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Resolvent Flows for Convex Functionals and p-Harmonic Maps

Abstract: We prove the unique existence of the (non-linear) resolvent associated to a coercive proper lower semicontinuous function satisfying a weak notion of p-uniform λ-convexity on a complete metric space, and establish the existence of the minimizer of such functions as the large time limit of the resolvents, which generalizing pioneering work by Jost for convex functionals on complete CAT(0)-spaces. The results can be applied to L^p-Wasserstein space over complete p-uniformly convex spaces. As an application, we solve an initial boundary value problem for p-harmonic maps into CAT(0)-spaces in terms of Cheeger type p-Sobolev spaces.

Keywords: CAT(0)-space; CAT(κ)-space; p-uniformly convex space; weak convergence; p-uniformly λ-convex function; Moreau-Yosida approximation; Hamilton-Jacobi semi-group; Hopf-Lax formula; resolvent; local slope; global slope; stationary point; Cheeger's energy; Cheeger type Sobolev space; p-harmonic map; L^p-Wasserstein space; generalized geodesics

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1 Introduction

The notion of p-uniformly convex space is a natural generalization of the notion of p-uniformly convex Banach space (see [23, 34]). Typical examples of p-uniformly convex spaces are L^p-spaces with $p \geq 2$, CAT(0)-spaces, like Hadamard manifolds and trees, and many others. An L^p-mapping space over a measurable space, with target space a p-uniformly convex space having the NPC property in the sense of Busemann, is also a p-uniformly convex geodesic space having the NPC property in the sense of Busemann. An energy functional defined in a suitable way on it becomes convex and lower semicontinuous. Thus, it is reasonable to consider that (H, d_H) a p-uniformly convex geodesic space instead of such an L^p-mapping space, and $E : H \to]-\infty, +\infty]$ a coercive convex lower semicontinuous function with $E \not\equiv +\infty$. For any $\tau > 0$ and $u \in H$, there exists a unique minimizer, say $J_\tau^E(u) \in H$, of $v \mapsto E(v) + \frac{1}{p\tau^{p-1}} d_H^p(u, v)$. This defines a map $J_\tau^E : H \to H$, called the *resolvent of* E (see Propositions 3.26 below and [10, 17, 22] for the case $p = 2$). The minimum $E^\tau(u) := \min_{v \in H}(E(v) + \frac{1}{p\tau^{p-1}} d_H^p(u, v))$ is called *Moreau-Yosida approximation*, *Hamilton-Jacobi semigroup* or *Hopf-Lax formula*. Note that if H is a Hilbert space and if E is a closed densely defined symmetric quadratic form on H, then we have $J_\tau^E = (I + \frac{\tau}{2}A)^{-1}$, where A is the infinitesimal generator associated with E. The one-parameter family $[0, +\infty[\ni \tau \mapsto J_\tau^E(u)$ gives a deformation of a given map $u \in H$ to a minimizer of E (or a p-harmonic map), provided the limit $\lim_{\tau_n \to +\infty} J_{\tau_n}^E(u)$ exists for a subsequence τ_n. Jost [12] studied the convergence of resolvents and Moreau-Yosida approximations on a fixed complete CAT(0)-space. On a complete CAT(0)-space as complete 2-uniformly convex space, the limit $T_t^E(u) := \lim_{l \to \infty}(J_{t/l}^E)^l(u)$ exists and call it (non-linear) semigroup or gradient flow associated to E, which was proved by Uwe-Mayer [22] (see also

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Jost [12]). The gradient flow studied in [22] has influenced the theory of harmonic maps between geometric singular spaces, for example, it was effectively applied to prove a fixed point theorem in terms of discrete groups acting on spaces and combinatorial harmonic maps between them (see Izeki-Nayatani [9]). The gradient flows in [22] were generalized in Ambrosio-Gigli-Savaré [1] as *p-curves of maximal slope* for coercive proper lower semicontinuous function E having a p-uniform λ-convexity along a continuous curve on a complete metric space in order to construct gradient flows for functionals on L^p-Wasserstein space over a Hilbert space. For $p = 2$, they proved that the gradient flow for such a function E having 2-strong λ-convexity along a continuous curve can be constructed as the local uniform limit of the discrete flow, and the constructed flow satisfies the 2-Evolution Variational Inequality (2-(EVI) in short):

$$\frac{\mathrm{d}}{\mathrm{d}t}d_H^2(u(t),v)+\lambda d_H^2(u(t),v)+E(u(t))\le E(v), \tag{1.1}$$

for a.e. $t > 0$ and $v \in D(E)$, which yields the contraction property of gradient flows:

$$d_H(u(t),v(t))\le e^{-\lambda t}d_H(u_0,v_0)\quad \text{for } t>0, \tag{1.2}$$

where $u(t) = T_t^E(u_0)$, $v(t) = T_t^E(v_0)$ with $u_0, v_0 \in \overline{D(E)}$, and the semigroup property: $T_{t+s}^E(u_0) = T_t^E(T_s^E(u_0)$ for $t, s \ge 0$ (see [1, Chapter 4]). It is not clear that the p-curves of maximal slope for the functionals treated in [1]) possesses the contraction property and the semigroup property. To establish the theory of p-harmonic maps, these properties are important as developed in [22].

To develop the theory of p-harmonic maps into p-uniformly convex space, we employ the resolvent flow $(J_\tau^E(u))_{\tau>0}$ for proper lower semicontinuous functions E satisfying a weaker notion of p-uniform λ-convexity on a complete geodesic space for general $p \in [2, +\infty[$, instead of the gradient flow for such functionals.

We prove that if the resolvent flow $(J_\tau^E(u))_{\tau>0}$ for proper coercive lower semicontinuous function $E(J_\tau^E(u))$ is bounded and its energy is lower bounded along a subsequence, then it converges to a minimizer and E is lower bounded provided $\lambda \ge 0$, and it converges to a unique minimizer under $\lambda > 0$ (Theorem 3.46). In Proposition 3.48, we give a generalization of a Poincaré type inequality as a sufficient condition for the boundedness of the resolvent flow.

We can apply our results to L^p-Wasserstein space $(\mathcal{P}^p(H), d_{W^p})$ over a complete separable p-uniformly convex space (H, d_H). We prove that any lower semicontinuous function $E : \mathcal{P}^p(H) \to]-\infty, +\infty]$, which is p-uniformly λ-convex along generalized geodesics of $(\mathcal{P}^p(H), d_{W^p})$, satisfies our convexity condition Assumption 3.22 (Theorem 4.11).

Another application is the resolvent flow for a Cheeger type p-energy on L^p-maps into complete p-uniformly convex space on any measurable space. In this framework, we develop the previous work [25] by Ohta.

The paper is structured as follows: In Section 2, we give the definition of a p-uniformly convex space and summarize several results on this notion. In Section 3, we construct resolvent flows and prove the results mentioned above. In Section 4, we show that our results can be applied to the L^p-Wasserstein space over complete separable p-uniformly convex spaces. In Section 5, we recall Cheeger type energy functions over L^p-maps into geodesic spaces and solve the initial boundary problem for p-harmonic maps into CAT(0)-spaces assuming $p \ge 2$.

2 p-uniformly convex spaces

Definition 2.1 (Geodesics). Let (Y, d_Y) be a metric space. A map $\gamma : I \to Y$ is said to be a *curve* if it is continuous, where $I = [a, b] \subset \mathbb{R}$ is a closed interval. The length $L(\gamma)$ of a curve $\gamma : I \to Y$ is defined to be

$$L(\gamma) := \sup\left\{\sum_{i=1}^n d_Y(\gamma(t_{i-1}),\gamma(t_i))\;\middle|\; a=t_0<t_1<\cdots<t_{n-1}<t_n=b\right\}.$$

A curve $\gamma : I \to Y$ is said to be a *minimal geodesic* if $L(\gamma|_{[s,t]}) = d_Y(\gamma_s, \gamma_t)$ holds for any $s, t \in I$, $s < t$, equivalently $d_Y(\gamma_r, \gamma_t) = d_Y(\gamma_r, \gamma_s) + d_Y(\gamma_s, \gamma_t)$ for any $r < s < t$. A curve $\gamma : I \to Y$ is said to be a *geodesic* if for any $s, t \in I$, $s < t$ with sufficiently small $|t - s|$, $L(\gamma|_{[s,t]}) = d_Y(\gamma_s, \gamma_t)$ holds. Any minimal geodesic $\gamma : I = [a, b] \to Y$ can be parametrized by arc-length, that is, there exists a reparametrization $\tilde{\gamma} : \tilde{I} \to Y$ such that $\tilde{\gamma} : \tilde{I} \to Y$ is an isometry and $\tilde{\gamma}(\tilde{I}) = \gamma(I)$. The image $\gamma(I)$ is called a *geodesic segment*. So for a given minimal segment joining γ_0 and γ_1, we may always take a minimal geodesic $\gamma : [0, 1] \to Y$ having the property $d_Y(\gamma_s, \gamma_t) = |t - s| d_Y(\gamma_0, \gamma_1)$ for $s, t \in [0, 1]$. A metric space (Y, d_Y) is called a *R-geodesic space* for $R \in]0, +\infty]$ if any two points in Y whose distance is strictly less than R can be joined by a geodesic segment. We simply say that (Y, d_Y) is a *geodesic space* if it is an $+\infty$-geodesic space. Throughout this paper, for given $x, y \in Y$, denote by $\gamma^{xy} : [0, 1] \to Y$ a minimal geodesic from $x =: \gamma_0^{xy}$ to $y =: \gamma_1^{xy}$ provided (Y, d_Y) is an R-geodesic space and $d_Y(x, y) < R$.

For $n \in \mathbb{N}$, we denote by $\mathbb{M}^n(\kappa)$ the n-dimensional space form of constant curvature $\kappa \in \mathbb{R}$. Let R_κ be the diameter of $\mathbb{M}^n(\kappa)$, that is, $R_\kappa := +\infty$ if $\kappa \le 0$ and $R_\kappa := \pi/\sqrt{\kappa}$ if $\kappa > 0$.

Definition 2.2 (CAT(κ)-inequality, see [5]). Let (Y, d_Y) be a metric space and $\triangle$ a geodesic triangle in Y with perimeter strictly less than $2R_\kappa$. Let $\widetilde{\triangle}$ be a comparison triangle of $\triangle$ in $\mathbb{M}^2(\kappa)$. We say that $\triangle$ satisfies CAT(κ)-inequality if all $p, q \in \triangle$ and its corresponding points $\tilde{p}, \tilde{q} \in \widetilde{\triangle}$ satisfy

$$d_Y(p, q) \le d_{\mathbb{M}^2(\kappa)}(\tilde{p}, \tilde{q}).$$

Definition 2.3 (CAT(κ)-space, see [5]). A metric space (Y, d_Y) is said to be a CAT(κ)-space if (Y, d_Y) is an R_κ-geodesic space and all geodesic triangles in Y with perimeter strictly less than $2R_\kappa$ satisfy CAT(κ)-inequality.

The following notion of p-uniformly convex space is proposed by Naor-Silberman [23] in the framework of geodesic spaces. It has been formulated in terms of the modulus of convexity for Banach spaces.

Definition 2.4 (p-uniformly convex space; cf. Naor-Silberman [23]). Fix $p \in]1, +\infty[$. A metric space (Y, d_Y) is called *p-uniformly convex with parameter $k > 0$* if (Y, d_Y) is a geodesic space and for any three points $x, y, z \in Y$, any minimal geodesic $\gamma := (\gamma_t)_{t\in[0,1]}$ in Y with $\gamma_0 = x$, $\gamma_1 = y$, and all $t \in [0, 1]$,

$$d_Y^p(z, \gamma_t) \le (1 - t)d_Y^p(z, x) + t d_Y^p(z, y) - \frac{k}{2} t(1 - t) d_Y^p(x, y). \tag{2.5}$$

Remark 2.6.

(1) By definition, $p \ge 2$ automatically holds in Definition 2.4 as noted in [23]. Details are due to Silberman [27]. Suppose that (Y, d_Y) is a p-uniformly convex with parameter $k > 0$ and assume $p \in]1, 2[$. Take $z, y \in Y$. We may assume $d_Y(z, y) = 1 + h$ with $h > 0$ by scaling (Y, d_Y). Take a point x on the minimal geodesic joining z to y with $d_Y(z, x) = 1$. Then (2.5) implies that, for all $h > 0$ and $t \in [0, 1]$,

$$\frac{k}{2} t(1 - t) h^p \le (1 - t) + t(1 + h)^p - (1 + th)^p =: f_t(h). \tag{2.7}$$

By Taylor expansion, we see that $f_t(h) = \frac{f_t''(\theta h)}{2} h^2$ for some $\theta = \theta_h \in [0, 1]$ from $f_t(0) = f_t'(0) = 0$. Dividing (2.7) by h^p, and letting $h \to 0$, we have a contradiction under $p \in]1, 2[$ because of $\lim_{s\to 0} f_t''(s) = t(1 - t)p(p - 1)$. Hence we have $p \in [2, +\infty[$.

(2) Putting $z = \gamma_t$, we have $k \in]0, c_p]$ with $c_p := 8/2^p$ for $p \in [2, +\infty[$ (see Proposition 2.5 in [14]).

(3) Let $W_p(t) := t(1 - t)^p + (1 - t)t^p$. Since $\frac{4}{2^p} t(1 - t) \le W_p(t) \le t(1 - t)$ for $t \in [0, 1]$ and $p \ge 2$, it is easy to see that a geodesic space (Y, d_Y) is p-uniformly convex with some parameter $k \in]0, c_p]$ if and only if there exists a constant $c \in]0, 1]$ such that for $x, y, z \in Y$ and any minimal geodesic γ joining $\gamma_0 = x$ to $\gamma_1 = y$,

$$d_Y^p(z, \gamma_t) \le (1 - t)d_Y^p(z, x) + t d_Y^p(z, y) - c W_p(t) d_Y^p(x, y). \tag{2.8}$$

(4) The inequality (2.5) or (2.8) provides the (strict) convexity of $Y \ni x \mapsto d_Y^p(z, x)$ for a fixed $z \in Y$.

Example 2.9 (Examples of p-uniformly convex spaces)**.**

(1) Any closed convex subset of a p-uniformly convex space is again a p-uniformly convex space with the same parameter.

(2) Any L^p space over a measurable space is p-uniformly convex with parameter $k = k_p := \frac{8}{2^p}\frac{1+t_p^{p-1}}{(1+t_p)^{p-1}}$ for $p > 2$ by Lim-Xu-Xu [20], Xu [34, (3.4)] and Thakur-Jung [32, Lemma 2], where t_p is the unique zero in $]1, +\infty[$ of the function $g_p(x) := -x^{p-1} + (p-1)x + p - 2$, and it is 2-uniformly convex with parameter $k = 2(p-1)$ provided $1 < p \le 2$ by [4, Theorem 1] and [34, (3.7)]. More precisely, for any L^p-space with $p \ge 2$, the inequality (2.8) holds for $c = \alpha_p := \frac{1+t_p^{p-1}}{(1+t_p)^{p-1}}$ and the constant $\alpha_p (= 2^p k_p/8)$ is the best possible in the sense that if (2.8) holds for $c \in]0, 1]$, then $c \le \alpha_p$ (see Lim [19, Theorem 2.1] with Smarzewski [28, 29]). We easily see $t_p \in]1, e^2[$, which implies

$$\lim_{p\to 2+0} k_p = \lim_{p\to 2+0} \frac{8(p-1)}{2^p(1+t_p)^{p-2}} = 2.$$

(3) Every CAT(0)-space is a p-uniformly convex space with parameter $k = k_p$ (resp. $k = 2$) for $p > 2$ (resp. $p = 2$), because $\mathbb{R}^2$ is isometrically embedded into $L^p([0, 1])$ for $p > 1$ (see [8, Theorem 2.1],[23]).

(4) Any L^p-maps into complete CAT(0)-space form a complete p-uniformly (resp. 2-uniformly) convex space with parameter $k = k_p$ (resp. $k = 2(p-1)$) for $p \in]2, +\infty[$ (resp. $p \in]1, 2]$) (see [15, Corollary 7.3]).

(5) Ohta [25] proved that for $\kappa > 0$ any CAT(κ)-space Y with $\mathrm{diam}(Y) < R_\kappa/2$ is a 2-uniformly convex space with parameter $2\sqrt{\kappa}\cdot \mathrm{diam}(Y)/\tan\{\sqrt{\kappa}\cdot \mathrm{diam}(Y)\}$.

Remark 2.10. A Banach space $(Y, \|\cdot\|)$ is said to be uniformly convex if the modulus of convexity of Y,

$$\delta_Y(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \;\middle|\; x, y \in Y, \|x\| = \|y\| = 1, \|x-y\| \ge \varepsilon\right\},$$

satisfies $\delta_Y(\varepsilon) > 0$ for $\varepsilon \in]0, 2]$. For $p > 1$, $(Y, \|\cdot\|)$ is said to be *p-uniformly convex* if there exists $c > 0$ such that $\delta_Y(\varepsilon) \ge c\varepsilon^p$ for $\varepsilon \in]0, 2]$. Xu [34, Theorem 1] proved that for a Banach space $(Y, \|\cdot\|)$, it is p-uniformly convex if and only if it is p-uniformly convex in the sense of Definition 2.4.

3 Resolvent flows for convex functions

In this section we fix $p \in]1, +\infty[$ and set $q := p/(p-1)$.

3.1 Resolvents

Throughout this subsection, we fix a complete metric space (H, d_H). Consider a proper function $E : H \to]-\infty, +\infty]$ and set $D(E) := \{x \in H \mid E(x) < +\infty\}$, where 'proper' means $D(E) \neq \emptyset$.

Definition 3.1 (Moreau-Yosida approximation, coercivity [10])**.** For $E : H \to]-\infty, +\infty]$ we define $E^\tau : H \to [-\infty, +\infty]$ by

$$E^\tau(x) := \inf_{y\in H}\left(E(y) + \frac{1}{p\tau^{p-1}} d_H^p(y, x)\right), \qquad x \in H,\ \tau > 0,$$

and call it *Moreau-Yosida approximation*, *Hamilton-Jacobi semi-group* or *Hopf-Lax formula* for E. Since E is proper, $E^\tau < +\infty$ on H. E is said to be *coercive* if there exists $\tau > 0$ and $x \in H$ such that $E^\tau(x) > -\infty$.

The following proposition is well-known for experts.

Proposition 3.2 (Semigroup property of Hopf-Lax formula)**.** *Let E be a proper function $E : H \to]-\infty, +\infty]$. Suppose that (H, d_H) is a geodesic space. For $\tau, \mu > 0$, we have*

$$(E^\tau)^\mu = E^{\tau+\mu}.$$

Proof.

$$(E^\tau)^\mu(x) = \inf_{y\in H}\left(E^\tau(y) + \frac{1}{p\mu^{p-1}}d_H^p(x,y)\right) = \inf_{y\in H}\left\{\inf_{z\in H}\left(E(z) + \frac{1}{p\tau^{p-1}}d_H^p(y,z)\right) + \frac{1}{p\mu^{p-1}}d_H^p(x,y)\right\}.$$

For each $z \in H$,

$$\inf_{y\in H}\left(\frac{1}{\tau^{p-1}}d_H^p(y,z) + \frac{1}{\mu^{p-1}}d_H^p(x,y)\right)$$

is realized by a unique point $y_0 \in H$, the point on the geodesic arc γ_{xz} from x to z with

$$d_H(x,y_0) = \frac{\mu}{\tau+\mu}d_H(x,z), \quad d_H(z,y_0) = \frac{\tau}{\tau+\mu}d_H(x,z).$$

The point y_0 satisfies

$$\frac{1}{\tau^{p-1}}d_H^p(y_0,z) + \frac{1}{\mu^{p-1}}d_H^p(x,y_0) = \frac{1}{(\tau+\mu)^{p-1}}d_H^p(x,z).$$

From the following elementary inequality:

$$(1-t)a^p + tb^p \geq \frac{t(1-t)}{\left(t^{\frac{1}{p-1}} + (1-t)^{\frac{1}{p-1}}\right)^{p-1}}(a+b)^p$$

for $a, b \geq 0$ and $p \in [2,\infty[$, with equality when $a^{p-1} : b^{p-1} = t : (1-t)$, we get,

$$(E^\tau)^\mu(x) = \inf_{z\in H}\left(E(z) + \frac{1}{p(\tau+\mu)^{p-1}}d_H^p(x,z)\right) = E^{\tau+\mu}(x).$$

□

Lemma 3.3. *Suppose that E is coercive. We set*

$$\tau_\star(E) := \sup\{\tau > 0 \mid E^\tau(x_0) > -\infty \textit{ for some } x_0 \in H\} \in]-\infty, +\infty].$$

Assume $E^{\tau_\star}(x_0) > -\infty$ for $x_0 \in H$ and $\tau_\star \in]0,+\infty[$. Then for any $\tau < \tau_\star \leq \tau_\star(E)$ and $x \in H$, we have

$$E^\tau(x) \geq E^{\tau_\star}(x_0) - \frac{1}{p}\left(\frac{2}{\tau_\star - \tau}\right)^{p-1} d_H^p(x,x_0) > -\infty.$$

In particular, for $\tau \in]0,\tau_\star[$, $x \mapsto E^\tau(x)$ is bounded below on any bonded set.

Proof. The proof is similar to [1, Lemma 2.2.1] by using the following elementary inequality:

$$(a+b)^p \leq (1+\varepsilon)^{p-1}a^p + (1+\varepsilon^{-1})^{p-1}b^p, \quad a,b \geq 0, \quad \varepsilon \in]0,+\infty[.$$

□

If $y \mapsto E(y) + \frac{1}{p\tau^{p-1}}d_H^p(y,x)$ has a unique minimizer $J_\tau(x)$, it is called the *resolvent* of E. Note that if H is a Hilbert space and $p = 2$, and if E is a closed densely defined non-negative quadratic form on H, then we have $J_\tau(f) = (I + \frac{\tau}{2}A)^{-1}(f) = \frac{2}{\tau}G_{\frac{2}{\tau}}(f)$ for $f \in H$. Here, I is the identity operator, A the infinitesimal generator associated with E, i.e., the non-negative self-adjoint operator on H such that $D(E) = D(\sqrt{A})$ and $E(f) = (\sqrt{A}f, \sqrt{A}f)_H$ for any $f \in D(E)$, where $(\cdot,\cdot)_H$ is the Hilbert inner product on H, and $G_\alpha = (\alpha + A)^{-1}$, $\alpha > 0$ is the resolvent operator associated with A.

From now on, for a given $\lambda \in \mathbb{R}$, we consider the following assumption for a function E on H:

Assumption 3.4. Fix $k \in]0, +\infty[$. For a given $\lambda \in \mathbb{R}$, we assume that for any $x, y \in D(E)$ there exists a curve $\gamma : [0, 1] \to H$ with $\gamma_0 = x$ and $\gamma_1 = y$ such that $d_H(\gamma_0, \gamma_t) \le t d_H(\gamma_0, \gamma_1)$ for all $t \in [0, 1]$, and $t \mapsto E(\gamma_t) + \frac{1}{p\tau^{p-1}} d_H^p(\gamma_0, \gamma_t)$ is p-uniformly $\left(\frac{k}{p\tau^{p-1}} + \lambda\right)$-convex for each $\tau \in]0, (k/p\lambda^-)^{q-1}[$: for $t \in [0, 1]$ and $\tau \in]0, (k/p\lambda^-)^{q-1}[$

$$E(\gamma_t) + \frac{1}{p\tau^{p-1}} d_H^p(\gamma_0, \gamma_t) \le (1-t)E(\gamma_0) + tE(\gamma_1) + \frac{t}{p\tau^{p-1}} d_H^p(\gamma_0, \gamma_1) - \frac{1}{2}\left(\frac{k}{p\tau^{p-1}} + \lambda\right) t(1-t) d_H^p(\gamma_0, \gamma_1).$$

Neglecting the second part of the left-hand side of the above inequality and dividing by t we also have

$$\frac{E(\gamma_t) - E(\gamma_0)}{t} \le E(\gamma_1) - E(\gamma_0) + \frac{1}{p\tau^{p-1}}\left[\left(1 - \frac{k}{2}(1-t)\right) - \frac{\lambda p \tau^{p-1}}{2}(1-t)\right] d_H^p(\gamma_0, \gamma_1). \tag{3.5}$$

Remark 3.6. Assumption 3.4 is an extension of [1, Assumption 2.4.5] in which $p = k = 2$ is imposed. Note here that the choice of the curve γ in Assumption 3.4 is independent of τ. In particular, multiplying the inequality above by $p\tau^{p-1}$ and letting $\tau \to 0+$, γ satisfies $d_H(\gamma_0, \gamma_t) \le t^{1/p} d_H(\gamma_0, \gamma_1)$. This is weaker than $d_H(\gamma_0, \gamma_t) \le t d_H(\gamma_0, \gamma_1)$, which can be deduced if $p = k = 2$. If $\lambda \ge 0$, by letting $\tau \to \infty$ we obtain the p-uniform λ-convexity of E along γ (see [1, Remark 2.4.6]).

Definition 3.7 (Local slope, global slope, stationary point)**.** The *local slope* of E at $x \in H$ is defined by

$$|\partial E|(x) := \begin{cases} \overline{\lim}_{y \to x} \frac{(E(x)-E(y))^+}{d_H(x,y)} & x \in D(E), \\ +\infty & x \notin D(E) \end{cases}$$

and set $D(|\partial E|) := \{x \in H \mid |\partial E|(x) < +\infty\}$. The *global slope* $\mathfrak{l}_E$ at $x \in H$ is defined by

$$\mathfrak{l}_E(x) := \sup_{y \neq x} \frac{(E(x) - E(y))^+}{d_H(x, y)}.$$

A point $x \in D(|\partial E|)$ is called a *stationary point* for E if $|\partial E|(x) = 0$.

The following extends [31, Propositions 5.2 and 5.3(ii)].

Proposition 3.8 (Minimizers are stationary points and vice versa)**.** *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty]$ satisfies Assumption 3.4 for some $\lambda \in \mathbb{R}$ and $k \in]0, +\infty[$. Then, for $x \in D(E)$*

$$|\partial E|(x) = \sup_{x \neq y}\left(\frac{E(x) - E(y)}{d_H(x, y)} - \left(\frac{\lambda^-}{k} - \frac{\lambda^+}{2}\right) d_H^{p-1}(x, y)\right)^+ = \sup_{x \neq y}\left(\frac{E(x) - E(y)}{d_H(x, y)} - \frac{\lambda^-}{k} d_H^{p-1}(x, y)\right)^+. \tag{3.9}$$

In particular, if $\lambda \ge 0$, then for $x \in D(E)$ we have $|\partial E|(x) = \mathfrak{l}_E(x)$. Consequently $x \in D(E)$ is a minimimzer of E if and only if x is a stationary point of E.

Proof. The proof follows the one of [1, (2.4.14) and (2.4.18)]. We shall recall it for readers convenience. First we prove

$$|\partial E|(x) \ge \left(\frac{E(x) - E(y)}{d_H(x, y)} - \frac{1}{p\tau^{p-1}}\left(1 - \frac{k}{2} - \frac{p\tau^{p-1}}{2}\lambda\right) d_H^{p-1}(x, y)\right)^+ \tag{3.10}$$

for $\tau \in]0, (k/p\lambda^-)^{q-1}[$. Indeed, it is not restrictive to suppose

$$x \in D(E), \quad x \neq y \text{ with } E(x) - E(y) - \frac{1}{p\tau^{p-1}}\left(1 - \frac{k}{2} - \frac{p\tau^{p-1}}{2}\lambda\right) d_H^p(x, y) > 0. \tag{3.11}$$

Applying (3.5) with $\gamma_0 = x$ and $\gamma_1 = y$ to get γ_t satisfying for every $0 < \tau < (k/p\lambda^-)^{q-1}$

$$\frac{E(x) - E(\gamma_t)}{d_H(x, \gamma_t)} \ge \left(\frac{E(x) - E(y)}{d_H(x, y)} - \frac{1}{p\tau^{p-1}}\left[\left(1 - \frac{k}{2}(1-t)\right) - \frac{\lambda p \tau^{p-1}}{2}(1-t)\right] d_H^{p-1}(x, y)\right) \frac{t d_H(x, y)}{d_H(x, \gamma_t)}.$$

Since $d_H(x,\gamma_t) \le t d_H(x,y)$, inequality (3.11) yields

$$|\partial E|(x) \ge \overline{\lim_{t\downarrow 0}} \frac{E(x)-E(\gamma_t)}{d_H(x,\gamma_t)} \ge \frac{E(x)-E(y)}{d_H(x,y)} - \frac{1}{p\tau^{p-1}}\left(1-\frac{k}{2}-\frac{\lambda p\tau^{p-1}}{2}\right) d_H^{p-1}(x,y).$$

Letting $\tau \uparrow (k/p\lambda^-)^{q-1}$ in (3.10), we get

$$|\partial E|(x) \ge \left(\frac{E(x)-E(y)}{d_H(x,y)} + \left(-\frac{\lambda^-}{k}+\frac{\lambda^+}{2}\right) d_H^{p-1}(x,y)\right)^+.$$

Thus

$$\begin{aligned}|\partial E|(x) &\ge \sup_{x\ne y}\left(\frac{E(x)-E(y)}{d_H(x,y)} - \left(\frac{\lambda^-}{k}-\frac{\lambda^+}{2}\right) d_H^{p-1}(x,y)\right)^+ \ge \sup_{x\ne y}\left(\frac{E(x)-E(y)}{d_H(x,y)} - \frac{\lambda^-}{k} d_H^{p-1}(x,y)\right)^+ \\ &\ge \overline{\lim_{y\to x}}\left(\frac{E(x)-E(y)}{d_H(x,y)} - \frac{\lambda^-}{k} d_H^{p-1}(x,y)\right)^+ = |\partial E|(x).\end{aligned}$$

□

The following extends [31, Proposition 5.3(i)].

Corollary 3.12 (Lower semi continuity of local slope)**.** *Suppose that a proper lower semicontinuous function $E : H \to]-\infty,+\infty]$ satisfies Assumption 3.4 for some $\lambda \in \mathbb{R}$ and $k \in]0,+\infty[$. Then $|\partial E|$ is lower semicontinuous on H.*

Proof. Let $\{x_i\}$ be a net converging to x and take $y \ne x$. Then by (3.9)

$$|\partial E|(x_i) \ge \frac{E(x_i)-E(y)}{d_H(x_i,y)} - \left(\frac{\lambda^-}{k}-\frac{\lambda^+}{2}\right) d_H^{p-1}(x_i,y).$$

The lower semi continuity of E shows

$$\underline{\lim}_i |\partial E|(x_i) \ge \frac{E(x)-E(y)}{d_H(x,y)} - \left(\frac{\lambda^-}{k}-\frac{\lambda^+}{2}\right) d_H^{p-1}(x,y).$$

Applying (3.9) again, we obtain $\underline{\lim}_i |\partial E|(x_i) \ge |\partial E|(x)$. □

Recalling the Definition 3.7 of global slope $\mathfrak{l}_E$, from (3.9) we easily get

$$|\partial E|(x) \le \mathfrak{l}_E(x) \le |\partial E|(x) + \frac{\lambda^-}{k}\mathrm{diam}(H)^{p-1} \text{ for } x \in D(|\partial E|). \tag{3.13}$$

The following corollary can be proved in the same way as [31, Lemma 5.4 and Corollary 5.5].

Corollary 3.14. *Suppose that a proper lower semicontinuous function $E : H \to]-\infty,+\infty]$ satisfying Assumption 3.4 for some $\lambda \in \mathbb{R}$ and $k \in]0,+\infty[$. Assume that E is bonded below on H. Then*

$$\inf_{x\in H} |\partial E|(x) = 0.$$

In particular, there exists a sequence in H such that the local slope approaches 0. If the sequence converges to a point, then the point minimizes E.

Proof. By replacing E with $E - \inf E$, we may assume that E is non-negative on H. Suppose that $C := \inf_{x\in H}|\partial E|(x) > 0$. Then

$$0 < C \le \sup_{y\ne x}\left(\frac{E(x)-E(y)}{d_H(x,y)} - \frac{\lambda^-}{k} d_H^{p-1}(x,y)\right)$$

holds for all $x \in H$ by Proposition 3.8. In particular, $E(x) > 0$ for $x \in H$. The rest of the proof is similar to the proof of [31, Lemma 5.4 and Corollary 5.5]. We omit the details. □

Definition 3.15 (Absolutely continuous curves)**.** We say that a curve $v :]a, b[\to H$ belongs to $AC^p(]a, b[; H)$, for $p \in [1, +\infty]$ if there exists $A \in L^p(]a, b[)$ such that

$$d_H(v(s), v(t)) \leq \int_s^t A(r)\mathrm{d}r \quad a < s \leq t < b. \tag{3.16}$$

In the case $p = 1$, we simply write $AC(]a, b[; H)$ instead of $AC^1(]a, b[; H)$. We say that a curve $v : [0, +\infty[\to H$ belongs to $AC^p_{\mathrm{loc}}([0, +\infty[; H)$, for $p \in [1, +\infty]$ if there exists $A \in L^p_{\mathrm{loc}}([0, +\infty[)$ such that (3.16) holds for any $b > a \geq 0$.

Definition 3.17 (Strong upper gradient)**.** Let $E : H \to]-\infty, \infty]$ be a proper function. A function $g : H \to [0, +\infty]$ is a *strong upper gradient for E* if for every absolutely continuous curve $v \in AC(]a, b[; H)$ the function $g \circ v$ is Borel and

$$|E(v(t)) - E(v(s))| \leq \int_s^t g(v(r))|v'|(r)\mathrm{d}r \quad \text{for } a < s < t < b. \tag{3.18}$$

Definition 3.19 (Weak upper gradient)**.** Let $E : H \to]-\infty, \infty]$ be a proper function. A function $g : H \to [0, +\infty]$ is a *weak upper gradient for E* if for every absolutely continuous curve $v \in AC(]a, b[; H)$ such that

(1) $g \circ v|v'| \in L^1(]a, b[)$;
(2) $E \circ v$ is a.e. equal in $]a, b[$ to a function φ with finite pointwise variation in $]a, b[$;

we have

$$|\varphi'(t)| \leq g(v(t))|v'|(t) \quad \text{for a.e. } t \in]a, b[.$$

In this case, if $E \circ v \in AC(]a, b[)$ then $\varphi = E \circ v$ and (3.18) holds.

Corollary 3.20. *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty]$ satisfies Assumption* 3.4 *for some $\lambda \in \mathbb{R}$ and $k \in]0, +\infty[$. Then $|\partial E|$ is a strong upper gradient for E.*

Proof. By [1, Theorem 1.2.5], $\mathfrak{l}_E$ is a strong upper gradient for E and $|\partial E|$ is a weak upper gradient for E. If $\lambda \geq 0$, then $|\partial E|$ is a strong upper gradient from (3.13). In the case $\lambda < 0$, we should check that for any curve $z \in AC(]a, b[; H)$ with $|\partial E|(z)|z'| \in L^1(]a, b[)$ the function $E \circ z$ is absolutely continuous. It is not restrictive to assume that $]a, b[$ is a bounded interval and the curve z is continuously extended to $[a, b]$. Let $H_0 := z([a, b])$ be a compact metric space with the metric induced from (H, d_H). We consider the related global slope of E, denoted by $\mathfrak{l}^0_E$. (3.9) yields

$$\mathfrak{l}^0_E(x) = \sup_{z \in H_0 \setminus \{y\}} \frac{(E(y) - E(z))^+}{d_H(y, z)} \leq |\partial E|(y) + \frac{\lambda^-}{k}\mathrm{diam}(H_0)^{p-1} \text{ for } y \in H_0.$$

In particular, $\mathfrak{l}_E(z)|z'| \in L^1(]a, b[)$ and therefore [1, Theorem 1.2.5] yields the desired absolute continuity. □

Assumption 3.21. Fix $k \in]0, +\infty[$. For a given $\lambda \in \mathbb{R}$, we assume that for any $z \in H$, $x, y \in D(E)$, there exists a curve $\gamma = \gamma^z : [0, 1] \to H$ with $\gamma_0 = x$ and $\gamma_1 = y$ such that $t \mapsto E(\gamma_t) + \frac{1}{p\tau^{p-1}} d^p_H(z, \gamma_t)$ is p-uniformly $\left(\frac{k}{p\tau^{p-1}} + \lambda\right)$-convex for each $\tau \in]0, (k/p\lambda^-)^{q-1}[$: for $t \in [0, 1]$ and $\tau \in]0, (k/p\lambda^-)^{q-1}[$

$$\begin{aligned} E(\gamma_t) + \frac{1}{p\tau^{p-1}} d^p_H(z, \gamma_t) \leq &(1 - t)E(\gamma_0) + tE(\gamma_1) + \frac{1 - t}{p\tau^{p-1}} d^p_H(z, \gamma_0) + \frac{t}{p\tau^{p-1}} d^p_H(z, \gamma_1) \\ &- \frac{1}{2}\left(\frac{k}{p\tau^{p-1}} + \lambda\right) t(1 - t) d^p_H(\gamma_0, \gamma_1) \end{aligned}$$

and the curve $\gamma^x : [0, 1] \to H$ with $\gamma^x_0 = x$ and $\gamma^x_1 = y$ satisfies $d_H(x, \gamma^x_t) \leq t d_H(x, y)$ for all $t \in [0, 1]$.

Assumption 3.22. Fix $k \in]0, c_p]$ and $\lambda \in \mathbb{R}$. We assume that for any $z, x, y \in H$ there exists a curve $\gamma = \gamma^z : [0, 1] \to H$ with $\gamma_0 = x$ and $\gamma_1 = y$ such that $t \mapsto d_H^p(z, \gamma_t)$ is $k \cdot d_H^p(x, y)$-convex, that is, $y \mapsto d_H^p(z, y)$ is p-uniformly k-convex along γ, and the curve $\gamma^x : [0, 1] \to H$ with $\gamma_0^x = x$ and $\gamma_1^x = y$ satisfies $d_H(x, \gamma_t^x) \le t d_H(x, y)$ for all $t \in [0, 1]$. Moreover, we assume that E is p-uniformly λ-convex along the curve $\gamma^z : [0, 1] \to H$ joining x to y for all given $z \in H$, $x, y \in D(E)$.

Remark 3.23.

(1) Assumption 3.21 is a slight extension of [1, Assumption 4.0.1] in which $p = k = 2$ and $z \in D(E)$ are imposed.
(2) For $p \in [2, +\infty[$, Assumption 3.22 is satisfied if (H, d_H) is a p-uniformly convex space with $k \in]0, c_p]$ and E is a p-uniformly geodesically λ-convex function on (H, d_H). More generally, let $(\mathcal{P}^p(H), d_{W^p})$ be the L^p-Wasserstein space over a complete separable p-uniformly convex space (H, d_H) with $k \in]0, c_p]$. Take a p-uniformly λ-convex function E along a generalized geodesic on $(\mathcal{P}^p(H), d_{W^p})$ (see Definition 4.2 below for generalized geodesics). Then Assumption 3.22 is satisfied (see Theorem 4.11 below). Note that the geodesic space $(\mathcal{P}^p(H), d_{W^p})$ is not necessarily a p-uniformly convex space in general.
(3) Assumption 3.21 (resp. Assumption 3.22) is stronger than Assumptoin 3.4 (resp. Assumption 3.21). Note that the curve $\gamma = \gamma^z : [0, 1] \to H$ is not necessarily a minimal geodesic in Assumptions 3.4, 3.21 and 3.22. In particular, even for $\lambda = 0$ and $E \equiv 0$, Assumptions 3.21 and 3.22 are weaker conditions than the p-uniform convexity of (H, d_H) for $p \in [2, +\infty[$.
(4) Assumptions 3.4, 3.21 and 3.22 are stable under the change from E to $E+c$ for any constant c, respectively.
(5) By Remark 3.6, there is no need to assume $d_H(x, \gamma_t^x) \le t d_H(x, y)$, $t \in [0, 1]$ in Assumptions 3.21 and 3.22 if $p = k = 2$.

The idea of the proof of following proposition is due to Yokota.

Proposition 3.24 (Coercivity of convex functions)**.** *Let $E : H \to]-\infty, +\infty]$ be a proper function on H, which is lower semicontinuous at some point of H. We have the following:*

(1) *Suppose that E satisfies Assumption 3.4 for some $\lambda \in \mathbb{R}$ and $k \in]0, +\infty[$. Then E is bounded below on each bounded subset of H.*
(2) *Suppose that E satisfies Assumption 3.21 for some $\lambda \in \mathbb{R}$ and $k \in]0, +\infty[$. Then for each $x \in H$ and $\tau \in]0, (k/p\lambda^-)^{q-1}[$, $E^\tau(x) > -\infty$, in particular, E is coercive.*
(3) *Suppose that for $x, y \in D(E)$ there exists a curve $\gamma : [0, 1] \to H$ with $\gamma_0 = x$ and $\gamma_1 = y$ such that $d_H(x, \gamma_t) \le t d_H(x, y)$ for all $t \in [0, 1]$. Assume that for $x, y \in D(E)$, E is p-uniformly λ-convex along the curve γ joining x and y. Then, E is lower bounded under $\lambda > 0$. In particular, if E satisfies Assumption 3.22 for some $\lambda > 0$ and $k \in]0, +\infty[$, then E is lower bounded.*

Proof. Let $x_0 \in H$ be a point at which the lower semi continuity of E holds. Then there exists some $r > 0$ such that E is bounded below on the closed ball $B := \{y \in H \mid d_H(x_0, y) \le r\}$ of radius $r > 0$ and center x_0.

We first prove (1). Suppose that E satisfies Assumption 3.4. For any $x \in H \setminus B$, let $\gamma : [0, 1] \to H$ be the curve with $\gamma_0 = x_0$ and $\gamma_1 = x$ such that $d_H(x_0, \gamma_t) \le t d_H(x_0, x)$ for all $t \in [0, 1]$, and $t \mapsto E(\gamma_t) + \frac{1}{p\tau^{p-1}} d_H^p(x_0, \gamma_t)$ is p-uniformly $\left(\frac{k}{p\tau^{p-1}} + \lambda\right)$-convex for each $\tau \in]0, (k/p\lambda^-)^{q-1}[$. Taking $t := r/d_H(x_0, x) \in]0, 1[$, we have $d_H(x_0, \gamma_t) \le r$, i.e. $\gamma_t \in B$. Then

$$
\begin{aligned}
\inf_{y \in B} E(y) \le E(\gamma_t) &\le E(\gamma_t) + \frac{1}{p\tau^{p-1}} d_H^p(x_0, \gamma_t) \\
&\le (1-t)E(x_0) + tE(x) + \frac{t}{p\tau^{p-1}} d_H^p(x_0, x) - \frac{1}{2}\left(\frac{k}{p\tau^{p-1}} + \lambda\right) t(1-t) d_H^p(x_0, x) \\
&\le (1-t)E(x_0) + tE(x) + \frac{t}{p\tau^{p-1}} d_H^p(x_0, x) \\
&= \left(1 - \frac{r}{d_H(x_0, x)}\right) E(x_0) + \frac{r}{d_H(x_0, x)} \left(E(x) + \frac{1}{p\tau^{p-1}} d_H^p(x_0, x)\right),
\end{aligned}
$$

which yields

$$E(x) \geq E(x_0) + \frac{\inf_B E - E(x_0)}{r} d_H(x, x_0) - \frac{1}{p\tau^{p-1}} d_H^p(x, x_0).$$

Therefore, we obtain the conclusion.

Next we prove (3). Suppose that E satisfies the assumption of (3). For any $x \in H \setminus B$, let $\gamma : [0, 1] \to H$ be the curve with $\gamma_0 = x_0$ and $\gamma_1 = x$ such that $d_H(x_0, \gamma_t) \leq t d_H(x_0, x)$ for all $t \in [0, 1]$, and $t \mapsto E(\gamma_t)$ is p-uniformly λ-convex. Taking $t := r/d_H(x_0, x) \in]0, 1[$, we have $d_H(x_0, \gamma_t) \leq r$, i.e. $\gamma_t \in B$. Then

$$\begin{aligned}\inf_{y\in B} E(y) &\leq E(\gamma_t) \leq (1-t)E(x_0) + tE(x) - \frac{1}{2}\lambda t(1-t) d_H^p(x_0, x)\\ &= \left(1 - \frac{r}{d_H(x_0, x)}\right) E(x_0) + \frac{r}{d_H(x_0, x)} E(x) - \frac{\lambda}{2} \cdot \frac{r}{d_H(x_0, x)} \left(1 - \frac{r}{d_H(x_0, x)}\right) d_H^p(x_0, x),\end{aligned}$$

which yields

$$E(x) \geq E(x_0) + \frac{\inf_B E - E(x_0)}{r} d_H(x, x_0) + \frac{\lambda}{2}\left(d_H^p(x_0, x) - r d_H^{p-1}(x_0, x)\right)$$

Since $\lambda > 0$ and $p > 1$, the right-hand side of the above inequality is lower bounded on H.

Finally we prove (2). Fix $x \in H$. Suppose that E satisfies Assumption 3.21. This implies that $y \mapsto E(y) + \frac{1}{p\tau^{p-1}} d_H^p(x, y)$ is p-strongly $\left(\frac{k}{p\tau^{p-1}} + \lambda\right)$-convex with $\frac{k}{p\tau^{p-1}} + \lambda > 0$ under $\tau \in]0, (k/p\lambda^-)^{q-1}[$. Then, this is lower bounded on H from (3), that is, $E^\tau(x) = \inf_{y\in H}\left(E(y) + \frac{1}{p\tau^{p-1}} d_H^p(x, y)\right) > -\infty$ for each $x \in H$. □

The following is an extension of [22, Lemma 1.7].

Corollary 3.25. *Let (H, d_H) be a complete p-uniformly convex space with parameter $k \in]0, c_p]$. Let $E : H \to]-\infty, +\infty]$ be a proper p-uniformly geodesically λ-convex function which is lower semicontinuous at some point of H. Then E is coercive. Moreover, if $\lambda > 0$, then E is lower bounded on H and there exists a unique minimizer of E.*

Proof. Under the assumption, E satisfies Assumption 3.21. We can apply Proposition 3.24(2). Moreover, if $\lambda > 0$, we can apply Proposition 3.24(3). The existence of the unique minimizer of E under $\lambda > 0$ can be obtained following the proof of Proposition 3.26. □

Owing to Proposition 3.24, we do not need to assume the coercivity condition for several results below. Under Assumption 3.21, we have the following:

Proposition 3.26 (Unique existence of resolvent under Assumption 3.21)**.** *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty]$ satisfies Assumption* 3.21 *for some $\lambda \in \mathbb{R}$ and $k \in]0, +\infty[$. Then for any $x \in H$ and $\tau \in]0, (k/p\lambda^-)^{q-1}[$ there exists a unique point, say $J_\tau(x) \in D(E)$, such that*

$$E^\tau(x) = E(J_\tau(x)) + \frac{1}{p\tau^{p-1}} d_H^p(x, J_\tau(x)).$$

This defines a map $J_\tau : H \to D(E)$ for $\tau \in]0, (k/p\lambda^-)^{q-1}[$, called the resolvent of E.

Proof. Let $\{y_n\} \subset H$ be a minimizing sequence for

$$E^\tau(x) = \inf_{y\in H}\left(E(y) + \frac{1}{p\tau^{p-1}} d_H^p(y, x)\right).$$

We set $y_{n,m} := \gamma_{1/2}$ the mid point along the curve γ joining $\gamma_0 := y_n$ to $\gamma_1 := y_m$. Then for $\tau \in]0, (k/p\lambda^-)^{q-1}[$, we have

$$E^\tau(x) \leq E(y_{n,m}) + \frac{1}{p\tau^{p-1}} d_H^p(y_{n,m}, x)$$

$$\le \frac{1}{2}\left(E(y_n) + \frac{1}{p\tau^{p-1}} d_H^p(y_n, x)\right) + \frac{1}{2}\left(E(y_m) + \frac{1}{p\tau^{p-1}} d_H^p(y_m, x)\right) - \frac{1}{8}\left(\frac{k}{p\tau^{p-1}} + \lambda\right) d_H^p(y_n, y_m).$$

We get that $\{y_n\}$ is a Cauchy sequence, because $\frac{k}{p\tau^{p-1}} + \lambda > 0$ under $\tau \in]0, (k/p\lambda^-)^{q-1}[$. The limit $J_\tau(x) := \lim_{n\to+\infty} y_n$ is the desired minimizer. The proof of the uniqueness is similar. □

Remark 3.27.

(1) By Remark 3.23(4), the resolvent J_τ is also associated to the function $E + c$ for any constant c.
(2) If E is a constant function on $H(= D(E))$, then E is p-uniformly 0-convex along any curve, and $J_\tau(x) = x$ for all $x \in H$ and $\tau \in]0, +\infty[$.

Proposition 3.28 (Slope estimate). *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty]$ satisfying Assumption 3.21 for some $\lambda \in \mathbb{R}$ and $k \in]0, +\infty[$. Let $J_\tau(x)$ be the resolvent associated to E for $x \in H$ and $\tau \in]0, (k/p\lambda^-)^{q-1}[$. For $x \in H$ and $\tau \in]0, (k/p\lambda^-)^{q-1}[$, we have $J_\tau(x) \in D(|\partial E|)$ and*

$$|\partial E|^q(J_\tau(x)) \le \left(\frac{d_H(J_\tau(x), x)}{\tau}\right)^p.$$

Proof. The proof is similar to the one of [1, Lemma 3.1.3] by using the following elementary inequality

$$|x^p - y^p| \le p|x - y|(x \vee y)^{p-1} \quad \text{for} \quad x, y \in [0, +\infty[. \tag{3.29}$$

□

Lemma 3.30. *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty]$ satisfies Assumption 3.21 for some $\lambda \in \mathbb{R}$ and $k \in]0, +\infty[$. Let $J_\tau(x)$ be the resolvent associated to E for $x \in H$ and $\tau \in]0, (k/p\lambda^-)^{q-1}[$. Then $]0, (k/p\lambda^-)^{q-1}[\ni \tau \mapsto d_H(J_\tau(x), x)$ is non-decreasing and $]0, (k/p\lambda^-)^{q-1}[\ni \tau \mapsto E(J_\tau(x))$ is non-increasing.*

Proof. Take $\tau, \mu \in]0, (k/p\lambda^-)^{q-1}[$ with $0 < \mu < \tau$. From

$$E(J_\mu(x)) + \frac{1}{p\mu^{p-1}} d_H^p(J_\mu(x), x) \le E(J_\tau(x)) + \frac{1}{p\mu^{p-1}} d_H^p(J_\tau(x), x),$$

we have

$$\begin{aligned} &E(J_\tau(x)) + \frac{1}{p\tau^{p-1}} d_H^p(J_\tau(x), x) \\ &\quad \ge E(J_\mu(x)) + \frac{1}{p\tau^{p-1}} d_H^p(J_\mu(x), x) + \left(\frac{1}{p\mu^{p-1}} - \frac{1}{p\tau^{p-1}}\right)(d_H^p(J_\mu(x), x) - d_H^p(J_\tau(x), x)). \end{aligned}$$

Inequality $d_H^p(J_\mu(x), x) - d_H^p(J_\tau(x), x) > 0$ contradicts that $J_\tau(x)$ is the minimizer of $y \mapsto E(y) + \frac{1}{p\tau^{p-1}} d_H^p(y, x)$. Thus we have $d_H(J_\mu(x), x) \le d_H(J_\tau(x), x)$, and so

$$E(J_\tau(x)) + \frac{1}{p\tau^{p-1}} d_H^p(J_\tau(x), x) \le E(J_\mu(x)) + \frac{1}{p\tau^{p-1}} d_H^p(J_\mu(x), x) \le E(J_\mu(x)) + \frac{1}{p\tau^{p-1}} d_H^p(J_\tau(x), x)$$

yields $E(J_\tau(x)) \le E(J_\mu(x))$. □

Proposition 3.31 (Slope estimate under Assumption 3.21). *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty]$ satisfying Assumption 3.21 for some $\lambda \in \mathbb{R}$ and $k \in]0, +\infty[$. If $x \in D(|\partial E|)$ and $\tau < (k/p\lambda^-)^{q-1}$, then*

$$\begin{aligned} |\partial E|^q(J_\tau(x)) &\le \left(\frac{d_H(J_\tau(x), x)}{\tau}\right)^p \le \frac{2}{k + \lambda p\tau^{p-1}} \cdot \frac{E(x) - E^\tau(x)}{\tau} \\ &\le \frac{2}{k + \lambda p\tau^{p-1}} \cdot \frac{1}{q}\left(\frac{k}{k - p\tau^{p-1}\lambda^-}\right)^{q-1} |\partial E|^q(x) \end{aligned}$$

$$= \begin{cases} \frac{2k^{q-1}}{q(k-p\tau^{p-1}\lambda^-)^q}|\partial E|^q(x) & \lambda < 0, \\ \frac{2}{q(k+p\tau^{p-1}\lambda)}|\partial E|^q(x) & \lambda \ge 0. \end{cases}$$

In particular,

$$E(x) - E(J_\tau(x)) - \frac{1}{p\tau^{p-1}} d_H^p(x, J_\tau(x)) \le \frac{\tau}{q}\left(\frac{k}{k - p\tau^{p-1}\lambda^-}\right)^{q-1} |\partial E|^q(x). \tag{3.32}$$

Proof. The first inequality is proved in Proposition 3.28. The second and the third inequalities can be shown as in the proof of [1, Theorem 3.1.6] under Assumption 3.4, where Proposition 3.8 is used in the proof for the third inequality. □

Corollary 3.33 (A stationary point is an invariant point)**.** *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty]$ satisfies Assumption* 3.21 *for some $\lambda \in \mathbb{R}$ and $k \in]0, +\infty[$. If $x \in D(|\partial E|)$ is a stationary point, then $J_\tau(x) = x$ for any $\tau \in]0, (k/p\lambda^-)^{q-1}[$.*

Proof. Suppose that $x \in D(|\partial E|)$ is a stationary point. Then $E(x) = E^\tau(x) = E(J_\tau(x)) + \frac{1}{p\tau^{p-1}} d_H^p(J_\tau(x), x)$ for $\tau \in]0, (k/p\lambda^-)^{q-1}[$ by (3.32). Since $J_\tau(x)$ is the unique minimizer of $y \mapsto E(y) + \frac{1}{p\tau^{p-1}} d_H^p(y, x)$, we obtain $J_\tau(x) = x$. □

Lemma 3.34. *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty]$ satsifies Assumption* 3.21 *for some $\lambda \in \mathbb{R}$ and $k \in]0, +\infty[$, and let J_τ be the associated resolvent for $\tau \in]0, (k/p\lambda^-)^{q-1}[$. Then, for $x, y \in H$ and $\tau \in]0, (k/p\lambda^-)^{q-1}[$, we have*

$$\begin{aligned} d_H^p(J_\tau(x), J_\tau(y)) &\le \left(\frac{2}{k + \lambda p\tau^{p-1}}\right)(d_H^p(J_\tau(x), y) + d_H^p(x, J_\tau(y)) - d_H^p(x, J_\tau(x)) - d_H^p(y, J_\tau(y))) \\ &\le \left(\frac{2p}{k + \lambda p\tau^{p-1}}\right) d_H(x, y)\{(d_H(x, J_\tau(y)) + d_H(x, y))^{p-1} + (d_H(J_\tau(x), y) + d_H(x, y))^{p-1}\}. \end{aligned} \tag{3.35}$$

Proof. Put $x' := J_\tau(x)$ and $y' := J_\tau(y)$. By Assumption 3.21, there exists a curve $\gamma = \gamma^x : [0, 1] \to H$ joining $\gamma_0 := x'$ and $\gamma_1 := y'$ such that $t \mapsto E(\gamma_t) + \frac{1}{p\tau^{p-1}} d_H^p(x, \gamma_t)$ is p-uniformly $\left(\frac{k}{p\tau^{p-1}} + \lambda\right)$-convex. Then we have

$$\begin{aligned} E^\tau(x) &\le E(\gamma_t) + \frac{1}{p\tau^{p-1}} d_H^p(x, \gamma_t) \\ &\le (1-t)E^\tau(x) + t\left(E(y') + \frac{1}{p\tau^{p-1}} d_H^p(x, y')\right) - \frac{1}{2}\left(\frac{k}{p\tau^{p-1}} + \lambda\right) t(1-t) d_H^p(x', y'). \end{aligned}$$

Dividing by t and letting $t \to 0+$, we get

$$\begin{aligned} \frac{1}{2}\left(\frac{k}{p\tau^{p-1}} + \lambda\right) d_H^p(x', y') &\le E(y') + \frac{1}{p\tau^{p-1}} d_H^p(x, y') - E^\tau(x) \\ &\le E^\tau(y) + \frac{1}{p\tau^{p-1}}(d_H^p(x, y') - d_H^p(y, y')) - E^\tau(x) \\ &\le \frac{1}{p\tau^{p-1}}(d_H^p(x', y) + d_H^p(x, y') - d_H^p(y, y') - d_H^p(x, x')). \end{aligned}$$

Then we obtain the first desired inequality. By the elementary inequality

$$|a^p - b^p| \le p|a - b|(a \vee b)^{p-1} \quad \text{for} \quad a, b \in [0, +\infty[, \tag{3.36}$$

we see

$$d_H^p(x, y') - d_H^p(y, y') \le p d_H(x, y)|d_H(x, y') \vee d_H(y, y')|^{p-1} \le p d_H(x, y)|d_H(x, y') + d_H(x, y)|^{p-1}$$

and

$$d_H^p(x', y) - d_H^p(x, x') \le p d_H(x, y)|d_H(x', y) \vee d_H(x, x')|^{p-1} \le p d_H(x, y)|d_H(x, x') + d_H(x, y)|^{p-1}.$$

Therefore, we obtain the second one. The last statements are easy to deduce. □

Corollary 3.37. *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty]$ satisfies Assumption 3.21 for some $\lambda \in \mathbb{R}$ and $k \in]0, +\infty[$. Let J_τ be the associated resolvent for $\tau \in]0, (k/p\lambda^-)^{q-1}[$. Assume that exist $\varepsilon \in]0, p[$ and $C_\varepsilon > 0$ such that for any $z \in H$ with $E(z) < +\infty$*

$$\inf_{w\in H}\left(E(w) + C_\varepsilon d_H^\varepsilon(w, z)\right) > -\infty. \tag{3.38}$$

Then, the following holds:

(1) If $\{x_i\}$ is a bounded sequence in H, then $\{(J_\tau)^n(x_i)\}$ is also a bounded sequence for all $\tau \in]0, (k/p\lambda^-)^{q-1}[$ and $n \in \mathbb{N}$.
(2) If $\{x_i\}$ is a Cauchy sequence in H, then $\{J_\tau(x_i)\}$ is also a Cauchy sequence for all $\tau \in]0, (k/p\lambda^-)^{q-1}[$.
(3) If $\{x_i\}$ converges to $x \in H$, then $\{J_\tau(x_i)\}$ converges to $J_\tau(x)$ for all $\tau \in]0, (k/p\lambda^-)^{q-1}[$.

Proof. (1): By induction, it suffices to show only the case $n = 1$. The idea of the proof is taken from [2, Theorem 4.1] or [3, Theorem 5.2.4]. Set $y_i := J_\tau(x_i)$ and $D := \inf_{w\in H}\left(E(w) + C_\varepsilon d_H^\varepsilon(w, z)\right) \in \mathbb{R}$. Then we have

$$D \le E(y_i) + C_\varepsilon d_H^\varepsilon(y_i, z)$$

and

$$E(y_i) + \frac{1}{p\tau^{p-1}} d_H^p(y_i, x_i) \le E(z) + \frac{1}{p\tau^{p-1}} d_H^p(z, x_i).$$

Since $d_H^\varepsilon(y_i, z) \le \max\{2^{\varepsilon-1}, 1\}(d_H^\varepsilon(y_i, x_i) + d_H^\varepsilon(x_i, z))$, we obtain the upper boundedness of

$$\frac{1}{p\tau^{p-1}} d_H^p(y_i, x_i) - C_\varepsilon \max\{2^{\varepsilon-1}, 1\} d_H^\varepsilon(y_i, x_i).$$

Therefore, $\{y_i\}$ is bounded. (2) and (3) are trivially obtained from (1) by (3.35). □

The following is a trivial consequence of Corollary 3.37.

Corollary 3.39. *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty]$ satisfies Assumption 3.21 for some $\lambda \in \mathbb{R}$ and $k \in]0, +\infty[$. Let J_τ be the associated resolvent for $\tau \in]0, (k/p\lambda^-)^{q-1}[$. Suppose that E is lower bounded. Then the conclusion of Corollary 3.37 holds.*

The following is a corollary of Lemma 3.34.

Corollary 3.40. *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty]$ satisfies Assumption 3.21 for some $\lambda \in \mathbb{R}$ and $k \in]0, +\infty[$, and let J_τ be the associated resolvent for $\tau \in]0, (k/p\lambda^-)^{q-1}[$. Then the following holds:*

(1) Suppose $\lambda > 0$ or $\lambda = 0$ with $k > 2p/q^p$. Then $\sup_{\tau>0} d_H(J_\tau(x), x) < \infty$ is equivalent to $\sup_{\tau>0} d_H(J_\tau(y), y) < \infty$ for $x, y \in H$.
(2) Suppose $\lambda \ge 0$. Assume that there exists a stationary point $y \in D(|\partial E|)$. Then $\{J_\tau(x)\}_{\tau>0}$ is bounded for each $x \in H$. Moreover, if $\lambda > 0$, then $\{J_\tau(x)\}$ converges to y as $\tau \to \infty$. In particular, a stationary point of E is unique under $\lambda > 0$.

Proof. We first prove (1). Suppose $\lambda > 0$. Set

$$c_0 := \left\{\frac{1}{\lambda p}\left(\frac{2p}{q^p} - k\right)^+\right\}^{\frac{1}{p-1}} + \varepsilon$$

for a fixed $\varepsilon > 0$. Since $\tau \mapsto d_H(J_\tau(x), x)$ is non-decreasing, it suffices to show the equivalence between $\sup_{\tau\in[c_0,+\infty]} d_H(J_\tau(x), x) < \infty$ and $\sup_{\tau\in[c_0,+\infty]} d_H(J_\tau(y), y) < \infty$. From (3.35) with $\lambda > 0$,

$$d_H(J_\tau(x), x) \le d_H(J_\tau(x), J_\tau(y)) + d_H(J_\tau(y), y) + d_H(y, x)$$

$$\le \left(\frac{2p}{k+\lambda p\tau^{p-1}}\right)^{\frac{1}{p}} d_H(x,y)^{\frac{1}{p}}\{(d_H(x,J_\tau(y))+d_H(x,y))^{\frac{1}{q}}+(d_H(y,J_\tau(x))+d_H(x,y))^{\frac{1}{q}}\}+d_H(J_\tau(y),y)+d_H(y,x)$$

$$\le \left(\frac{2p}{k+\lambda p\tau^{p-1}}\right)^{\frac{1}{p}} d_H(x,y)^{\frac{1}{p}}\{(d_H(y,J_\tau(y))+2d_H(x,y))^{\frac{1}{q}}+(d_H(x,J_\tau(x))+2d_H(x,y))^{\frac{1}{q}}\}$$
$$+d_H(J_\tau(y),y)+d_H(y,x)$$

$$\le \left(\frac{2p}{k+\lambda p\tau^{p-1}}\right)^{\frac{1}{p}} d_H(x,y)^{\frac{1}{p}}\{(d_H(y,J_\tau(y)))^{\frac{1}{q}}+(2d_H(x,y))^{\frac{1}{q}}+d_H(x,J_\tau(x))^{\frac{1}{q}}+(2d_H(x,y))^{\frac{1}{q}}\}$$
$$+d_H(J_\tau(y),y)+d_H(y,x)$$

$$=\left(\frac{2p}{k+\lambda p\tau^{p-1}}\right)^{\frac{1}{p}}\left[d_H(x,y)^{\frac{1}{p}}d_H(x,J_\tau(x))^{\frac{1}{q}}+d_H(x,y)^{\frac{1}{p}}d_H(y,J_\tau(y))^{\frac{1}{q}}+2^{1+\frac{1}{q}}d_H(x,y)\}\right]$$
$$+d_H(J_\tau(y),y)+d_H(y,x)$$

$$\le \left(\frac{2p}{k+\lambda p\tau^{p-1}}\right)^{\frac{1}{p}}\left[\frac{d_H(x,y)}{p}+\frac{d_H(x,J_\tau(x))}{q}+\frac{d_H(x,y)}{p}+\frac{d_H(y,J_\tau(y))}{q}+2^{1+\frac{1}{q}}d_H(x,y)\right]$$
$$+d_H(J_\tau(y),y)+d_H(y,x)$$

yields the assertion, because $\tau \ge c_0$ implies $\frac{1}{q}\left(\frac{2p}{k+\lambda p\tau^{p-1}}\right)^{\frac{1}{p}} \le \frac{1}{q}\left(\frac{2p}{k+\lambda pc_0^{p-1}}\right)^{\frac{1}{p}} < 1$ and

$$\frac{\left(\frac{2p}{k+\lambda p\tau^{p-1}}\right)^{1/p}}{1-\frac{1}{q}\left(\frac{2p}{k+\lambda p\tau^{p-1}}\right)^{1/p}} \le \frac{\left(\frac{2p}{k+\lambda pc_0{}^{p-1}}\right)^{1/p}}{1-\frac{1}{q}\left(\frac{2p}{k+\lambda pc_0^{p-1}}\right)^{1/p}}.$$

The proof for the case $\lambda = 0$ with $k > 2p/q^p$ is similar.

Next we prove (2). If $y \in D(|\partial E|)$ is a stationary point and $\lambda \ge 0$, then $J_\tau(y) = y$ holds for all $\tau > 0$ by Corollary 3.33 and y is a minimizer of E from Proposition 3.8. Then we see that

$$E(J_\tau(x))+\frac{1}{p\tau^{p-1}}d_H^p(J_\tau(x),x) \le E(y)+\frac{1}{p\tau^{p-1}}d_H^p(y,x)$$

implies

$$\frac{1}{p\tau^{p-1}}d_H^p(J_\tau(x),x) \le E(y)-E(J_\tau(x))+\frac{1}{p\tau^{p-1}}d_H^p(y,x) \le \frac{1}{p\tau^{p-1}}d_H^p(y,x).$$

Thus we have $\sup_{\tau>0} d_H(J_\tau(x),x) \le d_H(x,y) < \infty$.

Suppose further $\lambda > 0$. Since the third factor of the right hand side of (3.35) is bounded with respect to $\tau > 0$ by way of (1). Thus we see the convergence

$$\lim_{\tau\to\infty} d_H(J_\tau(x),y) = \lim_{\tau\to\infty} d_H(J_\tau(x),J_\tau(y)) = 0.$$

□

Theorem 3.41. *Assume that a proper lower semicontinuous function $E : H \to]-\infty,+\infty]$ satisfies Assumption 3.22 for some $\lambda \ge 0$ and $k \in]0,2]$. Let J_τ with $\tau \in]0,+\infty[$ be the resolvent of E. Then, for each $\varepsilon \in [1,p[$, there exists $C_\varepsilon > 0$ such that (3.38) holds for some $z \in H$ with $E(z) < +\infty$. In particular, the conclusion of Corollary 3.37 holds.*

Proof. The idea of the proof is taken from [2, Theorem 4.1] or [3, Theorem 5.2.4]. Suppose that for each $\varepsilon \in [1,p[$ (3.38) fails. Then, for any sufficiently large $k \in \mathbb{N}$ there exists a subsequence $w_{i_k} \in H$ such that

$$E(w_{i_k}) \le -k(d_H^\varepsilon(w_{i_k},z)+1).$$

Take $y \in D(E)$. Let u_{i_k} be the t_{i_k}-point of the curve $\gamma := \gamma^y : [0,1] \to H$ joining y to w_{i_k} appearing in Assumption 3.22:

$$u_{i_k} := \gamma_{t_{i_k}} \quad \text{with} \quad t_{i_k} := \frac{1}{\sqrt{k}(d_H^\varepsilon(w_{i_k},y)+1)}.$$

We also have

$$d_H(y, u_{i_k}) \le t_{i_k} d_H(y, w_{i_k}).$$

Then $u_{i_k} \in H$ strongly converges to $y \in H$. Indeed, for $\varepsilon \in [1, p[$, $d_H^\varepsilon(u_{i_k}, y) \le t_{i_k}^\varepsilon d_H^\varepsilon(y, w_{i_k}) \le t_{i_k} d_H^\varepsilon(y, w_{i_k}) \le \frac{1}{\sqrt{k}} \to 0$. By the p-uniform λ-convexity along γ, for $\tau \in]0, +\infty[$

$$\begin{aligned} E(u_{i_k}) &\le (1 - t_{i_k})E(y) + t_{i_k} E(w_{i_k}) - \frac{\lambda}{2} t_{i_k}(1 - t_{i_k}) d_H^p(y, w_{i_k}) \\ &\le (1 - t_{i_k})E(y) - t_{i_k} k(d_H^\varepsilon(w_{i_k}, z) + 1) \\ &\le (1 - t_{i_k})E(y) - \sqrt{k}. \end{aligned}$$

Hence,

$$E(y) \le \varliminf_k E(u_{i_k}) \le -\infty,$$

which is impossible. □

Corollary 3.42. *Take $p \in [2, +\infty[$. Let (H, d_H) be a complete p-uniformly convex space with $k \in]0, c_p]$. Let E be a proper lower semicontinuous (geodesically) convex function $E : H \to]-\infty, +\infty]$. Let J_τ with $\tau \in]0, +\infty[$ be the resolvent of E. Then, for each $\varepsilon \in [1, p[$, there exists $C_\varepsilon > 0$ such that* (3.38) *holds for some $z \in H$ with $E(z) < +\infty$. In particular, the conclusion of Corollary* 3.37 *holds.*

Proof. This is a trivial consequence of Theorem 3.41 with Corollary 3.25. □

Proposition 3.43 (Slope estimate under Assumption 3.22)**.** *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty]$ satisfies Assumption* 3.22 *for some $\lambda \in \mathbb{R}$ and $k \in]0, c_p]$. Then for $x \in D(|\partial E|)$ and $\tau \in]0, (k/p\lambda^-)^{q-1}[$, we have*

$$|\partial E|(J_\tau(x)) \le |\partial E|(x) + \left(\frac{k+2}{2k}\lambda^- - \frac{\lambda^+}{2}\right) d_H^{p-1}(x, J_\tau(x)), \tag{3.44}$$

in particular, $|\partial E|(J_\tau(x)) \le |\partial E|(x) - \frac{\lambda}{2} d_H^{p-1}(x, J_\tau(x))$ for $\tau \in]0, +\infty[$ if $\lambda \ge 0$.

Proof. Set $y := J_\tau(x)$ and we may assume $|\partial E|(y) > 0$. Then we can take $z \in H$ satisfying

$$d_H^{p-1}(y, z) < \frac{|\partial E|(y)}{\lambda^-} \quad \text{and} \quad \frac{E(y) - E(z)}{d_H(y, z)} > \frac{|\partial E|(y)}{2}.$$

Let $\gamma = \gamma^y : [0, 1] \to H$ be the curve joining y to z satisfying $d_H(y, \gamma_s^y) \le s d_H(y, z)$, $s \in [0, 1]$ appeared in Assumption 3.22. The p-uniform λ-convexity of E along γ^y implies

$$\begin{aligned} E(\gamma_s) &\le E(y) + s\left(E(z) - E(y) - \frac{\lambda}{2}(1 - s) d_H^p(y, z)\right) \\ &\le E(y) + s\left(-\frac{1}{2}|\partial E|(y) d_H(y, z) - \frac{\lambda}{2}(1 - s) d_H^p(y, z)\right) \\ &\le E(y) + \frac{s}{2} d_H^p(y, z)(-\lambda^- - \lambda(1 - s)) \\ &\le E(y) \end{aligned}$$

for $s \in [0, 1]$. Since $y = J_\tau(x)$ is the minimizer of $y \mapsto E(y) + \frac{1}{p\tau^{p-1}} d_H^p(x, y)$, we have $d_H(x, y) \le d_H(\gamma_s, x)$. Let $\gamma^s : [0, 1] \to H$ be the curve joining x to γ_s. Then $d_H(\gamma_t^s, x) \le d_H(x, y)$ for $t := d_H(x, y)/d_H(\gamma_s, x)$. From this, we have $E(y) \le E(\gamma_t^s)$. Applying the p-uniform λ-convexity of E along γ^s, we have

$$\begin{aligned} E(y) \le E(\gamma_t^s) &\le (1 - t)E(x) + tE(\gamma_s) - \frac{\lambda}{2} t(1 - t) d_H^p(x, \gamma_s) \\ &\le (1 - t)E(x) + t\left((1 - s)E(y) + sE(z) - \frac{\lambda}{2} s(1 - s) d_H^p(y, z)\right) - \frac{\lambda}{2} t(1 - t) d_H^p(x, \gamma_s). \end{aligned}$$

Hence

$$
\begin{aligned}
0 &\le (1-t)\left(E(x)-E(y)-\frac{\lambda}{2}td_H^p(x,\gamma_s)\right)-ts\left(E(y)-E(z)-\frac{\lambda}{2}(1-s)d_H^p(y,z)\right)\\
&=(1-t)\left(E(x)-E(y)-\frac{\lambda}{2}d_H(x,y)d_H^{p-1}(x,\gamma_s)\right)-ts\left(E(y)-E(z)-\frac{\lambda}{2}(1-s)d_H^p(y,z)\right)\\
&=\frac{d_H(\gamma_s,x)-d_H(x,y)}{d_H(x,\gamma_s)}\left(E(x)-E(y)-\frac{\lambda}{2}d_H(x,y)d_H^{p-1}(x,\gamma_s)\right)-ts\left(E(y)-E(z)-\frac{\lambda}{2}(1-s)d_H^p(y,z)\right).
\end{aligned}
$$

Dividing $s>0$ and letting $s\to 0$, which implies $\gamma_s\to y$ and $t\to 1$, we obtain

$$
\frac{E(y)-E(z)}{d_H(y,z)}-\frac{\lambda}{2}d_H^{p-1}(y,z)\le\frac{E(x)-E(y)}{d_H(x,y)}+\frac{\lambda^-}{2}d_H^{p-1}(x,y).
$$

Finally $z\to y$ yields

$$
|\partial E|(y)\le\frac{(E(x)-E(y))^+}{d_H(x,y)}+\frac{\lambda^-}{2}d_H^{p-1}(x,y).
$$

By Proposition 3.8,

$$
\frac{(E(x)-E(y))^+}{d_H(x,y)}-\left(\frac{\lambda^-}{k}-\frac{\lambda^+}{2}\right)d_H^{p-1}(x,y)\le|\partial E|(x).
$$

Therefore, we obtain (3.44). □

Proposition 3.45 (Resolvent identity)**.** *Suppose that a proper lower semicontinuous function $E:H\to]-\infty,+\infty]$ satisfies Assumption 3.21 for some $\lambda\in\mathbb{R}$ and $k\in]0,+\infty[$. Assume that (H,d_H) is a geodesic space. For $x\in H$ and $\tau,\mu\in]0,(k/p\lambda^-)^{q-1}[$ with $\mu\le\tau$, we have*

$$
J_\tau(x)=J_\mu\left(\frac{\mu}{\tau}x+\left(1-\frac{\mu}{\tau}\right)J_\tau(x)\right),
$$

where $\frac{\mu}{\tau}x+\left(1-\frac{\mu}{\tau}\right)J_\tau(x)$ is the point on the geodesic joining x to $J_\tau(x)$ such that $d_H(x,\frac{\mu}{\tau}x+\left(1-\frac{\mu}{\tau}\right)J_\tau(x))=\left(1-\frac{\mu}{\tau}\right)d_H(x,J_\tau(x))$.

Proof. For simplicity, we write $y_{s,\tau}:=\frac{\mu}{\tau}x+\left(1-\frac{\mu}{\tau}\right)J_\tau(x)$ with $s:=\left(1-\frac{\mu}{\tau}\right)\in[0,1]$. Since $d_H(x,y_{s,\tau}):d_H(y_{s,\tau},J_\tau(x))=s:(1-s)$, we have the equality:

$$
d_H^p(J_\tau(x),x)=\frac{1}{(1-s)^{p-1}}d_H^p(J_\tau(x),y_{s,\tau})+\frac{1}{s^{p-1}}d_H^p(x,y_{s,\tau}).
$$

Hence,

$$
\begin{aligned}
E^\tau(x)&=E(J_\tau(x))+\frac{1}{p\tau^{p-1}}d_H^p(J_\tau(x),x)\\
&=E(J_\tau(x))+\frac{1}{p((1-s)\tau)^{p-1}}d_H^p(J_\tau(x),y_{s,\tau})+\frac{1}{p(s\tau)^{p-1}}d_H^p(x,y_{s,\tau})\\
&\ge\inf_{z\in H}\left(E(z)+\frac{1}{p((1-s)\tau)^{p-1}}d_H^p(z,y_{s,\tau})\right)+\frac{1}{p(s\tau)^{p-1}}d_H^p(x,y_{s,\tau})\\
&=E(J_{(1-s)\tau}(y_{s,\tau}))+\frac{1}{p((1-s)\tau)^{p-1}}d_H^p(J_{(1-s)\tau}(y_{s,\tau}),y_{s,\tau})+\frac{1}{p(s\tau)^{p-1}}d_H^p(x,y_{s,\tau})\\
&\ge E(J_{(1-s)\tau}(y_{s,\tau}))+\frac{1}{p\tau^{p-1}}d_H^p(J_{(1-s)\tau}(y_{s,\tau}),x)\ge E^\tau(x).
\end{aligned}
$$

Then the equality holds and the uniqueness of the minimizer implies $J_\tau(x)=J_{(1-s)\tau}(y_{s,\tau})$. □

Theorem 3.46 (Existence of the minimizer as the limit of the resolvent flow under $\lambda\ge 0$)**.** *Suppose that a proper lower semicontinuous function $E:H\to]-\infty,+\infty[$ satisfies Assumption 3.21 for some $\lambda\ge 0$ and $k\in]0,+\infty[$. Let $\{J_\tau(x)\}_{\tau\ge 0}$ be the resolvent associated to E for a fixed $x\in H$. Assume that there exists a sequence $\{\tau_n\}$ with $\lim_{n\to\infty}\tau_n=+\infty$ such that $\inf_{n\in\mathbb{N}}E(J_{\tau_n}(x))>-\infty$ and $\sup_{n\in\mathbb{N}}d_H(J_{\tau_n}(x),x)<\infty$. Then E is lower bounded and there exists a minimizer $\bar{x}\in H$ of E depending on x such that*

$$
\lim_{\tau\to\infty}d_H(J_\tau(x),\bar{x})=0.
$$

Proof. The proof is a repetition of the proof of [11, Theorem 3.1.1]. We show it for readers convenience. By assumption, we can get $\inf_H E = \inf_{n\in\mathbb{N}} E(J_{\tau_n}(x))$. In particular, E is lower bounded, and the sequence $\{J_{\tau_n}(x)\}_{n\in\mathbb{N}}$ is a minimizing sequence for E. Indeed,

$$\begin{aligned}-\infty < \inf_{n\in\mathbb{N}} E(J_{\tau_n}(x)) &\le \varliminf_{n\to\infty} E(J_{\tau_n}(x))\\ &= \varliminf_{n\to\infty}\left(E(J_{\tau_n}(x)) + \frac{1}{p\tau_n^{p-1}} d_H^p(J_{\tau_n}(x), x)\right)\\ &= \varliminf_{n\to\infty}\left(E(v) + \frac{1}{p\tau_n^{p-1}} d_H^p(v, x)\right) \le E(v).\end{aligned}$$

From Lemma 3.30, we have $\lim_{\tau\to\infty} d_H(J_\tau(x), x) = \sup_{\tau>0} d_H(J_\tau(x), x) < +\infty$. Let $y_{\tau,\mu}$ be the mid-point at the curve joining $J_\tau(x)$ to $J_\mu(x)$ specified in the p-uniform λ-convexity of E in Assumption 3.21. Then

$$\begin{aligned}&E(J_\mu(x)) + \frac{1}{p\mu^{p-1}} d_H^p(J_\mu(x), x)\\ &\le E(y_{\tau,\mu}) + \frac{1}{p\mu^{p-1}} d_H^p(y_{\tau,\mu}, x)\\ &\le \frac{1}{2}E(J_\tau(x)) + \frac{1}{2}E(J_\mu(x)) + \frac{1}{p\mu^{p-1}}\left(\frac{1}{2}d_H^p(J_\tau(x), x) + \frac{1}{2}d_H^p(J_\mu(x), x) - \frac{k}{8}d_H^p(J_\tau(x), J_\mu(x))\right)\\ &\le E(J_\mu(x)) + \frac{1}{p\mu^{p-1}}\left(\frac{1}{2}d_H^p(J_\tau(x), x) + \frac{1}{2}d_H^p(J_\mu(x), x) - \frac{k}{8}d_H^p(J_\tau(x), J_\mu(x))\right)\\ &\le E(J_\mu(x)) + \frac{1}{p\mu^{p-1}}\left(\frac{1}{2}d_H^p(J_\tau(x), x) + \frac{1}{2}d_H^p(J_\mu(x), x) - \frac{k}{8}d_H^p(J_\tau(x), J_\mu(x))\right)\end{aligned}$$

implies that $\{J_\tau(x)\}_{\tau\ge0}$ forms a Cauchy sequence in (H, d_H). Hence it converges to some $\bar{x}\in H$ and it satisfies

$$E(\bar{x}) \le \varliminf_{\tau\to\infty} E(J_\tau(x)) = \inf_{y\in H} E(y).$$

□

Corollary 3.47. *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty[$ satisfies Assumption 3.21 for some $\lambda\in\mathbb{R}$ and $k\in]0, +\infty[$. Suppose further $\lambda \ge 0$. Let $\{J_\tau(x)\}_{\tau\ge0}$ be the resolvent associated to E for $x\in H$. Then the following are equivalent:*

(1) There exists a minimizer of E.
(2) There exists a sequence $\{\tau_n\}$ with $\lim_{n\to\infty}\tau_n = +\infty$ such that

$$\inf_{n\in\mathbb{N}} E(J_{\tau_n}(x)) > -\infty, \quad \sup_{n\in\mathbb{N}} d_H(J_{\tau_n}(x), x) < \infty \quad \textit{for some } x\in H.$$

(3) We have

$$\inf_{\tau>0} E(J_\tau(x)) > -\infty, \quad \sup_{\tau>0} d_H(J_\tau(x), x) < \infty \quad \textit{for all } x\in H.$$

In particular, if E is lower bounded, then the following are equivalent:

(1) There exists a minimizer of E.*
(2) There exists a sequence $\{\tau_n\}$ with $\lim_{n\to\infty}\tau_n = +\infty$ such that*

$$\sup_{n\in\mathbb{N}} d_H(J_{\tau_n}(x), x) < \infty \quad \textit{for some } x\in H.$$

(3) We have $\sup_{\tau>0} d_H(J_\tau(x), x) < \infty$ for all $x\in H$.*

Proof. (3)⇒(2) is trivial. (2)⇒(1) is proved in Theorem 3.46. (1)⇒(3) follows Proposition 3.8 and Corollary 3.40(2). □

Proposition 3.48. *Suppose that a proper lower semicontinuous function $E : H \to]-\infty, +\infty[$ satisfies Assumption* 3.21 *for some $\lambda \ge 0$ and $k \in]0, +\infty[$. Let $\{J_\tau(x)\}_{\tau\ge 0}$ be the resolvent associated to E for $x \in H$. Assume that for a fixed $o \in H$, there exists a non-decreasing function $\varphi : \mathbb{R} \to [0, +\infty[$ such that*

$$d_H(x, o) \le \varphi(E(x)) \quad \textit{for all} \quad x \in D(E).$$

Then $\sup_{\tau>0} d_H(J_\tau(y), y) < \infty$ for each $y \in H$, in particular, there exists a minimizer of E if E is lower bounded.

Proof. It is easy to see

$$\begin{aligned}\sup_{\tau>0} d_H(J_\tau(y), y) &= \sup_{\tau>1} d_H(J_\tau(y), y)\\ &\le \sup_{\tau>1} d_H(J_\tau(y), o) + d_H(y, o)\\ &\le \sup_{\tau>1} \varphi(E(J_\tau(y))) + d_H(y, o)\\ &\le \varphi(E(J_1(y))) + d_H(y, o), \quad y \in H.\end{aligned}$$

Thus we obtain the conclusion. □

4 L^p-Wasserstein space over p-uniformly convex space

In this section, we show that proper lower semicontinuous functionals on the L^p-Wasserstein space over a complete separable p-uniformly convex space, which are p-uniformly λ-convex along generalized geodesics, satisfy Assumption 3.22.

Let (H, d_H) be a complete separable metric space and $p \ge 1$. Denote by $\mathcal{P}^p(H)$ be the space of Borel probability measures having p-th moment. The L^p-Wasserstein distance between $\mu^1, \mu^2 \in \mathcal{P}^p(H)$ is defined by

$$d^p_{W^p}(\mu^1, \mu^2) := \inf\left\{ \int_{H\times H} d^p_H(x_1, x_2)\boldsymbol{\mu}(\mathrm{d}x_1\mathrm{d}x_2) \;\middle|\; \boldsymbol{\mu} \in \Pi(\mu^1, \mu^2)\right\}, \tag{4.1}$$

where $\Pi(\mu^1, \mu^2)$ is the set of Borel measures $\boldsymbol{\mu}$ satisfying $\boldsymbol{\mu}(A \times H) = \mu^1(A)$ and $\boldsymbol{\mu}(H \times B) = \mu^2(B)$ for all Borel subsets A, B of H. Such $\boldsymbol{\mu}$ is called the *coupling of μ^1 and μ^2* or the *transfer plan between μ^1 and μ^2*. There always exists a minimizer for $\Pi(\mu^1, \mu^2) \ni \boldsymbol{\mu} \mapsto \int_{H\times H} d^p_H(x_1, x_2)\boldsymbol{\mu}(\mathrm{d}x_1\mathrm{d}x_2)$ called the *optimal coupling of μ^1 and μ^2* or the *optimal transfer plan between μ^1 and μ^2*. Denote by $\Pi_o(\mu^1, \mu^2)$ the set of all optimal transfer plans between μ^1 and μ^2 for (4.1). It is known that $(\mathcal{P}^p(H), d_{W^p})$ is a complete separable metric space called the *L^p-Wasserstein space over* (H, d_H). $(\mathcal{P}^p(H), d_{W^p})$ is a geodesic space if (H, d_H) is so. Hereafter, we assume that (H, d_H) is always a geodesic space. For given $\mu^1, \mu^2 \in \mathcal{P}^p(H)$, the constant speed geodesic $(\mu_t)_{t\in[0,1]}$ joining $\mu_0 = \mu^1$ to $\mu_1 = \mu^2$ with respect to the geodesic space $(\mathcal{P}^p(H), d_{W^p})$ is called the *L^p-Wasserstein geodesic*, that is, $d_{W^p}(\mu_0, \mu_t) = t d_{W^p}(\mu_0, \mu_1)$ for all $t \in [0, 1]$.

Definition 4.2 (Generalized geodesics)**.** Let $\mu^1, \mu^2, \mu^3 \in \mathcal{P}^p(H)$. A *generalized geodesic joining μ^2 to μ^3 with base μ^1* is a curve of the type

$$\mu_t^{1,2\to 3} = (\pi_t^{2\to 3})_\sharp \boldsymbol{\mu} \quad t \in [0, 1], \tag{4.3}$$

where $\boldsymbol{\mu} \in \mathcal{P}^p(H^3)$ satisfies

$$\boldsymbol{\mu} \in \Pi(\mu^1, \mu^2, \mu^3) \quad \text{and} \quad \pi^{1,2}_\sharp \boldsymbol{\mu} \in \Pi_o(\mu^1, \mu^2), \quad \pi^{1,3}_\sharp \boldsymbol{\mu} \in \Pi_o(\mu^1, \mu^3). \tag{4.4}$$

Here $\pi_t^{2\to 3} := (1-t)\pi^2 + t\pi^3$ is the map corresponding to the t-point on the geodesic segment between x_2 and x_3 from $(x_1, x_2, x_3) \in H \times H \times H$ to H. $\pi^{1,2} : H^3 \to H^2$ (resp. $\pi^{1,3} : H^3 \to H^2$) is the projection map defined by $\pi^{1,2}(x_1, x_2, x_3) := (x_1, x_2)$ (resp $\pi^{1,3}(x_1, x_2, x_3) := (x_1, x_3)$).

Theorem 4.5 (p-uniform convexity along generalized geodesics). *Suppose $p \in [2, +\infty[$ and let (H, d_H) be a complete separable p-uniformly convex space with parameter $k \in]0, c_p]$. Take $\mu^1, \mu^2, \mu^3 \in \mathcal{P}^p(H)$. Then $\mathcal{P}^p(H) \ni \nu \mapsto d^p_{W^p}(\mu^1, \nu)$ is p-uniformly k-convex along the generalized geodesic $\mu_t^{1,2\to3}$. Moreover, let $\mu_t^{2,2\to3}$ be the generalized geodesic joining μ^2 to μ^3 with base μ^2, then $d_{W^p}(\mu^2, \mu_t^{2,2\to3}) \le t d_{W^p}(\mu^2, \mu^3)$ for all $t \in [0, 1]$.*

Proof. Suppose that $\boldsymbol{\mu} \in \mathcal{P}^p(H^3)$ satisfies (4.4) and set $\mu_t^{1,2\to3}$ as in (4.3). It is easy to see

$$d^p_{W^p}(\mu^2, \mu^3) \le \int_{H^3} d^p_H(x_2, x_3)\boldsymbol{\mu}(\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3).$$

We introduce the transfer plan

$$\boldsymbol{\mu}_t^{1,2\to3} := ((1-t)\pi^{1,2} + t\pi^{1,3})_\sharp \boldsymbol{\mu} \in \Pi(\mu^1, \mu_t^{1,2\to3}).$$

Let $\gamma^{x_2x_3} : [0, 1] \to H$ be the geodesic segment joining x_2 to x_3. Then the p-uniform convexity of (H, d_H) with the parameter k yields

$$\begin{aligned} d^p_{W^p}(\mu^1, \mu_t^{1,2\to3}) &\le \int_{H\times H} d^p_H(y_1, y_2)\boldsymbol{\mu}_t^{1,2\to3}(\mathrm{d}y_1\mathrm{d}y_2) = \int_{H^3} d^p_H(x_1, \gamma_t^{x_2x_3})\boldsymbol{\mu}(\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3) \\ &\le (1-t)d^p_{W^p}(\mu^1, \mu^2) + t d^p_{W^p}(\mu^1, \mu^3) - \frac{k}{2}t(1-t)\int_{H^3} d^p_H(x_2, x_3)\boldsymbol{\mu}(\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3) \\ &\le (1-t)d^p_{W^p}(\mu^1, \mu^2) + t d^p_{W^p}(\mu^1, \mu^3) - \frac{k}{2}t(1-t)d^p_{W^p}(\mu^2, \mu^3). \end{aligned}$$

Finally we show the last assertion. Let $\mu_t^{2,2\to3}$ be the generalized geodesic joining μ^2 to μ^3 with base $\mu^2 = \mu^1$. Then $\pi^{1,2}_\sharp\boldsymbol{\mu} = (i \times i)_\sharp\mu^2 \in \Pi_o(\mu^2, \mu^2)$ holds, where i is the identity map defined by $i(x) := x$ for $x \in H$. This yields $\boldsymbol{\mu}((H^2 \setminus \mathsf{diag}) \times H) = (\pi^{1,2}_\sharp\boldsymbol{\mu})(H^2 \setminus \mathsf{diag}) = 0$. Then we have

$$\begin{aligned} d^p_{W^p}(\mu^2, \mu_t^{2,2\to3}) = d^p_{W^p}(\mu^1, \mu_t^{1,2\to3}) &\le \int_{H^3} d^p_H(x_1, \gamma_t^{x_2x_3})\boldsymbol{\mu}(\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3) \\ &= \int_{H^3} d^p_H(x_1, \gamma_t^{x_1x_3})\boldsymbol{\mu}(\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3) \quad (\because \boldsymbol{\mu}((H^2 \setminus \mathsf{diag}) \times H) = 0) \\ &= \int_{H^2} d^p_H(x_1, \gamma_t^{x_1x_3})\pi^{1,3}_\sharp\boldsymbol{\mu}(\mathrm{d}x_1\mathrm{d}x_3) \\ &= t^p \int_{H^2} d^p_H(x_1, x_3)\pi^{1,3}_\sharp\boldsymbol{\mu}(\mathrm{d}x_1\mathrm{d}x_3) = t^p d^p_{W^p}(\mu^2, \mu^3). \end{aligned}$$

□

Definition 4.6 (p-uniform λ-convexity along geodesic). A functional $E : \mathcal{P}^p(H) \to]-\infty, +\infty]$ is said to be *p-uniformly λ-convex along geodesic* if and only if for any $\mu^1, \mu^2 \in D(E)$, there exists an optimal transfer plan $\boldsymbol{\mu} \in \Pi_o(\mu^1, \mu^2)$ such that

$$E(\mu_t^{1\to2}) \le (1-t)E(\mu^1) + tE(\mu^2) - \frac{\lambda}{2}t(1-t)d^p_{W^p}(\mu^1, \mu^2) \quad t \in [0, 1], \tag{4.7}$$

where $\mu_t^{1\to2} = (\pi_t^{1\to2})_\sharp\boldsymbol{\mu} = ((1-t)\pi^1 + t\pi^2)_\sharp\boldsymbol{\mu}$. Here π^1 and π^2 are the projections on the first and second coordinates in H^2, respectively.

Definition 4.8 (p-uniform λ-convexity along generalized geodesic). A functional $E : \mathcal{P}^p(H) \to]-\infty, +\infty]$ is said to be *p-uniformly λ-convex along generalized geodesic* if and only if for any $\mu^1, \mu^2, \mu^3 \in D(E)$, there exists a generalized geodesic $\mu_t^{1,2\to3}$ joining μ^2 to μ^3 with base μ^1 induced by a plan $\boldsymbol{\mu} \in \Pi(\mu^1, \mu^2, \mu^3)$ satisfying (4.4) such that

$$E(\mu_t^{1,2\to3}) \le (1-t)E(\mu^2) + tE(\mu^3) - \frac{\lambda}{2}t(1-t)d^p_{W^p}(\mu^2, \mu^3) \quad t \in [0, 1]. \tag{4.9}$$

Remark 4.10.

(1) Our definition on the p-uniform λ-convexity along generalized geodesics is weaker than what is defined in [1, Definition 9.2.4] for $p = 2$.
(2) If E is convex along any interpolating curve $\mu_t^{2\to3} := ((1-t)\widetilde{\pi}^2 + t\widetilde{\pi}^3)_\sharp\widetilde{\boldsymbol{\mu}}$ induced by $\widetilde{\boldsymbol{\mu}} \in \Pi(\mu^2,\mu^3)$, then E is trivially convex along generalized geodesics, where $\widetilde{\pi}^2 : H^2 \to H$ (resp. $\widetilde{\pi}^3 : H^2 \to H$) is the projection map defined by $\widetilde{\pi}^2(x_2,x_3) := x_2$ (resp. $\widetilde{\pi}^3(x_2,x_3) := x_3$). This is because any generalized geodesic $\mu_t^{1,2\to3}$ with base $\mu^1 \in \mathcal{P}^p(H)$ can be realized as an interpolating curve $\mu_t^{2\to3}$, that is, $\mu_t^{1,2\to3} = ((1-t)\pi^2 + t\pi^3)_\sharp\boldsymbol{\mu} = ((1-t)\widetilde{\pi}^2 + t\widetilde{\pi}^3)_\sharp(\pi_\sharp^{2,3}\boldsymbol{\mu})$ with $\pi_\sharp^{2,3}\boldsymbol{\mu} \in \Pi(\mu^2,\mu^3)$.
(3) The p-uniform λ-convexity along generalized geodesics is stronger than the p-uniform λ-convexity along geodesics, because $\mu_t^{3,2\to3}$ is the Wasserstein geodesic joining μ^2 to μ^3 (see [1, Remark 9.2.8]).

Finally, we can conclude the following:

Theorem 4.11. *Suppose $p \in [2,+\infty[$ and let (H, d_H) be a complete separable p-uniformly convex space with parameter $k \in]0, c_p]$. Let $E : \mathcal{P}^p(H) \to]-\infty,+\infty]$ be a p-uniformly λ-convex functional along generalized geodesics. Then E satisfies Assumption 3.22 (hence Assumptions 3.21 and 3.4). In particular, if further E is a proper lower semicontinuous functional being lower bounded, then E has a minimizer provided $\lambda \geq 0$ and for each $\mu_o \in \mathcal{P}^p(H)$ there exists a non-decreasing function $\varphi : \mathbb{R} \to [0,+\infty[$ such that*

$$d_{W_p}(\mu,\mu_o) \leq \varphi(E(\mu)) \quad \text{for all} \quad \mu \in \mathcal{P}^p(H). \tag{4.12}$$

Remark 4.13.

(1) (4.12) can be regarded as a variant of Talagrand inequality.
(2) If $\lambda > 0$ and μ_o is a minimizer for E, then (4.12) holds as a Talagrand inequality. Indeed, let $(\mu_t)_{t\in[0,1]}$ be a generalized geodesic joining $\mu_0 := \mu$ and $\mu_1 = \mu_o$. We then have

$$\begin{aligned} E(\mu_o) \leq E(\mu_t) &\leq (1-t)E(\mu_0) + tE(\mu_1) - \frac{\lambda}{2}t(1-t)d_{W^p}^p(\mu_0,\mu_1) \\ &\leq (1-t)E(\mu) + tE(\mu_o) - \frac{\lambda}{2}t(1-t)d_{W^p}^p(\mu,\mu_o). \end{aligned}$$

Hence (4.12) holds for $\varphi(s) := (2(s - E(\mu_o))^+/\lambda)^{1/p}$.

5 Cheeger type Sobolev space over L^p-maps

5.1 The space of L^p-maps

Let $(X, \mathcal{X}, \mathfrak{m})$ be a σ-finite measure space. Denote by $\mathcal{X}^{\mathfrak{m}}$ the completion of $\mathcal{X}$ with respect to $\mathfrak{m}$. In what follows, we simply say *measurable* (resp. $\mathfrak{m}$-*measurable*) for $\mathcal{X}$-measurable (resp. $\mathcal{X}^{\mathfrak{m}}$-measurable). A numerical function f on X is a map $f : X \to [-\infty,+\infty]$. For a measurable numerical function f on X, we set $\|f\|_p := \left(\int_X |f(x)|^p\mathfrak{m}(\mathrm{d}x)\right)^{1/p}$, $\|f\|_\infty := \inf\{\tau > 0 \mid |f(x)| \leq \tau \ \ \mathfrak{m}\text{-a.e. } x \in X\}$. For $p \in]0,+\infty]$, denote by $L^p(X;\mathfrak{m})$ the family of $\mathfrak{m}$-equivalence classes of $\mathcal{X}^{\mathfrak{m}}$-measurable functions finite with respect to $\|\cdot\|_p$. Denote by $L^0(X;\mathfrak{m})$ the family of $\mathfrak{m}$-equivalence classes of $\mathcal{X}^{\mathfrak{m}}$-measurable numerical functions $f : X \to [-\infty,+\infty]$ with $|f| < +\infty$ $\mathfrak{m}$-a.e.

Let (Y, d_Y) be a metric space. For $p \in [1,+\infty]$ and measurable maps $f, g : X \to Y$, define a distance $d_{L^p}(f,g)$ by $d_{L^p}(f,g) := \|d_Y(f,g)\|_p$. If $p \in [1,+\infty[$, then

$$d_{L^p}(f,g) := \left(\int_X d_Y^p(f(x),g(x))\mathfrak{m}(\mathrm{d}x)\right)^{1/p}.$$

If $p = +\infty$, then $d_\infty(f,g)$ is the $\mathfrak{m}$-essentially supremum of $x \mapsto d_Y(f(x),g(x))$. We say that f and g are $\mathfrak{m}$-equivalent if

$$f(x) = g(x) \ \ \mathfrak{m}\text{-a.e. } x \in X$$

and write $f \overset{m}{\sim} g$. For a fixed measurable map $h : X \to Y$, we set

$$L^p_h(X, Y; \mathfrak{m}) := \{f \in \mathcal{X}/\mathcal{B}(Y) \mid d_Y(f, h) \in L^p(X; \mathfrak{m})\}/\overset{\mathfrak{m}}{\sim}.$$

The map $h : X \to Y$ is called a *base map*. If $\mathfrak{m}(X) < +\infty$ and $h : X \to Y$ is bounded, then $L^p_h(X, Y; \mathfrak{m})$ is independent of the choice of such h.

The following theorem and corollary are proved in [15]:

Theorem 5.1 (Theorem 7.2(1) in [15]). *Let $p \geq 2$ and (Y, d_Y) be a complete p-uniformly convex space having the NPC property of Busemann type. Fix a measurable map $h : X \to Y$. Then, $(L^p_h(X, Y; \mathfrak{m}), d_{L^p})$ is a complete p-uniformly convex space having the NPC property of Busemann type.*

Corollary 5.2 (Corollary 7.3 in [15]). *Let (Y, d_Y) be a complete CAT(0)-space. Then, $(L^p_h(X, Y; \mathfrak{m}), d_{L^p})$ is a complete p-uniformly (resp. 2-uniformly) convex space with parameter $k = k_p$ (resp. $k = 2(p-1)$) if $p \in [2, +\infty[$ (resp. $p \in]1, 2]$) and it is a NPC space of Busemann type.*

5.2 Upper gradient and Cheeger's Sobolev spaces

In what follows, let (X, d_X) be a metric space, and $U \subset X$ be an open set, and $\mathfrak{m}$ be a Borel regular measure on X such that any ball with finite positive radius is of finite positive measure. Let (Y, d_Y) be a complete geodesic space.

Definition 5.3 (Upper gradient). A Borel function $g : U \to [0, +\infty]$ is called an *upper gradient* for a map $u : U \to Y$ if, for any unit speed curve $c : [0, \ell] \to U$, we have

$$d_Y(u(c(0)), u(c(\ell))) \leq \int_0^\ell g(c(s))ds.$$

Definition 5.4 (Upper pointwise Lipschitz constant function). For a map $u : U \to Y$ and a point $z \in U$, we define

$$Lip\, u(z) := \varlimsup_{r \to 0} \sup_{d_X(z,w)=r} \frac{d_Y(u(z), u(w))}{r},$$
$$\mathrm{Lip}\, u(z) := \varlimsup_{r \to 0} \sup_{0<d_X(z,w)<r} \frac{d_Y(u(z), u(w))}{d_X(z, w)}$$

and we put $Lip\, u(z) = \mathrm{Lip}\, u(z) = 0$ if z is an isolated point of U. Clearly $Lip\, u \leq \mathrm{Lip}\, u$ on X. We call $\mathrm{Lip}\, u$ the *upper pointwise Lipschitz constant function* for u.

Cheeger [6] proved that for a locally Lipschitz function $f : U \to \mathbb{R}$, then $Lip\, f$, hence $\mathrm{Lip}\, f$, is an upper gradient for f. The same conclusion also holds for locally Lipschitz maps $u : U \to \mathbb{R}$ in view of $Lip(d_Y(u, y))(z) \leq Lip(u)(z)$ for any $y \in Y$. We next define the Cheeger type Sobolev spaces. Fix a point $o = o_Y \in Y$ as a base point and $p \in [1, +\infty[$. Let $L^p_o(U, Y; \mathfrak{m})$ be the space of L^p-maps with the base map o defined by $o(x) := o$, $x \in U$. We write $L^p(U, Y; \mathfrak{m})$ instead of $L^p_o(U, Y; \mathfrak{m})$ for simplicity. Note that the constant map $o : U \to Y$ belongs to $L^p(U, Y; \mathfrak{m})$.

Definition 5.5 (Cheeger type Sobolev space). For $u \in L^p(U, Y; \mathfrak{m})$, we define the Cheeger type p-energy of u as

$$\mathsf{Ch}(u) := \inf_{\{(u_i, g_i)\}_{i=1}^{+\infty}} \varliminf_{i \to +\infty} \|g_i\|^p_{L^p(U;\mathfrak{m})},$$

where the infimum is taken over all sequences $\{(u_i, g_i)\}_{i=1}^{+\infty}$ such that $u_i \to u$ in $L^p(U, Y; \mathfrak{m})$ as $i \to +\infty$ and g_i is an upper gradient for u_i for each i. The *Cheeger type* $(1, p)$*-Sobolev space* is defined by

$$H^{1,p}(U, Y; \mathfrak{m}) := \{u \in L^p(U, Y; \mathfrak{m}) \mid \mathsf{Ch}(u) < +\infty\}.$$

By definition, if $u = v$ $\mathfrak{m}$-a.e. on U, then $\mathsf{Ch}(u) = \mathsf{Ch}(v)$.

The constant map $o : U \to Y$ also belongs to $H^{1,p}(U, Y; \mathfrak{m})$ with $\mathsf{Ch}(o) = 0$. Moreover, if $\mathfrak{m}(U) < \infty$, then any constant map $c : U \to Y$ defined by $c(x) := c$, $x \in U$ for a given point $c \in Y$ satisfies $c \in H^{1,p}(U, Y; \mathfrak{m})$ with $\mathsf{Ch}(c) = 0$.

For $f \in L^p(U; \mathfrak{m}) := L^p_0(U, \mathbb{R}; \mathfrak{m})$, we set

$$\|f\|_{H^{1,p}} := \|f\|_{L^p(U;\mathfrak{m})} + \mathsf{Ch}(f)^{\frac{1}{p}}.$$

It is easy to see that $H^{1,p}(U) := \{f \in L^p(U; \mathfrak{m}) \mid \|f\|_{H^{1,p}} < +\infty\}$ is a subspace of $L^p(U; \mathfrak{m})$ and $\|\cdot\|_{H^{1,p}}$ forms a norm on $H^{1,p}(U)$ and $(H^{1,p}(U), \|\cdot\|_{H^{1,p}})$ is a Banach space by [6, Theorem 2.7].

The following is proved in [24].

Theorem 5.6 (Lower semi continuity of energy, see Theorem 2.8 in [24]). *If a sequence* $\{u_i\}_{i=1}^{+\infty}$ *converges to* u *in* $L^p(U, Y; \mathfrak{m})$, *then* $\mathsf{Ch}(u) \le \underline{\lim}_{i\to\infty} \mathsf{Ch}(u_i)$.

Definition 5.7 (Generalized upper gradient). A function $g \in L^p(U; \mathfrak{m})$ is called a *generalized upper gradient* for $u \in H^{1,p}(U, Y; \mathfrak{m})$ if there exists a sequence $\{(u_i, g_i)\}_{i=1}^{+\infty}$ such that g_i is an upper gradient for u_i and $u_i \to u$, $g_i \to g$ in $L^p(U, Y; \mathfrak{m})$, $L^p(U; \mathfrak{m})$ respectively as $i \to +\infty$.

From the definition of the p-energy, $\mathsf{Ch}(u) \le \|g\|^p_{L^p(U;\mathfrak{m})}$ for any generalized upper gradient $g \in L^p(U; \mathfrak{m})$ for $u \in H^{1,p}(U, Y; \mathfrak{m})$.

Definition 5.8 (Minimal generalized upper gradient). A generalized upper gradient $g \in L^p(U; \mathfrak{m})$ for a map $u \in H^{1,p}(U, Y; \mathfrak{m})$ is said to be *minimal* if it satisfies $\mathsf{Ch}(u) = \|g\|^p_{L^p(U;\mathfrak{m})}$.

Hereafter, we always assume that (Y, d_Y) is a complete Busemann's NPC space. Then the distance function $d_Y : Y \times Y \to [0, +\infty[$ is convex. We know the following results:

Lemma 5.9 (See, [25, Lemma 3.1]). *Suppose that* (Y, d_Y) *is a Busemann's NPC space. Let* $u_1, u_2 : U \to Y$ *be maps. For any upper gradient* g_1, g_2 *for* u_1, u_2 *respectively and* $0 \le \lambda \le 1$. *The function* $g := (1-\lambda)g_1 + \lambda g_2$ *is an upper gradient for the map* $v := (1-\lambda)u_1 + \lambda u_2$. *In particular, for any* $u_1, u_2 \in H^{1,p}(U, Y; \mathfrak{m})$ *with* $1 \le p < +\infty$ *and for any* $0 \le \lambda \le 1$, *we have*

$$\mathsf{Ch}((1-\lambda)u_1 + \lambda u_2)^{1/p} \le (1-\lambda)\mathsf{Ch}(u_1)^{1/p} + \lambda \mathsf{Ch}(u_2)^{1/p}.$$

That is, $\mathsf{Ch}^{1/p}$ *is (geodesically) convex. Consequently,* Ch *satisfies Assumption* 3.22.

Theorem 5.10 (See, Theorem 3.2 in [24]). *Let* $p \in]1, +\infty[$. *Suppose that* (Y, d_Y) *is a Busemann's NPC space. Then for any* $u \in H^{1,p}(U, Y; \mathfrak{m})$, *there exists a unique minimal generalized upper gradient* g_u *for* u.

Recall that $o = o_Y \in Y$ is the base point of (Y, d_Y).

Lemma 5.11 (Contraction property). *Suppose that* (Y, d_Y) *is a Busemann's NPC space and* $p \in]1, +\infty[$.

(1) Consider another complete metric space (Z, d_Z) *with a base point* $o_Z \in Z$. *Take any* $u \in H^{1,p}(U, Y; \mathfrak{m})$ *and* $\psi \in C^{\mathrm{Lip}}(Y, Z)$ *with* $\psi(o_Y) = o_Z$. *Then* $\psi \circ u \in H^{1,p}(U, Z; \mathfrak{m})$ *and*

$$\mathsf{Ch}(\psi \circ u) \le \mathrm{Lip}(\psi)^p \mathsf{Ch}(u).$$

(2) *For $u, v \in H^{1,p}(U, Y; \mathfrak{m})$ and $q \in Y$, we have $d_Y(u, o)$, $d_Y(u, v)$, $d_Y(u, q) - d_Y(o, q) \in H^{1,p}(U, \mathbb{R}; \mathfrak{m})$ and*

$$\mathsf{Ch}(d_Y(u, o)) \le \mathsf{Ch}(u), \quad \mathsf{Ch}(d_Y(u, q) - d_Y(o, q)) \le \mathsf{Ch}(u), \tag{5.12}$$

$$\mathsf{Ch}(d_Y(u, v))^{1/p} \le \mathsf{Ch}(u)^{1/p} + \mathsf{Ch}(v)^{1/p}. \tag{5.13}$$

In particular, we have $d_Y(u, q) \in H^{1,p}(U, Y; \mathfrak{m})$ and $\mathsf{Ch}(d_Y(u, q)) \le \mathsf{Ch}(u)$ for $u \in H^{1,p}(U, Y; \mathfrak{m})$ and $q \in Y$ if $\mathfrak{m}(U) < \infty$.

(3) *If $f_1, f_2 \in H^{1,p}(U, \mathbb{R}; \mathfrak{m}) \cap L^\infty(U; \mathfrak{m})$, then $f_1 f_2 \in H^{1,p}(U, \mathbb{R}; \mathfrak{m}) \cap L^\infty(U; \mathfrak{m})$ and*

$$\mathsf{Ch}(f_1 f_2)^{\frac{1}{p}} \le \|f_1\|_\infty \mathsf{Ch}(f_2)^{\frac{1}{p}} + \|f_2\|_\infty \mathsf{Ch}(f_1)^{\frac{1}{p}}.$$

Proof. The proof is straightforward by using the existence of the minimal generalized upper gradient. For (3), we use [6, Lemma 1.7]. We omit the details. □

For $p \in]1, +\infty[$, we define a distance $d_{H^{1,p}}$ on $H^{1,p}(U, Y; \mathfrak{m})$: for $u, v \in H^{1,p}(U, Y; \mathfrak{m})$,

$$d_{H^{1,p}}(u, v) := d_{L^p}(u, v) + \|g_u - g_v\|_{L^p(U;\mathfrak{m})}, \tag{5.14}$$

where g_u, g_v is the minimal generalized upper gradient for $u, v \in H^{1,p}(U, Y; \mathfrak{m})$, respectively. Let $(\overline{H}^{1,p}(U, Y; \mathfrak{m}), d_{\overline{H}^{1,p}})$ be the completion of $(H^{1,p}(U, Y; \mathfrak{m}), d_{H^{1,p}})$.

The following assertion is not stated clearly in [24]. We provide its proof for completeness.

Theorem 5.15. *Suppose that (Y, d_Y) is a Busemann's NPC space. Let $p \in]1, +\infty[$. We have $\overline{H}^{1,p}(U, Y; \mathfrak{m}) = H^{1,p}(U, Y; \mathfrak{m})$.*

Remark 5.16. Theorem 5.15 does not necessarily imply the $d_{H^{1,p}}$-completeness of $H^{1,p}(U, Y; \mathfrak{m})$, that is, $d_{\overline{H}^{1,p}} = d_{H^{1,p}}$ on $H^{1,p}(U, Y; \mathfrak{m})$ may not hold.

Proof of Theorem 5.15. Any element in $\overline{H}^{1,p}(U, Y; \mathfrak{m})$ is an equivalence class of $d_{H^{1,p}}$-Cauchy sequence of the elements in $H^{1,p}(U, Y; \mathfrak{m})$, where the equivalence is defined to be $d_{H^{1,p}}(u_n, v_n) \to 0$ as $n \to +\infty$ for $d_{H^{1,p}}$-Cauchy sequences $\{u_n\}$ and $\{v_n\}$ in $H^{1,p}(U, Y; \mathfrak{m})$. In particular, for each $u \in \overline{H}^{1,p}(U, Y; \mathfrak{m})$, there exists a $d_{H^{1,p}}$-Cauchy sequence $\{u_n\}$ of $H^{1,p}(U, Y; \mathfrak{m})$ such that $\{u_n\}$ converges to u as $n \to +\infty$ in $L^p(U, Y; \mathfrak{m})$. Let g_{u_n} be the unique minimal generalized upper gradient for u_n. Then g_{u_n} converges to some $g \in L^p(U; \mathfrak{m})$ in $L^p(U; \mathfrak{m})$ as $n \to +\infty$. By Theorem 5.6, we have

$$\mathsf{Ch}(u) \le \varliminf_{n\to+\infty} \mathsf{Ch}(u_n) = \lim_{n\to+\infty} \|g_{u_n}\|^p_{L^p(U;\mathfrak{m})} = \|g\|^p_{L^p(U;\mathfrak{m})} < +\infty,$$

which means $u \in H^{1,p}(U, Y; \mathfrak{m})$. Therefore, we obtain $\overline{H}^{1,p}(U, Y; \mathfrak{m}) = H^{1,p}(U, Y; \mathfrak{m})$. □

For functions f, g on U, we define $f \vee g$ (resp. $f \wedge g$) by $(f \vee g)(x) := \max\{f(x), g(x)\}$ (resp. $(f \wedge g)(x) := \min\{f(x), g(x)\}$).

Lemma 5.17. *We have the following:*

(1) *For $f_1, f_2 \in H^{1,p}(U)$, $f_1 \wedge f_2, f_1 \vee f_2 \in H^{1,p}(U)$ and*

$$\begin{aligned} g_{f_1\wedge f_2} &= \mathbf{1}_{\{f_1\le f_2\}} g_{f_1} + \mathbf{1}_{\{f_1>f_2\}} g_{f_2} \quad \mathfrak{m}\text{-a.e. on } U, \\ &= \mathbf{1}_{\{f_1<f_2\}} g_{f_1} + \mathbf{1}_{\{f_1\ge f_2\}} g_{f_2} \quad \mathfrak{m}\text{-a.e. on } U, \\ g_{f_1\vee f_2} &= \mathbf{1}_{\{f_1\le f_2\}} g_{f_2} + \mathbf{1}_{\{f_1>f_2\}} g_{f_1} \quad \mathfrak{m}\text{-a.e. on } U, \\ &= \mathbf{1}_{\{f_1<f_2\}} g_{f_2} + \mathbf{1}_{\{f_1\ge f_2\}} g_{f_1} \quad \mathfrak{m}\text{-a.e. on } U, \end{aligned}$$

in particular, $g_{f_1} = g_{f_2}$ $\mathfrak{m}$-a.e. on $\{f_1 = f_2\}$.

(2) *Suppose that $\{f_n\} \subset H^{1,p}(U)$ (resp. $\{g_n\} \subset H^{1,p}(U)$) $d_{H^{1,p}}$-converges to $f \in H^{1,p}(U)$ (resp. $g \in H^{1,p}(U)$). Then $\{f_n \wedge g_n\}$ (resp. $\{f_n \vee g_n\}$) $d_{H^{1,p}}$-converges to $f \wedge g$ (resp. $f \vee g$).*

Proof. (1) is a direct consequence that for a Borel set A and $f_1, f_2 \in H^{1,p}(U)$

$$f_1 = f_2 \quad \mathfrak{m}\text{-a.e. on } A \quad \text{implies} \quad g_{f_1} = g_{f_2} \quad \mathfrak{m}\text{-a.e. on } A$$

(see [6, Proposition 1.5, Corollaries 2.25 and 2.26]). It suffices to prove (2) for the L^p-convergence of $\{g_{f_n\wedge g_n}\}$ to $g_{f\wedge g}$, because $g_{f_n\vee g_n} = g_{(-f_n)\wedge(-g_n)}$ $\mathfrak{m}$-a.e. on U. We may assume $\{g_{f_n}\}$ (resp. $\{g_{g_n}\}$) $\mathfrak{m}$-a.e. converges to g_f (resp. g_g) by taking a further subsequence. From (1), we have

$$\begin{aligned} g_{f_n\wedge g_n} - g_{f\wedge g} &= \mathbf{1}_{\{f_n\le g_n\}}(g_{f_n} - g_{g_n}) - \mathbf{1}_{\{f\le g\}}(g_f - g_g) + g_{g_n} - g_g \\ &= \mathbf{1}_{\{f_n\le g_n\}}(g_{f_n} - g_f - g_{g_n} + g_g) + (\mathbf{1}_{\{f_n\le g_n\}} - \mathbf{1}_{\{f\le g\}})(g_f - g_g) + g_{g_n} - g_g. \end{aligned}$$

Since

$$\begin{aligned} \overline{\lim_{n\to\infty}}\, \mathbf{1}_{\{f_n\le g_n\}}|g_f - g_g| \le \mathbf{1}_{\{f\le g\}}|g_f - g_g| &= \mathbf{1}_{\{f<g\}}|g_f - g_g| \\ &= \underline{\lim_{n\to\infty}}\, \mathbf{1}_{\{f_n<g_n\}}|g_f - g_g| \\ &\le \underline{\lim_{n\to\infty}}\, \mathbf{1}_{\{f_n\le g_n\}}|g_f - g_g| \quad \mathfrak{m}\text{-a.e. on } U, \end{aligned}$$

we have $\lim_{n\to\infty} \mathbf{1}_{\{f_n\le g_n\}}|g_f - g_g| = \mathbf{1}_{\{f\le g\}}|g_f - g_g|$ $\mathfrak{m}$-a.e. on U. Then, Lebesgue's dominated convergence theorem yields $\lim_{n\to\infty} \|(\mathbf{1}_{\{f_n\le g_n\}} - \mathbf{1}_{\{f\le g\}})(g_f - g_g)\|_{L^p} = 0$ so that $\lim_{n\to\infty} \|g_{f_n\wedge g_n} - g_{f\wedge g}\|_{L^p} = 0$. □

Let $H^{1,p}_0(U)$ be the $H^{1,p}$-closure of $\{f \in H^{1,p}(U) \mid \operatorname{supp}[f] \subset U\}$.

Lemma 5.18. *Take $f, g \in H^{1,p}(U)$. Suppose $0 \le f \le g$ $\mathfrak{m}$-a.e. on U and $g \in H^{1,p}_0(U)$. Then $f \in H^{1,p}_0(U)$.*

Proof. Let $\{g_i\} \subset H^{1,p}_0(U)$ be a $d_{H^{1,p}}$-approximating sequence to g satisfying $\operatorname{supp}[g_i] \subset U$. Then $\{f \wedge |g_i|\}$ is a $d_{H^{1,p}}$-approximating sequence to $f \wedge g = f$ satisfying $\operatorname{supp}[f \wedge |g_i|] \subset U$ by Lemma 5.17(2). □

For $v \in H^{1,p}(U, Y; \mathfrak{m})$, we define

$$H^{1,p}_v(U, Y; \mathfrak{m}) := \{u \in H^{1,p}(U, Y; \mathfrak{m}) \mid d_Y(u, v) \in H^{1,p}_0(U)\}.$$

In particular, we set

$$H^{1,p}_o(U, Y; \mathfrak{m}) := \{u \in H^{1,p}(U, Y; \mathfrak{m}) \mid d_Y(u, o) \in H^{1,p}_0(U)\}.$$

By Lemma 5.18, it is easy to see

$$H^{1,p}_o(U, Y; \mathfrak{m}) := \{u \in H^{1,p}(U, Y; \mathfrak{m}) \mid d_Y(u, q) - d_Y(o, q) \in H^{1,p}_0(U) \text{ for any } q \in Y\}.$$

Lemma 5.19. *Suppose that (Y, d_Y) is a NPC space of Busemann type and $p \in]1, +\infty[$. Then $H^{1,p}_v(U, Y; \mathfrak{m})$ is a closed convex subset of $H^{1,p}(U, Y; \mathfrak{m})$ with respect to $d_{H^{1,p}}$.*

Proof. Suppose that $\{u_n\} \subset H^{1,p}_v(U, Y; \mathfrak{m})$ converges to $u \in H^{1,p}(U, Y; \mathfrak{m})$ with respect to $d_{H^{1,p}}$-distance. By definition, $\{d_Y(u_n, v)\} \subset H^{1,p}_0(U)$ $L^p(U; \mathfrak{m})$-converges to $d_Y(u, v)$. Let $\mathsf{Ch}_U(f)$ be the extended Cheeger energy of function $f \in L^p(U; \mathfrak{m})$ defined by

$$\mathsf{Ch}_U(f) := \begin{cases} \mathsf{Ch}(f), & f \in H^{1,p}_0(U), \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the completeness of $(H^{1,p}_0(U), \|\cdot\|_{H^{1,p}})$ implies the lower semi continuity of the extended energy:

$$\begin{aligned} \mathsf{Ch}_U(d_Y(u, v))^{\frac{1}{p}} &\le \underline{\lim_{n\to\infty}}\, \mathsf{Ch}_U(d_Y(u_n, v))^{\frac{1}{p}} \\ &\le \underline{\lim_{n\to\infty}}\, \mathsf{Ch}(u_n)^{\frac{1}{p}} + \mathsf{Ch}(v)^{\frac{1}{p}} < \infty, \end{aligned}$$

which shows $d_Y(u, v) \in H^{1,p}_0(U)$, that is, $u \in H^{1,p}_v(U, Y; \mathfrak{m})$. Here we use Lemma 5.11. Next we prove the convexity of $H^{1,p}_v(U, Y; \mathfrak{m})$. Take $u_0, u_1 \in H^{1,p}_v(U, Y; \mathfrak{m})$. Let $u_t(x)$ be the geodesics in Y joining $u_0(x)$ and $u_1(x)$. Lemma 5.9 shows $u_t \in H^{1,p}(U, Y; \mathfrak{m})$. On the other hand, $d_Y(u_t, v) \le (1-t)d_Y(u_0, v) + t d_Y(u_1, v)$ implies $d_Y(u_t, v) \in H^{1,p}_0(U)$, hence $u_t \in H^{1,p}_v(U, Y; \mathfrak{m})$. □

It is easy to see $H^{1,p}_{\phi}(U,Y;\mathfrak{m}) = H^{1,p}_{\psi}(U,Y;\mathfrak{m})$ if $\phi \in H^{1,p}(U,Ym)$ and $\psi \in H^{1,p}_{\phi}(U,Y;\mathfrak{m})$.

Definition 5.20 (Harmonic map). A map $\phi \in H^{1,p}(U,Y;\mathfrak{m})$ is said to be *harmonic* if it satisfies $\mathsf{Ch}(\phi) = \inf_{u\in H^{1,p}_{\phi}(U,Y;\mathfrak{m})}\mathsf{Ch}(u)$.

Hereafter, we assume $\partial U \neq \emptyset$ and fix $\phi \in H^{1,p}(U,Y;\mathfrak{m})$. For $u_0 \in L^p_o(U,Y;\mathfrak{m})$, we put

$$\mathsf{Ch}^{\tau,\phi}(u_0) := \inf\left\{\mathsf{Ch}(w) + \frac{1}{p\tau^{p-1}} d^p_{L^p}(w,u_0) \;\middle|\; w \in H^{1,p}_{\phi}(U,Y;\mathfrak{m})\right\}. \tag{5.21}$$

Theorem 5.22 (Harmonic map for initial boundary value problem). *Suppose that (Y,d_Y) is a complete p-uniformly convex space with parameter $k \in]0,c_p]$ with $p \in [2,+\infty[$ having the NPC property of Busemann type. Take $\phi \in H^{1,p}(U,Y;\mathfrak{m})$. Then we have the following:*

(1) For $\tau > 0$ and $u_0 \in L^p(U,Y;\mathfrak{m})$, there exists a unique minimizer $J_\tau(u_0) \in H^{1,p}_{\phi}(U,Y;\mathfrak{m})$ for $\mathsf{Ch}^{\tau,\phi}(u_0)$.

(2) Take $u_0 \in L^p(U,Y;\mathfrak{m})$. Suppose that there exists a sequence $\{\tau_n\}$ satisfying $\tau_n \to \infty$ as $n \to \infty$ and that $\{d_{L^p}(J_{\tau_n}(u_0),u_0)\}$ is bounded. Then, for $u_0 \in L^p(U,Y;\mathfrak{m})$, $\bar{u}_0 = \lim_{\tau\to\infty} J_\tau(u_0)$ exists in $H^{1,p}_{\phi}(U,Y;\mathfrak{m})$ and $\mathsf{Ch}(\bar{u}_0) = \lim_{\tau\to\infty}\mathsf{Ch}(J_\tau(u_0))$, and it is a harmonic map in the sense of Definition 5.20.

(3) Suppose that there exists a non-decreasing upper semicontinuous function $\varphi : [0,+\infty[\to [0,+\infty[$ such that for any $f \in H^{1,p}_0(U)$,

$$\|f\|_{L^p(U;\mathfrak{m})} \le \varphi\left(\mathsf{Ch}(f)\right).$$

Then, the same conclusion as in (2) holds.

(4) Suppose that the target space (Y,d_Y) is a complete CAT(0)-space and for $u_0 \in L^p(U,Y;\mathfrak{m})$, there exists an $\varepsilon > 0$ such that

$$\|d_Y(u_0,\phi)\|_{L^\infty(U\setminus U_\varepsilon;\mathfrak{m})} < \infty,$$

where $U_\varepsilon := \{x \in U \mid d(x,U^c) \ge \varepsilon\}$. Then, the same conclusion as in (2) holds.

Remark 5.23. Theorem 5.22(3) extends [25, Theorem 9.6].

The following corollary is a direct consequence of Theorem 5.22, because any CAT(0)-space is p-uniformly convex for any $p \ge 2$ and is a NPC space of Busemann type.

Corollary 5.24. *Suppose that (Y,d_Y) is a complete CAT(0)-space and $p \ge 2$. Then the same conclusion as in Theorem 5.22 holds.*

Proof of Theorem 5.22. (1): Recall that $(L^p(U,Y;\mathfrak{m}),d_{L^p})$ is a complete p-uniformly convex space with parameter $k \in]0,c_p]$ with $p \in [2,+\infty[$ having the NPC property of Busemann type by Theorem 5.1. The minimizing sequence $\{u_i\} \subset H^{1,p}_{\phi}(U,Y;\mathfrak{m})$ for $\mathsf{Ch}^{\tau,\phi}(u_0)$ forms a d_{L^p}-Cauchy sequence, so that it converges to some $u_\tau \in L^p(U,Y;\mathfrak{m})$. By the proof of Theorem 5.10 (see [24, Theorem 3.2]), we obtain that $\{g_{u_i}\}$ is an L^p-Cauchy sequence, so that it converges to some $g \in L^p(U;\mathfrak{m})$. Remark that g is a generalized upper gradient for u_τ. Therefore we have $u_\tau \in H^{1,p}(U,Y;\mathfrak{m})$ and $\mathsf{Ch}(u_\tau) \le \|g\|^p_{L^p}$. If $\mathsf{Ch}(u_\tau) = \|g\|^p_{L^p}$, then $g = g_u$ $\mathfrak{m}$-a.e. on U from Theorem 5.10 and hence $u_i \to u_\tau$ as $i \to \infty$ with respect to $d_{H^{1,p}}$ so that $u_\tau \in H^{1,p}_{\phi}(U,Y;\mathfrak{m})$ by Lemma 5.19. Assume $\mathsf{Ch}(u_\tau) < \|g\|^p_{L^p}$ and take a sequence $\{(u_i',g_i)\}_{i=1}^\infty$ such that $u_i' \to u$, $g_i \to g$ in $L^p(U,Y;\mathfrak{m})$, $L^p(U;\mathfrak{m})$, respectively as $i \to \infty$, and that g_i is an upper gradient for u_i'. We may assume $d_{L^p}(u_i,u), d_{L^p}(u_i',u) < i^{-2}$ for any $i \in \mathbb{N}$. Let ϕ_i be a Lipschitz function on U such that $\phi_i \equiv 1$ on $U_{2i^{-1}}$, $\phi_i \equiv 0$ on $U \setminus U_{i^{-1}}$, and its Lipschitz constant is not greater than i. Put $u_i'' := (1-\phi_i)u_i + \phi_i u_i'$. Since $u_i'' = u_i$ on $U\setminus U_{i^{-1}}$, we have $u_i'' \in H^{1,p}_{\phi}(U,Y;\mathfrak{m})$. Moreover,

$$d_{L^p}(u_i'',u) \le \|d_Y(u_i,u) \vee d_Y(u_i',u)\|_{L^p} \to 0 \quad \text{as} \quad i \to \infty.$$

From [24, Lemma 3.3], for any $\varepsilon > 0$, we have

$$\|g_{u_i''}\|_{L^p} \le \|\mathrm{Lip}(\phi_i)\cdot d_Y(u_i,u_i') + (1-\phi_i+\varepsilon)g_{u_i} + (\phi_i+\varepsilon)g_i\|_{L^p}$$

$$\leq i d_{L^p}(u_i, u_i') + \varepsilon(\|g_{u_i}\|_{L^p} + \|g_i\|_{L^p}) + \|g_{u_i}\|_{L^p(U\setminus U_{2i^{-1}})} + \|g_i\|_{L^p(U_{i^{-1}})}.$$

Letting $\varepsilon \to 0$, we obtain

$$\|g_{u_i''}\|_{L^p} \leq 2i^{-1} + 2\|g_{u_i}\|_{L^p(U\setminus U_{2i^{-1}})} + \|g_i\|_{L^p(U_{i^{-1}})} \to \|g_u\|_{L^p(U)} \quad \text{as} \quad i \to \infty.$$

Since $\|g_u\|_{L^p}^p = \mathsf{Ch}(u) < \|g\|_{L^p}^p$, for sufficiently large i, it holds that

$$\mathsf{Ch}(u_i'') + \frac{1}{p\tau^{p-1}} d_{L^p}^p(u_i'', u_0) < \|g\|_{L^p}^p + \frac{1}{p\tau^{p-1}} d_{L^p}^p(u, u_0) = \mathsf{Ch}^{\tau,\phi}(u_0).$$

This contradicts $u_i'' \in H_\phi^{1,p}(U, Y; \mathfrak{m})$. Therefore $u_\tau \in H_\phi^{1,p}(U, Y; \mathfrak{m})$ and it attains $\mathsf{Ch}^{\tau,\phi}(u_0)$. The uniqueness is easily follows the strong convexity of $d_{L^p}^p(\cdot, u_0)$ together with the convexity of Ch.

(2): By assumption, the sequence $\{J_{\tau_n}(u_0)\}_{n\in\mathbb{N}}$ forms a minimizing sequence of Ch in $H_\phi^{1,p}(U, Y; \mathfrak{m})$ and $\{J_\tau(u_0)\}_{\tau>0}$ forms a Cauchy sequence as $\tau \to \infty$ in (L^p, d_{L^p}) as in the proof of Theorem 3.46 (see [11, Theorem 3.1.1]). Hence $\{J_\mu(u_0)\}_{n\in\mathbb{N}}$ converges to some $\bar{u}_0 \in L^p(U, Y; \mathfrak{m})$ $\mu \to +\infty$, and by (5.21) we have

$$\mathsf{Ch}(\bar{u}_0) \leq \varliminf_{\mu\to\infty} \mathsf{Ch}(J_\mu(u_0)) = \inf_{v\in H_\phi^{1,p}(U,Y;\mathfrak{m})} \mathsf{Ch}(v).$$

By the similar discussion of (1), we obtain that $\{g_{J_\mu(u_0)}\}$ is a Cauchy sequence in $L^p(U; \mathfrak{m})$, $g_{J_\mu(u_0)} \to g_{\bar{u}_0}$ in $L^p(U; \mathfrak{m})$ and $J_{\mu_n}(u_0) \to \bar{u}_0$ with respect to $d_{H^{1,p}}$. Consequently we have $\bar{u}_0 \in H_\phi^{1,p}(U, Y; \mathfrak{m})$.

(3): Take $\tau > 0$ and a sequence $\{u_i\} \subset H^{1,p}(U, Y; \mathfrak{m})$ such that $\mathrm{supp}[d_Y(u_i, \phi)] \subset U$ and $u_i \to J_\tau(u_0)$ with respect to $d_{H^{1,p}}$, so that $g_{d_Y(u_i,\phi)} \leq g_{u_i} + g_\phi$ $\mathfrak{m}$-a.e. on U from [24, Corollary 3.5]. Since $d_Y(u_i, \phi) \in H_0^{1,p}(U)$, by assumption, we have

$$d_{L^p}(u_i, \phi) \leq \varphi\left(\|g_{d_Y(u_i,\phi)}\|_{L^p(U;\mathfrak{m})}\right) \leq \varphi\left(\|g_{u_i}\|_{L^p(U;\mathfrak{m})} + \|g_\phi\|_{L^p(U;\mathfrak{m})}\right).$$

Letting $i \to \infty$ with the upper semi continuity of φ, we have

$$d_{L^p}(J_\tau(u_0), \phi) \leq \varphi\left(\mathsf{Ch}(J_\tau(u_0)) + \mathsf{Ch}(\phi)\right) \leq \varphi\left(2\mathsf{Ch}(\phi) + \frac{1}{p\tau^{p-1}} d_{L^p}^p(\phi, u_0)\right),$$

because $\mathsf{Ch}(J_\tau(u_0)) + \frac{1}{p\tau^{p-1}} d_{L^p}^p(J_\tau(u_0), u_0) \leq \mathsf{Ch}(\phi) + \frac{1}{p\tau^{p-1}} d_{L^p}^p(\phi, u_0)$. Thus, we obtain $\sup_{\tau>0} d_{L^p}(J_\tau(u_0), u_0) = \sup_{\tau\geq 1} d_{L^p}(J_\tau(u_0), u_0) < \infty$, which implies the assertion by (2).

(4): For $R > 0$, define $R^z : Y \to Y$ by $R^z(y) := y$ under $d_Y(y, u_0(z)) \leq R$ and $R^z(y) := (1 - \frac{R}{d_Y(y,u_0(z))})u_0(z) + \frac{R}{d_Y(y,u_0(z))} y$ under $d_Y(y, u_0(z)) > R$. Then by [24, Lemma 4.5], $d_Y(R^z(y_1), R^z(y_2)) \leq d_Y(y_1, y_2)$ for $y_1, y_2 \in Y$. By definition, $R^z(u_0(z)) = u_0(z)$. For $u \in L^p(U, Y; \mathfrak{m})$, we set $(Ru)(z) := R^z(u(z))$. Then $Ru \in L^p(U, Y; \mathfrak{m})$. Indeed, under the measurability (resp. Borel measurability) of u, the measurability (resp. Borel measurability) of Ru is obvious from the continuity of $(\gamma_0, \gamma_1) \mapsto \gamma_t$ for minimal geodesic $\gamma : [0, 1] \to Y$ and $d_{L^p}(Ru, u_0) \leq d_{L^p}(Ru, Ru_0) \leq d_{L^p}(u, u_0) < \infty$. Moreover, for $u \in H^{1,p}(U, Y; \mathfrak{m})$ we have $Ru \in H^{1,p}(U, Y; \mathfrak{m})$, $\mathsf{Ch}(Ru) \leq \mathsf{Ch}(u)$ and $g_{Ru} \leq g_u$ $\mathfrak{m}$-a.e. on U in the same way of [24, Lemma 4.6]. Set $R := \|d_Y(u_0, \phi)\|_{L^\infty(U\setminus U_\varepsilon;\mathfrak{m})}$. Then $(R\phi)(z) = R^z(\phi(z)) = \phi(z)$ for $\mathfrak{m}$-a.e. $z \in U \setminus U_\varepsilon$ implies $d_Y(R\phi, \phi) \in H_0^{1,p}(U)$. Then $d_Y(RJ_\tau(u_0), \phi) \leq d_Y(RJ_\tau(u_0), R\phi) + d_Y(R\phi, \phi) \leq d_Y(J_\tau(u_0), \phi) + d_Y(R\phi, \phi)$ with Lemma 5.18 yields $d_Y(RJ_\tau(u_0), \phi) \in H_0^{1,p}(U)$. Therefore $RJ_\tau(u_0) \in H_\phi^{1,p}(U, Y; \mathfrak{m})$ is the unique minimizer of (5.21) so that $RJ_\tau(u_0) = J_\tau(u_0)$ $\mathfrak{m}$-a.e. on U, hence $d_Y(J_\tau(u_0), u_0) \leq R = \|d_Y(u_0, \phi)\|_{L^\infty(U\setminus U_\varepsilon;\mathfrak{m})}$ $\mathfrak{m}$-a.e. on U. Therefore $\sup_{\tau>0} d_{L^p}(J_\tau(u_0), u_0) \leq \|d_Y(u_0, \phi)\|_{L^\infty(U\setminus U_\varepsilon;\mathfrak{m})}\mathfrak{m}(U)^{1/p} < \infty$. □

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