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Bi-Lipschitz Bijections of $\mathbb{Z}$

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Abstract: It is shown that every bi-Lipschitz bijection from $\mathbb{Z}$ to itself is at a bounded L_∞ distance from either the identity or the reflection. We then comment on the group-theoretic properties of the action of bi-Lipschitz bijections.

Keywords: Bi-Lipschitz; bijections

MSC: 26A16; 54E40

1 Introduction

Definition 1. A bi-Lipschitz bijection between two metric spaces (X, ρ_X) and (Y, ρ_Y) is a bijective map $f : X \to Y$ so that there are $0 < C_1 \leq C_2 < +\infty$, such that for all $x_1, x_2 \in X$

$$C_1 \rho_X (x_1, x_2) \leq \rho_Y (f(x_1), f(x_2)) \leq C_2 \rho_X (x_1, x_2).$$

Recall the definition of the Lipschitz constant of a map:

$$\|f\|_{\mathrm{Lip}} := \sup_{x_1 \neq x_2} \frac{\rho_Y (f(x_1), f(x_2))}{\rho_X (x_1, x_2)}.$$

A map f is Lipschitz if and only if $\|f\|_{\mathrm{Lip}}$ is finite, and bi-Lipschitz if and only if it is bijective and both $\|f\|_{\mathrm{Lip}}$ and $\|f^{-1}\|_{\mathrm{Lip}}$ are finite.

While the real line $\mathbb{R}$ admits a large family of bi-Lipschitz bijections, e.g. including any increasing function with derivative bounded away from 0 and ∞, bi-Lipschitz bijections of $\mathbb{Z}$ turn out to be much more rigid. Namely, we have

Theorem 1. *Let $f : \mathbb{Z} \to \mathbb{Z}$ be a bi-Lipschitz bijection ($\mathbb{Z}$ is equipped with its usual metric, namely $\rho(x, y) := |x - y|$). Then either*

$$\sup_{x \in \mathbb{Z}} |f(x) - x| < +\infty$$

or

$$\sup_{x \in \mathbb{Z}} |f(x) + x| < +\infty.$$

More precisely,

$$f(x) = \pm x + \mathrm{const} + r(x), \quad |r(x)| \leq \|f\|_{\mathrm{Lip}} \|f^{-1}\|_{\mathrm{Lip}}.$$

This result extends to spaces that are bi-Lipschitz isomorphic to $\mathbb{Z}$, like, for instance, products $\mathbb{Z} \times G$ with a finite graph G, equipped with the graph metric.

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The reason for different behavior of $\mathbb{Z}$ vs. $\mathbb{R}$ is that unlike $\mathbb{R}$, $\mathbb{Z}$ cannot be *squeezed and stretched*. In the proof below one of the arguments is a cardinality estimate. It is quite obvious that this argument fails in the continuum, and indeed for $\mathbb{R}$ the statement is just wrong. However, the analogy is restored if we equip our space with a measure and require the bijection to be measure preserving. This motivates the following

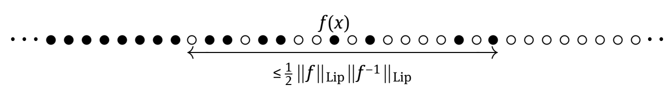
Question 1. *Let $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ be a bi-Lipschitz bijection. Can it be extended to a bi-Lipschitz Lebesgue measure preserving bijection $g : \mathbb{R}^2 \to \mathbb{R}^2$?*

Note that the two dimensional grid $\mathbb{Z}^2$ admits many bi-Lipschitz bijections. For example, let $g : \mathbb{Z} \to \mathbb{Z}$ be a Lipschitz function. Then $F(x, y) := (x, y + g(x))$ is a bi-Lipschitz bijection of $\mathbb{Z}^2$. This shows that a naive generalization of Theorem 1 fails for $\mathbb{Z}^2$: not every bi-Lipschitz bijection is at a bounded distance from an isometry.

For background on metric geometry see e.g. [1]. The group of bijections from $\mathbb{Z}$ to $\mathbb{Z}$ within a bounded L_∞ distance to the identity recently appeared in [2].

2 Proof of Theorem 1

The key to the result is to understand how the image sets $f((-\infty, x])$ may look like.



The "picture" above illustrates what we are going to prove. ∘'s are used to denote $y \in \mathbb{Z}$ such that $y \notin f((-\infty, x])$, and •'s for $y \in f((-\infty, x])$.

In the sequel we denote the constant $\|f\|_{\mathrm{Lip}}\|f^{-1}\|_{\mathrm{Lip}}$ by C.

Lemma 1. *One of the following two cases occurs: either*

$$\left(-\infty, \lfloor f(x) - C/2 \rfloor\right] \subset f((-\infty, x]) \subset \left(-\infty, \lfloor f(x) + C/2 \rfloor\right]$$

or

$$\left[\lceil f(x) + C/2 \rceil, +\infty\right) \subset f((-\infty, x]) \subset \left[\lceil f(x) - C/2 \rceil, +\infty\right).$$

for all $x \in \mathbb{Z}$.

Proof. Let $y \neq f(x)$ be such that $y \in f((-\infty, x])$ and $y + 1 \notin f((-\infty, x])$ (i.e. y is the position of a "•∘" on the "picture"). Then since $y \in f((-\infty, x])$, it follows that $f^{-1}(y) < x$. In the same way since $y + 1 \notin f((-\infty, x])$, we have $f^{-1}(y + 1) > x$. From the Lipschitz property of f it follows that

$$x - f^{-1}(y) \geq \frac{|f(x) - y|}{\|f\|_{\mathrm{Lip}}},$$

$$f^{-1}(y + 1) - x \geq \frac{|f(x) - y - 1|}{\|f\|_{\mathrm{Lip}}}.$$

Therefore,

$$f^{-1}(y + 1) - f^{-1}(y) \geq \frac{2\left|f(x) - y - \frac{1}{2}\right|}{\|f\|_{\mathrm{Lip}}}.$$

Now from the Lipschitz property of f^{-1} it follows that

$$1 = (y + 1) - y \geq \frac{f^{-1}(y + 1) - f^{-1}(y)}{\|f^{-1}\|_{\mathrm{Lip}}} \geq \frac{2\left|f(x) - y - \frac{1}{2}\right|}{\|f\|_{\mathrm{Lip}}\|f^{-1}\|_{\mathrm{Lip}}}.$$

In other words, the distance between $f(x)$ and any "•∘" is bounded by $\frac{1}{2}C$.

The same argument also applies to "○●": just replace $y+1$ by $y-1$ everywhere.

This proves that the characteristic function of the set $f((-\infty, x])$ does not change outside the region

$$\left[f(x) - \frac{1}{2}C, f(x) + \frac{1}{2}C\right].$$

Now since both $f((-\infty, x])$ and its complement $\mathbb{Z} \setminus f((-\infty, x]) = f([x+1, +\infty))$ must be infinite, only two possibilities are left: either $f((-\infty, x])$ is unbounded from below or it is unbounded from above, which obviously corresponds to the two possible conclusions of the lemma. □

Remark 1. Actually, by a little more careful application of the same argument one can show that the width of the region where the characteristic function of $f((-\infty, x])$ is nonconstant is bounded by $\frac{1}{2}C$.

Proof of Theorem 1. Let's assume that the images in Lemma 1 are unbounded from below (the other case can be treated analogously). Let $x_1, x_2 \in \mathbb{Z}$ be such that $x_2 - x_1 > C$. Then

$$\begin{aligned} f((x_1, x_2]) &= f((-\infty, x_2]) \setminus f((-\infty, x_1]) \subset \left(-\infty, f(x_2) + C/2\right] \setminus \left(-\infty, f(x_1) - C/2\right] \\ &= \left(f(x_1) - C/2, f(x_2) + C/2\right]. \end{aligned}$$

In the same way

$$f((x_1, x_2]) \supset \left(f(x_1) + C/2, f(x_2) - C/2\right].$$

Since f is a bijection, the cardinality of $f((x_1, x_2])$ must be $x_2 - x_1$. Therefore,

$$f(x_2) - f(x_1) - C \le x_2 - x_1 \le f(x_2) - f(x_1) + C.$$

Now if we fix $x_1 < 0$ and vary x_2, we see that for x in the interval $[x_1, +\infty)$

$$|f(x) - x - \mathrm{const}_{x_1}| \le C.$$

Note that x_1 can be arbitrary and the range of possible values of const_{x_1} is bounded independently of x_1 (e.g. $|\mathrm{const}_{x_1}| \le |f(0)| + C$). Therefore the bound holds on the whole $\mathbb{Z}$. □

3 Corollaries

As pointed out by the referee, our result implies that there is a remarkable difference between $\mathbb{Z}$ and higher-dimensional lattices in terms of the group-theoretic properties of the action of bi-Lipschitz bijections. In particular:

Corolary 1. *The group of bi-Lipschitz bijections of $\mathbb{Z}$ does not contain an infinite countable subgroup with property* (T).

Proof. The fact that the wobbling group of $\mathbb{Z}$ —i.e. the group of bijections that have finite ℓ^∞ distance from the identity – does not contain a countable property (T) subgroup follows from Theorem 4.1 in [3]. On the other hand, by our result, the wobbling group of $\mathbb{Z}$ is an index 2 subgroup of the group of bi-Lipschitz bijections. □

Note that Corollary 1 fails for $\mathbb{Z}^d$, $d \ge 3$, since $\mathrm{SL}(d, \mathbb{Z})$, $d \ge 3$ has property (T) and acts faithfully on $\mathbb{Z}^d$ by bi-Lipschitz bijections. We do not know what happens in the $d = 2$ case.

Question 2. *Does Corollary 1 hold for $\mathbb{Z}^2$?*

Another corollary concerns an amenability-like property:

Corolary 2. *There is a bi-Lipschitz invariant mean* (i.e. finitely additive probability measure) *on* $\mathbb{Z}$.

Proof. From Lemma 1 it follows that the sets $A_n := [-n, n]$ form a Følner sequence for the action of bi-Lipschitz bijections —i.e. for any particular bi-Lipschitz bijection f we have

$$\frac{|f(A_n) \cap A_n|}{|A_n|} \to 1, n \to \infty$$

Therefore, an invariant mean can be obtained by a standard argument, as a limiting point of the sequence of uniform measures on A_n with respect to the weak-* topology of $(\ell^\infty)^*$. □

On the other hand:

Proposition 1. *Corollary 2 fails for* $\mathbb{Z}^2$.

Proof. Let μ be a bi-Lipschitz invariant mean on $\mathbb{Z}^2$. Then the standard action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{Z}^2 \setminus \{0\}$ preserves the mean μ restricted to $\mathbb{Z} \setminus \{0\}$. This is impossible, since $\mathrm{SL}(2,\mathbb{Z})$ is non-amenable and acts on $\mathbb{Z}^2 \setminus \{0\}$ with amenable stabilizers. □

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