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Incidence Axioms for the Boundary at Infinity of Complex Hyperbolic Spaces

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Abstract: We characterize the boundary at infinity of a complex hyperbolic space as a compact Ptolemy space that satisfies four incidence axioms.

Keywords: complex hyperbolic spaces; Ptolemy spaces; incidence axioms

MSC: 53C35, 53C23

1 Introduction

We characterize the boundary at infinity of a complex hyperbolic space $\mathbb{C}\,\mathrm{H}^k$, $k \geq 1$, as a compact Ptolemy space that satisfies four incidence axioms.

Recall that a Möbius structure on a set X, or a Möbius space over X, is a class of Möbius equivalent metrics on X, where two metrics are equivalent if they have the same cross-ratios. A Ptolemy space is a Möbius space with the property that the metric inversion operation preserves the Möbius structure, that is, the function

$$d_\omega(x,y) = \frac{d(x,y)}{d(x,\omega)\cdot d(y,\omega)} \tag{1}$$

satisfies the triangle inequality for every metric d of the Möbius structure and every $\omega \in X$. A basic example of a Ptolemy space is the boundary at infinity of a rank one symmetric space M of noncompact type taken with the canonical Möbius structure, see sect. 2.1. In the case $M = \mathrm{H}^{k+1}$ is a real hyperbolic space, the boundary at infinity $\partial_\infty M$ taken with the canonical Möbius structure is Möbius equivalent to an extended Euclidean space $\widehat{\mathbb{R}}^k = \mathbb{R}^k \cup \{\infty\}$.

1.1 Incidence axioms

Basic objects of our axiom system are $\mathbb{R}$-circles, $\mathbb{C}$-circles and harmonic 4-tuples in a Möbius space X. Any $\mathbb{R}$-circle is Möbius equivalent to the extended real line $\widehat{\mathbb{R}}$ and any $\mathbb{C}$-circle is the square root of $\widehat{\mathbb{R}}$. In other words, an $\mathbb{R}$-circle can be identified with $\partial_\infty \mathrm{H}^2$, and a $\mathbb{C}$-circle with $\partial_\infty(\frac{1}{2}\mathrm{H}^2)$, where the hyperbolic plane $\frac{1}{2}\mathrm{H}^2$ has constant curvature -4.

An (ordered) 4-tuple $(x,z,y,u) \in X^4$ of pairwise distinct points is *harmonic* if

$$d(x,z)d(y,u) = d(x,u)d(y,z)$$

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for some and hence for any metric d of the Möbius structure.[1] We use the notation Harm_A for the set of harmonic 4-tuples in $A \subset X$. In the case $A = X$ we abbreviate to Harm. In the case $X = \partial_\infty M$ for $M = \mathrm{H}^2$ or $M = \frac{1}{2}\mathrm{H}^2$ a 4-tuple (x, z, y, u) is harmonic if and only if the geodesic lines $xy, zu \subset M$ intersect each other orthogonally.

We consider the following axioms.

(E) Existence axioms

($\mathrm{E}_\mathbb{C}$) Through every two distinct points in X there is a unique $\mathbb{C}$-circle.
($\mathrm{E}_\mathbb{R}$) For every $\mathbb{C}$-circle $F \subset X$, $\omega \in F$ and $u \in X \setminus F$ there is a unique $\mathbb{R}$-circle $\sigma \subset X$ through ω, u that intersects $F_\omega = F \setminus \{\omega\}$.

(O) Orthogonality axioms:
For every $\mathbb{R}$-circle σ and every $\mathbb{C}$-circle F with common distinct points o, ω the following holds

($\mathrm{O}_\mathbb{C}$) given $u, v \in \sigma$ such that $(o, u, \omega, v) \in \mathrm{Harm}_\sigma$, the 4-tuple (z, u, o, v) is harmonic for every $z \in F$,
($\mathrm{O}_\mathbb{R}$) given $x, y \in F$ such that $(o, x, \omega, y) \in \mathrm{Harm}_F$, the 4-tuple (z, x, ω, y) is harmonic for every $z \in \sigma$.

1.2 Main result

Our main result is the following

Theorem 1.1. *Let X be a compact Ptolemy space that satisfies axioms (E) and (O). Then X is Möbius equivalent to the boundary at infinity of a complex hyperbolic space $\mathbb{C}\mathrm{H}^k$, $k \geq 1$, taken with the canonical Möbius structure.*

In [2] we have obtained a Möbius characterization of the boundary at infinity of any rank one symmetric space assuming existence of sufficiently many Möbius automorphisms. In contrast to [2], there is no assumption in Theorem 1.1 on automorphisms of X, and one of the key problems with Theorem 1.1 is to establish existence of at least one nontrivial automorphism. In this respect, Theorem 1.1 is an analog for complex hyperbolic spaces of a result obtained in [5] for real hyperbolic spaces: every compact Ptolemy space such that through any three points there exists an $\mathbb{R}$-circle is Möbius equivalent to $\widehat{\mathbb{R}}^k = \partial_\infty \mathrm{H}^{k+1}$.

The model space $Y = \partial_\infty M$, $M = \mathbb{C}\mathrm{H}^k$, serves for motivation and illustration of various notions and constructions used in the proof of Theorem 1.1. These are reflections with respect to $\mathbb{C}$-circles, pure homotheties, orthogonal complements to $\mathbb{C}$-circles, suspensions over orthogonal complements, Möbius joins. In sect. 2, we describe the canonical Möbius structure on Y, and show that this structure satisfies axioms (E), (O).

Now, we briefly describe the logic of the proof. The first important step is Proposition 4.1 which leads to existence for every $\mathbb{C}$-circle $F \subset X$ of an involution $\varphi_F : X \to X$ whose fixed point set is F, and such that every $\mathbb{R}$-circle $\sigma \subset X$ intersecting F at two points is φ_F-invariant, $\varphi_F(\sigma) = \sigma$. The involution φ_F is called the reflection with respect to F.

In the second step, crucial for the paper, we obtain a distance formula, see Proposition 5.2, which is our basic tool. Using the distance formula, we establish existence of nontrivial Möbius automorphisms $X \to X$ beginning with vertical shifts, see sect. 7.1, and then proving that reflections with respect to $\mathbb{C}$-circles and pure homotheties are Möbius, sect. 7.4 and 7.5.

Next, in sect. 8 we introduce the notion of an orthogonal complement $A = (F, \eta)^\perp$ to a $\mathbb{C}$-circle $F \subset X$ at a Möbius involution $\eta : F \to F$ without fixed points. In the case of the model space Y, A is the boundary at infinity of the orthogonal complement to the complex hyperbolic plane $E \subset M$ with $\partial_\infty E = F$ at $a \in E$, $\eta = \partial_\infty s_a$, where $s_a : E \to E$ is the central symmetry with respect to a.

In the third step, we show that for any mutually orthogonal $\mathbb{C}$-circles $F, F' \subset X$ the intersection $A \cap A'$ of their orthogonal complements, when nonempty, satisfies axioms (E), (O), see sect. 8.4. This gives a possibility to proceed by induction on dimension.

1 This notion in the case of $\widehat{\mathbb{R}}$ was introduced by Karl von Staudt in his book “Geometrie der Lage”, 1847.

To do that, we introduce in sect. 9 the notion of a Möbius join $F * F' \subset X$ of a $\mathbb{C}$-circle F and its canonical orthogonal subspace $F' \subset (F,\eta)^{\perp}$. It turns out that $X = F * A$ for any $A = (F,\eta)^{\perp}$. All what remains to prove is that whenever F' is Möbius equivalent to $\partial_{\infty}\mathbb{C}\,\mathrm{H}^{k-1}$, the Möbius join $F * F'$ is equivalent to $\partial_{\infty}\mathbb{C}\,\mathrm{H}^{k}$. The distance formula works perfectly in the case $\dim F' = 1$, i.e., $F' \subset (F,\eta)^{\perp}$ is a $\mathbb{C}$-circle, however, if $\dim F' > 1$, it does not work directly. Thus we proceed in two steps. First, we prove that $F * F' = \partial_{\infty}\mathbb{C}\,\mathrm{H}^2$ for any mutually orthogonal $\mathbb{C}$-circles $F, F' \subset X$, in particular, this proves Theorem 1.1 in the case $\dim X = 3$. Then using this fact as a power tool, we establish that the base B_{ω} of the canonical foliation $X_{\omega} \to B_{\omega}$, $\omega \in X$, see sect. 6, is Euclidean. Using this, we finally prove the general case for $\dim F' > 1$.

1.3 Potential applications

Here we indicate a direction where a characterization as that presented in Theorem 1.1 could be useful. It is known that a number of important problems in low dimensional topology can be formulated in terms of Möbius geometry: given a group Γ with convergence action on the (topological) sphere S^k, $k = 1, 2$, show that the action of Γ is conjugate to one that preserves the canonical Möbius structure on $\widehat{\mathbb{R}}^k = \mathbb{R}^k \cup \{\infty\} \approx S^k$. This includes Nilsen realization problem, Seifert fiber space problem, Fuchsian group problem (the case $k = 1$), Cannon conjecture (the case $k = 2$), for more detail see e.g. P. Scott's review (MR1296353 96f:57011) of A. Casson and D. Jungreis' paper. The first three problems are solved, and their solutions were great achievements. The last problem asking whether a hyperbolic group with 2-spherical boundary at infinity is actually an isometry group of the real hyperbolic space H^3 is still open. Approaches related to a metric deformation (J. Cannon and coauthors, M. Bonk and B. Kleiner) did not succeeded as well as attempts to extend the action of Γ to the filling ball (V. Markovich).

On the other hand, there is a result by I. Mineyev [9] saying that the boundary at infinity of every Gromov hyperbolic group Γ possesses a Ptolemaic Möbius structure (PM-structure for brevity) invariant under Γ that induces on $\partial_{\infty}\Gamma$ the standard topology.

This result is a hint to approach Cannon conjecture by deforming a Möbius structure (e.g. constructed by Mineyev) to the canonical Möbius structure on $\widehat{\mathbb{R}}^2$. An obvious problem with this approach is how to recognize the canonical structure. In this respect, a characterization obtained in [5], which we mentioned above, looks very promising: it suffices to check that every three points lie on a Ptolemy circle.

Similarly, let Γ be a Gromov hyperbolic group with $(2k-1)$-spherical boundary at infinity, $k \geq 2$. By Mineyev's theorem, there is a Γ-invariant PM-structure $\mathcal{M}$ on $\partial_{\infty}\Gamma$ such that the $\mathcal{M}$-topology coincides with the standard one of S^{2k-1}. One can ask the following question. Assume $\mathcal{M}$ is sufficiently close (in a certain sense) to the canonical PM-structure of $\partial_{\infty}\mathbb{C}\,\mathrm{H}^k \approx S^{2k-1}$. Is it true (cp. [4]) that Γ is actually an isometry group of $\mathbb{C}\,\mathrm{H}^k$? Again, we expect that our characterization of $\partial_{\infty}\mathbb{C}\,\mathrm{H}^k$ given in Theorem 1.1 could be useful in this case.

1.4 Shortcut via the Tits' classification of 2-transitive actions

For an interested reader, we indicate how one could make a shortcut in the proof of Theorem 1.1 via the Tits' classification of 2-transitive actions. It follows from results of sect. 7.2, see Lemma 7.6, that the group $\mathrm{Aut}\,X$ of Möbius transformations of X is 2-transitive. Therefore, one could apply the Tits' classification of 2-transitive actions, see [8]. To this end, one needs to check that $\mathrm{Aut}\,X$, regarded as a topological group, is locally compact and σ-compact (that is, $\mathrm{Aut}\,X$ is a countable union of compact subsets). Then, to complete the proof of Theorem 1.1, one needs to show that the Möbius structure of X is uniquely determined by the respective appropriately normalized symmetric space.

However, there are problems with this application which we already described in [2, sect. 6.3]. The most serious one in our opinion is that the classification result does not provide understanding of the phenomenon, and this way to finish the proof is not satisfactory for us. We give instead a complete picture of the logical structure of Theorem 1.1 which otherwise would be hidden if one applies the Tits' classification.

2 The model space $\mathbb{C}\,\mathrm{H}^k$

Let $M = \mathbb{C}\,\mathrm{H}^k$, $k \geq 1$, be the complex hyperbolic space, $\dim M = 2k$. For $k = 1$ we identify $M = \frac{1}{2}\,\mathrm{H}^2$ with the hyperbolic plane of constant curvature -4. For $k \geq 2$ we choose a normalization of the metric so that the sectional curvatures of M are pinched as $-4 \leq K_\sigma \leq -1$.

We use the standard notation TM for the tangent bundle of M and UM for the subbundle of the unit vectors. For every unit vector $u \in U_aM$, $a \in M$, the eigenspaces $E_u(\lambda)$ of the *curvature operator* $\mathcal{R}(\cdot, u)u : u^\perp \to u^\perp$, where $u^\perp \subset T_oM$ is the subspace orthogonal to u, are parallel along the geodesic $\gamma(t) = \exp_a(tu)$, $t \in \mathbb{R}$, and the respective eigenvalues $\lambda = -1, -4$ are constant along γ. The dimensions of the eigenspaces are $\dim E_u(-1) = 2(k-1)$, $\dim E_u(-4) = 1$, $u^\perp = E_u(-1) \oplus E_u(-4)$.

Any $u \in U_aM$ and a unit vector $v \in E_u(-4)$ span a 2-dimensional subspace $L = L(u, v) \subset T_aM$ for which $\exp_a L \subset M$ is a totally geodesic subspace isometric to $\frac{1}{2}\,\mathrm{H}^2$ called a *complex* hyperbolic plane, while for every unit $v \in E_u(-1)$ the totally geodesic subspace $\exp_a L(u, v) \subset M$ is isometric to H^2 and called a *real* hyperbolic plane.

At every point $a \in M$ there is an isometry $s_a : M \to M$ with unique fixed point a, $s_a(a) = a$, such that its differential $ds_a : T_aM \to T_aM$ is the antipodal map, $ds_a = -\mathrm{id}$. The isometry s_a is called the *central symmetry* at a. For different $a, b \in M$ the composition $s_a \circ s_b : M \to M$ preserves the geodesic $\gamma \subset M$ through a, b and acts on γ as a shift whose differential is a parallel translation along γ. Any such isometry is called a *transvection*.

Furthermore, there is a complex structure $J : TM \to TM$, where $J_a : T_aM \to T_aM$ is an isometry with $J_a^2 = -\mathrm{id}$. Then for every $u \in U_aM$, the vectors u, $v = J_a(u)$ form an orthonormal basis of the tangent space to a complex hyperbolic plane $E \ni a$.

Remark 2.1. Given a complex hyperbolic plane $E \subset M$ and a point $a \in E$, the orthogonal complement $E^\perp \subset M$ to E at a is a totally geodesic subspace isometric to $\mathbb{C}\,\mathrm{H}^{k-1}$.

2.1 The canonical Möbius structure on $\partial_\infty \mathbb{C}\,\mathrm{H}^k$

We let $Y = \partial_\infty M$ be the geodesic boundary at infinity of a complex hyperbolic space M. For every $a \in M$ the function $d_a(\xi, \xi') = e^{-(\xi|\xi')_a}$ for $\xi, \xi' \in Y$ is a (bounded) metric on Y, where $(\xi|\xi')_a$ is the Gromov product based at a. For every $\omega \in Y$ and every Busemann function $b : M \to \mathbb{R}$ centered at ω the function $d_b(\omega, \omega) := 0$ and $d_b(\xi, \xi') = e^{-(\xi|\xi')_b}$, except for the case $\xi = \xi' = \omega$, is an extended (unbounded) metric on Y with infinitely remote point ω, where $(\xi|\xi')_b$ is the Gromov product with respect to b, see [3, sect.3.4.2]. Since M is a CAT(-1)-space, the metrics d_a, d_b satisfy the Ptolemy inequality and furthermore all these metrics are pairwise Möbius equivalent, see [6].

We let $\mathcal{M}$ be the *canonical* Möbius structure on Y generated by the metrics of type d_a, $a \in M$, i.e. any metric $d \in \mathcal{M}$ is Möbius equivalent to some metric d_a, $a \in M$. Then Y endowed with $\mathcal{M}$ is a compact Ptolemy space. Every extended metric $d \in \mathcal{M}$ is of type $d = d_b$ for some Busemann function $b : M \to \mathbb{R}$, while a bounded metric $d \in \mathcal{M}$ does not necessary coincide with λd_a, for some $a \in M$ and $\lambda > 0$, see [6]. We emphasize that metrics of $\mathcal{M}$ are neither Carnot-Carathéodory metrics nor length metrics.

2.2 Axioms (E) and (O) for $\partial_\infty \mathbb{C}\,\mathrm{H}^k$

Proposition 2.2. *The boundary at infinity $Y = \partial_\infty M$ of any complex hyperbolic space $M = \mathbb{C}\,\mathrm{H}^k$, $k \geq 1$, taken with the canonical Möbius structure, satisfies axioms (E), (O).*

Proof. Every $\mathbb{C}$-circle $F \subset Y$ is the boundary at infinity of a uniquely determined complex hyperbolic plane $E \subset M$, and every $\mathbb{R}$-circle $\sigma \subset Y$ is the boundary at infinity of a uniquely determined real hyperbolic plane $R \subset M$. Axioms (E), (O) are trivially satisfied for $\mathbb{C}\,\mathrm{H}^1 = \frac{1}{2}\,\mathrm{H}^2$. Thus we assume that $k \geq 2$.

Axiom ($E_{\mathbb{C}}$): given distinct $x, y \in Y$ there is a unique geodesic $\gamma \subset M$ with the ends x, y at infinity. This γ lies in the unique complex hyperbolic plane $E \subset M$. Then $F = \partial_\infty E \subset Y$ is a $\mathbb{C}$-circle containing x, y. The $\mathbb{C}$-circle F is uniquely determined by Lemma 3.1 below.

Axiom ($E_{\mathbb{R}}$): given a $\mathbb{C}$-circle $F \subset Y$, $\omega \in F$, $u \in Y \setminus F$, we let $E \subset M$ be the complex hyperbolic plane with $\partial_\infty E = F$, $a \in E$ the orthogonal projection of u to E, i.e. the geodesic ray $[au) \subset M$ is orthogonal to E at a. There is a unique geodesic $\gamma \subset E$ through a with ω as one of the ends at infinity. Then $[au)$ and γ span a real hyperbolic plane $R \subset M$, for which $\sigma = \partial_\infty R$ is an $\mathbb{R}$-circle in Y through ω, u that hits F_ω. By [2], every $\mathbb{R}$-circle in Y is the boundary at infinity of a totally geodesic subspace in M isometric to H^2, that is, the boundary of a real hyperbolic plane. Therefore, σ is uniquely determined.

Axioms (O): given an $\mathbb{R}$-circle $\sigma \subset Y$ and a $\mathbb{C}$-circle $F \subset Y$ with common distinct points o, ω, there are real hyperbolic plane $R \subset M$, complex hyperbolic plane $E \subset M$ with $\partial_\infty R = \sigma$, $\partial_\infty E = F$. Then R and E are mutually orthogonal along the geodesic $\gamma = R \cap E$.

($O_{\mathbb{C}}$): given $u, v \in \sigma$ such that $(o, u, \omega, v) \in \mathrm{Harm}_\sigma$, we can assume that $u \neq v$. Then the geodesic $uv \subset R$ is orthogonal to γ at $a = uv \cap \gamma$. For every $w \in F$ the geodesic rays $[aw) \subset E$, $[au) \subset R$ span a sector wau in a real hyperbolic plane R', which is isometric to the sectors $wav \subset R'$, $oau, oav \subset R$. Thus $d_a(o, u) = d_a(o, v) = d_a(w, u) = d_a(w, v)$ with respect to the metric $d_a(p, q) = e^{-(p|q)_a}$ on Y. It follows that $(w, u, o, v) \in \mathrm{Harm}$.

($O_{\mathbb{R}}$): given $x, y \in F$ such that $(o, x, \omega, y) \in \mathrm{Harm}_F$, we can assume that $x \neq y$. Then the geodesic $xy \subset E$ is orthogonal to γ at $a = xy \cap \gamma$. For $z \in \sigma$, let $b \in E$ be the orthogonal projection of z on E. Then $b \in o\omega$, and the geodesic rays $[bz) \subset R$, $[bx) \subset E$ span a sector zbx in a real hyperbolic plane R', which is isometric to the sector zby in another real hyperbolic plane R'', while the sectors $xb\omega, yb\omega \subset E$ are isometric. Thus $d_b(\omega, x) \cdot d_b(z, y) = d_b(\omega, y) \cdot d_b(z, x)$ with respect to the metric $d_b(p, q) = e^{-(p|q)_b}$ on Y. It follows that $(z, x, \omega, y) \in \mathrm{Harm}$. □

3 Spheres between two points

3.1 Briefly about Möbius geometry

Here we briefly recall basic notions of Möbius geometry. For more detail see [2], [5]. A quadruple $Q = (x, y, z, u)$ of points in a set X is said to be *admissible* if no entry occurs three or four times in Q. Two metrics d, d' on X are *Möbius equivalent* if for any admissible quadruple $Q = (x, y, z, u) \subset X$ the respective *cross-ratio triples* coincide, $\mathrm{crt}_d(Q) = \mathrm{crt}_{d'}(Q)$, where

$$\mathrm{crt}_d(Q) = (d(x, y)d(z, u) : d(x, z)d(y, u) : d(x, u)d(y, z)) \in \mathbb{R}P^2.$$

We consider *extended* metrics on X for which existence of an *infinitely remote* point $\omega \in X$ is allowed, that is, $d(x, \omega) = \infty$ for all $x \in X$, $x \neq \omega$. We always assume that such a point is unique if exists, and that $d(\omega, \omega) = 0$. We use notation $X_\omega := X \setminus \omega$ and the standard conventions for the calculation with $\omega = \infty$.

A *Möbius structure* on a set X, or a Möbius space over X, is a class $\mathcal{M} = \mathcal{M}(X)$ of metrics on X which are pairwise Möbius equivalent. A map $f : X \to X'$ between two Möbius spaces is called *Möbius*, if f is injective and for all admissible quadruples $Q \subset X$

$$\mathrm{crt}(f(Q)) = \mathrm{crt}(Q),$$

where the cross-ratio triples are taken with respect to some (and hence any) metric of the Möbius structure of X and of X'. If a Möbius map $f : X \to X'$ is bijective, then f^{-1} is Möbius, f is homeomorphism, and the Möbius spaces X, X' are said to be *Möbius equivalent*.

If two metrics of a Möbius structure have the same infinitely remote point, then they are homothetic, see [5]. We always assume that for every $\omega \in X$ the set X_ω is endowed with a metric of the structure having ω as infinitely remote point, and use notation $|xy|_\omega$ for the distance between $x, y \in X_\omega$. Sometimes we abbreviate to $|xy| = |xy|_\omega$.

A Möbius space X is Ptolemy, if it satisfies the *Ptolemy inequality*

$$|xz|\cdot|yu| \le |xy|\cdot|zu| + |xu|\cdot|yz|$$

for any admissible 4-tuple $(x,y,z,u) \subset X$ and for every metric of the Möbius structure. This is equivalent to the definition of a Ptolemy space given in sect. 1.

An $\mathbb{R}$-circle $\sigma \subset X$ satisfies the *Ptolemy equality*

$$|xz|\cdot|yu| = |xy|\cdot|zu| + |xu|\cdot|yz|$$

for every 4-tuple $(x,y,z,u) \subset \sigma$ (in this order) and for every metric of the Möbius structure. In particular, for a fixed $u \in \sigma$ we have $|xz|_u = |xy|_u + |yz|_u$, i.e. σ is a geodesic in the space X_u called an $\mathbb{R}$-*line*.

A $\mathbb{C}$-circle $F \subset X$ satisfies the squared Ptolemy equality

$$|xz|^2|yu|^2 = |xy|^2|zu|^2 + |xu|^2|yz|^2$$

for every 4-tuple $(x,y,z,u) \subset \sigma$ (in this order) and for every metric of the Möbius structure. In particular, for a fixed $u \in F$ we have $|xz|_u^2 = |xy|_u^2 + |yz|_u^2$, and F is called a $\mathbb{C}$-*line* in X_u.

3.2 Harmonic 4-tuples and spheres between two points

An admissible 4-tuple $(x,z,y,u) \subset X$ is *harmonic* if

$$\mathrm{crt}(x,z,y,u) = (1 : * : 1)$$

for some and hence any metric d of the Möbius structure. Note that if $x = y$ or $z = u$ for an admissible $Q = (x,z,y,u)$, then Q is harmonic.

We say that $x, x' \in X$ lie on a sphere between distinct $\omega, \omega' \in X$ if the 4-tuple (ω, x, ω', x') is harmonic. For a fixed ω, ω' this defines an equivalence relation on $X\backslash\{\omega,\omega'\}$, and any equivalence class $S \subset X\backslash\{\omega,\omega'\}$ is called a *sphere between* ω, ω'. Möbius maps preserve spheres between two points: if $f : X \to X$ is Möbius, then $f(S)$ is a sphere between $f(\omega), f(\omega')$. In the metric space X_ω the sphere S is a metric sphere centered at ω',

$$S = \{x \in X_\omega : |x\omega'|_\omega = r\}$$

for some $r > 0$. The points ω, ω' are *poles* of S.

3.3 The upper half-space model of $\mathbb{C}\,\mathrm{H}^k$

Let M be a complex hyperbolic space $\mathbb{C}\,\mathrm{H}^k$, $k \ge 1$, and $Y = \partial_\infty \mathbb{C}\,\mathrm{H}^k$ its boundary at infinity endowed with the canonical Möbius structure. The distance in M normalized as at the beginning of sect. 2 can be read out of Y as follows. Given distinct $s, t \in M$ let γ be the geodesic in M through s, t, and let $\omega, \omega' \in Y$ be the ends of γ at infinity. We identify s, t with the respective central symmetries $s, t : M \to M$. There are uniquely determined spheres $S, T \subset Y$ between ω, ω' such that $s(S) = S$, $t(T) = T$. We pick $x \in S$, $y \in T$ and set

$$\rho(s,t) = |\ln\langle \omega, x, y, \omega'\rangle|, \tag{2}$$

where $\langle \omega, x, y, \omega'\rangle = \frac{|\omega y|\cdot|x\omega'|}{|\omega x|\cdot|y\omega'|}$ is the cross-ratio of the quadruple (ω, x, y, ω'). This definition of the distance ρ in Y is independent of the choice $x \in S$, $y \in T$, see [2, sect. 8.3]. It is shown in [2, sect. 8] that ρ is the distance in M induced by the Riemannian metric.

We fix $\omega \in Y$. Then for every $s \in M$ there is a unique geodesic $\gamma_s \subset M$ through s one of the ends at infinity of which is ω. Then $\gamma = (\omega, \omega_s)$, where $\omega_s \in Y_\omega$ is the other end. There is a unique sphere $S \subset Y$ between ω, ω_s which is invariant under central symmetry s, $s(S) = S$. The sphere S is a metric sphere of radius $r_s > 0$ in the space Y_ω. In this way, we identify M with $Y_\omega \times \mathbb{R}_+$ by $s = (\omega_s, r_s)$. This is the *upper half-space model* of M.

For $s = (a, r)$, $s' = (a, r') \in M$ we have $\rho(s, s') = |\ln(r/r')|$ by definition of the distance ρ. It follows that $\gamma = a \times \mathbb{R}_+$ is the geodesic line in M with ends ω, a at infinity for every $a \in Y_\omega$.

Let N_ω be the subgroup of isometries of M preserving ω, $g(\omega) = \omega$ for every $g \in N_\omega$. Then N_ω leaves invariant every set $H_r = Y_\omega \times \{r\}$, $r > 0$. Since N_ω acts transitively on Y_ω, the set H_r is a horosphere in M centered at ω.

Lemma 3.1. *Given a $\mathbb{C}$-circle $F \subset Y$, there exists a complex hyperbolic plane $E \subset M$ with $\partial_\infty E = F$.*

Proof. For a fixed $\omega \in F$, the $\mathbb{C}$-circle F becomes a $\mathbb{C}$-line F_ω in Y_ω. The set $E = F_\omega \times \mathbb{R}_+ \subset M$ in the upper half-space model $M = Y_\omega \times \mathbb{R}_+$ is an isometric copy of $\mathbb{C}\mathrm{H}^1$ because $\partial_\infty E = F$ is Möbius equivalent to $\partial_\infty \mathbb{C}\mathrm{H}^1$, and the distance in M as well as that in $\mathbb{C}\mathrm{H}^1$ are determined from the boundary at infinity by formula (2). For every $a \in F_\omega$ the line $\{a\} \times \mathbb{R}_+ \subset E$ is a geodesic in M, thus E is a ruled surface in M. Therefore, the sectional curvature of M on the tangent to E spaces coincides with the Gauss curvature of E, which is -4.

For every $t > 0$ the intersection $E_t = H_t \cap E$ is a geodesic in the horosphere $H_t \subset M$ with respect to the intrinsic metric of H_t induced by M. This is because E_t is an isometric copy of a respective horocycle in $\mathbb{C}\mathrm{H}^1$. Thus E_t is uniquely determined by a speed vector at any point. It follows that E coincides with a complex hyperbolic plane $E' \subset M$ containing a geodesic $\gamma = \{a\} \times \mathbb{R}_+$, $a \in F_\omega$, because $E' \cap H_t = E_t$. □

3.4 First corollaries of axioms

Starting from this section, we assume that a Möbius space X satisfies the conditions of Theorem 1.1. The orthogonality axioms (O) tell us that if (o, u, ω, v) is harmonic on an $\mathbb{R}$-circle, then the $\mathbb{C}$-circle through o, ω lies in a sphere between u, v, and vice versa, if (x, o, y, ω) is harmonic on a $\mathbb{C}$-circle, then any $\mathbb{R}$-circle through x, y lies in a sphere between o, ω.

If we take some point on a sphere as infinitely remote, then the sphere becomes the *bisector* between its poles,

Lemma 3.2. *Let $S \subset X$ is a sphere between distinct $u, v \in X$. Then for every $\omega \in S$ the set S_ω is the bisector in X_ω between u, v, that is,*

$$S_\omega = \{x \in X_\omega : |xu|_\omega = |xv|_\omega\}.$$

Proof. For every $x \in X_\omega$ we have $\operatorname{crt}(x, u, \omega, v) = (|xu|_\omega : * : |xv|_\omega)$. Thus $x \in S_\omega$ if and only if $|xu|_\omega = |xv|_\omega$. □

Corollary 3.3. *Let σ, F be $\mathbb{R}$-circle, $\mathbb{C}$-circle respectively with distinct common points o, ω. Then*

(i) for any $(u, o, v, \omega) \in \mathrm{Harm}_\sigma$ the $\mathbb{C}$-line $F \subset X_\omega$ through o lies in the bisector between u, v, that is, $|zu|_\omega = |zv|_\omega$ for every $z \in F$,

(ii) for any $(x, o, y, \omega) \in \mathrm{Harm}_F$ any $\mathbb{R}$-line $\sigma \subset X_\omega$ through o lies in the bisector between x, y, that is, $|zx|_\omega = |zy|_\omega$ for every $z \in \sigma$.

In particular, if $x, y \in F$, $u, v \in \sigma$ with $|xo|_\omega = |yo|_\omega$ and $|uo|_\omega = |vo|_\omega$, then

$$|ux|_\omega = |vx|_\omega = |uy|_\omega = |vy|_\omega.$$

Proof. We have $\{o, \omega\} = F \cap \sigma$. Thus by axiom ($\mathrm{O}_\mathbb{C}$) the $\mathbb{C}$-line F lies in the bisector between u, v, and by axiom ($\mathrm{O}_\mathbb{R}$) the $\mathbb{R}$-line σ lies in the bisector between x, y. □

Lemma 3.4. *Any $\mathbb{C}$-circle F and any $\mathbb{R}$-circle σ have at most two points in common.*

Proof. Assume $o, \omega \in F \cap \sigma$. Choose $u, v \in \sigma$ such that $(o, u, \omega, v) \in \mathrm{Harm}_\sigma$. By Corollary 3.3 the $\mathbb{C}$-line $F \subset X_\omega$ lies in the bisector between u, v. Thus $\mathbb{R}$-line $\sigma \subset X_\omega$ intersects F_ω once (at o). □

Lemma 3.5. *If two $\mathbb{R}$-circles σ, γ have in common three distinct points, then they coincide, $\sigma = \gamma$.*

Proof. Assume $x, y, \omega \in \sigma \cap \gamma$. Then for the $\mathbb{C}$-circle F through x, ω we have $y \notin F$ by Lemma 3.4. By $(E_{\mathbb{R}})$, there is at most one $\mathbb{R}$-circle σ with $x, y, \omega \in \sigma$. □

Corollary 3.6. *Given a $\mathbb{C}$-circle $F \subset X$ and $\omega \in F$, there is a retraction $\mu_{F,\omega} : X \to F$ (continuous on X_ω), $\mu_{F,\omega}(u) = u$ for $u \in F$ and*

$$\mu_{F,\omega}(u) = \sigma \cap F_\omega$$

for $u \in X \setminus F$, where σ is the $\mathbb{R}$-circle through ω, u that hits F_ω.

Proof. By Lemma 3.4, the intersection $\sigma \cap F_\omega$ consists of a unique point, thus $\mu_{F,\omega}(u) \in F_\omega$ is well defined. Assume $u_i \to u \in X \setminus F$. We show that $p_i := \mu_{F,\omega}(u_i)$ is bounded. This then implies $\mu_{F,\omega}(u_i) \to \mu_{F,\omega}(u)$ by uniqueness. Assume to the contrary that $|op_i| \to \infty$ for some basepoint $o \in F$ in the metric of X_ω. Let $q_i \in F$ such that $(o, p_i, q_i, \omega) \in \mathrm{Harm}_F$. Then $|oq_i| \to \infty$. But this would imply that also $|ou_i| \to \infty$ because p_i and u_i lie on the same sphere between o and q_i. □

4 Involutions associated with complex circles

4.1 Reflections with respect to complex circles

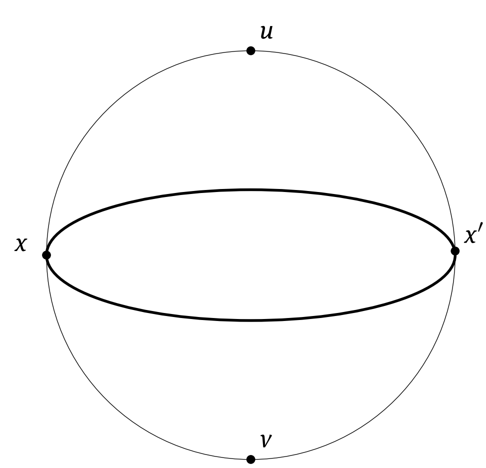
Let $F \subset X$ be a $\mathbb{C}$-circle. Then by Corollary 3.6, every $u \in X \setminus F$ defines an involution $\eta_u : F \to F$ without fixed points by

$$\eta_u(\omega) = \mu_{F,\omega}(u).$$

Proposition 4.1. *Given a $\mathbb{C}$-circle $F \subset X$ and $u \in X \setminus F$, there exists a unique $v \in X \setminus F$, $v \neq u$, such that $(x, u, \eta_u(x), v) \in \mathrm{Harm}_\sigma$ for every $x \in F$ and for the $\mathbb{R}$-circle $\sigma = \sigma_x$ through $x, u, \eta_u(x)$.*

This results defines for a $\mathbb{C}$-circle F an involution $\varphi_F : X \to X$ with fixed point set F in the following way: for $u \in X \setminus F$ let $v = \varphi_F(u)$ be the unique point defined by Proposition 4.1. For $u \in F$ define $\varphi_F(u) = u$. The involution φ_F is called the *reflection* with respect to F.

In the case that $u \in X \setminus F$ we say that u, v are *conjugate poles* of F. Note that $\eta_u = \eta_v$ for conjugate poles $u, v = \varphi_F(u)$ of F. Thus in a simplified way one can visualize F as equator of a 2-sphere, u, v as the poles, the map $\eta_u = \eta_v$ as antipodal map on the equator such that for all x on the equator the points $(x, u, \eta_u(x), v)$ lie in harmonic position on a circle as in the following picture, where $x' = \eta_u(x) = \eta_v(x)$.



For the proof of Proposition 4.1 we need Lemmas 4.2–4.5.

Lemma 4.2. *Given a $\mathbb{C}$-line $F_\omega \subset X_\omega$, $\mathbb{R}$-circle $\sigma \subset X_\omega$ and $(x, u, y, v') \in \mathrm{Harm}_\sigma$ with $x, y \in F_\omega$, we have*

$$|zu|_\omega = |zv'|_\omega$$

for all $z \in F_\omega$.

Proof. By Corollary 3.3(i), the $\mathbb{C}$-line F_y lies in the bisector in X_y between u, v', thus $|zu|_y = |zv'|_y$ for every $z \in F_y$. Taking the metric inversion (1) with respect to ω we obtain

$$|zu|_\omega = \frac{|zu|_y}{|z\omega|_y \cdot |u\omega|_y} = \frac{|zv'|_y}{|z\omega|_y \cdot |v'\omega|_y} = |zv'|_\omega$$

for every $z \in F_\omega$. □

Lemma 4.3. *Given $u \in X_\omega$ and a $\mathbb{C}$-line $F_\omega \subset X_\omega$, the distance function $d_u : F_\omega \to \mathbb{R}$, $d_u(z) = |zu|_\omega$, is symmetric with respect to $o = \mu_{F,\omega}(u)$, that is, $d_u(z) = d_u(z')$ for every $z, z' \in F_\omega$ with $|zo|_\omega = |z'o|_\omega$.*

Proof. If $|zo|_\omega = |z'o|_\omega$, then $(z, o, z', \omega) \in \mathrm{Harm}_F$. Thus by Corollary 3.3(ii), the $\mathbb{R}$-line $\sigma \subset X_\omega$ through u, o lies in the bisector in X_ω between z, z'. □

Lemma 4.4. *Assume for $u, v' \in X_\omega$ the distance functions $d_u, d_{v'} : F_\omega \to \mathbb{R}$ along a $\mathbb{C}$-line $F_\omega \subset X_\omega$ coincide, $d_u(z) = d_{v'}(z)$ for all $z \in F_\omega$, and d_u is symmetric with respect to $o \in F_\omega$, while $d_{v'}$ is symmetric with respect to $o' \in F_\omega$. Then $o = o'$.*

Proof. Let $\alpha, \alpha' : F_\omega \to F_\omega$ be isometric reflections with respect to o, o' respectively, $\beta = \alpha' \circ \alpha$. Using $d_u(\alpha x) = d_u(x)$ and $d_{v'}(\alpha' x) = d_{v'}(x)$ for all $x \in F_\omega$, we obtain

$$\begin{aligned} d_u(\beta^{i+1}(o)) &= d_{v'}(\alpha' \circ \alpha \circ \beta^i(o)) \\ &= d_{v'}(\alpha \circ \beta^i(o)) \\ &= d_u(\beta^i(o)). \end{aligned}$$

Thus by induction, the function d_u is constant along the sequence $\beta^i(o)$. If one admits that $o \neq o'$, then $\beta^i(o) \to \infty$ in X_ω and thus $d_u(\beta^i(o)) \to \infty$ as $i \to \infty$ being a distance function, a contradiction. Hence $o = o'$. □

We use abbreviation $|xy| := |xy|_\omega$ for the distance in X_ω up to the end of this section.

Lemma 4.5. *Assume that $\mu(u) = \mu(v') = o \in F_\omega$ for some $u, v' \in X_\omega$, where $\mu = \mu_{F,\omega} : X_\omega \to F_\omega$. If $|uv'| = |uo| + |ov'|$, then u, o, v' are contained in an $\mathbb{R}$-line.*

Proof. We can assume that $u, v' \neq o$. Let $\sigma, \sigma' : \mathbb{R} \to X_\omega$ be the unit speed parametrizations of the $\mathbb{R}$-lines through u, o and v', o respectively such that $\sigma(0) = o = \sigma'(0)$, $\sigma(|ou|) = u$, $\sigma'(|ov'|) = v'$. Then by the triangle inequality

$$d(s, t) := |\sigma(s)\sigma'(t)| \leq s + t$$

for all $s, t \geq 0$. We put $s_0 = |uo|$, $t_0 = |ov'|$. The function $d_{t_0}(s) = d(s, t_0)$ is convex and 1-Lipschitz, $d_{t_0}(0) = t_0$, $d_{t_0}(s_0) = |uv'| = s_0 + t_0$ by the assumption. Hence

$$d_{t_0}(s) = s + t_0$$

for all $s \geq 0$. Similarly, for a fixed $s \geq 0$ the function $d_s(t) = d(s, t)$ is convex and 1-Lipschitz, $d_s(0) = s$, $d_s(t_0) = d_{t_0}(s) = s + t_0$. Hence $d_s(t) = s + t$ for all $t \geq 0$. Therefore, $|\sigma(s)\sigma'(t)| = s + t$ for all $s, t \geq 0$. Thus the concatenation $\sigma|[0, \infty) \cup \sigma'|[0, \infty)$ is a geodesic line, which implies that u, o, v' are contained in an $\mathbb{R}$-line. □

Proof of Proposition 4.1. Let $o = \mu_{F,\omega}(u) \in F_\omega$, γ the $\mathbb{R}$-line in X_ω through u, o. We take $v \in \gamma$, $v \neq u$, such that $|vo| = |ou|$.

Given $x \in F$, we put $y = \eta_u(x) \in F$, and let σ be the $\mathbb{R}$-circle through x, u, y. We show that $v' = v$ for $v' \in \sigma$, $v' \neq u$, such that $(x, u, y, v') \in \mathrm{Harm}_\sigma$.

By Lemma 4.2 the distance functions $d_u, d_{v'} : F_\omega \to \mathbb{R}$, $d_u(z) = |zu|$, $d_{v'}(z) = |zv'|$, coincide, $|zu| = |zv'|$ for all $z \in F_\omega$. By Lemma 4.3, d_u is symmetric with respect to o, while $d_{v'}$ is symmetric with respect to $o' = \mu_{F,\omega}(v')$. Thus $o' = o$ by Lemma 4.4.

By the triangle inequality $|uv'| \le |uo| + |ov'| = 2|uo| = |uv|$. On the other hand, by the Ptolemy equality we have

$$|uv'| \cdot |xy| = |xu| \cdot |v'y| + |yu| \cdot |xv'|.$$

By Lemma 4.2, $|xv'| = |xu|$, and by Corollary 3.3(i), $|xu| = |xv|$. Similarly, $|yv'| = |yu| = |yv|$. By the Ptolemy inequality

$$|xu| \cdot |vy| + |yu| \cdot |xv| \ge |xy| \cdot |uv|.$$

We conclude that $|uv'| \ge |uv|$. Hence, $|uv'| = |uv| = |uo| + |ov'|$. From Lemma 4.5 we conclude that $v' \in \gamma \cap \sigma$, thus $v' = v$. □

4.2 A Möbius involution of a complex circle

Lemma 4.6. *Given a $\mathbb{C}$-circle F, $x, y \in F$, $u \in X \setminus F$, there exists $\omega \in F$ such that* $(x, \eta_u(\omega), y, \omega) \in \mathrm{Harm}_F$.

Proof. We put $x' = \eta_u(x)$, $y' = \eta_u(y)$. Then for every $\omega \in x'y'$ we have $o = \eta_u(\omega) \in xy$, where $xy \subset F$ is the arc between x, y that does not contain x', y', and $x'y' \subset F$ is the arc between x', y' that does not contain x, y. The cross-ratio function $f : x'y' \to \mathbb{R}$, defined by

$$f(\omega) = \frac{|xo| \cdot |y\omega|}{|x\omega| \cdot |oy|}$$

via a metric of the Möbius structure, is continuous and takes values $f(x') = 0$, $f(y') = \infty$. Then ω with $f(\omega) = 1$ does the job. □

Lemma 4.7. *Assume a sphere $S \subset X$ between ω, z' intersects the $\mathbb{C}$-circle F through ω, z' at x, x'. Then for every $u \in S$ there is an $\mathbb{R}$-circle γ' with $x, u, x' \in \gamma'$.*

Proof. By Lemma 4.6 there is $\widetilde{x} \in F$ such that $(z', \widetilde{x}, \omega, \widetilde{x}') \in \mathrm{Harm}_F$, where $\widetilde{x}' = \eta_u(\widetilde{x})$, in particular, there is an $\mathbb{R}$-circle $\gamma' \subset X$ with $\widetilde{x}, u, \widetilde{x}' \in \gamma'$. By axiom $\mathrm{O}_{\mathbb{R}}$, γ' lies on a sphere S' between z', ω. Since $u \in \gamma' \cap S$, we conclude that $S' = S$ and therefore $\{\widetilde{x}, \widetilde{x}'\} = \{x, x'\}$. □

Proposition 4.8. *For every $x, z \in F$, $u \in X \setminus F$ we have*

$$\mathrm{crt}(x, u, z, v) = \mathrm{crt}(\eta_u(x), v, \eta_u(z), u),$$

where $v \in X \setminus F$ is the conjugate to u pole of F.

Proof. We put $z' = \eta_u(z) \in F$. By Lemma 4.6 there is $\omega \in F$ such that $(x, o, z', \omega) \in \mathrm{Harm}_F$, where $o = \eta_u(\omega)$. By definition of η_u, there is an $\mathbb{R}$-circle σ through ω, u and o. Then F, σ are $\mathbb{C}$-line, $\mathbb{R}$-line respectively in the metric space X_ω, and o is the midpoint between x, z', $|xo| = |oz'|$, where $|xo| = |xo|_\omega$ is the distance in X_ω.

We denote by $\psi : F \to F$ the metric reflection with respect to o. Then ψ is an isometry with $\psi(o) = o$, in particular, ψ is Möbius. Note that F is in the bisector in X_ω between u and v, that is $|wu| = |wv|$ for every $w \in F$. Furthermore, σ is in the bisector between any $w \in F$ and $\psi(w)$. Thus $\psi(x) = z' = \eta_u(z)$. We show that $x' := \psi(z) = \eta_u(x)$.

Since $z' = \eta_u(z)$, there is an $\mathbb{R}$-circle $\gamma \subset X$ such that $(z, u, z', v) \in \mathrm{Harm}_\gamma$. Let $w \in F$ be the midpoint between z, z', $|zw| = |wz'| =: r$. The circle γ is a metric circle in X_ω centered at w, in particular, $|wu| = |wv| = r$.

We put $w' = \psi(w)$. Then $|w'u| = |w'v| = r = |w'x'|$ and $|xw'| = |wz'| = r$. It follows that x, u, x', v lie on the metric sphere of radius r centered at w'. Thus by Lemma 4.7 there is a circle $\gamma' \subset X$ through x, x' with $u \in \gamma'$. This means that $x' = \eta_u(x)$.

It follows

$$\begin{aligned}\operatorname{crt}(x,u,z,v) &= (|xu|\cdot|zv| : |xz|\cdot|uv| : |xv|\cdot|uz|)\\ &= \operatorname{crt}(\psi(x),u,\psi(z),v)\\ &= \operatorname{crt}(\eta_u(z),u,\eta_u(x),v) = \operatorname{crt}(\eta_u(x),v,\eta_u(z),u).\end{aligned}$$

□

Corollary 4.9. *For every $\mathbb{C}$-circle F and every $u \in X \setminus F$ the involution $\eta_u : F \to F$ is Möbius.*

Proof. Note that $\operatorname{crt}(x,u,y,v) = (1 : * : 1)$ for each $x, y \in F$, where v is the conjugate to u pole of F. Thus in the space X_v we have $|x'y'|_v = |xy|_v$ by Proposition 4.8, where $x' = \eta_u(x)$, $y' = \eta_u(y)$. Hence η_u preserves the cross-ratio triple of any admissible 4-tuple in F. □

5 A distance formula

Using Proposition 4.8, we derive here a distance formula (see Proposition 5.2) which plays a very important role in the paper.

Lemma 5.1. *Let $u, v \in X$ be conjugate poles of a $\mathbb{C}$-circle F. Then for every $x, z \in F$ we have*

$$|xz|_\omega \cdot |zy|_\omega = |zu|_\omega^2,$$

where $y = \eta_u(x)$, $\omega = \eta_u(z)$.

Proof. We consider a background metric $|\ |_v$ with infinitely remote point v and we define $|\ |_\omega$ and $|\ |_z$ as the metric inversions, i.e.

$$|ef|_\omega = \frac{|ef|_v}{|e\omega|_v \cdot |f\omega|_v}$$

and

$$|ef|_z = \frac{|ef|_v}{|ez|_v \cdot |fz|_v}.$$

Since F lies on a sphere between u, v, we have $|uw|_v =: \rho$ for every $w \in F$ and observe that $|z\omega|_v = 2\rho$. Thus we have $|uz|_\omega = |u\omega|_z = \frac{1}{2\rho}$.

Furthermore,

$$|xz|_\omega = \frac{|xz|_v}{|x\omega|_v \cdot |z\omega|_v} = \frac{|xz|_v}{2\rho|x\omega|_v}$$

and

$$|y\omega|_z = \frac{|y\omega|_v}{|yz|_v \cdot |z\omega|_v} = \frac{|y\omega|_v}{2\rho|yz|_v}.$$

By Proposition 4.8, $\operatorname{crt}(x,u,z,v) = \operatorname{crt}(y,v,\omega,u)$ and also $\operatorname{crt}(y,u,z,v) = \operatorname{crt}(x,v,\omega,u)$. Each of these cross-ratios is of the form $(1 : * : 1)$. Hence $|xz|_v = |y\omega|_v$ and $|yz|_v = |x\omega|_v$. It follows that $|xz|_\omega = |y\omega|_z$ and similarly $|yz|_\omega = |x\omega|_z$.

We denote by $a = |xz|_\omega$, $d = |yz|_\omega$, and consider the metric inversion

$$d_z(p,q) = \frac{ad|pq|_\omega}{|zp|_\omega \cdot |zq|_\omega}$$

with respect to z. We calculate $d_z(x,\omega) = \frac{ad}{|xz|_\omega} = d = |x\omega|_z$, $d_z(y,\omega) = \frac{ad}{|yz|_\omega} = a = |y\omega|_z$. Thus two metrics $d_z(p,q)$ and $|pq|_z$ with the common infinitely remote point z coincide, in particular, $d_z(u,\omega) = |u\omega|_z = |uz|_\omega$.

On the other hand, $d_z(u,\omega) = \frac{ad}{|uz|_\omega}$. Therefore $\frac{ad}{|uz|_\omega} = |uz|_\omega$ and hence $|xz|_\omega \cdot |yz|_\omega = |uz|_\omega^2$. □

Proposition 5.2. *Given $o, u \in X_\omega$, let $F \subset X_\omega$ be the $\mathbb{C}$-line through o (and ω), $z = \mu_{F,\omega}(u)$, and let $a = |zu|_\omega$, $b = |oz|_\omega$, $r = |ou|_\omega$. Then $r^4 = a^4 + b^4$.*

Proof. We take $x, y \in F$ with $|xo|_\omega = |oy|_\omega = r$. Without loss of generality we can assume that $o < z < x$. Then $|xz|^2_\omega = r^2 - b^2$, $|yz|^2_\omega = r^2 + b^2$. We have $\omega = \eta_u(z)$. The points x, y, u lie on the metric sphere of radius r centered at o. Thus by Lemma 4.7 there exists an $\mathbb{R}$-circle $\gamma \subset X_\omega$ through x, y, u, and therefore $y = \eta_u(x)$. Using that $|xz|^2_\omega \cdot |zy|^2_\omega = a^4$ by Lemma 5.1, we obtain $r^4 = a^4 + b^4$. □

Remark 5.3. As it is explained in [1], the distance formula $r^4 = a^4 + b^4$ gives rise to the Korányi gauge on the Heisenberg group $\mathbb{H}^{2k-1}$, see [1, Remark 14.4].

6 The canonical foliation of X_ω

We fix $\omega \in X$ and denote by B_ω the set of all the $\mathbb{C}$-circles in X through ω. By axiom ($E_\mathbb{C}$), for every $x \in X_\omega$ the is a unique $\mathbb{C}$-circle $F \in B_\omega$ with $x \in F$. This defines a map $\pi_\omega : X_\omega \to B_\omega$, $\pi_\omega(x) = F$, called the *canonical* projection, the set B_ω is called the *base* of π_ω, and the foliation of X_ω by the fibers of π_ω is said to be *canonical*. In this section we use notation $|xy|$ for the distance between $x, y \in X_\omega$ in the metric space X_ω.

6.1 Busemann functions on X_ω

Lemma 6.1. *Let $\sigma \subset X_\omega$ be an $\mathbb{R}$-line. Any Busemann function $b : X_\omega \to \mathbb{R}$ associated with σ is constant along any $\mathbb{C}$-line $F \subset X_\omega$ which meets σ.*

Proof. We parametrize $\sigma : \mathbb{R} \to X_\omega$ by arclength such that $\sigma(0) = \sigma \cap F$ and b decreases along σ. By Proposition 5.2 we have $|x\sigma(t)|^4 = t^4 + |x\sigma(0)|^4$ for every $x \in F$ and $t \in \mathbb{R}$. Therefore $b(x) = \lim_{t\to\infty}(|x\sigma(t)| - t) = 0$. □

Lemma 6.2. *For every $\mathbb{C}$-line $F \subset X_\omega$ the function $x \mapsto |x\mu_{F,\omega}(x)|$ is constant along any $\mathbb{C}$-line $F' \in X_\omega$.*

Proof. For $x' \in F'$ let $\sigma : \mathbb{R} \to X_\omega$ be the arclength parametrization of the uniquely determined $\mathbb{R}$-line with $\sigma(0) = \mu_{F,\omega}(x')$, $\sigma(a) = x'$, where $a = |x'\mu_{F,\omega}(x')|$. The Busemann function of σ is 1-Lipschitz and by Lemma 6.1 it is constant on F as well as on F'. It follows that $|yy'| \geq a$ for every $y \in F$, $y' \in F'$. In particular, $|y'\mu_{F,\omega}(y')| \geq a$ for all $y' \in F'$. By symmetry we get equality. □

6.2 Canonical metric on the base

Given $F, F' \in B_\omega$ we put $|FF'| := |x'\mu_{F,\omega}(x')|$ for some $x' \in F'$. By Lemma 6.2 this is well defined and moreover, it is a metric on B_ω. Indeed, given $F, F', F'' \in B_\omega$, we take $x' \in F'$ and put $x = \mu_{F,\omega}(x') \in F$, $x'' = \mu_{F'',\omega}(x') \in F''$. Then $|FF''| \leq |xx''| \leq |xx'| + |x'x''| = |FF'| + |F'F''|$. This metric on B_ω is said to be *canonical*.

The projection $\pi_\omega : X_\omega \to B_\omega$ is 1-Lipschitz and isometric if restricted to any $\mathbb{R}$-line in X_ω. This implies that every two points in B_ω lie on a geodesic line.

Lemma 6.3. *The base B_ω with the canonical metric is uniquely geodesic, i.e. between any two points there is a unique geodesic. In particular, B_ω is contractible.*

Proof. Given pairwise distinct $F, F', F'' \in B_\omega$ with $|FF''| = |FF'| + |F'F''|$, we take $x' \in F'$ and put $x = \mu_{F,\omega}(x') \in F$, $x'' = \mu_{F'',\omega}(x') \in F''$, $y = \mu_{F,\omega}(x'') \in F$. Then

$$|yx''| \leq |xx''| \leq |xx'| + |x'x''| = |FF''| = |yx''|.$$

Therefore $y = x$ by Proposition 5.2. Since $\mu(x) = \mu(x'') = x'$ for $\mu = \mu_{F',\omega}$ and $|xx''| = |xx'| + |x'x''|$, the points x, x', x'' are contained in an $\mathbb{R}$-line by Lemma 4.5. This line is unique by Lemma 3.5. The claim follows. □

Lemma 6.4. *For every $\mathbb{C}$-line $F \subset X_\omega$ the retraction $\mu_{F,\omega} : X_\omega \to F$ if restricted to any $\mathbb{C}$-line $F' \subset X_\omega$ is isometric.*

Proof. Given $x', y' \in F'$ we let $x = \mu_{F,\omega}(x')$, $y = \mu_{F,\omega}(y')$, $a = |FF'|$, $b = |xy|$. By Proposition 5.2, $|x'y|^4 = a^4 + b^4$. On the other hand, $y' = \mu_{F',\omega}(y)$. Thus again by Proposition 5.2, $|x'y|^4 = a^4 + |x'y'|^4$. Hence, $|xy| = |x'y'|$. □

6.3 Busemann functions are affine

Lemma 6.5. *Assume that $\omega_i \to \omega$ in X, and a point $x \in X$ distinct from ω is fixed. Then any $\mathbb{R}$-circle $l \subset X$ through ω, x is the (pointwise) limit of a sequence of $\mathbb{R}$-circles $l_i \subset X$ through ω_i, x.*

Proof. Let $o \in l$ be a point different from x, ω, and let $F = F(o, \omega) \subset X$ be the $\mathbb{C}$-circle through o, ω. For i we let $F_i = F(o, \omega_i)$ be a $\mathbb{C}$-circle through o, ω_i, which is uniquely determined for sufficiently large i as soon as $\omega_i \neq o$. Since X is compact, $F_i \to F$ pointwise and thus for i large enough $x \notin F_i$. By axiom ($\mathrm{E}_\mathbb{R}$) there is uniquely determined $\mathbb{R}$-circle $l_i \subset X$ through ω_i, x which hits $F_i \setminus \omega_i$. Then l_i subconverges to an $\mathbb{R}$-circle $\bar{l}$ through ω, x that hits $F \setminus \omega$. By axiom ($\mathrm{E}_\mathbb{R}$), $\bar{l} = l$ and $l_i \to l$ follows. □

The following result has been obtained in [5, Proposition 4.1], [2, Corollary 3.19] under different assumptions.

Proposition 6.6. *Given two $\mathbb{R}$-lines $l, l' \subset X_\omega$, the Busemann functions of l are affine on l'.*

Proof. Let $b : X_\omega \to \mathbb{R}$ be a Busemann function of l. We can write $b(x) = \lim_{i\to\infty}(|x\omega_i| - |o\omega_i|)$ for every $x \in X_\omega$, where $o \in l$ is some fixed point, $l \ni \omega_i \to \omega$.

Let $m \in l'$ be the midpoint between x, $y \in l'$, $|xm| = |my| = \frac{1}{2}|xy|$. We have to show that $b(m) = \frac{1}{2}(b(x) + b(y))$. Busemann functions in any Ptolemy space are convex, see [5, Proposition 4.1], thus $b(m) \le \frac{1}{2}(b(x) + b(y))$.

By Lemma 6.5, for every sufficiently large i there is an $\mathbb{R}$-circle $l_i \subset X_\omega$ through x and ω_i such that the sequence l_i converges pointwise to l'. Thus there are points $y_i \in l_i$ with $y_i \to y$. The points x, y_i divide l_i into two segments. Choose a point m_i in the segment that does not contain ω_i such that $|xm_i| = |m_iy_i|$. One easily sees that $m_i \to m$.

The points x, m_i, y_i, ω_i lie on the $\mathbb{R}$-circle l_i in this order. Thus

$$|xy_i| \cdot |m_i\omega_i| = |xm_i| \cdot |y_i\omega_i| + |m_iy_i| \cdot |x\omega_i|,$$

and since $|xm_i| = |m_iy_i| \ge \frac{1}{2}|xy_i|$, we see $|m_i\omega_i| \ge \frac{1}{2}(|x\omega_i| + |y_i\omega_i|)$. This implies in the limit $b(m) \ge \frac{1}{2}(b(x) + b(y))$. □

Lemma 6.7. *Busemann functions are constant on the fibers of $\pi_\omega : X_\omega \to B_\omega$.*

Proof. Let $b : X_\omega \to \mathbb{R}$ be a Busemann function associated with an $\mathbb{R}$-line $l \subset X_\omega$, and let $F \subset X_\omega$ be a $\mathbb{C}$-line, $x \in F$, $c = b(x) \in \mathbb{R}$. Using Lemma 6.1 we can assume that $l \cap F = \emptyset$. We take $y \in l$ with $b(y) = c$ and consider the $\mathbb{C}$-line $F' \subset X_\omega$ through y. Then $F' \cap F = \emptyset$. By Lemma 6.1, the function b takes the constant value c on F'. Let $z = \mu_{F',\omega}(x) \in F'$. Then $b(z) = b(x) = c$, and there is a uniquely determined $\mathbb{R}$-line $\sigma \subset X_\omega$ through x, z. By Proposition 6.6 the function b takes the constant value c along σ.

Given $x' \in F$, for the $\mathbb{R}$-line $\sigma' \subset X_\omega$ through x', $z' = \mu_{F',\omega}(x') \in F'$ we have $\pi_\omega(\sigma') = \pi_\omega(\sigma) \subset B_\omega$ by Lemma 6.3. Thus the values of b along σ' are uniformly bounded because b is Lipschitz and the distance of any $u \in \sigma'$ to σ equals $|x'x| = |z'z|$ by Lemma 6.4. Since $b|\sigma'$ is affine, we conclude that $b|\sigma' \equiv b(z') = c$, in particular, $b(x') = c$. □

6.4 Properties of the base

Proposition 6.8. *The base B_ω is isometric to a normed vector space of a finite dimension with a strictly convex norm.*

Proof. By Lemma 6.3, B_ω is a geodesic metric space such that through any two distinct points there is a unique geodesic line. We show that affine functions on B_ω separate points. Any Busemann function $b : X_\omega \to \mathbb{R}$ is affine on $\mathbb{R}$-lines by Proposition 6.6. By Lemma 6.7, b is constant on the fibers of π_ω, thus it determines a function $\overline{b} : B_\omega \to \mathbb{R}$ such that $\overline{b} \circ \pi_\omega = b$. Every geodesic line $\overline{l} \subset B_\omega$ is of the form $\overline{l} = \pi_\omega(l)$ for some $\mathbb{R}$-line $l \subset X_\omega$, and each unit speed parameterization $c : \mathbb{R} \to X_\omega$ of l induces the unit speed parameterization $\overline{c} = \pi_\omega \circ c$ of $\overline{l}$. Then $\overline{b} \circ \overline{c} = \overline{b} \circ \pi_\omega \circ c = b \circ c$ is an affine function on $\mathbb{R}$. Through any $x, x' \in B_\omega$, there is a geodesic line $\overline{l} = \pi_\omega(l)$. Let b be a Busemann function on X_ω associated with l. Then b takes different values on the fibers of π_ω over x, x' respectively. Thus the affine function $\overline{b}$ separates the points x, x', $\overline{b}(x) \neq \overline{b}(x')$. Then by [7], B_ω is isometric to a convex subset of a normed vector space with a strictly convex norm. Since B_ω is geodesically complete, i.e., through any two points there is a geodesic line, this subset is a subspace, and therefore B_ω is isometric to a normed vector space E. The Ptolemy space X is compact, thus B_ω is locally compact, and the dimension of E is finite. □

Corollary 6.9. *The space X is homeomorphic to sphere S^{k+1}, $k \geq 0$.*

Proof. By Proposition 6.8, B_ω is homeomorphic to an Euclidean space $\mathbb{R}^k$, and since $\pi_\omega : X_\omega \to B_\omega$ is a fibration over B_ω with fibers homeomorphic to $\mathbb{R}$, the space X_ω is homeomorphic to $\mathbb{R}^{k+1}$. Thus X is homeomorphic to S^{k+1}. □

Proposition 6.10. *The Ptolemy space X has the following property: Any 4-tuple $Q \subset X$ of pairwise distinct points lies on an $\mathbb{R}$-circle $\sigma \subset X$ provided three of the points of Q lie on σ and the Ptolemy equality holds for the cross-ratio triple* $\operatorname{crt}(Q)$.

Proof. We assume that $Q = (x, y, z, u)$, $x, y, z \in \sigma$ and

$$|xz| \cdot |yu| = |xy| \cdot |zu| + |xu| \cdot |yz|.$$

Choosing y as infinitely remote, we have $|xz|_y = |xu|_y + |uz|_y$ and σ is an $\mathbb{R}$-line in X_y. Recall that the canonical projection $\pi_y : X_y \to B_y$ is 1-Lipschitz and isometric on every $\mathbb{R}$-line. Thus

$$|\overline{x}\,\overline{z}| = |xz|_y = |xu|_y + |uz|_y \geq |\overline{xu}| + |\overline{uz}|,$$

where $\overline{x} = \pi_y(x)$. The triangle inequality in B_y implies $|\overline{x}\,\overline{z}| = |\overline{x}\,\overline{u}| + |\overline{u}\,\overline{z}|$. By Proposition 6.8, B_y is isometric to a normed vector space with a strictly convex norm. Thus we conclude that $\overline{u}$ lies between $\overline{x}$ and $\overline{z}$ on a line in B_y. This means that the $\mathbb{C}$-line $F \subset X_y$ through u hits σ, and for $o = \sigma \cap F$ we have $|xz|_y = |xo|_y + |oz|_y$. By Proposition 5.2, $|xo|_y < |xu|_y$ and $|oz|_y < |uz|_y$ unless $u = o$. We conclude that $u = o \in \sigma$. □

7 Möbius automorphisms

7.1 Vertical shifts

We fix $\omega \in X$ and change notation using the letter b for elements of the base B_ω and F_b for the respective fiber of π_ω. For any two $b, b' \in B_\omega$ we have by Lemma 6.4 the isometry $\mu_{bb'} : F_b \to F_{b'}$.

Lemma 7.1. *The isometries $\mu_{bb'} : F_b \to F_{b'}$ depend continuously of $b, b' \in B_\omega$, that is, for $b_i \to b'$ and for any $x \in F_b$, we have $\mu_{bb_i}(x) \to \mu_{bb'}(x)$.*

Proof. If a sequence of geodesic segments in a metric space pointwise converges, then the limit is also a geodesic segment. Together with uniqueness of $\mathbb{R}$-lines in X_ω and compactness of X, this implies the claim. □

We fix an orientation of F_b and define the orientation of $F_{b'}$ via the isometry $\mu_{bb'}$. This gives a simultaneously determined orientation O on all the fibers of π_ω.

Lemma 7.2. *The orientation O is well defined and independent of the choice of $b \in B_\omega$.*

Proof. By Lemma 6.3, the base B_ω is contractible. Using Lemma 7.1, we see that the orientation of $F_{b''}$ induced by $\mu_{bb''}$ coincides with that induced by $\mu_{b'b''} \circ \mu_{bb'}$ for each $b', b'' \in B$. Hence, the claim. □

We assume that a simultaneous orientation O of $\mathbb{C}$-lines in X_ω is fixed, and we call it the *fiber orientation*. Now we are able to produce nontrivial Möbius automorphisms of X. Using the fiber orientation O we define for every $s \in \mathbb{R}$ the map $\gamma = \gamma_s : X_\omega \to X_\omega$ which acts on every fiber F on π_ω as the shift by $|s|$ in the direction determined in the obvious way by the sign of s and O. The map γ is called a *vertical shift*.

Proposition 7.3. *Every vertical shift $\gamma : X_\omega \to X_\omega$ is an isometry.*

Proof. This immediately follows from definition of a vertical shift, Lemma 6.2 and Proposition 5.2. □

7.2 Homotheties and shifts

Recall that every Möbius map $\gamma : X \to X$ which fixes the point $\omega \in X$ is a homothety of X_ω, that is, $|\gamma(x)\gamma(y)|_\omega = \lambda|xy|_\omega$ for some $\lambda > 0$ and every $x, y \in X_\omega$.

Lemma 7.4. *For every $\mathbb{C}$-circle $F \subset X$, every Möbius involution $\gamma : F \to F$ without fixed points extends to a Möbius map $\overline{\gamma} : X \to X$.*

Proof. We identify $F = \partial_\infty Y$, where $Y = \frac{1}{2}\,\mathrm{H}^2$ is the hyperbolic plane of constant curvature -4. For a fixed $\omega \in F$ any vertical shift $\alpha : X_\omega \to X_\omega$ restricted to F is induced by a parabolic rotation $\widetilde{\alpha} : Y \to Y$, which is an isometry without fixed points in Y having the unique fixed point $\omega \in \partial_\infty Y$. One easily sees that parabolic rotations generate the isometry group of Y preserving orientation.

The involution γ is induced by a central symmetry of Y. Thus γ can be represented by a composition of vertical shifts with appropriate fixed points on F. Hence γ extends to a Möbius map $\overline{\gamma} : X \to X$. □

Lemma 7.5. *Given distinct $o, \omega \in X$, $\lambda > 0$, there is a homothety $\overline{\gamma} : X_\omega \to X_\omega$ with coefficient λ, $\overline{\gamma}(o) = o$, preserving the fiber orientation O.*

Proof. Let F be the $\mathbb{C}$-circle determined by o, ω. Using representation $F = \partial_\infty Y$ for $Y = \frac{1}{2}\,\mathrm{H}^2$ we write $\gamma = \alpha \circ \beta$ for any orientation preserving homothety $\gamma : F \to F$ with fixed o, ω, where $\alpha, \beta : F \to F$ are Möbius involutions without fixed points. By Lemma 7.4, γ extends to a Möbius $\overline{\gamma} : X \to X$. □

Assuming that $\omega \in X$ is fixed we denote $B = B_\omega$. For an isometry $\alpha : X_\omega \to X_\omega$ preserving the fiber orientation O, we use notation $\operatorname{rot}\alpha$ for the rotational part of the induced isometry $\overline{\alpha} : B \to B$. An isometry $\gamma : X_\omega \to X_\omega$ is called a *shift*, if γ preserves O and $\operatorname{rot}\gamma = \mathrm{id}$.

Lemma 7.6. *For any $x, x' \in X_\omega$ there is a shift $\eta = \eta_{xx'} : X_\omega \to X_\omega$ with $\eta(x) = x'$.*

Proof. Let $\mathbb{O} = \mathbb{O}(B)$ be the isometry group of B preserving a fixed point $\overline{o} \in B$. For $A \in \mathbb{O}$ we put $|A| = \sup_v |vA(v)|$, where the supremum is taken over all unit vectors $v \in B$. Note that $\mathbb{O}$ is compact and that $|A| \le 2$ for every $A \in \mathbb{O}$. We shall use the following well known (and obvious) fact. For every $\delta > 0$ there is $K(\delta) = K(B, \delta) \in \mathbb{N}$ such that for any $A \in \mathbb{O}$ there is an integer m, $1 \le m \le K(\delta)$, with $|A^m| < \delta$.

By Lemma 7.5, for every $\varepsilon > 0$, there are homotheties $\varphi, \psi : X_\omega \to X_\omega$ with coefficient $\lambda = 1/\varepsilon$, $\varphi(x) = x$, $\psi(x') = x'$. Next, there is an integer m, $1 \le m \le K(\varepsilon/K(\varepsilon))$, such that $|\operatorname{rot}\varphi^m| < \varepsilon/K(\varepsilon)$. Applying the same argument to ψ^m, we find an integer r, $1 \le r \le K(\varepsilon)$, such that $|\operatorname{rot}\psi^{mr}| < \varepsilon$. Then also $|\operatorname{rot}\varphi^{mr}| < \varepsilon$, and for $\varphi_\varepsilon = \varphi^{mr}$, $\psi_\varepsilon = \psi^{mr}$ we have $\varphi_\varepsilon(x) = x$, $|\operatorname{rot}\varphi_\varepsilon| < \varepsilon$ and $\psi_\varepsilon(x') = x'$, $|\operatorname{rot}\psi_\varepsilon| < \varepsilon$. For the isometry $\eta_\varepsilon = \psi_\varepsilon^{-1} \circ \varphi_\varepsilon$, we have $|\operatorname{rot}\eta_\varepsilon| < 2\varepsilon$, $|\eta_\varepsilon(x)x'| = |xx'|/\lambda_\varepsilon$, where $\lambda_\varepsilon = \lambda^{mr} \to \infty$ as $\varepsilon \to 0$. Thus $\eta_\varepsilon \to \eta$ as $\varepsilon \to 0$ with $\eta(x) = x'$, $\operatorname{rot}\eta = \mathrm{id}$. The isometry η preserves the fiber orientation O because the homotheties φ, ψ do. □

7.3 Lifting isometry

We fix $\omega \in X$ and use notations $B = B_\omega$, $\pi = \pi_\omega : X_\omega \to B$. Let $P \subset B$ be a *pointed* oriented parallelogram, i.e., we assume that an orientation and a vertex o of P are fixed. We have a map $\tau_P : F \to F$, where $F \subset X_\omega$ is the fiber of π over o, $F = \pi^{-1}(o)$. Namely, given $x \in F$, by axiom ($\mathrm{E}_\mathbb{R}$) there is a unique $\mathbb{R}$-line in X_ω through x that projects down by π to the first (according to the orientation) side of P containing o. In that way, we lift the sides of P to X_ω in the cyclic order according to the orientation and starting with o which is initially lifted to x. Then $\tau_P(x) \in F$ is the resulting lift of the parallelogram sides.

Lemma 7.7. *The map $\tau_P : F \to F$ is an isometry that preserves orientation and, therefore, it acts on F as a vertical shift.*

Proof. The map τ_P is obtained as a composition of four $\mathbb{C}$-line isometries of type $\mu_{bb'}$, see sect. 7.1. Any isometry $\mu_{bb'}$ preserves orientation, see Lemma 7.2. Thus $\tau_P : F \to F$ is an isometry preserving orientation. □

There is a unique vertical shift $\gamma : X_\omega \to X_\omega$ with $\gamma|F = \tau_P$. Furthermore, every shift $\eta : X_\omega \to X_\omega$ commutes with γ, thus the extension γ of τ_P coincides with that of $\tau_{P'}$ for any P' obtained from P by a shift of the base B. We use the same notation for the extension $\tau_P : X_\omega \to X_\omega$ and call it a *lifting isometry*.

Since the group of vertical shifts is commutative, we have $\tau_P \circ \tau_{P'} = \tau_{P'} \circ \tau_P$ for any (pointed oriented) parallelograms and even for any closed oriented polygons $P, P' \subset B$.

Let $Q \subset B$ be a closed, oriented polygon. Adding a segment $qq' \subset B$ between points $q, q' \in Q$ we obtain closed, oriented polygons P, P' such that $Q \cup qq' = P \cup P'$, the orientations of P, P' coincide with that of Q along Q, and the segment $qq' = P \cap P'$ receives from P, P' opposite orientations. In this case we use notation $Q = P \cup P'$.

Lemma 7.8. *In the notation above we have $\tau_Q = \tau_{P'} \circ \tau_P$.*

Proof. We fix $q \in Q \cap P \cap P'$ as the base point. Moving from q along Q in the direction prescribed by the orientation of Q, we also move along one of P, P' according to the induced orientation. We assume without loss of generality that this is the polygon P. In that way, we first lift P to X_ω starting with some point $o \in F$, where F is the fiber of the projection $\pi : X_\omega \to B$ over q, such that the side $q'q \subset P$ is the last one while lifting P. Now, we lift P' to X_ω starting with $o' = \tau_P(o) \in F$ moving first along the side $qq' \subset P'$. Then clearly the resulting lift of P' gives $\tau_Q(o) = \tau_{P'}(o') \in F$. Thus $\tau_Q = \tau_{P'} \circ \tau_P$. □

For a vertical shift $\gamma : X_\omega \to X_\omega$ we denote $|\gamma| = |x\gamma(x)|_\omega$ the displacement of γ. This is independent of $x \in X_\omega$. If γ, γ' are vertical shifts in the same direction, then $|\gamma \circ \gamma'|^2 = |\gamma|^2 + |\gamma'|^2$.

Lemma 7.9. *Given a representation $P = T \cup T'$ of an oriented parallelogram $P \subset B$ by oriented triangles T, T' with induced from P orientations, whose common side qq' is a diagonal of P, we have $\tau_T = \tau_{T'}$.*

Proof. For every $n \in \mathbb{N}$ we consider the subdivision $P = \cup_{a \in A} P_a$ of P into $|A| = n^2$ congruent parallelograms P_a with sides parallel to those of P. We assume that the orientation of each P_a is induced by that of P. The parallelograms $P_a, P_{a'}$ are obtained from each other by a shift of the base B, thus $\tau_a = \tau_{a'}$ for each $a, a' \in A$, where $\tau_a = \tau_{P_a}$.

By Lemma 7.8 we have $\prod_{a\in A}\tau_a=\tau_P$, hence $|\tau_P|^2=n^2|\tau_a|^2$ for every $a\in A$. Therefore $|\tau_a|^2=|\tau_P|^2/n^2\to 0$ as $n\to\infty$. We subdivide the set A into three disjoint subsets $A=C\cup C'\cup D$, where $a\in D$ if and only if the interior of P_a intersects the diagonal qq' of P, and $a\in C$ if $P_a\subset T$, $a\in C'$ if $P_a\subset T'$. Then $|C|=|C'|=\frac{n(n-1)}{2}$, $|D|=n$. We conclude that $\prod_{a\in C}\tau_a=\prod_{a\in C'}\tau_a$ and

$$\left|\prod_{a\in D}\tau_a\right|^2=n|\tau_P|^2/n^2\to 0$$

as $n\to\infty$. By Lemma 7.8, $\tau_P=\tau_T\circ\tau_{T'}$. It follows that

$$\tau_T=\lim_n\prod_{a\in C}\tau_a=\lim_n\prod_{a\in C'}\tau_a=\tau_{T'}.$$

□

The following Lemma will be used in sect. 10.7, see the proof of Proposition 10.22.

Lemma 7.10. *Let $T=vyz\subset B$ be an oriented triangle. Then for the triangle $P=xyz$ with $x\in[vz]$ we have*

$$|\tau_P|^2=\frac{|xz|}{|vz|}|\tau_T|^2.$$

Proof. Arguing as in Lemma 7.8 for the oriented parallelogram $Q=vyzw$ (for which vz is a diagonal) we find that $\tau_P=\tau_{P'}$ for $P=xyz$, $P'=vyx$ in the case x is the midpoint of $[vz]$. Therefore $|\tau_P|^2=\frac{1}{2}|\tau_T|^2=\frac{|xz|}{|vz|}|\tau_T|^2$ in this case. Next we obtain by induction the required formula for the case $|xz|/|vz|$ is a dyadic number. Then the general case follows by continuity. □

Lemma 7.11. *Let $P\subset B$ be an oriented parallelogram, $\lambda P\subset B$ the parallelogram obtained from P by a homothety $h:B\to B$ with coefficient $\lambda>0$, $h(o)=o$, $h(v)=\lambda v$. Then $|\tau_{\lambda P}|=\lambda|\tau_P|$.*

Proof. We can assume that the parallelograms P, λP have o as a common vertex, and that λ is rational. For a general λ one needs to use approximation. We choose $n\in\mathbb{N}$ such that $\lambda n\in\mathbb{N}$ and subdivide P into n^2 congruent parallelograms, $P=\cup_{a\in A}P_a$, $|A|=n^2$, as in Lemma 7.9. This also gives a subdivision of λP into λ^2n^2 parallelograms congruent to ones of the first subdivision, $\lambda P=\cup_{a\in A'}P_a$, $|A'|=\lambda^2n^2$. Then $\tau_P=\prod_{a\in A}\tau_a$, $|\tau_a|^2=|\tau_P|^2/n^2$ and

$$|\tau_{\lambda P}|^2=\lambda^2n^2|\tau_a|^2=\lambda^2|\tau_P|^2.$$

□

7.4 Reflections with respect to $\mathbb{C}$-circles

In sect. 4 we have defined for every $\mathbb{C}$-circle $F\subset X$ the reflection $\varphi_F:X\to X$ whose fixed point set is F and $v=\varphi_F(u)$ is conjugate to u pole of F for every $u\in X\setminus F$.

Proposition 7.12. *For every $\mathbb{C}$-circle $F\subset X$ the reflection $\varphi_F:X\to X$ is Möbius.*

Proof. We fix $\omega\in F$ and consider F as a $\mathbb{C}$-line in X_ω. By definition, $\varphi_F:X_\omega\to X_\omega$ preserves every $\mathbb{R}$-line $\sigma\subset X_\omega$ intersecting F and acts on σ as the reflection with respect to $\sigma\cap F$, in particular, $\varphi_F|\sigma$ is isometric. It follows from Lemma 6.4 that φ_F is isometric on every $\mathbb{C}$-line in X_ω, thus φ_F induces the central symmetry $\overline{\varphi}=\overline{\varphi}_F:B_\omega\to B_\omega$ with respect to $\overline{o}=\pi_\omega(F)\in B_\omega$.

Given $x,x'\in X_\omega$ we show that $|yy'|=|xx'|$ for $y=\varphi_F(x)$, $y'=\varphi_F(x')$. Let $F_x,F_y\subset X_\omega$ be the $\mathbb{C}$-lines through x, y respectively. We denote $z=\mu_{F_x,\omega}(x')\in F_x$, $z'=\mu_{F_y,\omega}(y')\in F_y$. By Proposition 5.2 we have

$$|xx'|^4=|x'z|^4+|zx|^4\quad\text{and}\quad|yy'|^4=|y'z'|^4+|z'y|^4.$$

Furthermore, $|x'z| = |y'z'|$ because $|x'z| = |\overline{x}'\overline{z}|$, $|y'z'| = |\overline{y}'\overline{z}'|$ and $\overline{\varphi}(\overline{x}') = \overline{y}'$, $\overline{\varphi}(\overline{z}) = \overline{z}'$, where "bar" means the projection by π_ω.

The triangles $\overline{o}\,\overline{x}'\overline{z}$ and $\overline{o}\,\overline{y}'\overline{z}'$ in B are symmetric to each other by $\overline{\varphi}$ and have the same orientation. It follows from Lemma 7.9 that any lift of the closed polygon $P = \overline{x}'\overline{y}'\overline{z}'\overline{z}\,\overline{x}' \subset B$ closes up in X_ω, that is, $\tau_P = \mathrm{id}$. Hence $\mu_{F_y,\omega}(z) = z'$ and therefore $|xz| = |yz'|$. We conclude that $|yy'| = |xx'|$. □

7.5 Pure homotheties

For an $\mathbb{R}$-line $\sigma \subset X_\omega$ we define the *semi-$\mathbb{C}$-plane* $R = R_\sigma \subset X_\omega$ as $R = \pi_\omega^{-1}(\pi_\omega(\sigma))$. Note that R has two foliations: one by $\mathbb{C}$-lines and another by $\mathbb{R}$-lines.

Given a $\mathbb{C}$-line $F \subset X_\omega$ and an $\mathbb{R}$-line $\sigma \subset X_\omega$ with $o = F \cap \sigma$, let $R \subset X_\omega$ be the semi-$\mathbb{C}$-plane *spanned* by F and σ. Every point $x \in R$ is uniquely determined by its projections the *vertical* to F, $x_F = \mu_{F,\omega}(x)$, and the *horizontal* to σ, $x_\sigma = \sigma \cap F_x$, where F_x is the $\mathbb{C}$-line through x.

For $o \in X_\omega$ and $\lambda > 0$ we define a map $h = h_{o,\lambda} : X_\omega \to X_\omega$ as follows. We put $h(o) = o$ and require that h preserves the $\mathbb{C}$-line F through o and every $\mathbb{R}$-line σ through o acting on F and σ as the homotheties with coefficient λ. Finally, h preserves every semi-$\mathbb{C}$-plane R containing F and acts on R by $h(x_F, x_\sigma) = (h(x_F), h(x_\sigma))$, where R is spanned by F and the $\mathbb{R}$-line σ through o.

Proposition 7.13. *The map $h : X_\omega \to X_\omega$ defined above is a homothety with coefficient λ, $|h(x)h(y)| = \lambda|xy|$ for every $x, y \in X_\omega$, in particular, h is Möbius.*

Proof. It follows from definition and Proposition 5.2 that the restriction $h|R$ is the required homothety for every semi-$\mathbb{C}$-plane $R \subset X_\omega$ containing the $\mathbb{C}$-line F through o. Thus the induced map $\overline{h} : B_\omega \to B_\omega$ is the homothety with coefficient λ.

Let F_x be the $\mathbb{C}$-line through x, σ the $\mathbb{R}$-line through o that intersect F_x. Similarly, let F_y be the $\mathbb{C}$-line through y, γ the $\mathbb{R}$-line through o that intersect F_y. We denote $z = \mu_{F_x,\omega}(y) \in F_x$ the projection of y to F_x. Then by Proposition 5.2 we have $|xy|^4 = |xz|^4 + |zy|^4$.

For the closed polygon $P = \overline{o}\,\overline{y}\,\overline{z}\,\overline{o} \subset B_\omega$, where "bar" means the projection by π_ω, we have $\tau_P(y_F) = z_F$, where $y = (y_F, y_\gamma)$, $z = (z_F, z_\sigma)$ are vertical and horizontal coordinates in respective semi-$\mathbb{C}$-planes.

Applying the map h we obtain $x' = h(x)$, $y' = h(y)$, $z' = h(z)$, $y'_F = h(y_F)$, $z'_F = h(z_F)$. Furthermore, the polygon $\lambda P = \overline{h}(P) = \overline{o}\,\overline{y}'\overline{z}'\overline{o}$ is obtained from P by the homothety $\overline{h} : B_\omega \to B_\omega$, $\overline{h}(\overline{o}) = \overline{o}$, in particular $|\overline{y}'\overline{z}'| = \lambda|\overline{y}\,\overline{z}| = \lambda|yz|$. Using the fact that $h \circ \tau_a = \tau_{\lambda a} \circ h$ for any vertical shift $\tau_a : F \to F$ with $|\tau_a| = a \geq 0$ and Lemmas 7.9, 7.11 we obtain $\tau_{\lambda P}(y'_F) = z'_F$. It means that $z' = \mu_{F_{x'},\omega}(y')$. Therefore, $|y'z'| = \lambda|yz|$ and $|x'y'|^4 = |x'z'|^4 + |z'y'|^4$ again by Proposition 5.2. Since $|x'z'| = \lambda|xz|$, we obtain $|x'y'| = \lambda|xy|$. □

By definition, the homothety $h : X_\omega \to X_\omega$, $h(o) = o$, preserves every $\mathbb{R}$-line $\sigma \subset X_\omega$ through o. Every homothety with this property is said to be *pure*.

8 Orthogonal complements to a $\mathbb{C}$-circle

8.1 Definition and properties

Let $F \subset X$ be a $\mathbb{C}$-circle. Every $u \in X \setminus F$ determines an involution $\eta_u : F \to F$ without fixed points, which is Möbius by Corollary 4.9. In other words, we have a map $\mathcal{F} : X \setminus F \to J$, $u \mapsto \eta_u$, where $J = J_F$ is the set of Möbius involutions of F without fixed points. We study fibers of this map, $\mathcal{F}^{-1}(\eta) =: (F, \eta)^\perp$. The set $(F, \eta)^\perp = \{u \in X \setminus F : \eta_u = \eta\}$ is called the *orthogonal complement* to F at η.

Given distinct $x, y \in F$ let $S_{x,y} \subset X$ be the set covered by all $\mathbb{R}$-circles in X through x, y. By Lemma 4.7, this set can be described as the sphere in X between $o, \omega \in F$ such that $(x, o, y, \omega) \in \mathrm{Harm}_F$. Then for every

$u \in S_{x,y} \setminus \{x, y\}$ we have $\eta_u(x) = y$. Thus

$$(F, \eta)^{\perp} = \bigcap_{x \in F} S_{x,\eta(x)} \tag{3}$$

Lemma 8.1. *Given a $\mathbb{C}$-circle $F \subset X$ and a Möbius involution $\eta : F \to F$ without fixed points, the orthogonal complement $A = (F, \eta)^{\perp}$ to F at η can be represented as*

$$A = S_{x,y} \cap S_{o,\omega}$$

for any $x, o \in F$, $o \neq x, y = \eta(x)$, where $\omega = \eta(o)$.

Proof. We have $A \subset S_{x,y} \cap S_{o,\omega}$ by Eq. (3). On the other hand, for every $u \in S_{x,y} \cap S_{o,\omega}$ we have $\eta_u = \eta$ along the 4-tuple (x, o, y, ω). Thus $\eta_u = \eta$ because any Möbius $\eta : F \to F$ is uniquely determined by values at three distinct points. Hence $u \in A$. □

Note that the set $(F, \eta)^{\perp}$ contains with every $u \in (F, \eta)^{\perp}$ the conjugate pole $v = \varphi_F(u)$ of F. Thus $\varphi_F : (F, \eta)^{\perp} \to (F, \eta)^{\perp}$ is an involution without fixed points. By Proposition 7.12 this involution is Möbius.

Lemma 8.2. *For each pair $u, v \in (F, \eta)^{\perp}$ of conjugate poles of F the $\mathbb{C}$-circle F' through u, v is contained in $(F, \eta)^{\perp}$.*

Proof. We take distinct $x, o \in F$, $o \neq x, y$, such that $(x, o, y, \omega) \in \mathrm{Harm}_F$, where $y = \eta_u(x)$, $\omega = \eta_u(o)$. There are $\mathbb{R}$-circles $\sigma, \gamma \subset X$ such that $(x, u, y, v) \in \mathrm{Harm}_{\sigma}$, $(o, u, \omega, v) \in \mathrm{Harm}_{\gamma}$. By axiom $\mathrm{O}_{\mathbb{C}}$ the $\mathbb{C}$-circle F' through u, v is contained in $S \cap S'$, where $S, S' \subset X$ are spheres between x, y and o, ω respectively that contain u, v. Since $(x, o, y, \omega) \in \mathrm{Harm}_F$, by axiom $\mathrm{O}_{\mathbb{R}}$ we have $\gamma \subset \widetilde{S}$ and $\sigma \subset \widetilde{S}'$, where $\widetilde{S}, \widetilde{S}'$ are spheres between x, y through o, ω and between o, ω through x, y respectively. Hence $\widetilde{S} = S$, $\widetilde{S}' = S'$. We conclude that $S = S_{o,\omega}$, $S' = S_{x,y}$. Then $S \cap S' = (F, \eta)^{\perp}$ by Lemma 8.1, hence $F' \subset (F, \eta)^{\perp}$. □

Lemma 8.3. *Given a $\mathbb{C}$-circle $F \subset X$ and a Möbius involution $\eta : F \to F$ without fixed points, the orthogonal complement $A = (F, \eta)^{\perp}$ is foliated by $\mathbb{C}$-circles F' through pairs $u, v \in A$ of conjugate poles of F, and $\varphi_F : A \to A$ preserves every fiber of this fibration.*

Proof. By Lemma 8.2, A is covered by $\mathbb{C}$-circles F'. We have $\varphi_F(F') = F'$ because φ_F permutes conjugate poles of F. Thus by axiom $\mathrm{E}_{\mathbb{C}}$ distinct $\mathbb{C}$-circles of the covering are disjoint, that is, the $\mathbb{C}$-circles F' form a fibration of A. □

The fibration of $(F, \eta)^{\perp}$ by $\mathbb{C}$-circles described in Lemma 8.3 is called *canonical*. We do not claim that every $\mathbb{C}$-circle in $(F, \eta)^{\perp}$ is a fiber of the canonical fibration, this is actually not true in general.

Lemma 8.4. *Given a Möbius involution $\eta : F \to F$ without fixed points of a $\mathbb{C}$-circle F, for every fiber F' of the canonical fibration of $A = (F, \eta)^{\perp}$ the reflection $\varphi' = \varphi_{F'} : X \to X$ preserves A and its canonical fibration.*

Proof. For every $u \in A$ and $x \in F$ there is an $\mathbb{R}$-circle $\sigma \subset X$ such that $(x, u, y, v) \in \mathrm{Harm}_{\sigma}$, where $y = \eta(x)$, $v = \varphi_F(u)$. Taking $u \in F'$ we observe that by definition φ' permutes x, y, thus φ' preserves F and $\varphi'|F = \eta$. Since φ' is Möbius, we have

$$\varphi'(A) = (\varphi'(F), \varphi' \circ \eta \circ \varphi'^{-1})^{\perp} = (F, \eta)^{\perp} = A.$$

For an arbitrary $u \in A$ we have $\varphi'(x, u, y, v) = (y, u', x, v') \in \mathrm{Harm}_{\sigma'}$, where $\sigma' = \varphi'(\sigma)$, and $u', v' \in A$ are conjugate poles of F. Thus φ' moves the fiber of the canonical fibration through u, v to the fiber through u', v'. □

8.2 Mutually orthogonal $\mathbb{C}$-circles

We say that distinct $\mathbb{C}$-circles $F, F' \subset X$ are *mutually orthogonal* to each other, $F \perp F'$, if $\varphi_F(F') = F'$ and $\varphi_{F'}(F) = F$. Note that then F, F' are disjoint because by Lemma 3.4 a $\mathbb{C}$-circle and an $\mathbb{R}$-circle have in common at most two points. If $\varphi_F(F') = F'$, then $\varphi_{F'}(F) = F$ automatically by definition of φ_F. If $\eta = \varphi_{F'}|F$ and $\eta' = \varphi_F|F'$, then $F \subset (F', \eta')^\perp$ and $F' \subset (F, \eta)^\perp$. We also note that if $F \perp F'$, then φ_F acts on F' as a Möbius involution without fixed points.

Lemma 8.5. *Assume a $\mathbb{C}$-circle $F \subset X$ is invariant under a Möbius $\varphi' : X \to X$. Then $\varphi = \varphi_F$ commutes with φ', $\varphi \circ \varphi' = \varphi' \circ \varphi$. In particular, $\varphi_F \circ \varphi_{F'} = \varphi_{F'} \circ \varphi_F$ for mutually orthogonal $\mathbb{C}$-circles F, F'.*

Proof. Since F is the fixed point set for φ, we have $\varphi \circ \varphi' = \varphi' \circ \varphi$ along F. Thus to prove the equality $\varphi \circ \varphi'(x) = \varphi' \circ \varphi(x)$ for an arbitrary $x \in X$, we can assume that $x \notin F$.

For an arbitrary $\omega \in F$ there is a (uniquely determined) $\mathbb{R}$-circle $\sigma \subset X$ such that $(x, z, y, \omega) \in \mathrm{Harm}_\sigma$, where $y = \varphi(x)$, $z = \eta_x(\omega) \in F_\omega \cap \sigma$. Then for the $\mathbb{R}$-circle $\sigma' = \varphi'(\sigma)$ we have $(x', z', y', \omega') = \varphi'(x, z, y, \omega) \in \mathrm{Harm}_{\sigma'}$, where $\omega', z' = \eta_{x'}(\omega') \in F$. This means that $y' = \varphi(x')$, that is, $\varphi' \circ \varphi(x) = \varphi \circ \varphi'(x)$. □

A collection $\{F_\lambda : \lambda \in \Lambda\}$ of $\mathbb{C}$-circles is said to be *orthogonal*, if $F_\lambda \perp F_{\lambda'}$ for each distinct $\lambda, \lambda' \in \Lambda$. We denote by $\varphi_\lambda = \varphi_{F_\lambda}$ the respective Möbius involutions. By Lemma 8.5, $\varphi_\lambda \circ \varphi_{\lambda'} = \varphi_{\lambda'} \circ \varphi_\lambda$ for each $\lambda, \lambda' \in \Lambda$.

Lemma 8.6. *Let $\{F_\lambda : \lambda \in \Lambda\}$ be an orthogonal collection of $\mathbb{C}$-circles. Then for distinct $\lambda, \lambda', \lambda'' \in \Lambda$ we have $\varphi_\lambda|F_{\lambda''} = \varphi_{\lambda'}|F_{\lambda''}$, in particular, $F_{\lambda''}$ lies in the fixed point set of $\varphi_\lambda \circ \varphi_{\lambda'}$.*

Proof. By definition, $\varphi_\lambda, \varphi_{\lambda'}$ preserve $F_{\lambda''}$ and therefore act on $F_{\lambda''}$ as Möbius involutions without fixed points. Since they commute, the composition $\varphi_\lambda \circ \varphi_{\lambda'}$ has a finite order. It follows that $\varphi_\lambda|F_{\lambda''} = \varphi_{\lambda'}|F_{\lambda''}$ because otherwise $\varphi_\lambda \circ \varphi_{\lambda'}$ would be of infinite order. □

8.3 Intersection of orthogonal complements

Proposition 8.7. *Assume*

$$(F, \eta)^\perp \cap (F', \eta')^\perp = \emptyset$$

for mutually orthogonal $\mathbb{C}$-circles $F, F' \subset X$, where $\eta = \varphi_{F'}|F$, $\eta' = \varphi_F|F'$. Then $\dim X = 3$.

Proof. We let $\dim X = k + 1$ with $k \geq 0$. Then $\dim A = \dim A' = k - 1$ for $A = (F, \eta)^\perp$, $A' = (F', \eta')^\perp$. By Lemma 8.3, A is foliated by $\mathbb{C}$-circles, thus k is even. If $k = 0$, then $X = \partial_\infty M$ for $M = \mathbb{C}\mathrm{H}^1$. We can assume that $k \geq 2$. Note that the codimension of A equals two and that F is not contractible in $X \setminus A$. Thus the assumption $A \cap A' = \emptyset$ implies by transversality argument that $2(k - 1) < k + 1$. Hence $k = 2$ and $\dim X = 3$. □

Proposition 8.8. *Let $F, F' \subset X$ be mutually orthogonal $\mathbb{C}$-circles, $F \perp F'$, $A = (F, \eta)^\perp$, $A' = (F', \eta')^\perp$, where $\eta = \varphi'|F$, $\eta' = \varphi|F'$ and $\varphi = \varphi_F$, $\varphi' = \varphi_{F'}$. Then for every $u \in A \cap A'$ we have $\varphi(u) = \varphi'(u)$.*

Proof. We denote $v = \varphi(u)$, $v' = \varphi'(u)$. We fix $o \in F$, $x \in F'$ and put $\omega = \varphi'(o) \in F$, $y = \varphi(x) \in F'$. Since $F' \subset A$, there is an $\mathbb{R}$-circle $\sigma \subset X$ such that $(x, o, y, \omega) \in \mathrm{Harm}_\sigma$. Since $u \in A$, there is an $\mathbb{R}$-circle $\gamma \subset X$ such that $(u, o, v, \omega) \in \mathrm{Harm}_\gamma$.

Lemma 8.9. *In the metric of X_ω we have $|vy|_\omega = |ux|_\omega$, $|vx|_\omega = |yu|_\omega$, $|ox|_\omega = |ou|_\omega = |ov|_\omega$.*

Proof. The $\mathbb{C}$-circle F is the fixed point set of the Möbius $\varphi : X \to X$. Thus φ acts on X_ω as an isometry. Using that $\varphi(u) = v$ and $\varphi(x) = y$, we obtain $|vy|_\omega = |ux|_\omega$, $|vx|_\omega = |yu|_\omega$. Recall that $u, v, x, y \in A$ and A lies in a sphere between o, ω. Thus $|ox|_\omega = |ou|_\omega = |ov|_\omega$. □

Since $u \in A'$, there is an $\mathbb{R}$-circle $\gamma' \subset X$ such that $(u, x, v', y) \in \mathrm{Harm}_{\gamma'}$.

Lemma 8.10. *In the metric of X_x we have $|v'\omega|_x = |ou|_x$, $|v'o|_x = |u\omega|_x$, $|v'y|_x = |y\omega|_x = |uy|_x$.*

Proof. The $\mathbb{C}$-circle F' is the fixed point set of the Möbius $\varphi' : X \to X$. Thus φ' acts on X_x as an isometry. Using that $\varphi'(u) = v'$ and $\varphi'(o) = \omega$, we obtain $|v'\omega|_x = |ou|_x$, $|v'o|_x = |u\omega|_x$. Recall that $u, v', o, \omega \in A'$ and A' lies in a sphere between x, y. Thus $|v'y|_x = |y\omega|_x = |uy|_x$. □

Using Lemma 8.10, we obtain

$$|v'y|_\omega = \frac{|v'y|_x}{|v'\omega|_x \cdot |y\omega|_x} = \frac{1}{|v'\omega|_x} = |v'x|_\omega,$$

and using Lemma 8.9, we obtain

$$|v'x|_\omega = \frac{1}{|v'\omega|_x} = \frac{1}{|ou|_x} = \frac{|ox|_\omega \cdot |ux|_\omega}{|ou|_\omega} = |ux|_\omega.$$

That is, $|v'y|_\omega = |v'x|_\omega = |ux|_\omega$.

Next we show that $|ov'|_\omega = |ov|_\omega$. Using Lemma 8.10, we obtain

$$|ov'|_\omega = \frac{|ov'|_x}{|o\omega|_x \cdot |v'\omega|_x} = \frac{|u\omega|_x}{|o\omega|_x \cdot |ou|_x} = \frac{|ox|_\omega}{|ux|_\omega} \cdot \frac{|ox|_\omega \cdot |ux|_\omega}{|ou|_\omega} = \frac{|ox|_\omega^2}{|ou|_\omega}.$$

By Lemma 8.9, $|ox|_\omega = |ou|_\omega = |ov|_\omega$, hence $|ov'|_\omega = |ov|_\omega$.

Since $(u, x, v', y) \in \mathrm{Harm}_{\gamma'}$, we have

$$|uv'|_\omega \cdot |xy|_\omega = |ux|_\omega \cdot |v'y|_\omega + |uy|_\omega \cdot |xv'|_\omega = 2|ux|_\omega \cdot |v'y|_\omega = 2|ux|_\omega^2.$$

Applying the Ptolemy inequality to the 4-tuple (u, x, v, y), we obtain

$$|uv|_\omega \cdot |xy|_\omega \le |ux|_\omega \cdot |vy|_\omega + |vx|_\omega \cdot |uy|_\omega = |ux|_\omega^2 + |uy|_\omega^2.$$

By Lemma 8.10,

$$|uy|_\omega = \frac{|uy|_x}{|u\omega|_x \cdot |y\omega|_x} = \frac{1}{|u\omega|_x} = |ux|_\omega.$$

Thus

$$|uv|_\omega \cdot |xy|_\omega \le 2|ux|_\omega^2 = |uv'|_\omega \cdot |xy|_\omega.$$

We conclude that $|uv|_\omega \le |uv'|_\omega$.

On the other hand, $|uv|_\omega = |uo|_\omega + |ov|_\omega$ and $|uv'|_\omega \le |uo|_\omega + |ov'|_\omega = |uo|_\omega + |ov|_\omega = |uv|_\omega$. Hence $|uv'|_\omega = |uo|_\omega + |ov'|_\omega$. Therefore, the 4-tuple (u, o, v', ω) satisfies the Ptolemy equality

$$|uv'| \cdot |o\omega| = |uo| \cdot |v'\omega| + |u\omega| \cdot |ov'|$$

and three of its entries u, o, ω lie on the $\mathbb{R}$-circle γ. By Proposition 6.10, $v' \in \gamma$ and hence $v' = v$. □

Proposition 8.11. *Let $F, F' \subset X$ be mutually orthogonal $\mathbb{C}$-circles, $F \perp F'$, $A = (F, \eta)^\perp$, $A' = (F', \eta')^\perp$, where $\eta = \varphi'|F$, $\eta' = \varphi|F'$ and $\varphi = \varphi_F$, $\varphi' = \varphi_{F'}$. Then the intersection $A \cap A'$ is the fixed point set of $\psi = \varphi' \circ \varphi$, and it carries a fibration by $\mathbb{C}$-circles, which coincides with the restriction of the canonical fibrations of A, A' to $A \cap A'$.*

Proof. Assume $u \in A \cap A'$. There is a $\mathbb{C}$-circle $G \subset A$ of the canonical fibration of A passing through u and $\varphi(u)$, and there is a $\mathbb{C}$-circle $G' \subset A'$ of the canonical fibration of A' passing through u and $\varphi'(u)$. By Proposition 8.8, $\varphi(u) = \varphi'(u)$. Hence $G = G' \subset A \cap A'$, i.e. $A \cap A'$ carries a fibration by $\mathbb{C}$-circles, which coincides with the restrictions of the canonical fibrations of A, A' to $A \cap A'$. Furthermore, since φ, φ' are involutions, we have $\psi(u) = \varphi' \circ \varphi(u) = u$, i.e. $A \cap A'$ lies in the fixed point set $\mathrm{Fix}\,\psi$ of ψ.

Assume $\psi(u) = u$ for some $u \in X$. Then $u \notin F \cup F'$ and $v = \varphi(u) = \varphi'(u) \ne u$. Thus there is a unique $\mathbb{C}$-circle F'' through u, v. Then $\varphi(F'') = F'' = \varphi'(F'')$, hence F, F', F'' is an orthogonal collection of $\mathbb{C}$-circles. By Lemma 8.6, $\varphi'' = \varphi_{F''}$ acts on F (F') as φ' (φ) does, $\varphi''|F = \varphi'|F$, $\varphi''|F' = \varphi|F'$, and $F'' \subset \mathrm{Fix}(\varphi' \circ \varphi) = \mathrm{Fix}\,\psi$. Therefore $F'' \subset A \cap A'$ is a $\mathbb{C}$-circle of the canonical fibrations of A, A'. Thus $A \cap A' = \mathrm{Fix}\,\psi$. □

8.4 An induction argument

Lemma 8.12. *Assume a subset $A \subset X$ foliated by $\mathbb{C}$-circles is the fixed point set of a Möbius $\psi : X \to X$. Then A satisfies axioms (E), (O).*

Proof. We only need to check the existence axioms (E). Given distinct $a, a' \in A$, there is a unique $\mathbb{C}$-circle $F \subset X$ through a, a'. Since $\psi(a) = a$, $\psi(a') = a'$, we have $\psi(F) = F$. Then in the space X_a, the Möbius $\psi : X_a \to X_a$ acts as a homothety preserving a'. On the other hand, X_a is foliated by $\mathbb{C}$-lines, one of which, F', lies by the assumption in Fix ψ. Hence, $\psi : X_a \to X_a$ is an isometry pointwise preserving F' and a'. This excludes a possibility that ψ acts on F_a as the reflection at a'. Therefore, $F \subset \text{Fix}\,\psi = A$, which is axiom ($E_{\mathbb{C}}$).

Given a $\mathbb{C}$-circle $F \subset A$, $\omega \in F$ and $u \in A \setminus F$, there is a unique $\mathbb{R}$-circle $\sigma \subset X$ through ω, u that hits F_ω. Thus at least three distinct points of σ lies in Fix ψ. We conclude that ψ pointwise preserves σ, i.e. $\sigma \subset A$, which is axiom ($E_{\mathbb{R}}$). □

Corollary 8.13. *Let $F, F' \subset X$ be mutually orthogonal $\mathbb{C}$-circles, $F \perp F'$, $A = (F, \eta)^\perp$, $A' = (F', \eta')^\perp$, where $\eta = \varphi'|F$, $\eta' = \varphi|F'$ and $\varphi = \varphi_F$, $\varphi' = \varphi_{F'}$. Then the intersection $A \cap A'$ satisfies axioms (E) and (O).*

Proof. Apply Lemma 8.12 to $A \cap A'$ which is by Proposition 8.11 the fixed point set of the Möbius $\psi = \varphi' \circ \varphi$ foliated by $\mathbb{C}$-circles. □

9 Möbius join

9.1 Canonical subspaces orthogonal to a $\mathbb{C}$-circle

Let (F, η) be a $\mathbb{C}$-circle in X with a Möbius involution $\eta : F \to F$ without fixed points. Let $F' \subset (F, \eta)^\perp$ be a nonempty subspace that satisfies axioms (E), (O), and is invariant under the reflection $\varphi_F : X \to X$, $\varphi_F(F') = F'$. In this case we say that F' is a *canonical* subspace orthogonal to F at η, COS for brevity. Note that F' carries a canonical fibration by $\mathbb{C}$-circles induced by φ_F, where the fiber through $x \in F'$ is the uniquely determined $\mathbb{C}$-circle through x, $\varphi_F(x)$. We use notation $\mathcal{F} = \mathcal{F}_{F'}$ for this fibration, and $H \in \mathcal{F}$ means that H is a fiber of $\mathcal{F}$.

Lemma 9.1. *For every $u \in F$, $x \in F'$ there are uniquely determined $v \in F$, $y \in F'$ such that* $(u, x, v, y) \in \text{Harm}_\sigma$ *for an $\mathbb{R}$-circle $\sigma \subset X$.*

Proof. By definition of $(F, \eta)^\perp$, there is a uniquely determined $\mathbb{R}$-circle $\sigma \subset X$ through u, x that hits F at $v = \eta(u)$. Then there is a unique $y = \varphi_F(x) \in \sigma$ such that $(u, x, v, y) \in \text{Harm}_\sigma$. Again, $y \in F'$ by definition of F'. □

We define the *(Möbius) join* $F * F'$ as the union of $\mathbb{R}$-circles $\sigma \subset X$ such that $(u, x, v, y) \in \text{Harm}_\sigma$ with $\{u, v\} = \sigma \cap F$, $\{x, y\} = \sigma \cap F'$, where $v = \eta(u)$, $y = \varphi_F(x)$. Every such a circle is called a *standard* $\mathbb{R}$-circle in $F * F'$.

Lemma 9.2. *For any different standard $\mathbb{R}$-circles $\sigma, \sigma' \subset F * F'$ we have $\sigma \cap \sigma' \subset F \cup F'$.*

Proof. We take $o \in \sigma \cap F$, $o' = \sigma' \cap F$, and put $\omega = \eta(o) \in \sigma \cap F$, $\omega' = \eta(o') \in \sigma' \cap F$. Then by Axiom ($O_{\mathbb{R}}$), for any $u', v' \in F$ such that $(u', o', v', \omega') \in \text{Harm}_F$ the $\mathbb{R}$-circle σ' lies in a sphere $S' \subset X$ between u', v'.

Assume $o' \in \{o, \omega\}$. Without loss of generality, $o' = o$. Then $\omega' = \omega$ and $\sigma \cap \sigma' = \{o, \omega\}$ since otherwise $\sigma = \sigma'$ by Lemma 3.5. Thus we can assume that $o' \neq o, \omega$.

We take $v' = \omega$ and $u' \in F$ such that $(u', o', v', \omega') \in \text{Harm}_F$. In the space X_ω, σ is an $\mathbb{R}$-line through o, and F is a $\mathbb{C}$-line through o, while S' is the metric sphere centered at the midpoint $u' \in F$ between o', ω', $|u'o'| = |u'\omega'| =: r$. The distance function $d_{u'} : \sigma \to \mathbb{R}$, $d_{u'}(x) = |u'x|$, is convex and by Corollary 3.3(i) it is symmetric with respect to o. Thus $d_{u'}$ takes the value r at most two times, that is, σ intersects S' at most two times (actually exactly two times because the pair (o', ω) separates the pair (o', ω') on F by properties of η).

On the other hand, σ intersects F' twice and $F' \subset S'$ because $(F, \eta)^{\perp} \subset S'$. Thus $\sigma \cap \sigma' \subset F \cup F'$. □

Proposition 9.3. *Assume (F, η) is a $\mathbb{C}$-circle in X with a free of fixed point Möbius involution $\eta : F \to F$, $F' \subset X$ a COS to F at η. If $\dim X = \dim F' + 2$, then $X = F \star F'$.*

Proof. We show that every point $u \in X$ lies on a standard $\mathbb{R}$-circle in $F \star F'$. Given $\omega \in F$, we put $o = \eta(\omega) \in F$ and use notation $|xy| = |xy|_\omega$ for the distance between x, y in X_ω. Then F is a $\mathbb{C}$-line in X_ω, and every standard $\mathbb{R}$-circle $\gamma \subset F \star F'$ through o, ω is an $\mathbb{R}$-line in X_ω. Moreover, every $\mathbb{R}$-line $\gamma \subset X_\omega$ through o intersects F' because $\dim X = \dim F' + 2$, furthermore $(o, u, \omega, v) \in \mathrm{Harm}_\gamma$ for $\{u, v\} = \gamma \cap F'$ and thus $\gamma \subset F \star F'$ is a standard $\mathbb{R}$-line. That is, for every $x \in F$ every $\mathbb{R}$-circle $\gamma \subset X$ through x, $y = \eta(x) \in F$ is a standard one in $F \star F'$. Let $w \in F$ be the midpoint between x, y, $|xw| = |wy| = r$. By Lemma 4.7 the sphere $S_r(w) \subset X_\omega$ centered at w of radius r is covered by $\mathbb{R}$-circles through x, y. Thus it suffices to show that for every $u \in X \setminus F$ there is $x \in F$ such that $u \in S_r(w)$, where $w \in F$ is the midpoint between x, $y = \eta(x)$, $r = |xw| = |wy|$.

Let $z = \mu_{F,\omega}(u)$ be the projection of u to F in X_ω, $a = |zu|$, $b = |zo|$. Let $\rho > 0$ be the radius of the sphere in X_ω centered at o that contains F'. For $x \in F$ the condition $y = \eta(x)$ is equivalent to

$$|xo| \cdot |oy| = \rho^2. \tag{4}$$

Assuming that z lies on F between o and x in X_ω, we also have $|xz|^2 = |xo|^2 - b^2$, $|yz|^2 = |yo|^2 + b^2$. By Lemma 5.1, $a^4 = |xz|^2 \cdot |yz|^2$, which gives another equation for $|xo|$, $|yo|$,

$$|xo|^2 - |yo|^2 = \frac{a^4 + b^4 - \rho^4}{b^2}. \tag{5}$$

The equations (4), (5) have a positive solution $(|xo|, |yo|)$, which allows us to find x, $y = \eta(x) \in F$ and the midpoint $w \in F$ between x, y, that is, $r := |wx| = |wy|$ with $|wu| = r$. Indeed, we have $|xz|^2 + |yz|^2 = 2r^2$, $|zw|^2 = r^2 - |xz|^2$. Using $a^4 = |xz|^2 \cdot |yz|^2$ and Proposition 5.2, we obtain

$$|wu|^4 = |zw|^4 + a^4 = r^4 + |xz|^2(|xz|^2 + |yz|^2 - 2r^2) = r^4.$$

This shows that $F \star F' = X$. □

10 Möbius join equivalence

Theorem 10.1. *Let (F, η) be a $\mathbb{C}$-circle in X with a fixed point free Möbius involution $\eta : F \to F$, $F' \subset X$ a canonical subspace orthogonal to F at η. Assume that F' is Möbius equivalent to $\partial_\infty \mathbb{C}\mathrm{H}^k$, $k \geq 1$, taken with the canonical Möbius structure. Then the join $F \star F'$ is Möbius equivalent to $\partial_\infty \mathbb{C}\mathrm{H}^{k+1}$.*

We start the proof with

Lemma 10.2. *Let F, G be Möbius spaces which are equivalent to $Y = \partial_\infty \mathbb{C}\mathrm{H}^k$, $k \geq 1$, taken with the canonical Möbius structure. Given Möbius involutions $\eta : F \to F$, $\eta' : G \to G$ without fixed points, there is a Möbius equivalence $g : F \to G$ that is equivariant with respect to η, η',*

$$\eta' \circ g = g \circ \eta.$$

Proof. If we identify F with Y, then there is $a \in M = \mathbb{C}\mathrm{H}^k$ such that the central symmetry $s_a : M \to M$ induces the involution η on F, $\partial_\infty s_a = \eta$. Similarly, $\eta' = \partial_\infty s_{a'}$ for $a' \in M$. Thus any isometry $f : M \to M$ with $f(a) = a'$ induces a required Möbius equivalence $g = \partial_\infty f$. □

Lemma 10.3. *Let $E \subset M = \mathbb{C}\mathrm{H}^{k+1}$ be a complex hyperbolic plane and $a \in E$. Let $E' \subset M$ be the orthogonal complement to E at a. Denote by $G = \partial_\infty E$, $G' = \partial_\infty E' \subset Y = \partial_\infty M$. Then $\varphi_G | G' = \psi | G'$, where $\psi : Y \to Y$ is the Möbius involution without fixed points induced by the central symmetry $s_a : M \to M$ at a, $\psi = \partial_\infty s_a$.*

Proof. We use the fact that every Möbius $Y \to Y$ is induced by an isometry $M \to M$. Recall that G is the fixed point set of the reflection $\varphi_G : Y \to Y$. Then E is the fixed point set for the isometry $\zeta : M \to M$ with $\partial_\infty \zeta = \varphi_G$, and ζ acts on E' as s_a does, $\zeta|E' = s_a|E'$. Hence $\varphi_G|G' = \psi|G'$. □

10.1 Constructing a map between Möbius joins

We fix a complex hyperbolic plane $E \subset M = \mathbb{C}\mathrm{H}^{k+1}$ and $a \in E$. The orthogonal complement $E' \subset M$ to E at a is isometric to $\mathbb{C}\mathrm{H}^k$. We denote by $G = \partial_\infty E$, $G' = \partial_\infty E'$.

Recall that the boundary at infinity $Y = \partial_\infty M$ taken with the canonical Möbius structure satisfies axioms (E), (O) (see Proposition 2.2). Thus all the notions involved in the definition of the Möbius join $G \star G'$ are well defined for Y.

The central symmetry $s_a : M \to M$ induces the Möbius involution $\psi = \partial_\infty s_a : Y \to Y$ without fixed points, for which G, G' are invariant, $\psi(G) = G$, $\psi(G') = G'$. Furthermore, $G' \subset Y$ is a COS to G at ψ and $\dim Y = \dim G' + 2$. Then by Proposition 9.3, $Y = G \star G'$.

By Lemma 10.2, there is a Möbius equivalence $g : G \to F$, which is equivariant with respect to ψ and η, $g \circ \psi|G = \eta \circ g$. Note that $\varphi_F : X \to X$ acts on F' as a Möbius involution without fixed points. By the assumption, $F' \subset X$ is Möbius equivalent to $G' \subset Y$. By Lemma 10.2 again, there is a Möbius equivalence $g' : G' \to F'$ which is equivariant with respect to ψ and $\psi' = \varphi_F|F'$, $g' \circ \psi|G' = \psi' \circ g'$. We define $\psi' : F \cup F' \to F \cup F'$ by $\psi'|F = \eta$, $\psi'|F' = \varphi_F|F'$, and $f : G \cup G' \to F \cup F'$ by $f|G = g$, $f|G' = g'$. Then f is equivariant with respect to ψ and ψ', $\psi' \circ f = f \circ \psi$. Furthermore, f maps the canonical fibration of G' by $\mathbb{C}$-circles to that of F'.

The intersection $\sigma \cap (F \cup F')$ of every standard $\mathbb{R}$-circle $\sigma \subset F \star F'$ with $F \cup F'$ is invariant under ψ'. Then ψ' uniquely extends to a Möbius $\sigma \to \sigma$, for which we use the same notation ψ'. By Lemma 9.2, different standard $\mathbb{R}$-circles in $F \star F'$ may have common points only in $F \cup F'$. Thus we have a well defined involution $\psi' : F \star F' \to F \star F'$ without fixed points, which is Möbius along F, F' and every standard $\mathbb{R}$-circle in $F \star F'$.

For every standard $\mathbb{R}$-circle $\sigma \subset G \star G'$, we have the map $f : \sigma \cap (G \cup G') \to F \star F' \subset X$, which is equivariant with respect to ψ and ψ'. The map f uniquely extends to a Möbius $f : \sigma \to F \star F'$. By Lemma 9.2 different standard $\mathbb{R}$-circles in $G \star G'$, $F \star F'$ may have common points only in $G \cup G'$, $F \cup F'$ respectively, thus this gives a well defined bijection $f : G \star G' \to F \star F'$ which is Möbius along G, G' and any standard $\mathbb{R}$-circle in Y. Moreover, f is equivariant with respect to ψ, ψ'.

We show that f is Möbius. We fix $\omega \in G$, put $o = \psi(\omega) \in G$, $\omega' = f(\omega)$, $o' = f(o) \in F$ and consider Y_ω, $X_{\omega'}$ with metrics normalized so that $f|G : G \to F$ is an isometry.

It suffices to check that $f : Y_\omega \to (F \star F')_{\omega'}$ is an isometry, $|u'v'|_{\omega'} = |uv|_\omega$ for each $u, v \in Y_\omega$, $u' = f(u)$, $v' = f(v') \in (F \star F')_{\omega'}$. For brevity, we use notation $f_\omega := f : Y_\omega \to (F \star F')_{\omega'}$ regarding f as a map between respective metric spaces.

During the proof we will also consider the maps $f_u : Y_u \to X_{u'}$, where $u' = f(u)$. We view f_u as a map between metric spaces, where on Y_u the metric is defined as the metric inversion (1) of $|\,|_\omega$ and on $X_{u'}$ the metric inversion of $|\,|_{\omega'}$.

Note that then $f|G_u$ is isometric, if $u \in G$.

10.2 Isometricity along standard objects

Lemma 10.4. *The map f_ω is isometric on every standard $\mathbb{R}$-line in Y_ω through o.*

Proof. Recall that Y is a compact Ptolemy space with axioms (E), (O). Thus G' lies in a sphere between o, ω, that is, in a metric sphere in Y_ω centered at o, while F' lies in a metric sphere in $X_{\omega'}$ centered at o', see sect. 8. The respective radii are called the *radii* of G', F' in Y_ω, $X_{\omega'}$ respectively.

There is $u \in G$ such that $(u, o, v, \omega) \in \mathrm{Harm}_G$ for $v = \psi(u)$. Any standard $\mathbb{R}$-circle $\sigma \subset Y$ through u, v lies in a metric sphere in Y_ω centered at o. Then $\sigma' = f(\sigma)$ lies in a metric sphere in $X_{\omega'}$ centered at $o' = f(o)$. Since $f|G$ preserves distances, the metric spheres in Y_ω, $X_{\omega'}$ centered at o, o' containing σ, σ' respectively have equal radii. Hence the radius of G' in Y_ω is the same as the radius of F' in $X_{\omega'}$, $|o'x'|_{\omega'} = |ox|_\omega = r$ for every $x \in G'$, $x' \in F'$.

Any standard $\mathbb{R}$-circle $\sigma \subset Y$ through o, ω hits G' at $x, y = \psi(x)$ and it is an $\mathbb{R}$-line in Y_ω. Thus $f|\sigma : \sigma \to \sigma'$ is a homothety with respect to the metrics of $Y_\omega, X_{\omega'}$. Since $|o'x'|_{\omega'} = r = |ox|_\omega$ for $x' = f(x)$, we observe that $f|\sigma$ is an isometry, that is, f_ω preserves distances along any standard $\mathbb{R}$-line through o. □

Lemma 10.5. *Assume for $u \in G$ the map f_u is isometric along $\mathbb{R}$-lines (not necessarily standard) through some $v \in G_u$ and preserves the distance between some x, y on those lines, $|x'y'|_{u'} = |xy|_u$, where $x' = f(x)$, $y' = f(y)$, $u' = f(u)$. Then $|x'y'|_{\omega'} = |xy|_\omega$.*

Proof. We can assume that $u \neq \omega$ since otherwise there is nothing to prove. Let $\sigma \subset Y_u$ be the $\mathbb{R}$-line through v, x. Then $\sigma' = f(\sigma)$ is an $\mathbb{R}$-line in $(F * F')_{u'}$ through $v' = f(v) \in F$. Recall that, G is a $\mathbb{C}$-line in Y_u. Then by Proposition 5.2 the distance $|x\omega|_u$ is uniquely determined by $|vx|_u, |v\omega|_u$ for every $x \in \sigma$, $|x\omega|_u^4 = |vx|_u^4 + |v\omega|_u^4$. Since f_u is isometric along σ and along G_u, we have $|v'x'|_{u'} = |vx|_u$, $|v'\omega'|_{u'} = |v\omega|_u$. Hence $|x'\omega'|_{u'} = |x\omega|_u$, and similarly $|y'\omega'|_{u'} = |y\omega|_u$. Since

$$|xy|_\omega = \frac{|xy|_u}{|x\omega|_u \cdot |y\omega|_u}$$

and a similar formula holds for $|x'y'|_{\omega'}$, we obtain $|x'y'|_{\omega'} = |xy|_\omega$. □

Corollary 10.6. *The map f_ω is isometric along every standard $\mathbb{R}$-circle $\sigma \subset Y$.*

Proof. We let $\{u, v\} = \sigma \cap G$. Then σ is an $\mathbb{R}$-line in Y_u, hence by Lemma 10.4, f_u is isometric along σ. By Lemma 10.5, f_ω is isometric along σ. □

Lemma 10.7. *For every $u \in G'$ the map f_u, is isometric on G.*

Proof. Given $x, y \in G$, there are standard $\mathbb{R}$-circles $\sigma, \gamma \subset Y$ through u with $x \in \sigma$, $y \in \gamma$. By the assumption, $|x'y'|_{\omega'} = |xy|_\omega$ for $x' = f(x)$, $y' = f(y)$. By Corollary 10.6, $|x'u'|_{\omega'} = |xu|_\omega$ and $|y'u'|_{\omega'} = |yu|_\omega$. Applying the metric inversions with respect to u, u' we obtain $|x'y'|_{u'} = |xy|_u$. □

Lemma 10.8. *The map f_ω is isometric on G'.*

Proof. For $x, y \in G'$ we put $z = \psi(x)$, $u = \psi(y) \in G'$. There are the standard $\mathbb{R}$-circles $\sigma, \gamma \subset Y$ such that $(x, o, z, \omega) \in \mathrm{Harm}_\sigma$, $(y, o, u, \omega) \in \mathrm{Harm}_\gamma$, in particular, σ, γ are $\mathbb{R}$-lines in Y_ω. Hence $|xz|_\omega = |yu|_\omega = 2r$, where $r > 0$ is the radius of G' in Y_ω. For $(x', y', z', u') = f(x, y, z, u) \subset F'$ we have $(x', o', z', \omega') \in \mathrm{Harm}_{\sigma'}$, $(y', o', u', \omega') \in \mathrm{Harm}_{\gamma'}$ for the $\mathbb{R}$-circles $\sigma' = f(\sigma)$, $\gamma' = f(\gamma)$. Hence $|x'z'|_{\omega'} = |y'u'|_{\omega'} = 2r$. Since $f|G' : G' \to F'$ is Möbius, $\mathrm{crt}(x', y', z', u') = \mathrm{crt}(x, y, z, u)$, and we conclude

$$|x'y'|_{\omega'} \cdot |z'u'|_{\omega'} = |xy|_\omega \cdot |zu|_\omega, \quad |x'u'|_{\omega'} \cdot |y'z'|_{\omega'} = |xu|_\omega \cdot |yz|_\omega.$$

On the other hand, $z' = \psi'(x')$, $u' = \psi'(y')$ because f is equivariant with respect to ψ, ψ'. Thus $z' = \varphi_F(x')$, $u' = \varphi_F(y')$. Furthermore, $z = \varphi_G(x)$, $u = \varphi_G(y)$ by Lemma 10.3. Recall that G (F) is the fixed point set of φ_G (φ_F), thus φ_G (φ_F) is an isometry of Y_ω ($X_{\omega'}$) and hence $|zu|_\omega = |xy|_\omega$, $|z'u'|_{\omega'} = |x'y'|_{\omega'}$. Therefore, $|x'y'|_{\omega'} = |xy|_\omega$ for any $x, y \in G'$, i.e. $f_\omega|G'$ is an isometry. □

10.3 $\mathbb{R}$- and COS-foliations of a suspension

Let $H \subset Y$ be a COS to a $\mathbb{C}$-circle $K \subset Y$. For every $u \in K$, $x \in H$ there is a uniquely determined $\mathbb{R}$-circle $\sigma \subset Y$ through u, x that meets K at another point v. We denote by $S_uH = S_{u,v}H$ the set covered by $\mathbb{R}$-circles in Y through u, v which meet H. Note that $\sigma \cap H = \{x, y\}$, $y = \varphi_K(x)$, for every such an $\mathbb{R}$-circle σ. Topologically, S_uH is a *suspension* over H. The points u, v are called the *poles* of S_uH. The set $S_uH \setminus \{u, v\}$ is foliated by $\mathbb{R}$-circles through u, v. Slightly abusing terminology, this foliation is called the $\mathbb{R}$-*foliation* of S_uH. In the metric of Y_u, the circles of the $\mathbb{R}$-foliation are $\mathbb{R}$-lines through v, and K is a $\mathbb{C}$-line.

Let $h = h_\lambda : Y_u \to Y_u$ be a pure homothety (see sect. 7.5) with coefficient $\lambda > 0$ centered at v, $h(v) = v$. The homothety h is induced by an isometry $\gamma : M \to M$, which is a transvection along the geodesic $uv \subset M$, $h = \partial_\infty \gamma$. Then h preserves every $\mathbb{R}$-line $\sigma \subset Y_u$ through v, acting on σ as the homothety with coefficient λ, which preserves orientations of σ. The image $H_\lambda = h(H) \subset Y_u$ is the boundary at infinity of $\widetilde{E}_\lambda = \gamma(\widetilde{E})$, where $\widetilde{E} \subset M$ is the subspace with $\partial_\infty \widetilde{E} = H$, and it is also a COS to K. In that way, the COS's H_λ form a foliation of $S_u H \setminus \{u, v\}$, which is called the *COS-foliation* of $S_u H$. In the case H is a $\mathbb{C}$-circle, we say about $\mathbb{C}$-foliation of $S_u H$.

Lemma 10.9. *Assume f_u is isometric along every R-line σ of the $\mathbb{R}$-foliation of $S_u H$ and along H. Then f_u is isometric along every COS H_λ of the $\mathbb{C}$-foliation.*

Proof. Note that $f_u|\sigma$ is equivariant with respect to h, h', where $h' = h'_\lambda : X_{u'} \to X_{u'}$ is the pure homothety (see sect. 7.5) with coefficient λ with the fixed point $v' = f(v)$. It follows that $f|H_\lambda : H_\lambda \to f(H)_\lambda = h' \circ f(H)$ is an isometry with respect to the metrics of Y_u, $X_{u'}$ because $f|H_\lambda = h' \circ f \circ h^{-1}|H_\lambda$. □

Lemma 10.10. *Let COS H to K be a $\mathbb{C}$-circle. Under the assumption of Lemma 10.9, the map f_u is isometric along the suspension $S_u H$.*

Proof. By the assumption, f_u is isometric along every $\mathbb{R}$-line in $S_u H$ through v. Thus taking $x, y \in S_u H$ we can assume that neither x nor y coincides with u or v. Then there are uniquely determined an $\mathbb{R}$-line $\sigma \subset S_u H$ of the $\mathbb{R}$-foliation through x and a $\mathbb{C}$-circle $H_\lambda \subset S_u H$ of the $\mathbb{C}$-foliation through y. The intersection $\sigma \cap H_\lambda$ consists of two points, say, z, w. Taking the metric inversion with respect to w we obtain

$$|xy|_w = \frac{|xy|_u}{|xw|_u \cdot |yw|_u}$$

and similar expressions for $|xz|_w$, $|yz|_w$. In the space X we have respectively

$$|x'y'|_{w'} = \frac{|x'y'|_{u'}}{|x'w'|_{u'} \cdot |y'w'|_{u'}}$$

and similar expressions for $|x'z'|_{w'}$, $|y'z'|_{w'}$, where “prime” means the image under f.

By the assumption and Lemma 10.9, f_u is isometric along σ and H_λ. Hence $|x'w'|_{u'} = |xw|_u$, $|y'w'|_{u'} = |yw|_u$, and $|x'z'|_{u'} = |xz|_u$, $|y'z'|_{u'} = |yz|_u$. On the other hand,

$$|xy|_w^4 = |xz|_w^4 + |yz|_w^4, \quad |x'y'|_{w'}^4 = |x'z'|_{w'}^4 + |y'z'|_{w'}^4$$

by Proposition 5.2, and we conclude that $|x'y'|_{w'} = |xy|_w$. Therefore $|x'y'|_{u'} = |xy|_u$. □

Lemma 10.11. *For every $u \in G$, every $v \in G'$, the map f_u is isometric along the suspension $S_v G = S_{v,w} G$, $w = \varphi_G(v)$.*

Proof. Every circle σ of the $\mathbb{R}$-foliation of $S_v G$ is standard in Y, and σ intersects G at two points p, $q = \psi(p)$. Then by Lemma 10.4 applied to $\omega = p$, f_p is isometric along σ. Thus f_v is isometric along σ because $v \in \sigma$. By Lemma 10.7, f_v is isometric along G. Using Lemmas 10.9 and 10.10, we obtain that f_v is isometric along $S_v G$, in particular, f is Möbius along $S_v G$. Note that $u \in S_v G$ because $G \subset S_v G$. Thus for any $x, z \in S_v G$ the 4-tuple (v, x, z, u) lies in $S_v G$. Hence $\mathrm{crt}(v', x', z', u') = \mathrm{crt}(v, x, z, u)$, where “prime” means the image under f. On the other hand, $\mathrm{crt}(v, x, z, u) = (|vx|_u : |vz|_u : |xz|_u)$, and $|v'x'|_{u'} = |vx|_u$ by Corollary 10.6 because v, x lie on a standard $\mathbb{R}$-circle. Therefore $|x'z'|_{u'} = |xz|_u$. □

10.4 Isometricity along fibers of the $\mathbb{R}$-foliation

Lemma 10.12. *The polynomial $g(s) = s^4 + c(s + b)^4 - d$ with positive b, c such that $cb^4 - d < 0$ has a unique positive root.*

Proof. We have $\frac{d^2g}{ds^2} = 12s^2 + 12c(s+b)^2 > 0$ for all $s \in \mathbb{R}$, and $\frac{dg}{ds}(0) = 4cb^3 > 0$. Thus g is convex and a unique minimum point $\widetilde{s}$ of g is negative, $\widetilde{s} < 0$. Since $g(\widetilde{s}) < g(0) = cb^4 - d < 0$, there is a unique positive root s_0 of g. □

Proposition 10.13. *Let $S_uH \subset Y$ be a suspension over a $\mathbb{C}$-circle H with the poles u, $v = \varphi_H(u)$, and let K be the $\mathbb{C}$-circle through u, v. Assume f_u is isometric along K and along every $\mathbb{C}$-circle H_λ of the $\mathbb{C}$-foliation of S_uH, and the following assumptions hold*

(i) $f \circ \mu_{K,u}(x) = \mu_{K',u'} \circ f(x)$ and $|v'x'|_{u'} = |vx|_u$ for every $x \in S_uH$, where $\mu_{K,u} : Y_u \to K$ and $\mu_{K',u'} : X_{u'} \to K' = f(K)$ are projections to the respective $\mathbb{C}$-lines,

(ii) for every $x \in S_uH$ and every $\mathbb{C}$-circle H_λ of the $\mathbb{C}$-foliation there is $y \in H_\lambda$ such that $|x'y'|_{u'} = |xy|_u$,

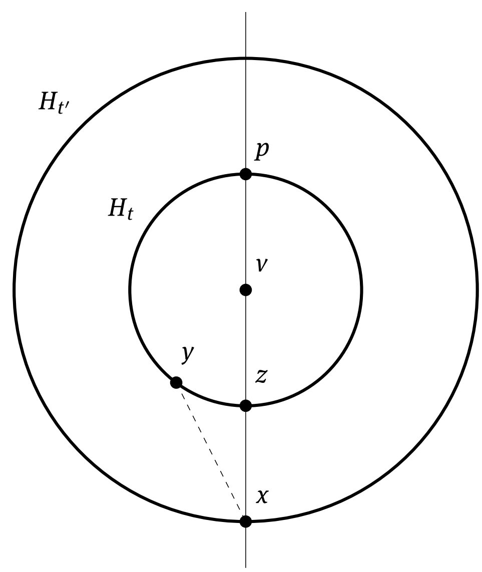
where "prime" means the image under f. Then f_u is isometric along every $\mathbb{R}$-line $\sigma \subset S_uH$ of the $\mathbb{R}$-foliation.

Proof. We fix an arclength parametrization $\sigma : \mathbb{R} \to Y_u$ of σ with $\sigma(0) = v$. We show that $f \circ \sigma : \mathbb{R} \to (F * F')_{u'}$ is an arclength parametrization of an $\mathbb{R}$-line $\sigma' \subset (F * F')_{u'}$ through $v' = f(v)$.

By assumption (i) we know that $f \circ \mu_{K,u}(\sigma(t)) = \mu_{K',u'} \circ f(\sigma(t)) = v'$ and $|f \circ \sigma(t)v'|_{u'} = |\sigma(t)v|_u = |t|$ for every $t \in \mathbb{R}$.

Given $t, t' \in \mathbb{R}$, we assume without loss of generality that $t' > 0$ and $0 < |t| < t'$. We put $x = \sigma(t')$, $z = \sigma(t)$, $p = \sigma(-t)$, and assume furthermore that $|xp|_u > |zp|_u$. Note that in the case $t > 0$ the last assumption follows from $|t| < t'$ because $|xp|_u = t' + t > 2t = |zp|_u$.

We let $H_t \subset S_uH$ be the $\mathbb{C}$-circle of the $\mathbb{C}$-foliation with $\sigma(t) \in H_t$, and note that $x \in H_{t'}$, $z, p \in H_t$.



By assumption (ii), there is $y \in H_t$ such that $|x'y'|_{u'} = |xy|_u$.

Remark 10.14. Since f_u is isometric along H_t, the triangle xyz has two sides $|xy|_u$, $|yz|_u$ preserved by f_u. Our aim is to show that the third distance $|xz|_u$ is also preserved by f_u. In the space Y_p the distance $|xz|_p$ is uniquely determined by $|xy|_p$, $|yz|_p$, $|xz|_p^4 + |yz|_p^4 = |xy|_p^4$ by Proposition 5.2. We can pass to Y_u by a metric inversion. The change of metrics involves the distances $|yp|_u$, $|zp|_u$ which are preserved by f_u, and the distance $|xp|_u = |xz|_u + |zp|_u$. In that way, we obtain an equation of 4th degree for $|xz|_u$ to which we apply Lemma 10.12.

For brevity, we use notation $|xy| = |xy|_u$ for $x, y \in Y_u$. The metric inversion with respect to p gives

$$|xy|_p = \frac{|xy|}{|xp| \cdot |yp|}, \quad |xz|_p = \frac{|xz|}{|xp| \cdot |zp|}, \quad |yz|_p = \frac{|yz|}{|yp| \cdot |zp|}.$$

Thus

$$\frac{|xy|^4}{|xp|^4\cdot|yp|^4} = \frac{|xz|^4}{|xp|^4\cdot|zp|^4} + \frac{|yz|^4}{|yp|^4\cdot|zp|^4}, \tag{6}$$

and we obtain

$$|xz|^4 + \frac{|yz|^4}{|yp|^4}(|xz|+|zp|)^4 = \frac{|xy|^4\cdot|zp|^4}{|yp|^4},$$

which we write as

$$s^4 + c(s+b)^4 = d, \tag{7}$$

where $s = |xz|$, $b = |zp|$, $c = \frac{|yz|^4}{|yp|^4}$, $d = \frac{|xy|^4\cdot|zp|^4}{|yp|^4}$. The coefficients b, c, d are positive and preserved by f. We show that $cb^4 - d < 0$. Using (6), we have

$$|xy|^4 - |yz|^4 = |yz|^4\left(\frac{|xp|^4}{|zp|^4} - 1\right) + \frac{|xz|^4\cdot|yp|^4}{|zp|^4} > 0$$

because $|xp| > |zp|$ by the assumption. Hence

$$cb^4 - d = \frac{|zp|^4}{|yp|^4}(|yz|^4 - |xy|^4) < 0.$$

By Lemma 10.12, the equation (7) has a unique positive solution r_0. Thus $r_0 = |xz|$. Since the same equation with the same coefficients b, c, d holds in the space $(F \star F')_{u'} \subset X_{u'}$ for $|x'z'| = |x'z'|_{u'}$, we obtain $|x'z'| = |xz|$, that is, f_u preserves the distance $|xz| = |xz|_u$.

Hence in the case $t > 0$ we have

$$|v'x'|_{u'} = |vx| = |vz| + |zx| = |v'z'|_{u'} + |z'x'|_{u'}.$$

By assumption (i), the points v', z', u' lie on an $\mathbb{R}$-circle $\sigma' \subset X$, and by the above, the Ptolemy equality holds for $\mathrm{crt}(v', z', x', u')$. Then $x' \in \sigma'$ by Proposition 6.10. This shows that f_u is isometric on the ray $\sigma([0,\infty)) \subset \sigma$, and similarly, f_u is isometric on the opposite ray $\sigma([0,-\infty) \subset \sigma$.

Assuming $t < 0$ with $0 < |t| < t'$ we observe that the equality $|x'z'|_{u'} = |xz|$ implies $|x'z'|_{u'} = |x'v'|_{u'} + |v'z'|_{u'}$ because $|x'v'|_{u'} = |xv|$, $|v'z'|_{u'} = |vz|$ and $|xz| = |xv| + |vz|$. Then by Lemma 4.5, the concatenation of the rays $f\circ\sigma([o,\infty)) \cup f\circ\sigma([0,-\infty))$ is an $\mathbb{R}$-line in $X_{u'}$. Thus f_u is isometric on σ. □

10.5 Isometricity along suspensions over $\mathbb{C}$-circles

Lemma 10.15. *For every $u \in G$ the map f commutes with the projections $\mu_{G,u} : Y_u \to G$, $\mu_{F,u'} : (F \star F')_{u'} \to F$ (in $X_{u'}$), $u' = f(u)$,*

$$\mu_{F,u'} \circ f(w) = f \circ \mu_{G,u}(w)$$

for every $w \in Y_u$. Moreover, $|w'z'|_{u'} = |wz|_u$, where $z = \mu_{G,u}(w)$, $w' = f(w)$, $z' = \mu_{F,u'}(w') = f(z)$.

Proof. We can assume that $w \notin G$. There is a standard $\mathbb{R}$-circle $\sigma \subset Y$ through w. We let $\{x,y\} = \sigma \cap G$, $v = \varphi_G(w) \in \sigma$. Then $(x,w,y,v) \in \mathrm{Harm}_\sigma$ and $(x',w',y',v') \in \mathrm{Harm}_{\sigma'}$ for $\sigma' = f(\sigma)$, $(x',w',v',y') = f(x,w,y,v)$. By Corollary 10.6, f_u is isometric along σ, thus we have $|w'v'|_{u'} = |wv|_u$, $|x'w'|_{u'} = |xw|_u$. Therefore,

$$|z''w'|_{u'} = \frac{1}{2}|w'v'|_{u'} = \frac{1}{2}|wv|_u = |zw|_u.$$

for $z'' = \mu_{F,u'}(w')$, and using Proposition 5.2 we obtain $|zx|_u = |x'z''|_{u'}$. Thus $z'' = z' = f(z)$. □

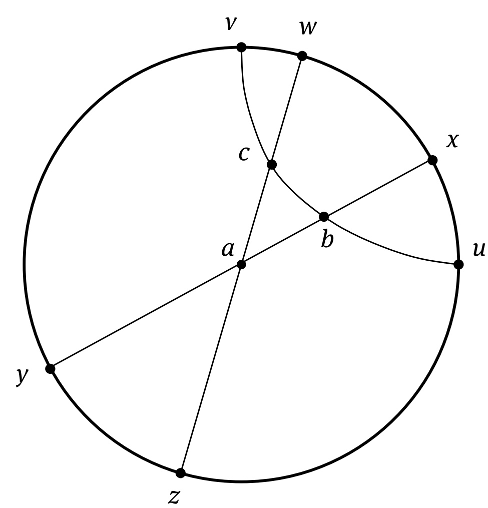
Recall that $G' \subset Y$ is a COS to G at ψ, and G' carries the canonical fibration $\mathcal{F}_{G'}$ by $\mathbb{C}$-circles, where the $\mathbb{C}$-circle $H \subset G'$ of the fibration through $x \in G'$ is determined by $\varphi_G(x) \in H$.

For distinct $u, v \in G$ the sphere $S_{u,v} \subset Y$ formed by all $\mathbb{R}$-circles in Y through u, v is foliated (except u, v) by COS's to G. One can visualize this picture by taking the geodesic $\gamma = uv \subset E$ and regarding the orthogonal projection $\mathrm{pr}_E : M \to E$. Then $S_{u,v} = \partial_\infty \mathrm{pr}_E^{-1}(\gamma)$, while the fibers $G_b = \partial_\infty \mathrm{pr}_E^{-1}(b)$, $b \in \gamma$, are COS's to G, which foliate $S_{u,v} \setminus \{u, v\}$. In the case $v = \psi(u)$, i.e., the geodesic uv passes through a, the $\mathbb{R}$-circles through u, v are standard, and the sphere $S_{u,v}$ is called *standard*. Every standard sphere is a suspension over G'. If $S_{u,v}$ is not standard, then for every $b \in uv \subset E$ the fiber G_b is the intersection $G_b = S_{u,v} \cap S_w$ for the uniquely determined standard sphere $S_w = S_{w,\psi(w)}$, $w \in G$: one takes the geodesic $\gamma_b \subset E$ through a, b, then $S_w = \partial_\infty \operatorname{pr}_E^{-1}(\gamma_b)$, and $w, \psi(w) \in G$ are the ends at infinity of γ_b.

Every COS G_b to G, $b \in uv$, carries the canonical fibration $\mathcal{F}_b = \mathcal{F}_{G_b}$ by $\mathbb{C}$-circles, where the $\mathbb{C}$-circle $H \subset G_b$ through $x \in G_b$ is determined by $\varphi_G(x) \in H$. Note that the $\mathbb{C}$-circles G, H are mutually orthogonal, $G \perp H$, and $v = \varphi_H(u)$. The suspension $S_{u,v}H = S_uH$ is a 2-dimensional sphere in $S_{u,v}$.

Lemma 10.16. *Given distinct $u, v \in G$ with $v \neq \psi(u)$ and a $\mathbb{C}$-circle $H \in \mathcal{F}_{G'}$, there is a uniquely determined suspension $S_u\widehat{H} = S_{u,v}\widehat{H}$, $v = \varphi_{\widehat{H}}(u)$, over a $\mathbb{C}$-circle $\widehat{H} \subset Y$ such that every fiber $\widehat{H}_c$ of the $\mathbb{C}$-foliation of $S_u\widehat{H}$ is represented as $\widehat{H}_c = S_u\widehat{H} \cap S_wH$ for some $w \in G$.*

Proof. Let $uv \subset E$ be the geodesic with end points at infinity u, v. By the assumption, $a \notin uv$. We take $b \in uv$, consider the geodesic $\gamma_b \subset E$ through a, b, put $\{x, y\} = \partial_\infty \gamma_b \subset G$, and let $S_xH = S_{x,y}H$ be the Möbius suspension over H with poles x, y. The suspension S_xH intersects the COS G' over H. We put $\widehat{H} = S_xH \cap G_b$, the fiber of the $\mathbb{C}$-foliation of S_uH over b.



For an arbitrary $c \in uv$ let $\gamma_c \subset E$ be the geodesic through a, c, $\{w, z\} = \partial_\infty \gamma_c \subset G$, and let S_wH be the suspension over H with poles w, z.

Every transvection along a geodesic $\gamma \subset M$ induces a Möbius automorphism of Y, which is a pure homothety of Y_u for any end $u \in \partial_\infty \gamma$. For brevity, we speak about the action of transvections on Y.

The fiber $\widehat{H}_c = S_u\widehat{H} \cap G_c$ of the $\mathbb{C}$-foliation of $S_u\widehat{H}$ over c is obtained from $\widehat{H}$ by the transvection along uv, which moves b into c, and $\widehat{H}$ is obtained from H by the transvection along γ_b, which moves a into b. Thus $\widehat{H}_c$ is obtained from H by the composition of these transvections.

On the other hand, the complex structure J of M is parallel, thus the holonomy of the normal bundle $E^\perp$ along any loop in E preserves any complex hyperbolic plane in $E^\perp$. It follows that $\widehat{H}_c$ is obtained from H by the transvection along γ_c that moves a into c. Therefore, $\widehat{H}_c = S_u\widehat{H} \cap S_wH$. □

Lemma 10.17. *For every distinct $u, v \in G$ the map f_ω is isometric along any suspension $S_u\widehat{H} = S_{u,v}\widehat{H}$, $v = \varphi_{\widehat{H}}(u)$, over a $\mathbb{C}$-circle $\widehat{H}$ that is orthogonal to G.*

Proof. As above we use a "prime" notation for images under f, and the notation $f_u := f : Y_u \to (F * F')_{u'}$, $u \in Y$.

We first note that for every $u \in G$, the map f_ω is isometric along any *standard* suspension $S_uH = S_{u,v}H$, where $H \subset G'$ is a $\mathbb{C}$-circle of the canonical fibration and $v = \psi(u) = \varphi_H(u)$. Indeed, the $\mathbb{R}$-foliation of S_uH consists of standard $\mathbb{R}$-circles, and by Lemma 10.4, f_u is isometric along every of them. By Lemma 10.8, f_u is isometric on H. Thus by Lemma 10.10, f_u is isometric on S_uH. Then by Lemma 10.5, f_ω is isometric on S_uH.

Now, we show that f_ω is isometric along every suspension $S_u\widehat{H} = S_{u,v}\widehat{H}$, $u \in G$, where $v = \varphi_{\widehat{H}}(u) \neq \psi(u)$. We check that the assumptions of Proposition 10.13 are satisfied. In this case, $K = G$, and f_u is isometric along G because f_ω is assumed to be isometric along G and $u \in G$. By the first part of the argument, f_u is isometric on every $\mathbb{C}$-circle of the $\mathbb{C}$-foliation of any standard suspension S_wH, $w \in G$. Thus by Lemma 10.16, f_u is isometric on every $\mathbb{C}$-circle of the $\mathbb{C}$-foliation of $S_u\widehat{H}$. Property (i) follows from Lemma 10.15.

To check property (ii) of Proposition 10.13, we take $x \in S_u\widehat{H}$ and a $\mathbb{C}$-circle $\widehat{H}_\lambda$ of the $\mathbb{C}$-foliation of $S_u\widehat{H}$. There is a $\mathbb{C}$-circle $\widehat{H}_{\lambda'}$ of the $\mathbb{C}$-foliation with $x \in \widehat{H}_{\lambda'}$. By Lemma 10.16, $\widehat{H}_{\lambda'} = S_u\widehat{H} \cap S_{w'}H$ for some $w' \in G$, and there is an $\mathbb{R}$-circle $\sigma' \subset S_{w'}H$ of the $\mathbb{R}$-foliation with $x \in \sigma'$. The suspension $S_{w'}H$ is standard, and the intersection $\sigma' \cap H$ consists of two points. We take one of them, $q \in \sigma' \cap H$. Again, $\widehat{H}_\lambda = S_u\widehat{H} \cap S_wH$ for some $w \in G$, and there is a (standard) $\mathbb{R}$-circle $\sigma \subset S_wH$ of the $\mathbb{R}$-foliation with $q \in \sigma$. The intersection $\sigma \cap \widehat{H}_\lambda$ consists of two points, and we take one of them, $y \in \sigma \cap \widehat{H}_\lambda$. Note that σ, σ' are standard $\mathbb{R}$-circles with $q \in \sigma \cap \sigma'$. Thus the suspension S_qG over G with poles q, $\varphi_G(q)$ contains x, y. By Lemma 10.11, f_u is isometric on S_qG, hence $|x'y'|_{u'} = |xy|_u$. That is, the assumptions of Proposition 10.13 are satisfied. By that Proposition f_u is isometric along every $\mathbb{R}$-circle of the $\mathbb{R}$-foliation of $S_u\widehat{H}$. Applying Lemma 10.10, we see that f_u is isometric along $S_u\widehat{H}$. Then by Lemma 10.5, f_ω is isometric along $S_u\widehat{H}$. □

Proposition 10.18. *For every $\mathbb{C}$-circle $H \in \mathcal{F}_{G'}$ the map f_ω is isometric along the Möbius join $G * H \subset Y$.*

Proof. Given $x, y \in G * H$, we show that there is a suspension $S_u\widehat{H} = S_{u,v}\widehat{H}$, $u, v = \varphi_{\widehat{H}}(u) \in G$ with $x, y \in S_u\widehat{H}$.

There are standard suspensions S_pH, $S_qH \subset G * H$, $p, q \in G$, over H with $x \in S_pH$, $y \in S_qH$. We can assume that $\overline{x} = \mathrm{pr}_E(x)$, $\overline{y} = \mathrm{pr}_E(y) \in E$ are distinct since otherwise we can take $S_pH = S_qH$ as the required suspension. Then there is a unique geodesic $\gamma \subset E$ through $\overline{x}, \overline{y}$. Let $u, v \in G$ be the ends at infinity of γ. We can assume that $v \neq \psi(u)$, since otherwise again $S_pH = S_qH$. Then by Lemma 10.16, there is a uniquely determined suspension $S_u\widehat{H}$ such that every fiber $\widehat{H}_c$ of the $\mathbb{C}$-foliation of $S_u\widehat{H}$ is represented as $\widehat{H}_c = S_u\widehat{H} \cap S_wH$ for some $w \in G$. Then $x \in S_u\widehat{H} \cap S_pH$, $y \in S_u\widehat{H} \cap S_qH$, in particular, $x, y \in S_u\widehat{H}$.

By Lemma 10.17, $|x'y'|_{\omega'} = |xy|_\omega$, that is, f_ω is isometric along $G * H$. □

Remark 10.19. Proposition 10.18 implies Theorem 10.1 and Theorem 1.1 in the case $\dim X = 3$ because $Y = G * H$ by Proposition 9.3 in that case. Moreover, it implies Theorem 10.1 in the case F, F' are mutually orthogonal $\mathbb{C}$-circles in X.

10.6 Properties of the base revisited

Here we make a digression to prove an important fact, Proposition 10.21.

Let $M = \mathbb{C}\mathrm{H}^2$, $Y = \partial_\infty M$. We fix $o \in M$ and consider a complex reflection $h : M \to M$ with respect to a complex hyperbolic plane $E \subset M$ through o. It acts on Y as a reflection with respect to the $\mathbb{C}$-circle $F = \partial_\infty E \subset Y$. The tangent space $V = T_oM$ is a 2-dimension complex vector space with a hermitian form generated by the Riemannian metric, and $g = dh : V \to V$ is a unitary involution whose fixed point set is the complex line $T_oE \subset V$. We formalize this situation as follows.

Let $V = \mathbb{C}^2$ be the 2-dimensional complex vector space with the standard hermitian form. A unitary involution $g : V \to V$ is said to be *reflection* if $\dim_{\mathbb{C}} \mathrm{Im}(1 - g) = 1$. Note that $\det g = -1$. Thus any product $g \cdot g'$ of unitary involutive reflections lies in $S\mathcal{U}(2)$. We need the following simple

Lemma 10.20. *Let $\mathcal{F} \subset \mathcal{U}(2)$ be the set of unitary involutive reflections $V \to V$. Then the subgroup $G \subset S\mathcal{U}(2)$ generated by products $g \cdot g'$ with $g, g' \in \mathcal{F}$ is transitive on the unit sphere $S \subset V$, and for $g \in \mathcal{F}$ there is a dense subset $\widetilde{\mathcal{F}} \subset \mathcal{F}$ such that $g' \cdot g$ has a finite order for every $g' \in \widetilde{\mathcal{F}}$.*

Proof. We identify $\mathcal{F}$ with the set of complex lines in V and fix $g \in \mathcal{F}$. Then we have a decomposition $V = \mathbb{C}\oplus\mathbb{C}$ such that $\mathbb{C} \oplus \{0\} = \operatorname{Fix} g$ is the fixed point set of g, and g acts on V as $g(a, b) = (a, -b)$. Note that g preserves any real 2-subspace $L \subset V$ spanned by $(a, 0)$, $(0, b)$ with $a, b \in \mathbb{C}$, $|a| = |b| = 1$, acting on L as the reflection with respect to $L \cap \operatorname{Fix} g$.

Each pair of opposite points in $L \cap S$ is represented as $L \cap g' \cap S$ for some $g' \in \mathcal{F}$, $g' \cdot g$ acts on L by a rotation, and the subgroup in G preserving L acts transitively on $L \cap S$. If $(g' \cdot g)^k$ is identical on L, then it is identical, $(g' \cdot g)^k = \mathrm{id}$. Thus there is a dense subset $\widetilde{\mathcal{F}} \subset \mathcal{F}$ such that $g' \circ g$ has a finite order for every $g' \in \widetilde{\mathcal{F}}$.

Given $u, v \in S$, $u = (a_u, b_u)$, $v = (a_v, b_v)$, we can find $g_u, g_v \in G$ such that $g_u(u) = (a_u, 0)$, $g_v(v) = (0, b_v)$. Let $L \subset V$ be the real 2-subspace $L \subset V$ spanned by $(a_u, 0)$, $(0, b_v)$. Then there is $h \in G$ with $h((a_u, 0)) = (0, b_v)$. Hence $g_v^{-1} \circ h \circ g_u(u) = v$. □

Proposition 10.21. *For every $\omega \in X$ the canonical metric on the base B_ω of the projection $\pi_\omega : X_\omega \to B_\omega$ is an Euclidean one, and the dimension of B_ω is even.*

Proof. We fix a $\mathbb{C}$-circle $F \subset X$ through ω and a Möbius involution $\eta : F \to F$ without fixed points. We show that the group G of isometries $X_\omega \to X_\omega$ preserving $o = \eta(\omega)$ acts on $(F, \eta)^\perp$ transitively.

There is $x \in F$, $x \neq \omega, o$, such that $(x, o, y, \omega) \in \operatorname{Harm}_F$, where $y = \eta(x)$. By Lemma 8.1 the orthogonal complement $A = (F, \eta)^\perp \subset X$ can be represented as $A = S_{x,y} \cap S_{o,\omega}$, where $S_{x,y}$ is the sphere in X between o, ω through x, y and $S_{o,\omega}$ is the sphere in X between x, y through o, ω. In the space X_ω the sphere $S_{o,\omega}$ consists of $\mathbb{R}$-lines through o, and $A = \{a \in S_{o,\omega} : |oa| = r\}$ for some $r > 0$, where $|oa| = |oa|_\omega$. For simplicity of notation we assume that $r = 1$.

By Lemma 8.3 there is a canonical fibration $\mathcal{F}$ of A by $\mathbb{C}$-circles each of which is invariant under the reflection $\varphi_F : X_\omega \to X_\omega$ with respect to F, in particular, $\dim A \geq 1$ is odd. Since $\pi_\omega|A : A \to \pi_\omega(A)$ is homeomorphism, the unit sphere $\overline{A} = \pi_\omega(A) \subset B = B_\omega$ centered at $\overline{o} = \pi_\omega(o)$ has an odd dimension ≥ 1. It follows that $\dim B$ is even.

Every $H \in \mathcal{F}$ and F are mutually orthogonal, $F \perp H$. Proposition 10.18 implies that the Möbius join $F * H \subset X$ is Möbius equivalent to the standard model space $Y = \partial_\infty \mathbb{C}\mathrm{H}^2$. If $\dim A = 1$ then $A = H$, $X = F * H$ is Möbius equivalent to $Y = \partial_\infty \mathbb{C}\mathrm{H}^2$, and there is nothing to prove. Thus we assume that $\dim A \geq 3$.

The reflection $\varphi_H : X \to X$ with respect to H preserves F and $\varphi_H|F = \eta$, furthermore, by Lemma 8.4, φ_H preserves A and its fibration $\mathcal{F}$ for every $H \in \mathcal{F}$. Thus the composition $\varphi_{H'} \circ \varphi_H : X \to X$ preserves A and its fibration $\mathcal{F}$, and it acts on F identically for each $\mathbb{C}$-circles $H, H' \in \mathcal{F}$. In particular, $\varphi_{H'} \circ \varphi_H : X_\omega \to X_\omega$ is an isometry.

We fix a $\mathbb{C}$-circle $H \in \mathcal{F}$. For every $H' \in \mathcal{F}$ that is orthogonal to H, $H' \subset (H, \varphi_F|H)^\perp \cap A$, the Möbius join $Z = H * H' \subset X$ is Möbius equivalent to Y by Proposition 10.18, hence Z satisfies axioms (E) and (O). We show that $Z \subset A$. Note that Z is invariant under the reflection φ_F, $\varphi_F(Z) = Z$, because H, H' are. Thus Z carries a fibration $\mathcal{F}_Z$ by $\mathbb{C}$-circles invariant under φ_F, in particular, every $\mathbb{C}$-circle $K \in \mathcal{F}_Z$ is orthogonal to F, $K \perp F$, and $\varphi_K(F) = F$.

To see that $\varphi_K : X \to X$ preserves Z, we note that there is a (unique) $\mathbb{C}$-circle $K' \in \mathcal{F}_Z$ that is orthogonal to K at $\varphi_F|K$, actually, $K' = (K, \varphi_F|K)^\perp$ in Z. Then $\varphi_K(K') = K'$, and since φ_K acts on K identically, we see that $\varphi_K : X \to X$ preserves Z, $\varphi_K(Z) = Z$, because Z can be represented as the Möbius join $Z = K * K'$. Using conjugation via a Möbius isomorphism $Y \to Z$ and the fact that $Y = \partial_\infty \mathbb{C}\mathrm{H}^2$, we can consider φ_K as a unitary involutive reflection of $V = \mathbb{C}^2$. Applying Lemma 10.20, we find a dense subset $\widetilde{\mathcal{F}}_Z \subset \mathcal{F}_Z$ such that the composition $g = \varphi_K \circ \varphi_H$, when restricted to Z, has a finite order, $(g|Z)^k = \mathrm{id}_Z$, for every $\mathbb{C}$-circle $K \in \widetilde{\mathcal{F}}_Z$ and some $k \in \mathbb{N}$ depending on K. Note that $g|F : F \to F$ being a composition of two Möbius involutions $\varphi_K|F$, $\varphi_H|F$ without fixed points has an infinite order or is identical. In the first case there are two fixed points u, $v \in F$ for $g|F$, and g acts on X_u as a nontrivial homothety. This contradicts the fact that $(g|Z)^k = \mathrm{id}_Z$. Therefore, $g|F = \mathrm{id}_F$ and thus $\varphi_K|F = \varphi_H|F = \eta$. Consequently, $K \subset A$. By continuity, this holds for every $K \in \mathcal{F}_Z$, hence $Z \subset A$.

For every $K, K' \in \mathcal{F}_Z$ the product $g = \varphi_K \circ \varphi_{K'} : X \to X$ is identical on F, hence $g : X_\omega \to X_\omega$ is an isometry. By Lemma 10.20 isometries of this type act on Z transitively. Varying $H \in \mathcal{F}$, $H' \in (H, \varphi_F|H)^\perp \cap A$, we see that the Möbius joins $Z = H * H'$ cover A. Thus the group G acts on A transitively. It follows that the group of isometries of the base B preserving $\overline{o}$ acts on the unit sphere $\overline{A} \subset B$ centered at $\overline{o}$ transitively. Using the standard argument with Lövner ellipsoid, we obtain that the unit sphere $\overline{A}$ is an ellipsoid and thus the metric of B is Euclidean. □

10.7 Suspension over a COS

We turn back to the proof of Theorem 10.1 and to notations of sect. 10.1.

Given distinct $u, v \in Y$, let $K \subset Y$ be the $\mathbb{C}$-circle through u, v. The sphere $S_{u,v} \subset Y$ formed by all $\mathbb{R}$-circles in Y through u, v carries $\mathbb{R}$- and COS-foliations, see sect. 10.3. Note that every fiber H of the COS-foliation of $S_{u,v}$ satisfies axioms (E) and (O) because H is the boundary at infinity of an orthogonal complement in $M = \mathbb{C}\mathrm{H}^{k+1}$ to the complex hyperbolic plane $E \subset M$ with $\partial_\infty E = K$.

Now we are able to prove the following generalization of Lemma 10.10.

Proposition 10.22. *Given distinct $u, v \in Y$ assume f_u is isometric along every $\mathbb{R}$-circle of the $\mathbb{R}$-foliation of $S = S_{u,v}$ and every fiber of the COS-foliation of S. Then f_u is isometric along S.*

Proof. We take $x, y \in S$ and show that $|x'y'|_{u'} = |xy|_u$, where we use "prime" notation for images under f.

Every fiber H of the COS-foliation of S lies in a sphere between u, v, that is, there is $r = r(H) > 0$ such that $|vh|_u = r$ for every $h \in H$. In the space Y_u every $\mathbb{R}$-circle σ of the $\mathbb{R}$-foliation of S is an $\mathbb{R}$-line through v. Since f_u is isometric along σ, we can assume that neither x nor y coincides with u or v. Then there are uniquely determined an $\mathbb{R}$-line $\sigma \subset S$ of the $\mathbb{R}$-foliation through x and a fiber $H \subset S$ of the COS-foliation through y, and we also assume that $x \notin H$, $y \notin \sigma$, since otherwise there is nothing to prove. The intersection $\sigma \cap H$ consists of two points, say, z, w such that the pair z, w separate the pair u, v on σ. The points v, z, u, w subdivide σ into four arcs, and we assume without loss of generality that x lies on the arc $vz \subset \sigma$.

Since H satisfies axioms (E) and (O), the $\mathbb{C}$-circle $K \subset Y$ through w, y lies in H, and we assume that $z \notin K$ because otherwise Lemma 10.10 applies. Since K is a $\mathbb{C}$-line in Y_w, we have $\overline{y} \neq \overline{z}$, where "bar" means the image under $\pi_w : Y_w \to B_w$.

It follows from Lemma 3.2 that H together with K lies in the bisector in Y_w between u, v. Thus $|uz|_w = |vz|_w$, $|uk|_w = |vk|_w$ for every $k \in K$ and therefore $|\overline{u}\,\overline{y}| = |\overline{v}\,\overline{y}|$. Note that σ is an $\mathbb{R}$-line also in Y_w, thus $\overline{u}\,\overline{v} = \overline{\sigma}$, $\overline{x} \in \overline{u}\,\overline{v}$ and $\overline{z} \in \overline{u}\,\overline{v}$ is the midpoint between $\overline{u}$, $\overline{v}$. The metric on the base B_w is Euclidean, thus the line $\overline{z}\,\overline{y} \subset B_w$ through $\overline{z}$, $\overline{y}$ is orthogonal to the line $\overline{u}\,\overline{v}$. Therefore $|\overline{x}\,\overline{y}|^2 = |\overline{x}\,\overline{z}|^2 + |\overline{z}\,\overline{y}|^2$.

In the space X, we have the sphere $S_{u',v'}$ with poles u', v', the image $S' = f(S) \subset S_{u',v'}$, the $\mathbb{R}$-circle $\sigma' = f(\sigma) \subset S'$ with $v', x', z', u', w' \in \sigma'$ (in this cyclic order) and $H' = f(H)$ with $y', z', w' \in H'$. By our assumption on f_u we have $|h'v'|_{u'} = r$ for every $h' \in H'$, and $K' = f(K) \subset H'$ is a $\mathbb{C}$-circle through w', y'. Thus H' lies in a sphere in X between u', v', hence H' together with the $\mathbb{C}$-line K' lies in the bisector in $X_{w'}$ between u' and v'.

By Proposition 10.21, the base $B_{w'}$ of the canonical foliation $\pi_{w'} : X_{w'} \to B_{w'}$ is also Euclidean. Thus we obtain as above $|\overline{x}'\overline{y}'|^2 = |\overline{x}'\overline{z}'|^2 + |\overline{z}'\overline{y}'|^2$. Note that f_w is isometric along σ, hence $|\overline{x}\,\overline{z}| = |xz|_w = |x'z'|_{w'} = |\overline{x}'\overline{z}'|$. Since f_u is isometric along H, we see that f_w is also isometric along H applying the metric inversion with respect to w, thus $|\overline{z}\,\overline{y}| = |\overline{z}'\overline{y}'|$. It follows that $|\overline{x}'\overline{y}'| = |\overline{x}\,\overline{y}|$.

Using that $|v'h'|_{u'} = r = |vh|_u$ for every $h \in H$ and applying the metric inversion with respect to w, we obtain $|vh|_w = \frac{1}{|hw|_u} = |uh|_w$, $|v'h'|_{w'} = \frac{1}{|h'w'|_{u'}} = |u'h'|_{w'}$. Hence the map f_w preserves the distances of u, v to points of K, $|v'h'|_{w'} = |u'h'|_{w'} = |vh|_w = |uh|_w$, in particular, $|v'y'|_{w'} = |vy|_w$.

Let $\widetilde{v} = \mu_{K,w}(v)$, $\widetilde{x} = \mu_{K,w}(x)$, $\widetilde{z} = \mu_{K,w}(z) \in K$ be the projections in Y_w of v, x, z to the $\mathbb{C}$-line K. Since the map f_w preserves the distances of v to points of K, we see using the distance formula that $f(\widetilde{v}) = \mu_{K',w'}(v') =: \widetilde{v}'$, i.e. the projections $\mu_{K,w} : Y_w \to K$, $\mu_{K',w'} : X_{w'} \to K'$ commute with f at v. Similarly, $f(\widetilde{z}) = \mu_{K',w'}(z') =: \widetilde{z}'$.

On the other hand, the point $\widetilde{v}$ can be obtained from $\widetilde{z}$ by lifting isometry (see sect. 7.3) of the oriented triangle $T = \overline{v}\,\overline{y}\,\overline{z} \subset B_w$, $\widetilde{v} = \tau_T(\widetilde{z})$. Analogously, in the space $X_{w'}$ the point $\widetilde{v}'$ can be obtained from $\widetilde{z}'$ by lifting isometry of the oriented triangle $T' = \overline{v}'\overline{y}'\overline{z}' \subset B_{w'}$, $\widetilde{v}' = \tau_{T'}(\widetilde{z}')$. Thus $|\tau_{T'}| = |\widetilde{z}'\widetilde{v}'|_{w'} = |\widetilde{z}\,\widetilde{v}|_w = |\tau_T|$.

Using Lemma 7.10 we obtain

$$|\tau_P|^2 = \frac{|xz|}{|vz|}|\tau_T|^2 = \frac{|x'z'|}{|v'z'|}|\tau_{T'}|^2 = |\tau_{P'}|^2$$

for the oriented triangles $P = \overline{x}\,\overline{y}\,\overline{z} \subset B_w$, $P' = \overline{x}'\overline{y}'\overline{z}' \subset B_{w'}$. We assume that a fiber orientation of $X_w \to B_w$ is fixed and that the fiber orientation of $X_{w'} \to B_{w'}$ is induced by f_w. Then $f(\widetilde{x}) = f \circ \tau_P(\widetilde{z}) = \tau_{P'} \circ f(\widetilde{z})$ and we conclude that the projections $\mu_{K,w}, \mu_{K',w'}$ commute with f at x, $f(\widetilde{x}) = \widetilde{x}'$ for $\widetilde{x}' = \mu_{K',w'}(x')$. Hence

$$|x'y'|_{w'}^4 = |\overline{x}'\overline{y}'|^4 + |y'\widetilde{x}'|_{w'}^4 = |\overline{x}\,\overline{y}|^4 + |y\widetilde{x}|_w^4 = |xy|_w^4$$

by Proposition 5.2. Applying the metric inversion with respect to u and using that $|u'x'|_{w'} = |ux|_w$, $|u'y'|_{w'} = |uy|_w$, we finally obtain $|x'y'|_{u'} = |xy|_u$. □

10.8 Proof of Theorems 10.1 and 1.1

Proof of Theorem 10.1. We show that the map $f_\omega = f : Y_\omega \to (F * F')_{\omega'}$ is isometric.

First, we note that for every $u \in G$ the map f_u is isometric along the standard sphere $S_u = S_{u,v}$, $v = \psi(u)$. Indeed, $S_u = S_u G'$ is a suspension over G', and f_u is isometric along the fibers of the $\mathbb{R}$-foliation of $S_u G'$ because they are standard $\mathbb{R}$-lines in Y_u, and along G' by Lemma 10.8. Thus by Lemma 10.9, f_u is isometric along every fiber of the COS-foliation of $S_u G'$. By Proposition 10.22, f_u is isometric along $S_u = S_u G'$. It follows from Lemma 10.5 that f_ω is isometric along S_u.

Given $x, y \in Y$ we can assume that the points $\overline{x} = \mathrm{pr}_E(x)$, $\overline{y} = \mathrm{pr}_E(y) \in E$ are distinct and neither of them coincides with o, since otherwise x, y lie in a standard sphere S_u for some $u \in G$, hence $|x'y'|_{\omega'} = |xy|_\omega$. Then there is a unique geodesic $\gamma \subset E$ through $\overline{x}, \overline{y}$. Let $u, v \in G$ be the ends at infinity of γ, $v \neq \psi(u)$ by our assumption. Then $x, y \in S_{u,v} = \mathrm{pr}_E^{-1}(\gamma)$.

The map f_u is isometric along the fibers of the COS-foliation of the sphere $S_{u,v}$ because every such a fiber is represented as the fiber $S_{u,v} \cap S_z$ of the COS-foliation of a standard sphere S_z for some $z \in G$. Furthermore, $S_{u,v} = S_{u,v}\widehat{G}$ is a suspension over a fiber $\widehat{G}$ of the COS-foliation of $S_{u,v}$, and every fiber σ of the $\mathbb{R}$-foliation of $S_{u,v}$ is a fiber of the $\mathbb{R}$-foliation of $S_{u,v}\widehat{H}$ for a $\mathbb{C}$-circle $\widehat{H}$ of the canonical fibration of $\widehat{G}$. Thus as in Lemma 10.17 we see that f_u is isometric along σ. By Proposition 10.22, f_u is isometric along $S_{u,v}$. By Lemma 10.5 again, f_ω is isometric along $S_{u,v}$. Hence $|x'y'|_{\omega'} = |xy|_\omega$. This completes the proof of Theorem 10.1. □

Proof of Theorem 1.1. It follows from Proposition 10.21 that $\dim X$ is odd, $\dim X = 2k - 1$, $k \geq 1$. If $k = 1$, then $X = \partial_\infty \mathbb{C}\mathrm{H}^1$ is a $\mathbb{C}$-circle. In the case $k = 2$, Theorem 1.1 is already proved, see Remark 10.19. Thus we assume that $k \geq 3$ and argue by induction over dimension. We take any mutually orthogonal $\mathbb{C}$-circles $F, F' \subset X$. By Proposition 8.7 their respective orthogonal complements A, A' have nonempty intersection $X' = A \cap A'$. By Corollary 8.13, X' satisfies axioms (E) and (O), hence $X' = \partial_\infty \mathbb{C}\mathrm{H}^{k-2}$ by the inductive assumption. By Theorem 10.1 the join $X'' = F' * X'$ is Möbius equivalent to $\partial_\infty \mathbb{C}\mathrm{H}^{k-1}$. Since $F', X' \subset A$, we see as in the proof of Proposition 10.21 that $X'' \subset A$, and therefore X'' is a COS to F. Applying Theorem 10.1 once again to $F * X''$, we obtain that $X = F * X''$ is Möbius equivalent to $\partial_\infty \mathbb{C}\mathrm{H}^k$. □

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