DE GRUYTER OPEN Anal. Geom. Metr. Spaces 2015; 3:317–324

Research Article Open Access

Barry Minemyer*

Isometric Embeddings of Pro-Euclidean Spaces

DOI 10.1515/agms-2015-0019
Received May 27, 2015; accepted October 9, 2015

Abstract: In [12] Petrunin proves that a compact metric space $\mathcal{X}$ admits an intrinsic isometry into $\mathbb{E}^n$ if and only if $\mathcal{X}$ is a pro-Euclidean space of rank at most n, meaning that $\mathcal{X}$ can be written as a "nice" inverse limit of polyhedra. He also shows that either case implies that $\mathcal{X}$ has covering dimension at most n. The purpose of this paper is to extend these results to include both embeddings and spaces which are proper instead of compact. The main result of this paper is that any pro-Euclidean space of rank at most n is proper and admits an intrinsic isometric embedding into $\mathbb{E}^{2n+1}$. Since every n-dimensional Riemannian manifold is a pro-Euclidean space of rank at most n, this result is a partial generalization of (the C^0 version of) the famous Nash isometric embedding theorem from [10].

Keywords: differential geometry; discrete geometry; metric geometry; Euclidean polyhedra; polyhedral space; intrinsic isometry

MSC: 51K10, 51F99, 52B11, 54C25

1 Introduction

A general, but difficult, question is to determine whether or not a given metric space admits an isometric embedding into Euclidean space $\mathbb{E}^n$. A closely related problem is to find necessary and sufficient conditions under which such an isometric realization exists. A common technique used in all branches of mathematics is to prove a desired result for some class of "nice" objects, and then to generalize this result to more general objects by expressing them as some sort of "limit" of these nice objects. This is the method of attack that we will employ in this paper.

The most famous results in this area of mathematics are the Nash isometric embedding theorems for Riemannian manifolds [10] and [11]. One could then try to use these results in the limiting idea mentioned above, and indeed both Petrunin [12] and Le Donne [6] observed that one can use the C^1 Nash isometric embedding theorem to prove that every n-manifold equipped with a sub-Riemannian metric admits an isometric embedding into $\mathbb{E}^{2n+1}$. But in general, it can be difficult to express a general metric space as a limit of Riemannian manifolds because of both the nice local topology of manifolds and because of the nice metric structure provided by the smooth Riemannian metric.

A *Euclidean polyhedron* $\mathcal{P}$ is a metric space endowed with a locally finite (simplicial) triangulation $\mathcal{T}$ such that each n-dimensional simplex $\tau \in \mathcal{T}$ is affinely isometric to an n-simplex in $\mathbb{E}^n$. Euclidean polyhedra are sometimes referred to as *polyhedral spaces* in the literature. Such spaces are "nice" due to the local constant curvature, but they provide a little more flexibility than manifolds because the triangulation may contain singularities of arbitrary codimension. There is a very nice string of results concerning piecewise-linear (pl) isometries of n-dimensional Euclidean polyhedra in $\mathbb{E}^n$, and the author would like to list references [14],

***Corresponding Author: Barry Minemyer:** Department of Mathematics, The Ohio State University, Columbus, Ohio 43210, E-mail: minemyer.1@osu.edu

[1], and [5] for the interested reader. These results, specifically a result due to Arseniy Akopyan, led to the following result of Petrunin found in [12]:

Theorem 1 (Petrunin). *A compact metric space $\mathcal{X}$ admits an intrinsic isometry into $\mathbb{E}^n$ if and only if $\mathcal{X}$ is a compact pro-Euclidean space of rank at most n. Either of these statements implies that dim($\mathcal{X}$) ≤ n, where dim($\mathcal{X}$) denotes the (Lebesgue) covering dimension of $\mathcal{X}$.*

A preliminary version of Akopyan's result, in Russian, can be found at: http://www.moebiuscontest.ru/files/2007/akopyan.pdf. An English translation can be found in [8].

A metric space $\mathcal{X}$ is said to be a *compact pro-Euclidean space of rank at most n* if it can be represented as an inverse limit of a sequence of Euclidean polyhedra $\{\mathcal{P}_i\}_{i=1}^{\infty}$, where each polyhedron has dimension $\leq n$. This inverse limit is taken in the category of compact metric spaces and short (1-Lipschitz) maps. A *pro-Euclidean space* refers to an inverse limit as above, but where the polyhedra are locally finite instead of compact. Also, an "intrinsic isometry" is a slightly more general notion of the more common "path isometry". For more precise definitions, please see section 2.

Note that Theorem 1 fits the model described above. It proves the existence of an isometry of a metric space into $\mathbb{E}^n$ by considering it as an appropriate limit of "nice spaces" (Euclidean polyhedra) which we already have results about. The above results about Euclidean polyhedra all deal with pl isometries that are, in general, not embeddings though. This of course can be seen by the dimension of the ambient Euclidean space. One can think of these results as "origami type" foldings of the polyhedra into the same dimensional Euclidean space. Obviously, $\mathbb{E}^n$ is the smallest dimensional space that one could construct such foldings into.

In [7] the author used Akopyan's result to prove the following:

Theorem 2. *Let $\mathcal{P}$ be an n-dimensional Euclidean polyhedron, let $f : \mathcal{P} \to \mathbb{E}^N$ be a short map, and let $\varepsilon > 0$ be arbitrary. Then there exists an intrinsic isometric embedding $h : \mathcal{P} \to \mathbb{E}^N$ which is an ε-approximation of f, meaning $|f(x) - h(x)| < \varepsilon$ for all $x \in \mathcal{X}$, provided $N \geq 2n + 1$.*

Remark 3. *Theorem 2 can be slightly generalized to allow the ε to vary as you move away from some fixed point. This generalization can also be found in [7]. But this fact will only play a small role in this paper, so we present the current version for simplicity.*

The main purpose of this short note is to use Theorem 2 to extend Petrunin's result to include (intrinsic) isometric embeddings. The secondary purpose is to extend from the setting of compact to proper metric spaces, which is mostly trivial but has a subtle detail at one spot.

The following is the main result of this paper:

Main Theorem. *Let $\mathcal{X}$ be a pro-Euclidean space of rank at most n. Then $\mathcal{X}$ is proper, dim($\mathcal{X}$) ≤ n, and $\mathcal{X}$ admits an intrinsic isometric embedding into $\mathbb{E}^{2n+1}$.*

Of course, the conclusion that dim($\mathcal{X}$) $\leq n$ is directly from Petrunin's result. It is well known (see [2] and [3]) that a complete metric space which is an ultralimit of proper metric spaces is also proper. Then since a pro-Euclidean space is, in particular, an ultralimit of locally-finite polyhedra, what needs to be shown is that any pro-Euclidean space is complete. This is reasonably clear since all of the maps involved are Lipschitz, but in any case this will be discussed further in section 2. So the real essence of our Main Theorem is that every pro-Euclidean space of rank at most n admits an intrinsic isometric embedding into $\mathbb{E}^{2n+1}$. This will be proved in section 3.

In Theorem 1 Petrunin provides necessary and sufficient conditions for a compact metric space to be realized isometrically as a subspace of $\mathbb{E}^n$. So in this vein, one would like a converse to our Main Theorem. Petrunin's method of proof in [12] is to take finer and finer cubulations of $\mathbb{E}^n$, and to construct the polyhedra in the inverse limit by gluing together isometric copies of these cubes along their common boundaries. But one needs to insert multiple copies of each cube corresponding to how many times the image of $\mathcal{X}$ intersects this cube. Since Theorem 1 only considers the case of $\mathcal{X}$ compact, there is no issue here. But when trying to extend to a wider range of spaces, the assumption of $\mathcal{X}$ being proper is necessary to ensure that the corresponding polyhedra are locally finite.

Applying this method to our setting only provides a partial converse because the dimension of the corresponding polyhedra will be $2n + 1$ instead of n. But if we define a *pro-Euclidean space of finite rank* to be a pro-Euclidean space where the dimensions of the polyhedra in the inverse limit are bounded above by some natural number N, then we have:

Corollary 4. *A proper metric space $\mathcal{X}$ admits an intrinsic isometric embedding into $\mathbb{E}^N$ for some N if and only if $\mathcal{X}$ is a pro-Euclidean space of finite rank.*

Let $\mathcal{Y}$ denote $\mathbb{R}^2$ with the origin removed, equipped with its natural length metric. Since $\mathcal{Y}$ is not complete, $\mathcal{Y}$ is not proper and therefore is not a pro-Euclidean space of finite rank. But $\mathcal{Y}$ obviously admits an intrinsic isometric embedding into $\mathbb{R}^2$. This demonstrates that the assumption of being proper is necessary in Corollary 4.

Assume that $\mathcal{X}$ is proper, $\dim(\mathcal{X}) \leq n$, and that $\mathcal{X}$ admits an intrinsic isometric embedding into $\mathbb{E}^{2n+1}$. Then using a variant of Whitney's triangulation technique from [13], one can approximate the image of $\mathcal{X}$ by n-dimensional Euclidean polyhedra. But it is not clear how to construct the short maps between the various spaces, or even if the metrics of the polyhedra converge to the metric of $\mathcal{X}$ in the limit. At the time of this writing, the author can ensure that the metrics of these polyhedra converge to a metric on $\mathcal{X}$ which is bi-Lipschitz equivalent to the original metric on $\mathcal{X}$. An open question then is whether or not such a space $\mathcal{X}$ is a pro-Euclidean space of rank at most n. This is not obviously true (at least to the author), in that a priori there may exist proper n-dimensional subspaces of $\mathbb{E}^{2n+1}$ which can only be approximated (in the metric sense) by polyhedra whose dimension is greater than n.

One last note is that, in [4], Gromov proves that every n-dimensional Riemannian manifold M admits a path isometry into $\mathbb{E}^n$ (note the same dimension). Then using Theorem 1 it can be seen that M is a pro-Euclidean space of rank at most n. Our Main Theorem then is a partial generalization of (the C^0 version of) the famous Nash isometric embedding theorem found in [10] which states: every n-dimensional Riemannian manifold admits a C^0 isometric embedding into $\mathbb{E}^{2n+1}$.

Remark 5. *In this paper, the term embedding simply means a continuous injective map. If the domain is compact, then such a map is clearly a topological embedding. But for proper spaces this is not always the case. Such a map is clearly a local embedding though, and the only issues that can arise are with the image of the limit set of the space approaching the image of the interior.*

2 Preliminaries

2.1 Pullback Metrics and Intrinsic Isometries

Parts of this subsection closely follow [12].

Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces and $f : \mathcal{X} \to \mathcal{Y}$ a continuous map. f is *short* if for any two points $x, x' \in \mathcal{X}$ we have that $d_{\mathcal{Y}}(f(x), f(x')) \leq d_{\mathcal{X}}(x, x')$, and f is *strictly short* if $d_{\mathcal{Y}}(f(x), f(x')) < d_{\mathcal{X}}(x, x')$ for any $x, x' \in \mathcal{X}$ with $x \neq x'$.

Now, given two points $x, x' \in \mathcal{X}$, a sequence of points $x = x_0, x_1, \ldots, x_{k-1}, x_k = x'$ is called an *ε-chain* from x to x' if $d_{\mathcal{X}}(x_{i-1}, x_i) \leq \varepsilon$ for any i. Define:

$$\mathrm{pull}_{f,\varepsilon}(x, x') := \inf \left\{ \sum_{i=1}^{k} d_{\mathcal{Y}}(f(x_{i-1}), f(x_i)) \right\}$$

where the infimum is taken over all ε-chains from x to x'. It is not hard to see that for any $\varepsilon > 0$ $\mathrm{pull}_{f,\varepsilon}$ is almost a metric on $\mathcal{X}$. The only issue is the possibility that $\mathrm{pull}_{f,\varepsilon}(x, x') = 0$ for some $x \neq x'$. But $\mathrm{pull}_{f,\varepsilon}$ is clearly monotone nonincreasing with respect to ε. So it makes sense to define the limit

$$\mathrm{pull}_f(x, x') := \lim_{\varepsilon \to 0} \mathrm{pull}_{f,\varepsilon}(x, x')$$

where this limit may be infinite. pull_f is called the *pullback metric* for f.

A map $f : \mathcal{X} \to \mathcal{Y}$ is an *intrinsic isometry* if

$$d_{\mathcal{X}}(x, x') = \mathrm{pull}_f(x, x')$$

for all $x, x' \in \mathcal{X}$.

The following Lemma is the key ingredient used by Petrunin to prove Theorem 1. It is stated without proof in [12], but since it plays such an important role we will give a short proof here for completeness.

Lemma 6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces with $\mathcal{X}$ compact and let a continuous map $f : \mathcal{X} \to \mathcal{Y}$ be such that*

$$\sup_{x,x' \in \mathcal{X}} \mathit{pull}_f(x, x') < \infty.$$

Then for any $\lambda > 0$ there exists $\delta = \delta(f, \lambda) > 0$ such that for any short map $h : \mathcal{X} \to \mathcal{Y}$ satisfying

$$d_{\mathcal{Y}}(f(x), h(x)) < \delta \text{ for any } x \in \mathcal{X}$$

we have that

$$\mathit{pull}_f(x, x') < \mathit{pull}_h(x, x') + \lambda$$

for any $x, x' \in \mathcal{X}$

The key to proving Lemma 6 is the following Lemma, which is proved in [12].

Lemma 7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces, $\mathcal{X}$ compact, and $f, h : \mathcal{X} \to \mathcal{Y}$ be two continuous maps. Let $\delta > 0$ and assume that $d_{\mathcal{Y}}(f(x), h(x)) < \delta$ for all $x \in \mathcal{X}$. Then for all $x, x' \in \mathcal{X}$ we have that*

$$\mathit{pull}_{f,\varepsilon}(x, x') \leq \mathit{pull}_{h,\varepsilon}(x, x') + 4\delta \mathit{pack}_{\varepsilon}\mathcal{X}$$

where $\mathit{pack}_{\varepsilon}\mathcal{X}$ denotes the maximal number of points in $\mathcal{X}$ at a distance more than ε from each other.

Proof of Lemma 6. Let $\lambda > 0$. Because $\mathcal{X}$ is compact and since $\sup_{x,x' \in \mathcal{X}} \mathrm{pull}_f(x, x') < \infty$, there exists $\varepsilon > 0$ with

$$\mathrm{pull}_f(x, x') < \mathrm{pull}_{f,\varepsilon}(x, x') + \frac{\lambda}{2}.$$

for all $x, x' \in \mathcal{X}$. Choose $\delta < \dfrac{\lambda}{8\mathrm{pack}_{\varepsilon}\mathcal{X}}$. Using Lemma 7 we have that for all $x, x' \in \mathcal{X}$:

$$\mathrm{pull}_f(x, x') < \mathrm{pull}_{f,\varepsilon}(x, x') + \frac{\lambda}{2} < \mathrm{pull}_{h,\varepsilon}(x, x') + 4\delta\mathrm{pack}_{\varepsilon}\mathcal{X} + \frac{\lambda}{2} < \mathrm{pull}_{h,\varepsilon}(x, x') + \frac{\lambda}{2} + \frac{\lambda}{2} \leq \mathrm{pull}_h(x, x') + \lambda.$$

□

Let $(\mathcal{X}, d)$ be any metric space and let $\alpha : [a, b] \to \mathcal{X}$ be a path. The *length* of α, denoted $\ell(\alpha)$, is defined by

$$\ell(\alpha) = \sup \sum_{i=1}^{n} d(\alpha(t_{i-1}), \alpha(t_i))$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$ (and with no bound on n). A map $f : \mathcal{X} \to \mathcal{Y}$ between metric spaces is called a *path isometry* if f "preserves the length of paths", i.e. if $\ell(\alpha) = \ell(f \circ \alpha)$ for all paths α in $\mathcal{X}$. Clearly, the notions of 'path isometry' and 'intrinsic isometry' are very similar. In [12] Petrunin justifies the statement that, in general, being an intrinsic isometry is slightly stronger than being a path isometry. Nevertheless, Le Donne proves in [6] that the two notions are equivalent if both $\mathcal{X}$ and $\mathcal{Y}$ are proper geodesic metric spaces. As we will soon see, this is the setting that we will be dealing with in section 3. So there is no ambiguity if the reader just considers path isometries instead of intrinsic isometries in the proof of the Main Theorem.

One last comment about the length functional ℓ. It is well-known that ℓ is *lower semicontinuous*, meaning that if (α_n) is a sequence of paths which uniformly converge to the rectifiable path α (meaning that $\ell(\alpha) < \infty$), then $\underline{\lim}_{n\to\infty} \ell(\alpha_n) \geq \ell(\alpha)$. To see that this inequality can be strict, consider a "zig-zag" sequence of paths which have constant length but converge to a path with strictly shorter length. The point here though is that Lemma 6 is the analogue to this statement in the setting of a "sequence of maps between spaces" instead of a "sequence of paths within a space". Lemma 6 implies that if $f : \mathcal{X} \to \mathcal{Y}$ is a "rectifiable map" with $\mathcal{X}$ compact, and if a sequence of maps (h_i) from $\mathcal{X}$ to $\mathcal{Y}$ converges uniformly to f, then $\mathrm{pull}_f(x, x') \leq \underline{\lim}_{i\to\infty} \mathrm{pull}_{h_i}(x, x')$. What tends to be difficult is proving the reverse inequality. The reason that we require our maps to be short in what follows is in order to circumvent this difficulty.

2.2 Euclidean polyhedra and pro-Euclidean spaces

Parts of this subsection again directly follow [12], and moreover the interested reader should consult [12] for some useful comments and insights not presented here.

A metric space $\mathcal{P}$ is called a *Euclidean polyhedron* if there exists a locally finite simplicial triangulation of $\mathcal{P}$ such that each k-dimensional simplex is affinely isometric to a non-degenerate k-simplex in $\mathbb{E}^k$. Such spaces are sometimes called *polyhedral spaces* in the literature, although this term may also refer to polyhedra whose simplices have some (fixed) curvature different from 0. The dimension of $\mathcal{P}$ is equal to the dimension of the maximal simplex in any triangulation of $\mathcal{P}$. It is clear that, due to being locally finite, Euclidean polyhedra are proper metric spaces (meaning that closed bounded sets are compact). Also, it was proved by Moussong in [9] that Euclidean polyhedra are necessarily geodesic metric spaces, meaning that the distance between any two points is realized by the length of some path between those two points.

A collection $(\varphi_{l,k}, \mathcal{X}_k)_{k,l=0}^{\infty}$ is an *inverse system* of proper metric spaces $\mathcal{X}_k$ and short maps $\varphi_{l,k} : \mathcal{X}_l \to \mathcal{X}_k$ (for $l \geq k$) if:

1. $\varphi_{m,l} \circ \varphi_{l,k} = \varphi_{m,k}$ for any triple $m \geq l \geq k$
2. for all k, the function $\varphi_{k,k}$ equals the identity map on $\mathcal{X}_k$.

A metric space $\mathcal{X}$ is the *inverse limit of the system* $(\varphi_{l,k}, \mathcal{X}_k)$, denoted $\mathcal{X} = \varprojlim \mathcal{X}_k$, if both of the following two conditions hold:

1. $\mathcal{X}$ consists of all sequences (x_k) such that $x_k \in \mathcal{X}_k$ for all k and $\varphi_{l,k}(x_l) = x_k$ for all $l \geq k$
2. The distance in $\mathcal{X}$ between $x = (x_k)$ and $x' = (x'_k)$ is defined by

$$d_{\mathcal{X}}(x, x') = \lim_{k\to\infty} d_{\mathcal{X}_k}(x_k, x'_k).$$

If $\mathcal{X} = \varprojlim \mathcal{X}_k$ then there exist *projection maps* $\psi_k : \mathcal{X} \to \mathcal{X}_k$ defined by $(x_k) \mapsto x_k$. By definition, $\psi_k = \varphi_{l,k} \circ \psi_l$ for all $l \geq k$. Also note that, by necessity, each projection map must be short.

Finally, a metric space $\mathcal{X}$ is called a *pro-Euclidean space of rank at most n* if $\mathcal{X}$ is the inverse limit of the system $(\varphi_{l,k}, \mathcal{P}_k)$ where each $\mathcal{P}_k$ is a Euclidean polyhedron with dimension $\leq n$. Note that every pro-

Euclidean space is complete. To see this, consider a Cauchy sequence $(x_k) \subseteq \mathcal{X}$. Since ψ_i is Lipschitz for all i, the sequence $(\psi_i(x_k)) \subseteq \mathcal{P}_i$ is also Cauchy. But $\mathcal{P}_i$ is complete, and so this sequence has some limit $y_i \in \mathcal{P}_i$. Then $y = (y_i)$ is a well-defined point of $\mathcal{X}$, and $\lim_{k\to\infty} x_k = y$.

Finally, it is well known that a complete space that is the ultralimit of proper geodesic spaces is also proper and geodesic (see either [2] or [3]). One can also consult the previous two references for the definition of an ultralimit, but all that is important here is that being a pro-Euclidean space is stronger than being an ultralimit of polyhedra. Therefore, this proves the following Lemma.

Lemma 8. *Every pro-Euclidean space is proper and geodesic.*

3 Proof of the Main Theorem

Proof of the Main Theorem. Since $\mathcal{X}$ is a pro-Euclidean space, we know that $\mathcal{X}$ is proper. We first proceed under the assumption that $\mathcal{X}$ is compact. Afterward, we indicate how to alter the proof for when $\mathcal{X}$ is proper.

Let $\mathcal{X}$ be a pro-Euclidean space of rank at most n and let $(\mathcal{P}_i, \varphi_{j,i})$ be the inverse system associated to $\mathcal{X}$, where $\mathcal{P}_i$ is an n-dimensional (locally finite) Euclidean polyhedron for all i. For each i let $\psi_i : \mathcal{X} \to \mathcal{P}_i$ be the projection map, and recall that every map associated with this system is short. For a picture, see the upper half of Figure 1.

Given $\varepsilon_{i+1} > 0$ and a pl intrinsic isometric embedding $f_i : \mathcal{P}_i \to \mathbb{E}^{2n+1}$, by Theorem 2 there exists a pl intrinsic isometric embedding $f_{i+1} : \mathcal{P}_{i+1} \to \mathbb{E}^{2n+1}$ such that

$$|f_{i+1}(x) - (f_i \circ \varphi_{i+1,i})(x)| < \varepsilon_{i+1}$$

for all $x \in \mathcal{P}_{i+1}$.

Then, for all i, define $h_i := f_i \circ \psi_i$. A diagram of all of the aforementioned maps is given in Figure 1. What needs to be shown is that the values for ε_i can be chosen in such a way that the sequence (h_i) converges uniformally to an intrinsic isometric embedding.

To see that the sequence (ε_i) can be chosen so that the sequence (h_i) converges to an embedding we use a technique similar to one used by Nash in [10]. Consider the collection of sets

$$\Omega_i := \{(x, x') \in \mathcal{X} \times \mathcal{X} \mid d_{\mathcal{X}}(x, x') \geq 2^{-i}\}.$$

Since $\mathcal{X} = \varprojlim \mathcal{P}_i$ and because Ω_i is compact, for every $i \in \mathbb{N}$ there exists i' such that $\psi_{i'}(x) \neq \psi_{i'}(x')$ for all $(x, x') \in \Omega_i$. We also require that $i' > (i-1)'$ which satisfies the above. Thus, $h_{i'}(x) \neq h_{i'}(x')$ for all $(x, x') \in \Omega_i$. Then set

$$\mu_i := \inf\{|h_{i'}(x) - h_{i'}(x')| \mid (x, x') \in \Omega_i\}.$$

Observe that $\mu_i > 0$. If we require that

$$\varepsilon_i < \frac{1}{8}\min\{\mu_i, \varepsilon_{i-1}\} \tag{3.1}$$

then no point pair in Ω_i can come together in the limit. Eventually any pair of distinct points is contained in some Ω_i, which completes the proof that the map h is injective.

The fact that the sequence (ε_i) can be chosen so that the sequence (h_i) converges uniformally to an intrinsic isometry is identical to the proof by Petrunin in [12], but will be included here for completeness.

Since each of the maps f_i and $\varphi_{i+1,i}$ are short by assumption, we have that each h_i is short. It is therefore clear that h is short, i.e. that $\text{pull}_h(x, x') \leq d_{\mathcal{X}}(x, x')$ for all $x, x' \in \mathcal{X}$. To show the reverse inequality, at each stage of the limiting process choose $\varepsilon_{i+1} > 0$ so that

$$\varepsilon_{i+1} < \frac{1}{2}\min\left\{\varepsilon_i, \delta\left(f_i, \frac{1}{i}\right)\right\} \tag{3.2}$$

where $\delta\left(f_i, \frac{1}{i}\right)$ is as in Lemma 6. We then see that

$$|h(x) - h_i(x)| = |h(x) - f_i \circ \psi_i(x)| < \sum_{j=i+1}^{\infty} \varepsilon_j < \delta\left(f_i, \frac{1}{i}\right).$$

Therefore, by Lemma 6 we have that

$$d_{\mathcal{P}_i}(\psi_i(x), \psi_i(x')) \le \mathrm{pull}_{f_i \circ \psi_i}(x, x') < \mathrm{pull}_h(x, x') + \frac{1}{i}.$$

Then since $\lim\limits_{i \to \infty} d_{\mathcal{P}_i}(\psi_i(x), \psi_i(x')) = d_{\mathcal{X}}(x, x')$, we have that $d_{\mathcal{X}}(x, x') \le \mathrm{pull}_h(x, x')$ which completes the proof for the case when $\mathcal{X}$ is compact.

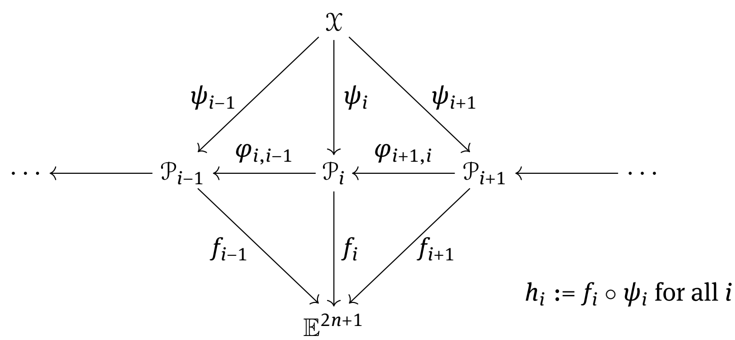

Figure 1: Diagram for the proof of the Main Theorem.

For $\mathcal{X}$ proper, we perform the usual trick of exhausting $\mathcal{X}$ by a collection of compact subspaces. More precisely, fix a base point $b \in \mathcal{X}$ and let $B_k(b)$ denote the closed ball of radius $k \in \mathbb{N}$ about b. Since $\mathcal{X}$ is proper, each $B_k(b)$ is compact. So the above proof goes through in $B_k(b)$ for some positive sequence $(\varepsilon_i^k)_{i=1}^{\infty}$. We add the requirement that $\varepsilon_i^{k+1} < \varepsilon_i^k$ for all i and k. Then by remark 3 we can vary the epsilon's in equations (3.1) and (3.2) as we move away from $\psi_{i+1}(b)$ to ensure that the sequence (h_i) still converges to an intrinsic isometric embedding.

□

Acknowledgement: This research was partially supported by the NSF grant of Tom Farrell and Pedro Ontaneda, DMS-1103335.

References

[1] U. Brehm, Extensions of distance reducing mappings to piecewise congruent mappings on $\mathbb{R}^m$, J. Geom., 16 (1981), no. 2, 187-193.
[2] M. Bridson A. Haefliger, Metric Spaces of Non-Positive Curvature, Springer-Verlag Berlin Heidelberg, 1999.
[3] D. Burago, Y. Burago, S. Ivanov, A course in metric geometry, Graduate Studies in Mathematics, Vol. 33, American Mathematical Society, Providence, RI, 2001.
[4] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, Birkhäuser, 2001.
[5] S. Krat, Approximation problems in length geometry, PhD thesis, The Pennsylvania State University, State College, PA, 2004.
[6] E. Le Donne, Lipschitz and path isometric embeddings of metric spaces, Geom. Dedicata, 166 (2012), 47-66.
[7] B. Minemyer, Isometric Embeddings of Polyhedra into Euclidean Space, J. Topol. Anal. (in press), DOI:10.1142/S179352531550020X.
[8] B. Minemyer, Isometric Embeddings of Polyhedra, PhD thesis, The State University of New York at Binghamton, Binghamton, NY, 2013.
[9] G. Moussong, Hyperbolic Coxeter Groups, PhD thesis, The Ohio State University, Columbus, OH, 1988.

[10] J. Nash, C^1 Isometric Imbeddings, Ann. of Math. (2), 60 (1954), 383-396.
[11] J. Nash, The Imbedding Problem for Riemannian Manifolds, Ann. of Math. (2), 63 (1956), 20-63.
[12] A. Petrunin, On Intrinsic Isometries to Euclidean Space, St. Petersburg Math. J., 22 (2011), 803-812.
[13] H. Whitney, Geometric integration theory, Princeton University Press, 1957.
[14] V. A. Zalgaller, Isometric imbedding of polyhedra, Dokl. Akad. Nauk (in Russian), 123 (1958), 599-601.