinct inverse images of $\hat{\pi}^{-1}(e_j')$ by the unique isometry such that the map $\hat{\pi}$ factors through the quotient map $\sigma : \hat{Y}_1 \to Y_1$ i.e. $\hat{\pi} = \pi \circ \sigma$ for some $\pi$. Then if we equip $Y_1$ with the induced metric on edges and push-forward measure, $\sigma_\star(\hat{\nu}_1) = \nu_1$, we obtain a new admissible edge inverse $(Y_0, Y_1, \pi)$.

Note that with the obvious identification of inverse images of endpoints of $[0, 1]$, we have

$$\mathcal{D}(\ell) \,|\, \pi^{-1}(\ell) = \mathcal{D}(\ell) \,|\, \hat{\pi}^{-1}(\ell) \,. \tag{8.16}$$

We also can also identity a pair of edges in $\hat{\pi}^{-1}([0, 1/m])$ provided they have the same left-hand endpoint or a pair in $\hat{\pi}^{-1}([(m-1)/m, 1])$ if they have the right-hand endpoint, and do same the construction.

We refer to any edge inverse which is obtained by starting with $(Y_0, \hat{Y}_1, \hat{\pi})$ and iterating the above constructions a *quotient of* $(Y_0, \hat{Y}_1, \hat{\pi})$.

Similarly, the above argument can be repeated by identifying vertices in the inverse images of interior vertices of $Y_0'$ in place of edges. We also refer to the result as a *quotient of* $(Y_0, Y_1, \pi)$.

In particular, the quotient construction can be applied to a an admissible edge inverse as in Proposition 8.7. More importantly, it can be applied to "special admissible edge inverse" as defined in the next section. In fact, we will show that every admissible edge inverse arises as a quotient of a special one.

*Remark* 8.17. It is easy to verify that both $(Y_1, d_1, \nu_1) \xrightarrow{\sigma} (Y_0, d_0, \nu_0)$ and $(\hat{Y}_1, \hat{d}_1, \hat{\nu}_1) \xrightarrow{\sigma} (Y_1, d_1, \nu_1)$, satisfy Axioms (1)–(6).

## 8.7 Special admissible edge inverses

In this section we define a class of admissible edge inverses (called "special") whose combinatorial and metric classification can be reduced to the problem of describing the supports of all probability matrices with specified marginals. For the case in which the marginals take rational values, this can be done in terms of the Birkoff-Von Neumann theorem. For each possible support, the Birkoff-Von Neumann theorem also provides a canonical representative probability matrix whose entries have a definite lower bound. This is required to control the measure of the associated special edge inverse.

It will be clear that the cardinality of the collection of combinatorially distinct admissible edge inverses with specified marginals will be arbitrarily large if the parameters on which the associated matrix depends are sufficienly large. Moreover, by taking quotients as in the last section one obtains a much larger class of combinatorially distinct examples. In a subsequent subsection we will see that all examples of admissible edge inverses arise as quotients of special ones.

A *special edge inverse* is an edge inverse such that:

1. Each component of $\pi^{-1}((0,1))$ is an open interval $\gamma$. (Thus, the closures of two such components can intersect only at some point of $\pi^{-1}(0)$ and/or some point of $\pi^{-1}(1)$.)
2. If $\gamma$ is a component of $\pi^{-1}((0,1))$ then $\mathcal{D}(\pi(w))(w)$ is the same for all $w \in \gamma$.

For $w \in \gamma$ as above, we call $\mathcal{D}(\pi(w))(w)$ the *weight* of $\gamma$.

Suppose we are given a special admissible edge inverse. Let $n_1, n_2$ denote the cardinalities of $\pi^{-1}(0) = \{w_{0,t}\}$ and $\pi^{-1}(1) = \{w_{1,s}\}$ respectively. Define an $n_1 \times n_2$ probability matrix $P_{s,t}$, whose $s,t$-th entry is the sum of the weights of all those $\gamma$ as above with initial point $w_{0,t}$ and final point $w_{1,s}$. Then $P_{s,t}$ has the property that its marginals are given by $\mathcal{D}(0)(w_{0,t})$ and $\mathcal{D}(1)(w_{1,s})$.

Conversely, suppose we are given an $n_1 \times n_2$ probability matrix $P_{s,t}$ and positive integers $c_{s,t}$ for each nonzero entry $p_{s,t} > 0$. Then there is a unique special admissible edge inverse with $c_{s,t}$ paths $\gamma$ connecting $w_{0,t}$ to $w_{1,s}$ for each $(s,t)$, such that each such $\gamma$ connecting $w_{0,s}$ and $w_{1,t}$ has weight $p_{s,t}/c_{s,t}$. The resulting special edge inverse has the property that $\mathcal{D}(0)(w_{0,t})$ and $\mathcal{D}(1)(w_{1,s})$ are given by the marginals of $P_{s,t}$.

Therefore, we get the following.

**Proposition 8.18.** *The combinatorial classification of special admissible edge inverses with a specified $\mathcal{D}$ on the inverse images of the endpoints, is equivalent to the classification of the supports of probability matrices with specified marginals.*

Consider the simplest special case treated at the beginning of this section, in which $n_1 = n_2 = n$ and marginals, all equal to $\frac{1}{n}$. In that case, $P_{s,t}$ is a so called doubly stochastic matrix and there is a representation theorem, the Birkoff-Von Neumann theorem, which describes all such matrices.

**Theorem 8.19.** *(Birkoff-Von Neumann) The space of all doubly stochastic matrices has dimenson* $(n-1)\times(n-1)$. *Any such matrix is a convex combination of permutation matrices.*

*Remark* 8.20. Note that while the combinatorial and metric structure of the associated special admissible edge inverse is determined by the support of the corresponding probability matrix $P_{s,t}$, a bound on $\mathcal{D}$ (or equivalently on the ratio of $\nu_1$ to Lebesgue measure) is determined by a lower bound on the actual entries and the constants $c_{s,t}$, (which are bounded in terms of $N_2$).

For the case of doubly stochastic matrices the support is determined just by the collection of nozero coefficients representation in the representation supplied by the Birkoff-Von Neumann theorem. By choosing all such coefficients to be equal, we obtain matrix with the given support and a definite lower bound on the entries. Note that in the application to edge inverses, it is the entries which determine $\mathcal{D}_{k+1}$. Therefore, in what follows, we will always assume without further mention that this canonical choice has been made.

Below we will show that the classification of probability matrices with rational entries can also be reduced to the case of doubly stochastic matrices described above. Therefore, we have canonical representatives with a lower bound on the entries for each possible support in this case as well.

Given a $d \times d$ doubly stochastic matrix, for some integer $a$ replace the first $a$ rows by a single row which is equal to their sum and whose column marginal remains unchanged. By suitably iterating this operation we obtain a matrix whose row marginals are any sequence of length $< d$, of positive rational numbers with denominator $d$ whose sum is equal to 1. Then we can repeat the same operations with columns in place of rows. In this way we can obtain a matrix with any specified row and column marginals all of whose entries are rational numbers with denominator $d$. (We do not assume that these fractions are in lowest terms.)

In fact, every probability matrix with rational marginals such that every entry has denominator $d$ arises in this way. To see this, let $P = (p_{s,t})$ denote an $n_1 \times n_2$ probability matrix with rational entries and marginals $(\rho_s)$ and $(\tau_t)$. Let $d$ denote the least common denominator for $\{\rho_s\} \cup \{\tau_t\}$. Write $\rho_s = \alpha_s/d$, $\tau_t = \beta_t/d$. For each $s$, replace the $s$-th row by $\alpha_s$ identical rows, each with entries $p_{s,t}/\alpha_s$. This operation yields a $d \times n_2$ probability matrix whose row marginal has entries $1/d$ and whose column marginal remains unchanged. Now by repeating this operation with columns in place of rows, we obtain a doubly stochastic $d \times d$ probability matrix $\tilde{P}$ i.e. all entries of the row and column marginals are equal to $1/d$. Clearly, the original matrix $P_{s,t}$ can be obtained from the doubly stochastic matrix $\tilde{P}$ as in the previous paragraph.

In this sense, we have reduced the representation of arbitrary probability matrices with rational marginals to the Birkoff-Von Neumann theorem.

*Remark* 8.21. Suppose we are given the support of an $n_1 \times n_2$ probability matrix and a specified row marginal $(\rho_s)$. Then there is a unique probability matrix $P$ with the given row marginal such that all entries in any given row are the same.

As a consequence, given $X_k$ and a maximal collection of disjoint edges $\mathcal{C} = \{e_k\}$, the metric measure structure of the special edge inverses over these $e_k$ and in particular, the combinatorial structure, can be specified arbitrarily, the only caveat being that when necessary, we choose an arbitrary element of $\mathcal{G}_0$, $\mathcal{G}_1$ or $\mathcal{G}_{0,1}$; see Remark 8.10 and compare Remark 8.15. The corresponding collection of row and column marginals determins $\mathcal{D}_k$ on $\pi_k^{-1}(v_k)$, all vertices $v_k$ of $X_k$. Then the edge inverses of the remaining edges can be chosen as in the general inductive step. (The required $\tilde{C}m^{-k}$-dense set of connected edge inverses can be chosen from either $\mathcal{C}$ or its complement.)

## 8.8 Arbitrary edge inverses are quotients of special ones

We now show:

**Proposition 8.22.** *For any admissible edge inverse* $(Y_1, d_1, \nu_1) \xrightarrow{\pi} (Y_0, d_0, \nu_0)$*, there is a (canonically associated) special admissible edge inverse,* $(\hat{Y}_1, \hat{d}_1, \hat{\nu}_1) \xrightarrow{\hat{\pi}} (Y_0, d_0, \nu_0)$*, of which* $(Y_1, d_1, \nu_1) \xrightarrow{\pi} (Y_0, d_0, \nu_0)$ *is the quotient.*

*Proof.* Regard, $Y_0'$ as a path $\gamma_0'$, and let $\Gamma$ denote the collection of lifts to $Y_1$, as in Section 6. For each $\gamma_1 \in \Gamma$ take a copy $I_{\gamma_1}$ of $Y_0'$ and form the quotient space $\hat{Y}_1$ of $\bigcup_{\gamma_1 \in \Gamma} I_{\gamma_1}$ by the equivalence relations generated as follows: For all $\gamma_{1,1}, \gamma_{1,2} \in \Gamma$, identify $I_{\gamma_{1,1}}(0)$ with $I_{\gamma_{1,2}}(0)$ if and only if $\gamma_{1,1}(0) = \gamma_{1,2}(0)$. Similarly, identify $I_{\gamma_{1,1}}(1)$ with $I_{\gamma_{1,2}}(1)$ if and only if $\gamma_{1,1}(1) = \gamma_{1,2}(1)$. Give $\hat{Y}_1$ the path metric on components. There is a natural projection $\sigma : \hat{Y}_1 \to Y_1$. Put $\hat{\pi} = \sigma \circ \pi$. Then the restriction of $\sigma$ to $\hat{\pi}^{-1}(0) \cup \hat{\pi}^{-1}(1)$ is 1-1 and onto $\pi^{-1}(0) \cup \pi^{-1}(1)$.

It should be clear that the only remaining point is to specify the measure $\hat{\nu}_1$ such that $\sigma_\star(\hat{\nu}_1) = \nu_1$. To this end, we use an appropriate continuous fuzzy section $\hat{\mathcal{D}}_0$ of $\hat{\pi}$ defined as follows. For all $y_0'$ in the interior of $Y_0'$, $\gamma_1 \in \Gamma$ and $y_1 \in \hat{\pi}^{-1} \cap I_{\gamma_1}$, we put

$$\hat{\mathcal{D}}(y_0')(y_1) = \Omega(\gamma_1)\,, \tag{8.23}$$

where $\Omega$ is the probability measure on $\Gamma$ defined in (6.8). Then there is a unique extension of $\hat{\mathcal{D}}_0$ to a continuous fuzzy section of $\hat{\pi}$ on all of $Y_0'$. It then follows from (6.11) that $\sigma_\star(\hat{\mathcal{D}}_0) = \mathcal{D}_0$, which implies $\sigma_\star(\hat{\nu}_1) = \nu_1$. This suffices to complete the proof. □

# 9 Analytic dimension 1

In this section, we assume familiarity with certain material from [1] (see in particular Sections 2 and 4) including the fact that a PI space $(X, d, \mu)$ has a measurable cotangent bundle $TX^*$. In particular, there is a $\mu$-a.e. well defined fibre $TX_x^*$ whose dimension is sometimes called the analytic dimension. We also use the Sobolev spaces $H_{1,p}$ and the fact that they are reflexive.

We show:

**Theorem 9.1.** *If $(X_\infty, d_\infty, \mu_\infty)$ is the measure Gromov-Hausdorff limit of an admissible inverse system, then the dimension of the fibre of the cotangent bundle is 1 $\mu$-a.e..*

*Proof.* Without essential loss of generality, we can assume $X_0 = \mathbb{R}$. (Otherwise, we restrict attention to the inverse image of each individual open edge in $X_0$.) According to the definition of the cotangent bundle, it suffices to show the existence of a Lipschitz function $f_\infty$: $X_\infty \to \mathbb{R}$ such that if $h : X_\infty \to \mathbb{R}$ is Lipschitz, then at almost all points, the differential $dh$ is a bounded measurable function times $df_\infty$.

Let $f : \mathbb{R} \to \mathbb{R}$ denote the identity map viewed as a 1-Lipschitz function on $\mathbb{R}$. Let $f_i = f \circ \pi_{i-1} : X_i \to \mathbb{R}$. From Axioms (1) and (2) in the definition of admissible inverse systems, it is clear that $df_i$ defines a trivialization of the cotangent bundle of $X_i$. Let $\pi_i^\infty : X_\infty \to X_i$ denote the natural projection and set $f_\infty = f \circ \pi_i^\infty$.

It is easy to see that any $L$-Lipschitz function $h : X_\infty \to \mathbb{R}$, is the uniform limit as $i \to \infty$ of $2L$-Lipschitz functions of the form $h_i = \tilde{h}_i \circ \pi_i^\infty$ where $\tilde{h}_i$ is a $2L$-Lipschitz function on $X_i$. It follows that $d\tilde{h}_i$ is a bounded measurable function times $df_i$ and hence, that $dh_i$ is a bounded measurable function times $df_\infty$. Clearly, the same holds for any finite linear combination of the $h_i$.

By the reflexivity of the Sobolev space $H_{1,p}$ (Theorem 4.48 of [1]) it follows that there is a a sequence $\hat{h}_i$ of such combinations which converges to $h$ in $H_{1,p}$. It follows that $dh$ is a bounded measurable function times $df_\infty$, which suffices to complete the proof. □

# 10 Bi-Lipschitz nonembedding in Banach spaces with the RNP

Recall that a Banach space $\mathcal{V}$ is said to have the Radon-Nikodym Property if every Lipschitz map $f : \mathbb{R} \to \mathcal{V}$ is differentiable almost everywhere. Separable dual spaces such as $L_p$ for $1 < p < \infty$ and $\ell_1$ have the Radon-Nikodym Property but $L_1$ does not.

In this section we show that except in degenerate cases, the Gromov-Hausdorff limit $(X_\infty, d_\infty)$ of an admissible inverse system does not bilipschitz embed in any Banach spaces with the Radon-Nikodym property. However it follows directly from the main result of [7] these spaces do bilipschitz embed in $L_1$.

Since by Theorem 1.1, $(X_\infty, d_\infty, \mu_\infty)$ is a PI space, according to [4], it will suffice to give conditions on $(X_\infty, d_\infty, \mu_\infty)$ which guarantee that for a subset of positive $\mu_\infty$ measure, some tangent cone is not bilipschitz to a Euclidean space. According to the following lemma, in our situation, the only possibility for the dimension of this Euclidean space is 1; compare Corollary 2.16.

Let $(X_i, \pi_i, \mu_i)$ denote an admissible inverse system with subdivision parameter $m \geq 2$. Let $V_i^{\geq 3} \subset X_i$ denote the set of vertices of $X_i$ with degree at least 3. Given a vertex $v_i \in X_i$, we define the *halfstar of $v_i$ in $X_i$* to be the union $\mathrm{St}_{\frac{1}{2}}(v_i, X_i) \subset X_i$ of the segments of length $\frac{1}{2} m^{-i}$ emanating from $v_i$.

**Lemma 10.1.** *Let $(X_i, \pi_i, \mu_i)$ denote an admissible inverse system with subdivision parameter $m \geq 2$.*

1. *Let $x_\infty \in X_\infty$ and assume $\pi_i^\infty(x_\infty)$ is a vertex of $X_i$. Then there is a subset $Y_\infty \subset X_\infty$ which projects isometrically under $\pi_i^\infty$ to the halfstar $\mathrm{St}_{\frac{1}{2}}(\pi_i^\infty(x_\infty), X_i)$.*
2. *Let $x_\infty \in X_\infty$. Then every tangent cone of $X_\infty$ at $x_\infty$ is homeomorphic to $\mathbb{R}$ if and only if every such tangent cone is isometric to $\mathbb{R}$. This holds if and only if*

$$\liminf_{i\to\infty} \; m^i \cdot d_i(\pi_i^\infty(p_\infty), V_i^{\geq 3}) = \infty \,.$$

3. *For all $x_\infty \in X_\infty$, every tangent cone at $x_\infty$ has topological dimension* 1.

*Proof.* (1). Let $Y_i = \mathrm{St}_{\frac{1}{2}}(\pi_i^\infty(x_\infty), X_i)$. Given a geodesic path of length $\frac{1}{2}m^{-i}$ emanating from $\pi_i^\infty(x_\infty)$, we can lift it to a path in $X_{i+1}$ starting at $\pi_{i+1}^\infty(x_\infty)$; see the discussion of Axiom (2) in Section 1. By taking the union of one such lift for each path, we obtain a lift $Y_{i+1}$ of $Y_i$. Iterating this produces a compatible sequence $\{Y_j \subset X_j\}_{j\geq i}$ that projects isometrically to $\mathrm{St}_{\frac{1}{2}}(\pi_i^\infty(x_\infty), X_i)$ under the projections $\pi_i^j : X_j \to X_i$. Then the inverse limit of $\{Y_j\}$ is the desired subset.

(2). If $\liminf_{i\to\infty}\ m^i \cdot d_i(\pi_i^\infty(p_\infty), V_i^{\geq 3}) = D < \infty$, then using path lifting one gets sequences $i_j \to \infty$, $\{x_{j,\infty}\} \subset X_\infty$, such that $\pi_{i_j}^\infty(x_{j,\infty}) \in Y_{i_j}^{\geq 3}$, and $d(x_{j,\infty}, p_\infty) < 2Dm^{-i_j}$. Then by (1), for every $j$ the rescaled pointed space $(X_\infty, m^{i_j}d_\infty, p_\infty)$ contains an isometric copy of a "tripod" of size $\frac{1}{2}$ within the ball $B(p, 2(D+1)) \subset (X_\infty, m^{i_j}d_\infty)$. (By a tripod of size $\frac{1}{2}$, we mean 3 line segments, each of length $\frac{1}{2}$, emanating from a single point, equipped with the path metric.) Therefore any pointed Gromov-Hausdorff limit of a subsequence of the sequence $\{(X_\infty, m^{i_j}d_\infty, p_\infty)\}_j$ will contain an isometric copy of such a tripod, and hence cannot be homeomorphic to $\mathbb{R}$.

Suppose conversely, that $\liminf_{i\to\infty}\ m^i \cdot d_i(\pi_i^\infty(p_\infty), V_i^{\geq 3}) = \infty$. Let $D_i = m^i \cdot d_i(\pi_i^\infty(p_\infty), V_i^{\geq 3})$. Then $D_i \to \infty$, so we can choose sequences $\{j_i\}$, $\{R_i\}$ such that:

- $j_i - i \to \infty$ and $R_i \to \infty$ as $i \to \infty$.
- $B_{m^{-i}R_i}(\pi_{j_i}^\infty(p_\infty)) \subset X_{j_i}$ contains only degree 2 vertices and is therefore isometric to an interval.

It follows that the pointed sequence $\{(X_{j_i}, m^i d_{j_i}, \pi_{j_i}^\infty(p_\infty))\}$ converges to $(\mathbb{R}, 0)$ in the pointed Gromov-Hausdorff topology, and also to any tangent cone at $(X_\infty, p_\infty)$, since the projection map $\pi_{j_i}^\infty : (X_\infty, m^i d_\infty) \to (X_{j_i}, m^i d_{j_i})$ is a $Cm^{i-j_i}$-Hausdorff approximation.

(3). It is easy to see that up to rescaling of the metric, a tangent cone at a point of $X_\infty$ is itself the pointed Gromov-Hausdorff limit of an admissible inverse system. Then, by Corollary 2.16, it follows that every such tangent cone has topological dimension 1. □

Thus we obtain the following:

**Theorem 10.2.** *If $\{(X_i, d_i, \mu_i)\}$ is an admissible inverse system, and a positive $\mu_\infty$ measure set of points $x_\infty \in X_\infty$ satisfy*

$$\liminf_{i\to\infty}\ m^i \cdot d_i(\pi_i^\infty(x_\infty), V_i^{\geq 3}) < \infty\,, \tag{10.3}$$

*then $(X_\infty, d_\infty)$ does not bilipschitz embed in any Banach space with the Radon-Nikodym Property.*

*Proof.* By Lemma 10.1, any tangent cone at such a point $x_\infty$ has topological dimension 1, and contains an isometric copy of a tripod. Therefore it cannot be homeomorphic to $\mathbb{R}^n$ for any $n$. Now Theorem 1.6 of [4] implies that $X_\infty$ does not bilipschitz embed in any Banach space with the Radon-Nikodym Property. □

*Remark* 10.4. Examples which fail to satisfy (10.3) are "degenerate" in an obvious sense.

# 11 Higher dimensional inverse systems

In this section we consider higher dimensional inverse systems, where each $X_i$ is a cube complex. We would like to point out that there are other ways of generalizing to higher dimension; in particular, one can construct examples of inverse systems where $X_0$ is the Heisenberg group with the Carnot metric, the mappings $\pi_i : X_{i+1} \to X_i$ are "branched mappings", and the inverse limit is a PI space.

We recall that the *star* of a face $C$ in a polyhedral complex $X$ is the union $\mathrm{St}(C, X)$ of the faces containing it. A *gallery* in an $n$-dimensional polyhedral complex is a sequence $C_0, \ldots, C_N$ of top dimensional faces where $C_{i-1} \cap C_i$ is a codimension 1 face for all $1 \leq i \leq N$.

Fix $n \geq 1$. We consider an inverse system

$$X_0 \xleftarrow{\pi_0} \cdots \xleftarrow{\pi_{i-1}} X_i \xleftarrow{\pi_i} \cdots \tag{11.1}$$

such that each $X_i$ is a connected cube complex equipped with a path metric $d_i$ and a measure $\mu_i$, such that the following conditions hold, for some constants $2 \leq m \in \mathbb{Z}$, $\Delta$, $\theta$, $C \in (0, \infty)$ and every $i \in \mathbb{Z}$:

1. (Bounded local metric geometry) $(X_i, d_i)$ is a nonempty connected cube complex that is a union of $n$-dimensional faces isometric to the $n$-cube $[0, m^{-i}]^n$ (with respect to the path metric $d_i$), such that every link contains at most $\Delta$ faces.
2. (Simplicial projections are open) If $X'_i$ denotes the cube complex obtained by subdividing each cube of $X_i$ into $m^n$ subcubes isometric to $[0, m^{-(i+1)}]^n$, then $\pi_i$ induces a map $\pi_i : (X_{i+1}, d_{i+1}) \to (X'_i, d_i)$ which is open, cellular (with respect to the cube structure), and an isometry on every face.
3. (Gallery diameter of fibers is controlled) For every $x_i \in X'_i$, any two points in the inverse image $\pi_i^{-1}(x_i) \subset X_{i+1}$ can be joined by a gallery of $n$-cubes $C_0, \ldots, C_N$, where $N \leq \Delta$.
4. (Bounded local metric measure geometry.) The measure $\mu_i$ restricts to a constant multiple of Lebesgue measure on each $n$-cube $C_i \subset X_i$, and $\frac{\mu_i(C_{i,1})}{\mu_i(C_{i,2})} \in [C^{-1}, C]$ for any two adjacent $n$-cubes $C_{i,1}, C_{i,2} \subset X_i$.
5. (Compatibility with projections)
$$(\pi_i)_\star(\mu_{i+1}) = \mu_i\,,$$
where $(\pi_i)_\star(\mu_{i+1})$ denotes the pushforward of $\mu_{i+1}$ under $\pi_i$.
6. (Continuity across codimension 1 faces) For every pair of codimension 1 faces $c'_i \subset X'_i$, and $c_{i+1} \subset \pi_i^{-1}(v'_i)$, the quantity
$$\frac{\mu_{i+1}(\pi_i^{-1}(C'_i) \cap \mathrm{St}(c_{i+1}, X_{i+1}))}{\mu_i(C'_i)} \tag{11.2}$$
is the same for all $n$-cubes $C'_i \subset \mathrm{St}(c'_i, X'_i)$.

The biggest difference between the axioms above and Definition 2.10 is in Axiom (3) above, where path diameter has been replaced by gallery diameter. Note that the gallery diameter is the same as the path diameter in the case of graphs. A bound on the path diameter would be sufficient to verify most of the properties that hold for admissible inverse systems of graphs. However, it is not sufficient to recover the main result — the $(1, 1)$-Poincare inequality as the following example illustrates.

**Example 11.3.** Consider the 2-dimensional inverse system with subdivision parameter $m = 2$, where:

- $X_0$ is the unit square $[0, 1]^2$.
- $X_1$ is obtained by taking two copies of the subdivided complex $X'_0$ and gluing them together along their central vertices.
- All projection maps $\pi_i : X_{i+1} \to X'_i$ with $i > 0$ are isomorphisms.

Then $X_\infty$ is isometric to $X_1$, and does not satisfy a $(1, 1)$-Poincare inequality; this is because the gluing locus —a singleton— has zero 1-capacity.

Let $X_\infty$ be the inverse limit of an inverse system satisfying (1)-(6) above. The proof of the Poincaré inequality for $X_\infty$ using path families carries over in a straightforward way, when one uses geodesic paths that intersect each $n$-cube $C$ in a segment parallel to an edge of $C$. So does the proof using continuous fuzzy sections.

*Remark* 11.4. What is essential in Axioms (1) and (4) is that they imply that $X_i$ is doubling and satisfies a $(1, 1)$-Poincaré inequality on scale $m^{-i}$. In the above example, this doesn't hold. However, if Axiom (4) is appropriately modified, then Axiom (3) can be left as is.

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