

Research Article

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Poincaré Inequalities for Mutually Singular Measures

Abstract: Using an inverse system of metric graphs as in [3], we provide a simple example of a metric space X that admits Poincaré inequalities for a continuum of mutually singular measures.

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1 Introduction

Overview

In this note we provide a simple example of a metric measure space X that satisfies abstract Poincaré inequalities in the sense of Heinonen-Koskela [5] for a 1-parameter family of mutually singular measures. By a recent announcement of M. Csörnyei and P. Jones the classical Rademacher's Theorem is *sharp* in the sense that if its *conclusion* holds for the metric measure space (\mathbb{R}^n, μ) , then μ must be absolutely continuous with respect to the Lebesgue measure. From this, using the theory of *differentiability spaces* (compare [2, Sec. 14]), it follows that if μ is a doubling measure on \mathbb{R}^n such that the metric measure space (\mathbb{R}^n, μ) admits an abstract Poincaré inequality, then the measure μ is absolutely continuous with respect to the Lebesgue measure. The example presented here shows that for more general metric measure spaces a similar phenomenon does not hold; in particular, the measure class for which Cheeger's generalization of Rademacher's Theorem holds is not uniquely determined. More precisely, the result of this note is the following:

Theorem 1.1. *There is a compact geodesic metric space X and there is a family of doubling probability measures $\{\mu_w\}_{w \in (0,1)}$ defined on X such that:*

- Each metric measure space (X, μ_w) supports a $(1, 1)$ -Poincaré inequality;
- If $w \neq w'$ the measures μ_w and $\mu_{w'}$ are mutually singular.

Background

We now recall some facts about the classical Rademacher Theorem and Cheeger's extension of it to metric spaces, which help to put Theorem 1.1 in perspective.

The classical Rademacher Theorem asserts that a Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is differentiable at \mathcal{L}^n -a.e. point, where \mathcal{L}^n denotes Lebesgue measure. Let now $k = 1$ and let $\{x_i\}_{i=1}^n$ denote the standard coordinate functions on \mathbb{R}^n . Even though the differentiability of f at a point p can be formulated in several equivalent

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ways, the one which is more suitable for the following discussion is this: a Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at p if and only if there is a *unique* $a \in \mathbb{R}^n$ such that:

$$\limsup_{\substack{q \rightarrow p \\ q \neq p}} \frac{|f(q) - f(p) - \sum_{i=1}^n a_i (x_i(q) - x_i(p))|}{d(p, q)} = 0. \quad (1.2)$$

In [2] Cheeger introduced a generalization of Rademacher's Theorem, which we will informally call *Cheeger's Rademacher*, for real-valued Lipschitz functions defined on a metric measure space (X, μ) . The starting point is the following generalization of (1.2).

Definition 1.3. Let f and $\{\psi_i\}_{i=1}^n$ be real-valued Lipschitz functions defined on a metric space X , and let x be a point of X which not isolated. We say that f is differentiable at x with respect to the $\{\psi_i\}_{i=1}^n$ if there is a *unique* $a \in \mathbb{R}^n$ such that:

$$\limsup_{\substack{y \rightarrow x \\ y \neq x}} \frac{|f(y) - f(x) - \sum_{i=1}^n a_i (\psi_i(y) - \psi_i(x))|}{d(x, y)} = 0. \quad (1.4)$$

In this case we say that a is the derivative of f and x with respect to the $\{\psi_i\}_{i=1}^n$.

By analogy with the case of differentiable manifolds one is then led to introduce a notion of *chart*:

Definition 1.5. Let (X, μ) be a metric measure space and $U \subset X$ Borel with $\mu(U) > 0$. Let $\{\psi_i\}_{i=1}^n$ be real-valued Lipschitz functions defined on X . We say that $(U, \{\psi_i\}_{i=1}^n)$ is a (*differentiability*) *chart* if each real-valued Lipschitz function f is differentiable at μ -a.e. $x \in U$ with respect to the $\{\psi_i\}_{i=1}^n$.

Finally one defines differentiability spaces by requiring the existence of a countable atlas:

Definition 1.6. A metric measure space (X, μ) is a *differentiability space* if there are countably many charts $\left\{ \left(U_\alpha, \{\psi_{i,\alpha}\}_{i=1}^{N_\alpha} \right) \right\}_\alpha$ such that:

$$\mu \left(X \setminus \bigcup_\alpha U_\alpha \right) = 0 \quad (1.7)$$

$$\sup_\alpha N_\alpha = N < \infty; \quad (1.8)$$

the integer N is called the *differentiability dimension*. If (X, μ) is a differentiability space we will also (informally) say that Cheeger's Rademacher holds for (X, μ) .

Note that Cheeger's Rademacher is a genuine generalization of the classical Rademacher Theorem because $(\mathbb{R}^n, \mathcal{L}^n)$ is a differentiability space with a single chart $(\mathbb{R}^n, \{x_i\}_{i=1}^n)$. Moreover, [2] contains also a new self-contained proof of the classical Rademacher Theorem (see [2, Rmk. 8.14]). The class of differentiability spaces is very rich as Cheeger showed that Cheeger's Rademacher holds for all doubling metric measure spaces which admit a $(1, p)$ -Poincaré inequality ($p \in [1, \infty)$) in the sense of Heinonen and Koskela [5].

In the study of the differentiability properties of Lipschitz functions defined on \mathbb{R}^n an important question has been the following:

(Q1): Is there a non-trivial Radon measure μ on \mathbb{R}^n such that μ is singular with respect to \mathcal{L}^n and the conclusion of Rademacher's Theorem holds for μ ? Equivalently, is there a Radon measure μ on \mathbb{R}^n which is singular with respect to \mathcal{L}^n and such that Cheeger's Rademacher holds for (\mathbb{R}^n, μ) ?

That the answer to **(Q1)** is NO for $n = 1$ has been known for a long time [9]¹. More recently, Alberti, Csörnyei and Preiss showed that the answer is NO for $n = 2$ (an account can be found in [1]). Recently, Csörnyei and Jones have announced that the answer is NO also for $n \geq 3$. We are thus led to introduce the following definition:

Definition 1.9. Let X be a metric space; we say that Cheeger's Rademacher is *sharp* for X if there is a Radon measure μ_X on X such that:

1. Cheeger's Rademacher holds for (X, μ_X) ;
2. If ν is another Radon measure on X such that Cheeger's Rademacher holds for (X, ν) , then ν is absolutely continuous with respect to μ_X .

In some sense, if Cheeger's Rademacher is sharp for X , there is a canonical measure class defined on X . Therefore, Theorem 1.1 can be rephrased as saying that Cheeger's Rademacher is not sharp for the example X . We also point out other interesting features of this example. First, for any choice of $w \in (0, 1)$ the differentiability dimension of (X, μ_w) is always one, which should be contrasted with the fact that the answer to **(Q1)** for the case $n = 1$ has been known for a long time. Secondly, each (X, μ_w) admits a $(1, 1)$ -Poincaré inequality, which is the strongest kind of a Poincaré inequality that a metric measure space admits as a $(1, p)$ -Poincaré inequality always implies a $(1, p + \Delta)$ -Poincaré inequality for $\Delta > 0$.

A drawback of the example X is that μ_w is never Ahlfors-regular. Note that the lack of Ahlfors-regular measures on X depends really on the geometry of the space. However, in Remark 2.14 we show how to modify the example so that $\mu_{\frac{1}{2}}$ becomes Ahlfors-regular.

2 The Example

The goal of this Section is to prove Theorem 1.1. The metric space X that we consider is the example [8, pg. 290], compare also [4, Exa. 1.2]. We briefly recall the construction. We build a sequence of graphs $\{X_n\}_{n \geq 0}$ starting with X_0 , which consists of a single edge which we identify with the interval $[0, 1]$. The graph X_{n+1} is obtained from X_n by subdividing each edge of X_n into four equal parts and replacing it by a rescaled copy of the diamond graph (Figure 1, where we have labelled the edges for future reference). Each edge of X_n has length 4^{-n} and the map which collapses the diamond graph to a segment gives rise to 1-Lipschitz maps $\pi_{n+1, n} : X_{n+1} \rightarrow X_n$. In this way, the graphs $\{X_n\}_{n \geq 0}$ fit into an inverse system and, for each pair (k, n) of nonnegative integers satisfying $k \leq n$, one has a 1-Lipschitz map $\pi_{n, k} : X_n \rightarrow X_k$. The maps $\{\pi_{n, k}\}_{n \geq 0, k \geq 0}$ satisfy the compatibility relations:

$$\pi_{k, m} \circ \pi_{n, k} = \pi_{n, m}. \quad (2.1)$$

Having equipped the graphs X_n with the length distance, the space X is the Gromov-Hausdorff limit of the sequence $\{X_n\}_{n \geq 0}$; by the properties of the inverse system one also gets 1-Lipschitz maps $\pi_{\infty, n} : X \rightarrow X_n$

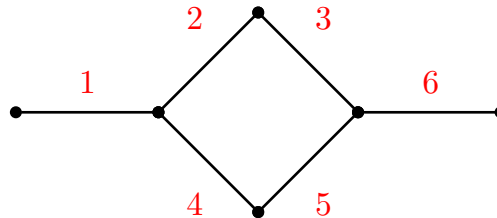


Figure 1: Diamond graph with labelled edges

¹ This is the first reference we are aware of, but this result was probably known before

which satisfy the compatibility conditions:

$$\pi_{n,k} \circ \pi_{\infty,n} = \pi_{\infty,k}. \tag{2.2}$$

We will let $\text{Edge}(n)$ and $\text{Vertex}(n)$ denote, respectively, the sets of edges and vertices of X_n .

We now turn to the construction of the family of measures $\{\mu_w\}_{w \in (0,1)}$. For $w \in (0, 1)$, the measure μ_w is the unique probability measure on X which satisfies the following requirements:

- (R1)** For each nonnegative integer n , one has $\pi_{\infty,n\#}(\mu_w) = \mu_{w,n}$ where $\mu_{w,n}$ is a probability measure on X_n ;
- (R2)** The measure $\mu_{w,n}$ is a multiple of arclength on each edge of X_n and μ_0 is identified with the Lebesgue measure on $[0, 1]$;
- (R3)** For each edge $e_n \in \text{Edge}(n)$, denoting by $\{e_{n+1,i}\}_{i=1,\dots,6} \subset \text{Edge}(n+1)$ the edges of X_{n+1} whose union is $\pi_{n+1,n}^{-1}(e_n)$ (labelling as in Figure 1), one has:

$$\mu_{w,n+1} \llcorner e_{n+1,i} = \begin{cases} \frac{\mu_{w,n}(e_n)}{4^{-n}} \mathcal{H}^1 \llcorner e_{n+1,i} & \text{if } i = 1, 6 \\ w \frac{\mu_{w,n}(e_n)}{4^{-n}} \mathcal{H}^1 \llcorner e_{n+1,i} & \text{if } i = 2, 3 \\ (1-w) \frac{\mu_{w,n}(e_n)}{4^{-n}} \mathcal{H}^1 \llcorner e_{n+1,i} & \text{if } i = 4, 5, \end{cases} \tag{2.3}$$

where \mathcal{H}^1 denotes the 1-dimensional Hausdorff measure.

By the main result of [3, Thm. 1.1] each metric measure space (X, μ_w) admits a $(1, 1)$ -Poincaré inequality. Note that the class of spaces considered in [3] is much broader and in this example one can also prove the $(1, 1)$ -Poincaré inequality directly by using pencils of curves similarly as in [7]. More specifically, one can appeal to a result of Keith [6, Thm. 2(4)] after having chosen, for each pair of points $x_1, x_2 \in X$, a suitable probability measure $P_{x_1,x_2,w}$ on a pencil of curves joining x_1 to x_2 . Note also that each measure μ_w is doubling, and that the doubling constant blows up as $w \rightarrow 1$ or $w \rightarrow 0$.

We now turn to a probabilistic description of the points in X . Let $V \subset X$ be the set of points which project to some vertex:

$$V = \{p \in X : \exists n \geq 0 : \pi_{\infty,n}(p) \in \text{Vertex}(n)\}; \tag{2.4}$$

then $\mu_w(V) = 0$ by conditions **(R1)** and **(R2)**. Let \mathcal{C} denote the Cantor set $\{1, \dots, 6\}^{\mathbb{N}}$ and let $A : \mathcal{C} \rightarrow X \setminus V$ be the map defined as follows: given $\tilde{p} \in \mathcal{C}$, $A(\tilde{p})$ is the unique point $p \in X \setminus V$ such that, for each $n \geq 1$, $\pi_{n+1}(p)$ belongs to the edge labelled by the integer $\tilde{p}(n)$ among those in $\pi_{n+1,n}^{-1}(e_n)$ (see Figure 1), where e_n is the unique edge of X_n containing $\pi_n(p)$. We now define the probability measure ν_w on $\{1, \dots, 6\}$ as follows:

$$\nu_w(i) = \begin{cases} \frac{1}{4} & \text{if } i = 1, 6 \\ \frac{w}{4} & \text{if } i = 2, 3 \\ \frac{1-w}{4} & \text{if } i = 4, 5; \end{cases} \tag{2.5}$$

we then let P_w denote the probability measure on \mathcal{C} which is the product of countably many copies of the measure ν_w . We observe that $A\#P_w = \mu_w$ and we let $T_{i,n}$ denote the random variable:

$$T_{i,n}(\tilde{p}) = \begin{cases} 1 & \text{if } \tilde{p}(n) = i \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Lemma 2.7. *Let $S_{i,n} = \sum_{k \leq n} T_{i,k}$; then one has P_w -a.s.:*

$$\lim_{n \rightarrow \infty} \frac{S_{i,n}}{n} = \begin{cases} \frac{1}{4} & \text{if } i = 1, 6 \\ \frac{w}{4} & \text{if } i = 2, 3 \\ \frac{1-w}{4} & \text{if } i = 4, 5. \end{cases} \tag{2.8}$$

Proof. This follows from the Strong Law of Large Numbers as the random variables $\{T_{i,n}\}_{n \geq 1}$ are i.i.d. □

We now complete the proof of Theorem 1.1:

Lemma 2.9. *If $w \neq w'$ the measures μ_w and $\mu_{w'}$ are mutually singular.*

Proof. We show that the Radon-Nikodym derivative $\frac{d\mu_w}{d\mu_{w'}}$ is null. We let $\text{Reg}(w')$ denote the set of points of \mathbb{C} such that (2.8) holds (for the parameter w'). By an application of measure differentiation, if we consider $p \in X \setminus V$ and let $e_n(p) \in \text{Edge}(n)$ denote the unique edge of X_n containing $\pi_n(p)$, then for $\mu_{w'}$ -a.e. $p \in A(\text{Reg}(w'))$ we have:

$$\frac{d\mu_w}{d\mu_{w'}}(p) = \lim_{n \rightarrow \infty} \frac{\mu_{w,n}(e_n(p))}{\mu_{w',n}(e_n(p))} = \lim_{n \rightarrow \infty} \underbrace{\left(\frac{w}{w'}\right)^{S_{2,n}(A^{-1}(p))+S_{3,n}(A^{-1}(p))} \left(\frac{1-w}{1-w'}\right)^{S_{4,n}(A^{-1}(p))+S_{5,n}(A^{-1}(p))}}_{c_n}; \quad (2.10)$$

note that here we used that the set $\pi_n^{-1}(e_n(p))$ is comparable to the ball of radius 4^{-n} about p . We now prove that $\lim_{n \rightarrow \infty} c_n = 0$ by showing that $\frac{\ln c_n}{n}$ converges to a negative number. Applying (2.8) we find:

$$\lim_{n \rightarrow \infty} \frac{\ln c_n}{n} = \frac{1}{2} \left(w' \ln \frac{w}{w'} + (1-w') \ln \frac{1-w}{1-w'} \right); \quad (2.11)$$

as the function \ln is strictly concave and as $w \neq w'$,

$$\lim_{n \rightarrow \infty} \frac{\ln c_n}{n} < \frac{1}{2} \ln \left(w' \frac{w}{w'} + (1-w') \frac{1-w}{1-w'} \right) = 0. \quad (2.12)$$

□

Remark 2.13. Note that each (X, μ_w) is a differentiability space in the sense of Cheeger [2] with a unique one-dimensional chart $(X, \pi_{\infty,0})$.

Remark 2.14. For any choice of $w \in (0, 1)$, the metric measure space (X, μ_w) is never Ahlfors-regular. We briefly sketch how to obtain examples which are Ahlfors-regular for some choice of the parameter w . We use the original examples of Laakso [7], and refer to [7, Sec. 1] for the notation and the terminology. On the Cantor set $K(t)$, having identified it with $\{0, 1\}^{\mathbb{N}}$, we consider the probability measure P_w which is obtained as a product of countably many copies of $\nu_w = w\delta_0 + (1-w)\delta_1$. The metric space $Y(t)$ is obtained as a quotient of $[0, 1] \times K(t)$ as in [7]. The measure μ_w is the push forward under the quotient map of the measure $\mathcal{H}^1 \llcorner [0, 1] \times P_w$. The original examples of Laakso correspond to the choice $w = \frac{1}{2}$, which yield metric measure spaces $(Y(t), \mu_{1/2})$ which are Ahlfors-regular of dimension $1 + \log_{1/t} 2$.

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