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Weak Chord-Arc Curves and Double-Dome Quasisymmetric Spheres

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Abstract: Let Ω be a planar Jordan domain and $\alpha > 0$. We consider double-dome-like surfaces $\Sigma(\Omega, t^\alpha)$ over $\bar{\Omega}$ where the height of the surface over any point $x \in \bar{\Omega}$ equals $\text{dist}(x, \partial\Omega)^\alpha$. We identify the necessary and sufficient conditions in terms of Ω and α so that these surfaces are quasisymmetric to \mathbb{S}^2 and we show that $\Sigma(\Omega, t^\alpha)$ is quasisymmetric to the unit sphere \mathbb{S}^2 if and only if it is linearly locally connected and Ahlfors 2-regular.

Keywords: quasisymmetric spheres; double-dome-like surfaces; chord-arc property; Ahlfors 2-regularity

MSC: Primary: 30C65. Secondary: 30C62

1 Introduction

A metric space which is quasisymmetric to the standard n -sphere \mathbb{S}^n is called a quasisymmetric n -sphere. Quasisymmetric circles were completely characterized by Tukia and Väisälä in [9]. Bonk and Kleiner [4] identified a necessary and sufficient condition for metric 2-spheres to be quasisymmetric spheres. A consequence of their main theorem is that if a metric 2-sphere is linearly locally connected (or LLC) and Ahlfors 2-regular then it is a quasisymmetric 2-sphere. Although the LLC property is necessary, there are examples of quasisymmetric 2-spheres constructed by snowflaking procedures that fail the 2-regularity [3], [5], [7].

In this paper we consider a special case of the double dome-like surfaces constructed over planar Jordan domains Ω in [12]. For a number $\alpha > 0$ and a Jordan domain $\Omega \subset \mathbb{R}^2$ consider the 2-dimensional surface in \mathbb{R}^3

$$\Sigma(\Omega, t^\alpha) = \{(x, z) : x \in \bar{\Omega}, z = \pm(\text{dist}(x, \partial\Omega))^\alpha\}.$$


As it turns out, for these surfaces the 2-regularity is necessary for quasisymmetric parametrization by \mathbb{S}^2 .

Theorem 1.1. *The surface $\Sigma(\Omega, t^\alpha)$ is quasisymmetric to \mathbb{S}^2 if and only if it is linearly locally connected and 2-regular.*

What conditions on Ω and α ensure that $\Sigma(\Omega, t^\alpha)$ is a quasisymmetric 2-sphere? When $\alpha > 1$, Theorem 1.1 is trivial since, for any Jordan domain Ω , the surface $\Sigma(\Omega, t^\alpha)$ is not linearly locally connected and hence not quasisymmetric to \mathbb{S}^2 . If $\alpha = 1$, $\Sigma(\Omega, t)$ is quasisymmetric to \mathbb{S}^2 if and only if $\partial\Omega$ is a quasicircle [12, Theorem 1.1]. This result, combined with the fact that the projection of $\Sigma(\Omega, t)$ on $\bar{\Omega}$ is a bi-Lipschitz mapping, and the fact that $\bar{\Omega}$ is 2-regular if $\partial\Omega$ is a quasicircle, gives Theorem 1.1 when $\alpha = 1$.

In the case $\alpha \in (0, 1)$, the part of $\Sigma(\Omega, t^\alpha)$ near $\partial\Omega \times \{0\}$ resembles the product of $\partial\Omega$ with an interval. Väisälä [10] has shown that the product $\gamma \times I$ of a Jordan arc γ and an interval I is quasisymmetric embeddable in \mathbb{R}^2 if and only if γ satisfies the chord-arc condition (2.2). In view of this result, it is expected that the chord-arc property of $\partial\Omega$ is necessary for $\Sigma(\Omega, t^\alpha)$ to be a quasisymmetric sphere. Moreover, as the double-dome-like

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surface envelops the interior Ω above and below, the following condition on Ω is needed to ensure the linear local connectedness of the surfaces $\Sigma(\Omega, t^\alpha)$. We say that Ω has the *level quasicircle property* (or LQC property) if there exist $\epsilon_0 > 0$ and $C > 1$ such that for all $\epsilon \in [0, \epsilon_0]$, the ϵ -level set of Ω

$$\gamma_\epsilon = \{x \in \overline{\Omega} : \text{dist}(x, \partial\Omega) = \epsilon\}$$

is a C -quasicircle. A consequence of Theorem 1.2 in [12] is that *if a planar Jordan domain Ω has the LQC property and $\partial\Omega$ is a chord-arc curve then $\Sigma(\Omega, t^\alpha)$ is quasisymmetric to \mathbb{S}^2 for all $\alpha \in (0, 1)$.*

For these surfaces, the LQC property of Ω is essential: if a Jordan domain Ω does not have the LQC property then $\Sigma(\Omega, t^\alpha)$ is not quasisymmetric to \mathbb{S}^2 for any $\alpha \in (0, 1)$; see Lemma 4.6. However, the chord-arc condition of $\partial\Omega$ is not necessary. Contrary to the intuition based on Väisälä's result [10], we construct in Section 5.1 a Jordan domain Ω whose boundary is a non-rectifiable curve and $\Sigma(\Omega, t^\alpha)$ is a quasisymmetric sphere for all $\alpha \in (0, 1)$.

Instead, only a weak form of the chord-arc condition is needed: a metric circle Γ is said to have the *weak chord-arc condition* if there exists $N_0 > 1$ such that every subarc $\Gamma' \subset \Gamma$ with $\text{diam } \Gamma' \leq 1$ can be covered by at most $N_0(\text{diam } \Gamma')^{-1}$ subarcs $\Gamma_i \subset \Gamma'$ of diameter at most $(\text{diam } \Gamma')^2$. Under this terminology we have the following.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain and $\alpha \in (0, 1)$. The following are equivalent.*

1. *The surface $\Sigma(\Omega, t^\alpha)$ is a quasisymmetric sphere.*
2. *The surface $\Sigma(\Omega, t^\alpha)$ is LLC and 2-regular.*
3. *Ω has the LQC property and $\partial\Omega$ is a weak chord-arc curve.*

An immediate consequence of Theorem 1.2 is that the existence of a quasisymmetric parametrization of $\Sigma(\Omega, t^\alpha)$ by \mathbb{S}^2 does not depend on $\alpha \in (0, 1)$. In particular, if $\Sigma(\Omega, t^\alpha)$ is quasisymmetric to \mathbb{S}^2 for some $\alpha \in (0, 1)$ then it is quasisymmetric to \mathbb{S}^2 for all $\alpha \in (0, 1)$.

Although not necessarily rectifiable, any weak chord-arc quasicircle has Hausdorff dimension equal to 1. In fact, a slightly stronger result holds true.

Theorem 1.3. *If a quasicircle satisfies the weak chord-arc property then it has Assouad dimension equal to 1.*

The example in Section 5.2, however, shows that the converse is false. Moreover, since the Assouad dimension is larger than the Hausdorff, upper box counting and lower box-counting dimensions, the conclusion of Theorem 1.3 holds for any of these dimensions. A consequence of Theorem 1.3 is that if Ω is a Jordan domain and $\partial\Omega$ has Assouad dimension greater than 1, then the surface $\Sigma(\Omega, t^\alpha)$ is not quasisymmetric to \mathbb{S}^2 for any $\alpha \in (0, 1)$.

In Section 3 we define an index that measures how much a curve Γ deviates from being a chord-arc curve on a certain scale, and we discuss the weak chord-arc condition. The proofs of Theorem 1.2 and Theorem 1.3 are given in Section 4. Finally, two examples, based on homogeneous snowflakes, illustrating the weak chord-arc condition are presented in Section 5.

2 Preliminaries

2.1 Definitions and notation

An embedding f of a metric space (X, d_X) into a metric space (Y, d_Y) is said to be η -quasisymmetric if there exists a homeomorphism $\eta: [0, \infty) \rightarrow [0, \infty)$ such that for all $x, a, b \in X$ and $t > 0$ with $d_X(x, a) \leq td_X(x, b)$,

$$d_Y(f(x), f(a)) \leq \eta(t)d_Y(f(x), f(b)).$$

A metric n -sphere \mathbb{S} that is quasisymmetrically homeomorphic to \mathbb{S}^n is called a *quasisymmetric sphere* when $n \geq 2$, and a *quasisymmetric circle* when $n = 1$.

A doubling metric circle γ is called a *C-quasicircle* for some $C > 1$ if it satisfies the *2-point condition*:

$$\text{for all } x, y \in \gamma, \text{ diam } \gamma(x, y) \leq C|x - y|, \quad (2.1)$$

where $\gamma(x, y)$ denotes the subarc of γ connecting x and y of smaller diameter, or either subarc when both have the same diameter. Tukia and Väisälä [9] proved that quasisymmetric circles are exactly the quasicircles. Ahlfors [1] showed that a planar Jordan curve $\gamma \subset \mathbb{R}^2$ is a C -quasicircle if and only if, there exists an η -quasisymmetric homeomorphism $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that maps \mathbb{S}^1 onto γ .

A rectifiable metric circle or arc γ is called a *c-chord-arc curve* for some $c > 1$ if

$$\text{for all } x, y \in \gamma, \ell(\gamma'(x, y)) \leq c|x - y|, \quad (2.2)$$

where $\gamma'(x, y)$ is the shortest component of $\gamma \setminus \{x, y\}$ and $\ell(\cdot)$ denotes length. It is easy to see that a rectifiable metric circle or arc γ is a chord-arc curve if and only if $\ell(\gamma(x, y)) \leq c'|x - y|$ for some $c' > 1$; here constants c and c' are quantitatively related.

A Jordan arc is any proper subarc of a Jordan curve and a quasiair is any proper subarc of a quasicircle. For the rest, all quasicircles, quasiaircs and chord-arc curves are assumed to be subsets of \mathbb{R}^2 .

The notion of linear local connectivity generalizes the 2-point condition (2.1) on curves to general spaces. A metric space X is λ -linearly locally connected (or λ -LLC) for $\lambda \geq 1$ if the following two conditions are satisfied.

1. (λ -LLC₁) If $x \in X, r > 0$ and $y_1, y_2 \in B(x, r) \cap X$, then there exists a continuum $E \subset B(x, \lambda r) \cap X$ containing y_1, y_2 .
2. (λ -LLC₂) If $x \in X, r > 0$ and $y_1, y_2 \in X \setminus B(x, r)$, then there exists a continuum $E \subset X \setminus B(x, r/\lambda)$ containing y_1, y_2 .

A metric space X is said to be *Ahlfors Q-regular* if there is a constant $C > 1$ such that the Q -dimensional Hausdorff measure \mathcal{H}^Q of every open ball $B(a, r)$ in X satisfies

$$C^{-1}r^Q \leq \mathcal{H}^Q(B(a, r)) \leq Cr^Q, \quad (2.3)$$

when $0 < r \leq 2 \text{ diam } X$.

Bonk and Kleiner found in [4] an intrinsic characterization of quasisymmetric 2-spheres and then derived a readily applicable sufficient condition.

Theorem 2.1 ([4, Theorem 1.1, Lemma 2.5]). *Let X be an Ahlfors 2-regular metric space homeomorphic to \mathbb{S}^2 . Then X is quasisymmetric to \mathbb{S}^2 if and only if X is LLC.*

For $x \in \mathbb{R}^n$ and $r > 0$, define $B^n(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$. For any $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, denote by

$$\pi(a) = (a_1, a_2, 0)$$

the projection of a on the plane $\mathbb{R}^2 \times \{0\}$. For $x, y \in \mathbb{R}^2$, denote by $[x, y]$ the line segment having x, y as its end points.

In the following, we write $u \lesssim v$ (resp. $u \simeq v$) when the ratio u/v is bounded above (resp. bounded above and below) by positive constants. These constants may vary, but are described in each occurrence.

2.2 Geometry of level sets

For a planar Jordan domain Ω and some $\epsilon > 0$ define the ϵ -level set

$$\gamma_\epsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) = \epsilon\}.$$

In general the sets γ_ϵ need not be connected and if connected need not be curves; see [11, Figure 1]. We say that Ω has the *level quasicircle property* (or *LQC property*), if there exist $\epsilon_0 > 0$ and $C \geq 1$ such that the level set γ_ϵ is a C -quasicircle for every $0 \leq \epsilon \leq \epsilon_0$. Sufficient conditions for a domain Ω to satisfy the LQC property have been given in [11] in terms of the *chordal flatness* of $\partial\Omega$, a scale-invariant parameter measuring the local deviation of subarcs from their chords.

We list some properties of the level sets from [11] which are used in the proof of Theorem 1.2.

Lemma 2.2 ([11, Lemma 4.1]). *Suppose that Ω is a Jordan domain and for some $\epsilon > 0$ the set γ_ϵ is a Jordan curve. If σ is a closed subarc of $\partial\Omega$ then the set $\sigma' = \{z \in \Omega : \text{dist}(z, \partial\Omega) = \text{dist}(z, \sigma) = \epsilon\}$ is either empty or a subarc of γ_ϵ .*

Lemma 2.3 ([11, Lemma 6.1]). *There is a universal constant $c_0 > 1$ such that if σ is a Jordan arc and $\epsilon > 3 \text{ diam } \sigma$ then the set $\sigma' = \{x \in \mathbb{R}^2 : \text{dist}(x, \sigma) = \epsilon\}$ is a c_0 -chord-arc curve.*

Lemma 2.4 ([11, Theorem 1.3]). *Let Ω be a Jordan domain that has the LQC property. If $\partial\Omega$ is a c -chord-arc curve then there exist $\epsilon_0 > 0$ and $c' > 1$ such that γ_ϵ is a c' -chord-arc curve for all $\epsilon \in [0, \epsilon_0]$.*

3 A weak-chord arc property

3.1 A chord-arc index

Let Γ be a Jordan curve or arc. A *partition* \mathcal{P} of Γ is a finite set of mutually disjoint (except for their endpoints) subarcs of Γ whose union is all of Γ . We denote with $|\mathcal{P}|$ the number of elements a partition \mathcal{P} has. A δ -*partition* of Γ , for $\delta \in (0, 1)$, is a partition \mathcal{P} of Γ such that for each $\Gamma' \in \mathcal{P}$

$$\frac{\delta}{2} \text{diam } \Gamma \leq \text{diam } \Gamma' \leq \delta \text{diam } \Gamma.$$

Standard compactness arguments show that every Jordan curve or arc has δ -partitions for all $\delta \in (0, 1)$. Moreover, if \mathcal{P} is a δ -partition of Γ then $|\mathcal{P}| \geq 1/\delta$. In the next lemma, we show that if Γ is a quasiarc and \mathcal{P} is a δ -partition then, $|\mathcal{P}| \lesssim \delta^{-2}$.

Lemma 3.1. *For each $C > 1$ there exists a number $C' > 1$ depending only on C such that, if Γ is a C -quasiarc and $\lambda \in (0, 1]$ then, each partition $\{\Gamma_1, \dots, \Gamma_N\}$ with $\text{diam } \Gamma_i \geq \lambda \text{diam } \Gamma$ satisfies $N \leq C'/\lambda^2$.*

Proof. We may assume that $\text{diam } \Gamma = 1$. Fix $\lambda \in (0, 1)$ and let $\{\Gamma_1, \dots, \Gamma_N\}$ be a partition of Γ with $\text{diam } \Gamma_i \geq \lambda$. Let $A = \{x_0, x_1, \dots, x_N\}$ be the set of endpoints of $\Gamma_1, \dots, \Gamma_N$. The 2-point condition of Γ implies that $|x - y| \geq \lambda/C$ for all $x, y \in A$. Since all points of A lie in \mathbb{R}^2 which is a doubling, space, there is a universal constant $C_0 > 1$ such that $N \leq C_0(\lambda/C)^{-2} = C_0 C^2 / \lambda^2$. \square

For a partition \mathcal{P} of Γ define

$$M(\Gamma, \mathcal{P}) = \frac{1}{\text{diam } \Gamma} \sum_{\Gamma' \in \mathcal{P}} \text{diam } \Gamma'.$$

The finite number $M(\Gamma, \mathcal{P})$ measures, on the scale of \mathcal{P} , the deviation of Γ from being a chord-arc curve. When $\ell(\Gamma)$ is finite, the number $M(\Gamma, \mathcal{P})$ is an estimation of $\ell(\Gamma)/\text{diam } \Gamma$: for any $\epsilon > 0$, there exists a partition \mathcal{P} of Γ such that $|M(\Gamma, \mathcal{P}) - \ell(\Gamma)/\text{diam } \Gamma| < \epsilon$. On the other hand, if $\ell(\Gamma)$ is infinite then for all $M > 0$ there exists a partition \mathcal{P} of Γ such that $M(\Gamma, \mathcal{P}) > M$. The properties of the gauge $M(\Gamma, \mathcal{P})$ are summarized in the following lemma.

Lemma 3.2. *Suppose that Γ is a Jordan curve or arc.*

1. *If \mathcal{P} is a δ -partition then $\frac{1}{2}|\mathcal{P}|\delta \leq M(\Gamma, \mathcal{P}) \leq |\mathcal{P}|\delta$.*

2. Suppose that $\mathcal{P} = \{\Gamma_1, \dots, \Gamma_N\}$ is a partition of Γ and for each $i = 1, \dots, N$, \mathcal{P}_i is a partition of Γ_i . Then,

$$M(\Gamma, \bigcup_{i=1}^N \mathcal{P}_i) \geq M(\Gamma, \mathcal{P}).$$

3. Assume that Γ is a C -quasiarc and $\delta \in (0, 1)$. There exists $M_1 > 1$ depending on C such that $M(\Gamma, \mathcal{P})/M(\Gamma, \mathcal{P}') \leq M_1$ for any two δ -partitions $\mathcal{P}, \mathcal{P}'$ of Γ .

Proof. Property (1) is an immediate consequence of the definition. For property (2) simply note that $\text{diam } \Gamma_i \leq \sum_{\Gamma' \in \mathcal{P}_i} \Gamma'$. Finally, to show (3), let $\mathcal{P} = \{\Gamma_1, \dots, \Gamma_N\}$ and $\mathcal{P}' = \{\Gamma'_1, \dots, \Gamma'_{N'}\}$ be two δ -partitions of Γ . By Lemma 3.1 there exists $N_0 > 0$ depending on C such that each Γ_i can contain at most N_0 subarcs Γ'_j . Thus, $N' \leq N_0 N$ and (3) follows from (1). \square

In the next lemma we show that if a quasiarc resembles a chord-arc curve for partitions of small scale then it should also do that for partitions of larger scale.

Lemma 3.3. *Suppose that Γ is a C -quasiarc, $N_0 > 0$ and \mathcal{P} is a δ -partition of Γ with $\delta \in (0, 1)$ and $|\mathcal{P}| \leq N_0/\delta$. There exists N_1 depending on C, N_0 such that for all $\delta' \in [\delta, 1)$ and all δ' -partitions \mathcal{P}' of Γ we have $|\mathcal{P}'| \leq N_1/\delta'$.*

Proof. We prove the lemma for $N_1 = 5M_1N_0$ where M_1 is as in the third part of Lemma 3.2. Contrary to the claim, suppose that there exist a number $\delta' \in [\delta, 1)$ and a δ' -partition $\mathcal{P}' = \{\Gamma_1, \dots, \Gamma_N\}$ such that $N > N_1/\delta'$. For each $i = 1, \dots, N$ consider a $\frac{\delta \text{diam } \Gamma}{\text{diam } \Gamma_i}$ -partition \mathcal{P}_i of Γ_i ; for those i that $\text{diam } \Gamma_i \leq \delta \text{diam } \Gamma$ set $\mathcal{P}_i = \{\Gamma'_i\}$. Note that $|\mathcal{P}_i| \geq \frac{\text{diam } \Gamma_i}{\delta \text{diam } \Gamma}$ and that $\mathcal{P}^* = \bigcup_{i=1}^N \mathcal{P}_i$ is a δ -partition of Γ . By Lemma 3.2,

$$M(\Gamma, \mathcal{P}^*) \geq (|\mathcal{P}_1| + \dots + |\mathcal{P}_N|) \frac{\delta}{2} > M_1 N_0.$$

But then $M(\Gamma, \mathcal{P}^*)/M(\Gamma, \mathcal{P}) > M_1$ which is a contradiction. \square

The following lemma is used in the proof of Theorem 1.3.

Lemma 3.4. *Let Γ be a Jordan arc and $0 < \delta < \delta' < 1$. Suppose that each δ -partition of Γ has at least N elements for some $N > 1$. Then, there exists a subarc $\Gamma' \subset \Gamma$ and a δ' -partition \mathcal{P}' of Γ' with $|\mathcal{P}'| \geq N \frac{\log \delta' + 1}{\log(\delta\delta'/2)}$.*

Proof. Contrary to the lemma, suppose that for each subarc $\sigma \subset \Gamma$, every δ' -partition \mathcal{P} of σ has $|\mathcal{P}| \leq N \frac{\log \delta' + 1}{\log(\delta\delta'/2)}$. We construct a partition of Γ as follows.

Let $k_0 \in \mathbb{N}$ be the smallest integer that is greater than $\frac{\log(\delta/2)}{\log \delta'}$. Let $\mathcal{P}_0 = \{\Gamma_1, \dots, \Gamma_{N_0}\}$ be a δ' -partition of Γ . For each $i = 1, \dots, N_0$ let $\mathcal{P}_i = \{\Gamma_{i1}, \dots, \Gamma_{iN_i}\}$ be a δ' -partition of Γ_i . Inductively, suppose that Γ_w has been defined where $w = i_1 i_2 \dots i_k, i_j \in \mathbb{N}$ and $k < k_0$. Then let $\mathcal{P}_w = \{\Gamma_{w1}, \dots, \Gamma_{wN_w}\}$ be a δ' -partition of Γ_w .

Define $\mathcal{P}^* = \bigcup_w \mathcal{P}_w$ where the union is taken over all words $w = i_1 \dots i_{k_0}$. Note that for each $\Gamma_w \in \mathcal{P}^*$,

$$\left(\frac{\delta'}{2}\right)^{k_0} \leq \frac{\text{diam } \Gamma_w}{\text{diam } \Gamma} \leq (\delta')^{k_0} < \frac{\delta}{2}.$$

Moreover, by our assumptions, for all partitions \mathcal{P}_w defined,

$$|\mathcal{P}_w| \leq N \frac{\log \delta' + 1}{\log(\delta\delta'/2)}.$$

Consequently, since $k_0 \leq \frac{\log(\delta/2)}{\log \delta'} + 1$,

$$|\mathcal{P}^*| \leq \left(N \frac{\log \delta' + 1}{\log(\delta\delta'/2)}\right)^{k_0} < \left(N \frac{\log \delta'}{\log(\delta\delta'/2)}\right)^{k_0} \leq N.$$

If \mathcal{P} is a δ -partition of Γ then, for each $\Gamma' \in \mathcal{P}$ and each $\Gamma_w \in \mathcal{P}^*$, $\text{diam } \Gamma_w < \text{diam } \Gamma'$. Thus, $|\mathcal{P}| \leq |\mathcal{P}^*| < N$ which is a contradiction. \square

3.2 A weak chord-arc property

Note that if a curve Γ is a chord-arc curve then there exists $M_0 > 1$ such that for all subarcs $\Gamma' \subset \Gamma$, all $\delta > 0$ and all δ -partitions \mathcal{P} of Γ' , we have $M(\Gamma', \mathcal{P}) < M_0$.

A curve or arc Γ is said to have the weak chord-arc property if there exists $M_0 > 0$ such that for all subarcs $\Gamma' \subset \Gamma$ with $\text{diam } \Gamma' < 1$, there exists a $\text{diam } \Gamma'$ -partition \mathcal{P} of Γ' satisfying $M(\Gamma', \mathcal{P}) \leq M_0$.

In other words, a curve Γ is a weak chord-arc curve if there exists $M_0 \geq 1$ such that any subarc Γ' of Γ can be partitioned to at most $2M_0/\text{diam } \Gamma'$ subarcs $\Gamma_1, \dots, \Gamma_N$ of diameters comparable to $(\text{diam } \Gamma')^2$. It is clear that a chord-arc curve is a weak chord-arc curve but the converse fails; see Section 5.1. The third claim of Lemma 3.2 implies that, for quasicircles, the weak chord-arc property is equivalent to a stronger condition.

Lemma 3.5. *Suppose that Γ is a C -quasicircle that satisfies the weak chord-arc property with constant $M_0 > 1$. For each $\alpha \in (0, 1)$ there exists $M_\alpha > 1$ depending only on C, M_0, α such that for all subarcs $\Gamma' \subset \Gamma$ of diameter less than 1 and all $(\text{diam } \Gamma')^{\frac{1}{\alpha}-1}$ -partitions \mathcal{P} we have $M(\Gamma', \mathcal{P}) \leq M_\alpha$.*

Proof. If $\alpha \in [\frac{1}{2}, 1)$ then the claim follows immediately from Lemma 3.3 with $M_\alpha = 10M_0M_1$ where M_1 is as in Lemma 3.2.

Suppose now that $\alpha \in (0, \frac{1}{2})$ and let k_0 be the smallest integer with $2^{-k_0} \leq \alpha$. Fix a subarc $\Gamma_0 \subset \Gamma$ with $\text{diam } \Gamma_0 < 1$ and a $\text{diam } \Gamma_0$ -partition $\mathcal{P}_0 = \{\Gamma_1, \dots, \Gamma_{N_0}\}$ of Γ_0 with $|\mathcal{P}_0| \leq 2M_0$. For each $i = 1, \dots, N_0$ let \mathcal{P}_i be a $\frac{(\text{diam } \Gamma_0)^4}{\text{diam } \Gamma_i}$ -partition of Γ_i . The weak chord-arc property of Γ_i and Lemma 3.3 imply $|\mathcal{P}_i| \leq 10M_0M_1(\text{diam } \Gamma_0)^{-2}$. Inductively, suppose that Γ_w has been defined where $w = i_1 \cdots i_k$, $i_j \in \mathbb{N}$ and $k < k_0$. Then, let $\mathcal{P}_w = \{\Gamma_{w1}, \dots, \Gamma_{wN_w}\}$ be a $\frac{(\text{diam } \Gamma_0)^{2^{k+1}}}{\text{diam } \Gamma_w}$ -partition of Γ_w . Again, note that $|\mathcal{P}_w| \leq 10M_0M_1(\text{diam } \Gamma_0)^{-2^k}$.

Define $\mathcal{P}^* = \bigcup_w \mathcal{P}_w$ where the union is taken over all words $w = i_1 \cdots i_{k_0}$ constructed as above. Note that \mathcal{P}^* is a $(\text{diam } \Gamma_0)^{2^{k_0}-1}$ -partition of Γ_0 with

$$M(\Gamma_0, \mathcal{P}^*) \leq \frac{(\text{diam } \Gamma_0)^{2^{k_0}}}{\text{diam } \Gamma_0} \sum_{w=i_1 \cdots i_{k_0}} |\mathcal{P}_w| \leq (10M_0M_1)^{k_0}.$$

Since $(\text{diam } \Gamma_0)^{1/\alpha} \geq (\text{diam } \Gamma_0)^{2^{k_0}} \geq \text{diam } \Gamma_w$ for all $\Gamma_w \in \mathcal{P}^*$, it follows from Lemma 3.3 that every $(\text{diam } \Gamma')^{\frac{1}{\alpha}-1}$ -partition \mathcal{P} of Γ_0 satisfies $M(\Gamma_0, \mathcal{P}) \leq M_\alpha$ with $M_\alpha = (10M_0M_1)^{k_0+1}$. \square

By interchanging the roles of $\frac{1}{2}$ and α , and applying the arguments in the proof of Lemma 3.5, the following converse can be obtained.

Remark 3.6. *Suppose that Γ is a C -quasicircle and that there exist $\alpha \in (0, 1)$ and $M_\alpha > 1$ such that every subarc $\Gamma' \subset \Gamma$ with $\text{diam } \Gamma' < 1$ has a $(\text{diam } \Gamma')^{\frac{1}{\alpha}-1}$ -partition \mathcal{P} satisfying $M(\Gamma', \mathcal{P}) \leq M_\alpha$. Then Γ has the weak chord-arc property.*

4 Proofs of the main results

The proof of Theorem 1.2 is given in Section 4.1 and Section 4.2. By the theorem of Bonk and Kleiner, it is clear that (2) implies (1).

To show that (3) implies (2) we first show in Lemma 4.6 that if Ω satisfies the LQC property then $\Sigma(\Omega, t^\alpha)$ is LLC for all $\alpha \in (0, 1)$. Then, in Proposition 4.1 we prove that if Ω has the LQC property and $\partial\Omega$ is a weak chord-arc curve then $\Sigma(\Omega, t^\alpha)$ is 2-regular.

To prove that (1) implies (3), we show in Lemma 4.6 that if $\Sigma(\Omega, t^\alpha)$ is LLC for some $\alpha \in (0, 1)$ then Ω satisfies the LQC property. Then, in Proposition 4.7 we show that if Ω has the LQC property and $\Sigma(\Omega, t^\alpha)$ is quasisymmetric to \mathbb{S}^2 then $\partial\Omega$ is a weak chord-arc curve.

The proof of Theorem 1.3 is given in Section 4.3.

4.1 Ahlfors 2-regularity

The following proposition connects the weak chord-arc property of $\partial\Omega$ with the 2-regularity of $\Sigma(\Omega, t^\alpha)$.

Proposition 4.1. *Suppose that Ω has the LQC property, $\partial\Omega$ has the weak chord-arc property and $\alpha \in (0, 1)$. Then, $\Sigma(\Omega, t^\alpha)$ is 2-regular.*

For the rest of Section 4.1 we assume that there exist $C_0 > 1$ and $\epsilon_0 \in (0, \frac{1}{8C_0})$ such that for any $\epsilon \in [0, \epsilon_0]$, the level set γ_ϵ is a C_0 -quasicircle. To show (2.3) we first apply some reductions on $a \in \Sigma(\Omega, t^\alpha)$ and $r > 0$.

Reduction 1. Let $\Sigma(\Omega, t^\alpha)^+ = \Sigma(\Omega, t^\alpha) \cap \{x \in \mathbb{R}^3 : x_3 \geq 0\}$. If $a \in \Sigma(\Omega, t^\alpha)^+$ and $r > 0$ then by symmetry of $\Sigma(\Omega, t^\alpha)$ with respect to $\mathbb{R}^2 \times \{0\}$ we have

$$\mathcal{H}^2(B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+) \leq \mathcal{H}^2(B^3(a, r) \cap \Sigma(\Omega, t^\alpha)) \leq 2\mathcal{H}^2(B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+).$$

Therefore, it is enough to verify

$$C^{-1}r^2 \leq \mathcal{H}^2(B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+) \leq Cr^2, \quad (4.1)$$

for some $C > 1$ and for all $a \in \Sigma(\Omega, t^\alpha)^+$ and $r \leq 2 \operatorname{diam} \Sigma(\Omega, t^\alpha)$.

Reduction 2. Let $r_0 = \min\{\epsilon_0/3, \frac{1}{180C_0^2} \operatorname{diam} \Omega\}$. By the doubling property of \mathbb{R}^3 , there exists $c_1 > 1$ depending only on ϵ_0 , $\operatorname{diam} \Omega$ and α such that each ball $B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+$ with $r \geq r_0$ can be covered by at most c_1 balls $B^3(a_i, r_i) \cap \Sigma(\Omega, t^\alpha)^+$ with $a_i \in \Sigma(\Omega, t^\alpha)^+$ and $r_i < r_0$. Hence, it suffices to show (4.1) for $r \leq r_0$.

Reduction 3. We claim that it is enough to verify (4.1) only for those points $a \in \Sigma(\Omega, t^\alpha)^+$ whose projection satisfies $\operatorname{dist}(\pi(a), \partial\Omega) \leq \epsilon_0$. Indeed, following the notation in [12], let $\Delta_{\epsilon_0/3}$ be the set of all points in Ω whose distance from $\partial\Omega$ is greater than $\epsilon_0/3$ and $\Delta_{\epsilon_0/3}^+$ be the subset of $\Sigma(\Omega, t^\alpha)^+$ whose projection on $\mathbb{R}^2 \times \{0\}$ is $\Delta_{\epsilon_0/3}$. Since $\partial\Delta_{\epsilon_0/3} = \gamma_{\epsilon_0/3}$ is a quasicircle, the domain $\Delta_{\epsilon_0/3}$ is 2-regular. Therefore, the surface $\Delta_{\epsilon_0/3}^+$, which is the graph of a Lipschitz function on $\Delta_{\epsilon_0/3}$, is 2-regular as well. Thus, if $a \in \Sigma(\Omega, t^\alpha)^+$ satisfies $\operatorname{dist}(\pi(a), \partial\Omega) > \epsilon_0$ and $r \in (0, \epsilon_0/3)$ then $B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+ \subset \Delta_{\epsilon_0/3}^+$ and $\mathcal{H}^2(B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+) \simeq r^2$.

Reduction 4. We show in Lemma 4.2 that it is enough to check the Hausdorff 2-measure of certain subsets on $\Sigma(\Omega, t^\alpha)$ defined as follows. Suppose that x_1, y_1, x_2, y_2 are points in $\Sigma(\Omega, t^\alpha)^+$ satisfying the following properties:

- (i) $\pi(x_1), \pi(y_1) \in \gamma_{t_1}$ and $\pi(x_2), \pi(y_2) \in \gamma_{t_2}$ for some $0 \leq t_2 < t_1 \leq \epsilon_0$,
- (ii) $|\pi(x_1) - \pi(x_2)| = \operatorname{dist}(\pi(x_1), \gamma_{t_2}) = |\pi(y_1) - \pi(y_2)| = \operatorname{dist}(\pi(y_1), \gamma_{t_2}) = t_1 - t_2$,
- (iii) $\frac{1}{20C_0} |x_1 - y_1| \leq t_1 - t_2 + t_1^\alpha - t_2^\alpha \leq \frac{1}{3} |x_1 - y_1| \leq \frac{1}{10C_0} \operatorname{diam} \Omega$.

Property (i) implies that for each $i = 1, 2$, x_i is on the same horizontal plane as y_i . Property (ii) implies that $\pi(x_2), \pi(y_2)$ are the points of γ_{t_2} which are closest to $\pi(x_1), \pi(y_1)$ respectively. Property (iii) implies that $|x_i - y_j|$, $|x_1 - x_2|$ and $|y_1 - y_2|$ are all comparable and sufficiently small. If $x_0, y_0 \in \partial\Omega$ are the closest points to $\pi(x_1), \pi(y_1)$ respectively, then $|\pi(x_2) - x_0| = |\pi(y_2) - y_0| = t_2$. Therefore, $\pi(x_1), \pi(x_2), x_0$ are colinear and the line segment $[\pi(x_1), \pi(x_2)]$ joining the points $\pi(x_1)$ and $\pi(x_2)$ is entirely in Ω . Similarly, $[\pi(y_1), \pi(y_2)] \subset \Omega$.

Denote with $D = D(x_1, y_1, x_2, y_2)$ the subset on $\Sigma(\Omega, t^\alpha)^+$ whose projection on \mathbb{R}^2 is the quadrilateral bounded by $[\pi(x_1), \pi(x_2)]$, $\gamma_{t_1}(\pi(x_1), \pi(y_1))$, $[\pi(y_1), \pi(y_2)]$, $\gamma_{t_2}(\pi(x_2), \pi(y_2))$. We call D a *square piece* on $\Sigma(\Omega, t^\alpha)^+$.

The following lemma is a corollary of [12, (6.4)] and [12, Remark 6.1].

Lemma 4.2. *There exist $C_1, C_2 > 1$ depending on $C_0, \epsilon_0, \operatorname{diam} \Omega$ such that*

1. $\operatorname{diam} D \leq C_1 |x_1 - y_1|$ for every square piece $D = D(x_1, y_1, x_2, y_2)$,
2. for all $a \in \Sigma(\Omega, t^\alpha)^+$ and $r > 0$ as above, there exist square pieces D_1 and D_2 such that $1 < \operatorname{diam} D_2 / \operatorname{diam} D_1 \leq C_2$ and

$$D_1 \subset B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+ \subset D_2.$$

Thus, by (iii), choosing ϵ_0 small enough, we may assume from now on that $\text{diam } D \leq (4C_0)^{-1}$ for all square pieces D . By Lemma 4.2 and the discussion above, Proposition 4.1 is now equivalent to the following lemma.

Lemma 4.3. *There exists $C > 1$ depending on C_0 and the weak chord-arc constant M_0 such that*

$$C^{-1}(\text{diam } D)^2 \leq \mathcal{H}^2(D) \leq C(\text{diam } D)^2 \tag{4.2}$$

for all square pieces D on $\Sigma(\Omega, t^\alpha)^+$ defined as above.

The lower bound in (4.2) was shown in [12, Section 6.2.1]. For the upper bound, the following lemma is used; we only give a sketch of its proof since it is similar to the discussion in [12, Section 6.2.1].

Lemma 4.4. *Let S be a closed subset of $\Sigma(\Omega, t^\alpha)^+$ and $0 \leq t_2 \leq t_1$ be such that $\pi(S)$ intersects with γ_t if and only if $t \in [t_2, t_1]$. Suppose that, for all $t, t' \in [t_2, t_1]$, the Hausdorff distance between $\pi(S) \cap \gamma_t$ and $\pi(S) \cap \gamma_{t'}$ is less than $c_1|t - t'|$ for some $c_1 > 1$, and $\pi(S) \cap \gamma_t$ is a c_2 -chord-arc curve for some $c_2 > 1$. Then $\mathcal{H}^2(S) \leq C(\text{diam } S)^2$ for some C depending on c_1, c_2 .*

Proof. Fix $\epsilon > 0$. Let $t_2 = \tau_1 < \dots < \tau_N = t_1$ be such that the sets $S_i = S \cap (\gamma_{\tau_i} \times \{\tau_i^\alpha\})$ satisfy $\epsilon/4 \leq \text{dist}(S_i, S_{i+1}) \leq \epsilon/2$. By the first assumption of the lemma, it is straightforward to check that $N \leq N_1 \text{diam } S/\epsilon$ for some N_1 depending on c_1 . The second assumption implies that each S_i contains points $x_{i,1}, \dots, x_{i,n_i}$ satisfying $|x_{i,j} - x_{i,j+1}| \leq \epsilon/2$ and $n_i \leq N_2 \text{diam } S_i/\epsilon \leq N_2 \text{diam } S/\epsilon$ for some N_2 depending on c_2 . Thus, S can be covered by at most $N_1 N_2 (\text{diam } S)^2/\epsilon^2$ balls of radius ϵ and the lemma follows. \square

For the upper bound of (4.2) we consider two cases. The first case is an application of Lemma 4.4 while in the second case we use the weak chord-arc condition to subdivide D into smaller pieces on which the first case applies.

Proof of Lemma 4.3. As mentioned already, the lower bound of (4.2) follows from the discussion in [12, Section 6.2.1] even without the weak chord-arc assumption of $\partial\Omega$. It remains to show the upper bound. By Lemma 3.3, there exists $M_\alpha > 1$ such that for any subarc $\Gamma \subset \partial\Omega$ with $\text{diam } \Gamma < 1$ and for any $(\text{diam } \Gamma)^{\frac{1}{\alpha}-1}$ -partition \mathcal{P} of Γ we have $M(\Gamma, \mathcal{P}) \leq M_\alpha$.

Fix a square piece $D = D(x_1, y_1, x_2, y_2)$ where x_1, y_1, x_2, y_2 satisfy equations (i)-(iii). Let $x_0, y_0 \in \partial\Omega$ be such that

$$|x_0 - \pi(x_1)| = |y_0 - \pi(y_1)| = t_1 \text{ and } |x_0 - \pi(x_2)| = |y_0 - \pi(y_2)| = t_2$$

and set Γ_0 to be the subarc of $\partial\Omega$, of smaller diameter, with endpoints x_0, y_0 . The choice of ϵ_0 implies $\text{diam } \Gamma_0 \leq C_0|x_0 - y_0| \leq C_0(\text{diam } D + 2t_1) \leq C_0(\text{diam } D + 2\epsilon_0) < 1/2$.

Case 1. Suppose that $(\text{diam } \Gamma_0)^{\frac{1}{\alpha}} \leq t_2/10$. We claim that there is $C > 1$ depending on C_0, M_α such that for each $\epsilon \in [t_2, t_1]$, the arc $\gamma_\epsilon \cap \pi(D)$ is a C -chord-arc curve. Assuming the claim, the upper bound of (4.2) follows from Lemma 4.4. To prove the claim, it suffices to show that $\gamma_{t_2} \cap \pi(D)$ is a C -chord-arc curve. Then, using the fact that for each $\epsilon > t_2$, the set γ_ϵ is the $(\epsilon - t_2)$ -level set of γ_{t_2} , the claim follows from Lemma 2.4. Let σ be a subarc of $\gamma_{t_2} \cap D$ and x, y be the endpoints of σ . Since γ_{t_2} is a quasicircle, it suffices to show that there exists $C > 1$ such that $\ell(\sigma) \leq C \text{diam } \sigma$. Let $x', y' \in \Gamma_0$ be such that $|x - x'| = |y - y'| = t_2$ and set $\sigma' = \Gamma_0(x', y')$.

If $\text{diam } \sigma' \leq t_2/10$ then, by Lemma 2.3, σ is a c_0 -chord-arc curve for some universal $c_0 > 1$.

If $\text{diam } \sigma' \in [t_2/10, 10t_2]$ then apply Lemma 3.1 to get a 10^{-2} -partition $\mathcal{P} = \sigma'_1, \dots, \sigma'_N$ of σ' with N bounded above by a positive constant depending on C_0 . For each $n = 1, \dots, N$ let σ_n be the set of all points z in σ that satisfy $\text{dist}(z, \sigma'_n) = t_2$. By Lemma 2.2, $\{\sigma_n\}$ are subarcs of σ , perhaps not mutually disjoint. Thus, σ is a union of N c_0 -chord-arc curves, hence $\ell(\sigma) \leq c_1 \text{diam } \sigma$ for some $c_1 > 1$ depending on C_0 .

Finally, if $\text{diam } \sigma' > 10t_2$ note that $\text{diam } \sigma \simeq \text{diam } \sigma'$. Since $(\text{diam } \sigma')^{\frac{1}{\alpha}} \leq (\text{diam } \Gamma_0)^{\frac{1}{\alpha}} \leq t_2/10$, Lemma 3.3 and the weak chord-arc property of σ' imply that there exists a $(\frac{t_2}{10 \text{diam } \sigma'})$ -partition $\mathcal{P} = \{\sigma'_1, \dots, \sigma'_N\}$ of σ' with $N \leq N_0 \text{diam } \sigma'/t_2$ and N_0 depending on C_0, M_α . Define $\sigma_1, \dots, \sigma_N$ as above and note that each σ_n is a

c_0 -chord-arc curve and satisfies $\text{diam } \sigma_n \leq \text{diam } \sigma'_n + t_2 \lesssim t_2$. Hence,

$$\ell(\sigma) \leq \sum_{n=1}^N \ell(\sigma_n) \leq \sum_{n=1}^N c_1 \text{diam } \sigma_n \lesssim N t_2 \lesssim \text{diam } \sigma' \simeq \text{diam } \sigma.$$

Case 2. Suppose that $(\text{diam } \Gamma_0)^{\frac{1}{\alpha}} > t_2/10$. Let $H = 10^{2/\alpha}$. For a subarc $\Gamma' \subset \Gamma_0$ define $D(\Gamma')$ to be the set of all points $(z, (\text{dist}(z, \partial\Omega))^\alpha)$ such that $z \in \overline{\Omega}$, $\text{dist}(z, \partial\Omega) = \text{dist}(z, \Gamma')$ and

$$0 \leq \text{dist}(z, \partial\Omega) \leq H(\text{diam } \Gamma')^{1/\alpha}.$$

Note that $\text{diam } D(\Gamma') \leq 2(H^\alpha \text{diam } \Gamma' + H(\text{diam } \Gamma')^{1/\alpha} + \text{diam } \gamma_{t_1}(x_1, y_1)) \lesssim \text{diam } \Gamma'$. From the middle inequality of (iii), it is easy to see that $t_1 < H(\text{diam } \Gamma_0)^{1/\alpha}$ and thus $D \subset D(\Gamma_0)$. Let

$$E_0 = \{x \in D(\Gamma_0) : \text{dist}(\pi(x), \partial\Omega) \geq 10(\text{diam } \Gamma_0)^{2/\alpha}\}$$

and note that $\text{diam } E_0 \leq \text{diam } D(\Gamma_0) \lesssim \text{diam } \Gamma_0$.

We claim that $\mathcal{H}^2(E_0) \leq C(\text{diam } \Gamma_0)^2$ for some $C > 1$ depending on C_0, M_α . As in *Case 1*, it is enough to show that $\gamma = \gamma_t \cap \pi(E_0)$ is a chord-arc curve when $t = 10(\text{diam } \Gamma_0)^{\frac{2}{\alpha}}$. The claim then follows from Lemma 2.4 and Lemma 4.4. Let $\sigma \subset \gamma$ and define $\sigma' \subset \Gamma_0$ as in *Case 1*. If $\text{diam } \sigma' \leq 10t$ then the claim follows from Lemma 2.3 and Lemma 3.3 as in *Case 1*. If $\text{diam } \sigma' \geq 10t$ then let $\{\sigma'_1, \dots, \sigma'_m\}$ be a $(\text{diam } \sigma')^{\frac{1}{\alpha}-1}$ -partition of σ' with $m \leq 2M_\alpha$. For each $i = 1, \dots, m$ let $\{\sigma'_{i1}, \dots, \sigma'_{iN_i}\}$ be a $(t(\text{diam } \sigma')^{-\frac{1}{\alpha}})$ -partition of σ'_i . Define subarcs $\{\sigma_{ij}\}$ of σ as in *Case 1*. Since $\text{diam } \sigma'_{ij} \leq t < 10t$, each σ_{ij} is a c_1 -chord-arc curve for some $c_1 > 1$ depending on C_0 . Moreover, by the weak chord-arc property and Lemma 3.5, $N_1 + \dots + N_m \leq 10M_1M_\alpha^2 \text{diam } \sigma'/t$ where M_1 is as in Lemma 3.2. Since $\text{diam } \sigma' \simeq \text{diam } \sigma$,

$$\ell(\sigma) \leq \sum_{i,j} \ell(\sigma_{ij}) \leq c_1 \sum_{i,j} \text{diam } \sigma_{ij} \lesssim t \sum_{i=1}^m N_i \lesssim \text{diam } \sigma' \lesssim \text{diam } \sigma.$$

and the claim follows.

Let $\mathcal{P}_1 = \{\Gamma_1, \dots, \Gamma_N\}$ be a $(\text{diam } \Gamma_0)^{\frac{1}{\alpha}-1}$ -partition of Γ_0 with $N \leq 2M_\alpha(\text{diam } \Gamma_0)^{1-\frac{1}{\alpha}}$. Define $D(\Gamma_i), E_i$ as above and note that the choice of H yields $D(\Gamma_0) \subset E_0 \cup \bigcup_{i=1}^N D(\Gamma_i)$. Similarly as above, $\mathcal{H}^2(E_i) \leq C(\text{diam } \Gamma_0)^{\frac{2}{\alpha}}$.

For each $i = 1, \dots, N$ let $\{\Gamma_{i1}, \dots, \Gamma_{iN_i}\}$ be a $\frac{(\text{diam } \Gamma_0)^{1/\alpha^2}}{\text{diam } \Gamma_i}$ -partition of Γ_i and set $\mathcal{P}_2 = \{\Gamma_{ij}\}$. The weak chord-arc condition and Lemma 3.1 imply that $N_i \leq 10M_\alpha M_1 (\text{diam } \Gamma_i)^{1-\frac{1}{\alpha}} \leq 10M_\alpha M_1 2^{\frac{1}{\alpha}-1} (\text{diam } \Gamma_0)^{\frac{1}{\alpha}-\frac{1}{\alpha^2}}$ where M_1 is as in Lemma 3.2. Thus, $|\mathcal{P}_2| \leq (m_0 2^{\frac{1}{\alpha}-1})^2 (\text{diam } \Gamma_0)^{1-\frac{1}{\alpha^2}}$ with $m_0 = 10M_\alpha M_1$. Again, the choice of H yields

$$D(\Gamma_0) \subset E_0 \cup \bigcup_{i=1}^N E_i \cup \bigcup_{i=1}^N \bigcup_{j=1}^{N_i} D(\Gamma_{ij}).$$

Inductively, we obtain partitions $\mathcal{P}_k = \{\Gamma_{i_1 \dots i_k}\}$ with

$$|\mathcal{P}_k| \leq (m_0 2^{\frac{1}{\alpha}-1})^k (\text{diam } \Gamma_0)^{1-\frac{1}{\alpha^k}}$$

and square-like pieces $E_{i_1 \dots i_k}$ with

$$D(\Gamma_0) \subset E_0 \cup \bigcup_i E_i \cup \bigcup_{i_1, i_2} E_{i_1, i_2} \cup \dots \cup \bigcup_{\Gamma_{i_1 \dots i_k} \in \mathcal{P}_k} D(\Gamma_{i_1 \dots i_k})$$

and $\mathcal{H}^2(E_{i_1 \dots i_k}) \leq C(\text{diam } \Gamma_0)^{\frac{2}{\alpha^k}}$. Therefore, since $D \subset D(\Gamma_0)$,

$$\begin{aligned} \mathcal{H}^2(D) &\leq \mathcal{H}^2(E_0) + \sum_i \mathcal{H}^2(E_i) + \sum_{i_1, i_2} \mathcal{H}^2(E_{i_1, i_2}) + \dots \\ &\lesssim (\text{diam } \Gamma_0)^2 \sum_{k=0}^{\infty} (m_0 2^{\frac{1}{\alpha}-1})^k (\text{diam } \Gamma_0)^{\frac{1}{\alpha^k}-1} \\ &\leq (\text{diam } \Gamma_0)^2 \sum_{k=0}^{\infty} \frac{(m_0 2^{\frac{1}{\alpha}-1})^k}{2^{\frac{1}{\alpha^k}-1}}. \end{aligned}$$

It is easy to see that the latter series converges. Since $\text{diam } \Gamma_0 \leq \text{diam } D$, we conclude that $\mathcal{H}^2(D) \lesssim (\text{diam } D)^2$ and the proof is complete. \square

4.2 The LQC property and Väisälä's method

The connection between the LQC property of Ω and the LLC property of $\Sigma(\Omega, t^\alpha)$ is established in the following proposition from [12].

Proposition 4.5 ([12, Proposition 5.1, Lemma 5.6]). *Suppose that $\alpha \in (0, 1)$ and Ω is a Jordan domain whose boundary $\partial\Omega$ is a quasicircle. Then $\Sigma(\Omega, t^\alpha)$ is LLC if and only if Ω has the LQC property.*

It turns out that the quasicircle assumption of $\partial\Omega$ can be dropped in Proposition 4.5.

Lemma 4.6. *Suppose that Ω is a Jordan domain and $\alpha \in (0, 1)$. Then, $\Sigma(\Omega, t^\alpha)$ is LLC if and only if Ω has the LQC property.*

Proof. Suppose that $\Sigma = \Sigma(\Omega, t^\alpha)$ is λ -LLC for some $\lambda > 1$. We show that $\partial\Omega$ is a $(4\lambda^2)$ -quasicircle and the proof follows then from Proposition 4.5.

Let $x, y \in \partial\Omega$ and γ, γ' be the two components of $\partial\Omega \setminus \{x, y\}$. By the λ -LLC₁ property, the points x, y are contained in a continuum E in $B^3(x, 2\lambda|x-y|) \cap \Sigma$. Then the projection $\pi(E)$ is a continuum in $\bar{\Omega}$ containing x, y . Suppose that there exist points z, z' in γ, γ' respectively, which lie outside of $B^3(x, 2\lambda^2|x-y|)$. The λ -LLC₂ property implies that there exists a continuum $E' \subset \Sigma \setminus B^3(x, 2\lambda|x-y|)$ that contains z, z' . But then, the projection $\pi(E')$ is a continuum in $\bar{\Omega}$ containing z, z' . It follows that $\pi(E) \cap \pi(E') \neq \emptyset$ and, since Σ is symmetric with respect to $\mathbb{R}^2 \times \{0\}$, E' intersects $B^3(x, 2\lambda|x-y|)$ which is a contradiction. Therefore, at least one of γ, γ' lies in $\bar{B}^3(x, 2\lambda^2|x-y|)$ and $\min\{\text{diam } \gamma, \text{diam } \gamma'\} \leq 4\lambda^2|x-y|$. \square

We now show that the weak chord-arc condition of $\partial\Omega$ is necessary for $\Sigma(\Omega, t^\alpha)$ to be quasisymmetric to \mathbb{S}^2 . This concludes the proof of Theorem 1.2. The proof follows closely that of [12, Proposition 6.2]. The main idea used is due to Väisälä from [10].

Proposition 4.7. *Suppose that Ω has the LQC property and $\alpha \in (0, 1)$. If $\Sigma(\Omega, t^\alpha)$ is quasisymmetric to \mathbb{S}^2 then $\partial\Omega$ is a weak chord-arc curve.*

Proof. By our assumptions, there exist $\epsilon_0 > 0$ and $C > 1$ such that, for all $\epsilon \in [0, \epsilon_0]$, the set γ_ϵ satisfies (2.1) with constant C . Set $\partial\Omega = \Gamma$.

Suppose that the claim is false. Then, by Remark 3.6, for each $n \in \mathbb{N}$, there exists a subarc $\Gamma_n \subset \Gamma$, of diameter less than 1, and a $(\text{diam } \Gamma_n)^{\frac{1}{\alpha}-1}$ -partition $\mathcal{P}_n = \{\Gamma_{n,1}, \dots, \Gamma_{n,N_n}\}$ with $M(\Gamma_n, \mathcal{P}_n) > 4Cn$. By Lemma 3.2, the latter implies that

$$N_n > 4Cn(\text{diam } \Gamma_n)^{1-\frac{1}{\alpha}}.$$

Let $\{x_{n,0}, x_{n,1}, \dots, x_{n,N_n}\}$ be the endpoints of the arcs $\Gamma_{n,1}, \dots, \Gamma_{n,N_n}$, ordered consecutively according to the orientation in Γ_n with $x_{n,0}, x_{n,N_n}$ being the endpoints of Γ_n . For these points, (2.1) implies that

$$\frac{1}{2C}(\text{diam } \Gamma_n)^{\frac{1}{\alpha}} \leq |x_{n,i} - x_{n,i-1}| \leq (\text{diam } \Gamma_n)^{\frac{1}{\alpha}}.$$

By adding more points from each subarc $\Gamma_{n,i}$ in this collection, if necessary, we may further assume that

$$\frac{1}{4C}(\text{diam } \Gamma_n)^{\frac{1}{\alpha}} \leq |x_{n,i} - x_{n,i-1}| \leq \frac{1}{2C}(\text{diam } \Gamma_n)^{\frac{1}{\alpha}}.$$

The existence of these new points follows easily from the doubling property of \mathbb{R}^2 and the 2-point condition of Γ . As we may have increased the number of points, it follows that

$$\sum_{i=1}^{N_n} |x_{n,i-1} - x_{n,i}| \geq N_n \frac{1}{4C}(\text{diam } \Gamma_n)^{\frac{1}{\alpha}} > n \text{diam } \Gamma_n.$$

Set $d_n = (10C^2)^{-1} \min_{1 \leq i \leq N_n} |x_{n,i} - x_{n,i-1}|$ and note that $d_n^\alpha \simeq \text{diam } \Gamma_n$.

The rest of the proof is identical to that of [12, Proposition 5.1] by setting $\varphi(t) = t^\alpha$ therein. We sketch the remaining steps for the sake of completeness.

Fix $n \in \mathbb{N}$ and write $N_n = N$ and $x_{n,i} = x_i$. Assume that there exists an η -quasisymmetric map F from $\Sigma(\Omega, t^\alpha)$ onto \mathbb{S}^2 . Composing F with a suitable Möbius map we may assume that $F(\Sigma(\Omega, t^\alpha)^+)$ is contained in the unit disc \mathbb{B}^2 . Since $\partial\Omega$ is a quasicircle, we can find points w_0, \dots, w_N on Γ and points w'_0, \dots, w'_N on $\gamma_d = \{x \in \Omega : \text{dist}(x, \partial\Omega) = d\}$ which follow the orientation of $\{x_0, \dots, x_N\}$ such that $|w_i - x_i| \in [d, 3C_0d]$ and $|w_i - w'_i| = d$. Hence, $|w_i - w_{i+1}| \simeq d$. Let D be the square-like piece on $\Sigma(\Omega, t^\alpha)^+$ whose projection on $\mathbb{R}^2 \times \{0\}$ is the Jordan domain bounded by $\Gamma(w_0, w_N), \gamma_d(w'_0, w'_N), [w_0, w'_0], [w_N, w'_N]$.

The partition of $\Gamma(w_0, w_N)$ into the subarcs $\Gamma(w_{i-1}, w_i), i = 1, \dots, N$ induces a partition of D into N tall and narrow strips D_i with height in the magnitude d^α and width in the magnitude d . Each D_i is further partitioned by planes parallel to $\mathbb{R}^2 \times \{0\}$ into k square-like pieces D_{ij} with $k \simeq d^{\alpha-1}$. The quasismetry of F implies that $(\text{diam } F(D_{ij}))^2 \leq c_1 \mathcal{H}^2(D_{ij})$ with $c_1 > 1$ depending on η, C . Summing first over j and then over i , and applying Hölder's inequality twice, we obtain $(\text{diam } F(D))^2 \leq Nkc_1 \mathcal{H}^2(D) \leq c_2 n \mathcal{H}^2(D)$ with $c_2 > 1$ depending on η, C . On the other hand, the quasismetry of F on D implies that $\mathcal{H}^2(F(D)) \leq c_3 (\text{diam } F(D))^2$ with $c_3 > 1$ depending on η, C . Since $\text{diam } F(D) \neq 0$, letting $n \rightarrow \infty$, we obtain a contradiction. \square

4.3 Assouad dimension

The Assouad dimension of a metric space (X, d) , introduced in [2], is the infimum of all $s > 0$ that satisfy the following property: there exists $C > 1$ such that for any $Y \subset X$ and $\delta \in (0, 1)$, the set Y can be covered by at most $C\delta^{-s}$ subsets of diameter at most $\delta \text{diam } Y$. In a sense, the main difference between Hausdorff and Assouad dimension of a space X is that the former is related to the average small scale structure of X , while the latter measures the size of X in all scales. The doubling metric spaces are exactly the metric spaces of finite Assouad dimension. See [6] for a detailed survey of the concept.

Remark 4.8. *If X is a quasicircle, then all sets in the definition of the Assouad dimension can be replaced by subarcs of X .*

The claim of the remark becomes evident after noticing that for all subsets Y of a C -quasicircle Γ there exists a subarc $\Gamma' \subset \Gamma$ containing Y , such that $\text{diam } \Gamma' \leq C \text{diam } Y$.

We now turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. Suppose that Γ is a C -quasicircle with Assouad dimension greater than 1; in particular, greater than $1 + \epsilon$ for some fixed $\epsilon \in (0, 1)$. We claim that Γ does not have the weak chord-arc property.

Contrary to the claim, assume that Γ satisfies the weak chord-arc condition for some $M_0 > 1$. By Lemma 3.3 there exists $N_0 > 1$ depending on M_0, C such that for all $\Gamma' \subset \Gamma$ with $\text{diam } \Gamma' < 1$ and all $\text{diam } \Gamma'$ -partitions \mathcal{P} of Γ' we have $|\mathcal{P}| < N_0$. By our assumption on the Assouad dimension of Γ , there exists a subarc $\Gamma' \subset \Gamma$ and a number $\delta \in (0, 1)$ such that all δ -partitions \mathcal{P} of Γ' satisfy $|\mathcal{P}| \geq M\delta^{-1-\epsilon}$ where $M = 10N_0M_1M_\alpha, M_1 > 1$ is as in the third claim of Lemma 3.2, $M_\alpha > 1$ is the number in Lemma 3.5 associated to $\alpha = \frac{1-\sqrt{\beta}}{1+\sqrt{\beta}}$ and $\beta = \frac{1+\epsilon/2}{1+\epsilon}$.

The subarc Γ' can be chosen small enough so that $\text{diam } \Gamma' < \min\{(2M_0M_1)^{-2/\epsilon}, e^{-1/(1-\sqrt{\beta})}\}$.

Case 1. Suppose that $\text{diam } \Gamma' \leq 2\delta^{(1-\sqrt{\beta})/\sqrt{\beta}}$. Assume first that $\delta > \text{diam } \Gamma'$. Let \mathcal{P} be a δ -partition of Γ' . The weak chord-arc property of Γ' and Lemma 3.3 yield $|\mathcal{P}| \leq N_0\delta^{-1} < M\delta^{-1-\epsilon}$ which is false.

Assume now that $\delta \leq \text{diam } \Gamma'$. Let $\alpha' \in (0, 1)$ be such that $(\text{diam } \Gamma')^{\frac{1}{\alpha'}-1} = \delta$. The assumption on $\text{diam } \Gamma', \delta$ and the fact that $\text{diam } \Gamma' < 1/2$ yield $\alpha' \geq \frac{1-\sqrt{\beta}}{1+\sqrt{\beta}} = \alpha$. By Lemma 3.3 and Lemma 3.5, every δ -partition \mathcal{P} of Γ' satisfies $|\mathcal{P}| \leq 10M_\alpha M_1 \delta^{-1} < M\delta^{-1-\epsilon}$ which is also false.

Case 2. Suppose that $\text{diam } \Gamma' > 2\delta^{(1-\sqrt{\beta})/\sqrt{\beta}}$. By our assumptions on δ and $\text{diam } \Gamma'$ we have

$$\begin{aligned} \frac{\log(\text{diam } \Gamma') + 1}{\log(\frac{\delta}{2} \text{diam } \Gamma')} &= \frac{\log(\text{diam } \Gamma') + 1}{\log(\text{diam } \Gamma')} \frac{\log \delta}{\log(\frac{\delta}{2} \text{diam } \Gamma')} \frac{\log(\text{diam } \Gamma')}{\log \delta} \\ &\geq \beta \frac{\log(\text{diam } \Gamma')}{\log \delta}. \end{aligned}$$

Apply Lemma 3.4 for Γ' with $\delta' = \text{diam } \Gamma$ and $N = \delta^{-1-\epsilon}$. There exists a subarc $\Gamma'' \subset \Gamma'$ and a $\text{diam } \Gamma'$ -partition \mathcal{P}' such that

$$|\mathcal{P}'| \geq \delta^{(-1-\epsilon)\beta \frac{\log(\text{diam } \Gamma')}{\log \delta}} \geq (\text{diam } \Gamma')^{-(1+\epsilon/2)}.$$

We create a $\text{diam } \Gamma''$ -partition of Γ'' as follows. For each $\sigma \in \mathcal{P}'$ let $\mathcal{P}(\sigma)$ be a $\frac{(\text{diam } \Gamma'')^2}{\text{diam } \sigma}$ -partition of σ ; for those $\sigma \in \mathcal{P}'$ that satisfy $\text{diam } \sigma < (\text{diam } \Gamma'')^2$ set $\mathcal{P}(\sigma) = \{\sigma\}$. Define \mathcal{P}'' to be the union of all partitions $\mathcal{P}(\sigma)$. It is easy to see that \mathcal{P}'' is a $\text{diam } \Gamma''$ -partition of Γ'' and Lemma 3.2 gives

$$M(\Gamma'', \mathcal{P}'') \geq M(\Gamma'', \mathcal{P}') \geq \frac{1}{2}(\text{diam } \Gamma')^{-\epsilon/2} > M_0 M_1.$$

The latter, however, is false by Lemma 3.2 and the proof is complete. □

5 Examples from homogeneous snowflakes

Let $N \geq 4$ be a natural number and $p \in (1/4, 1/2)$. A homogeneous (N, p) -snowflake in \mathbb{R}^2 is constructed as follows. Let S_0 be a regular N -gon, of diameter equal to $1/2$. At the n -th step, the polygon S_{n+1} is constructed by replacing *all* of the $N4^n$ edges of S_n by the *same* rescaled and rotated copy of one of the two polygonal arcs of Figure 1, in such a way that the polygonal regions are expanding. The curve \mathcal{S} is obtained by taking the limit of S_n , just as in the construction of the usual von Koch snowflake. It is easy to verify that every homogeneous snowflake satisfies (2.1) for some C depending on N, p , and as a result is a quasicircle. See [8] for relevant results.



Figure 1

Let E be an edge of some S_n towards the construction of \mathcal{S} . Denote with \mathcal{S}_E the subarc of \mathcal{S} , of smaller diameter, having the same endpoints as E . The next lemma will ease some of the computations in the rest.

Lemma 5.1. *A homogeneous (N, p) -snowflake \mathcal{S} is a weak chord-arc curve if and only if there exists $M^* > 1$ such that every subarc \mathcal{S}_E has a $\text{diam } \mathcal{S}_E$ -partition \mathcal{P} with $M(\mathcal{S}_E, \mathcal{P}) \leq M_0$.*

Proof. The necessity is clear. For the sufficiency, fix a subarc $\Gamma \subset \mathcal{S}$ and let n_0 be the greatest integer n for which Γ is contained in \mathcal{S}_E for some edge E of S_n . Assume for now that $n_0 > 0$. Denote by $E_i, i = 1, \dots, 4$, the oriented four segments constructed after E in the $n_0 + 1$ step, that is $\mathcal{S}_E = \bigcup_{i=1}^4 \mathcal{S}_{E_i}$. Inductively, if $w = i_1 \dots i_k$ with $i_j \in \{1, \dots, 4\}$, let E_{w_i} be the oriented four segments constructed after E_w in the $n_0 + k + 1$ step.

Suppose that Γ contains a subarc $\mathcal{S}_{E_i}, i = 1, \dots, 4$. Then $\text{diam } \Gamma \geq \frac{1}{4} \text{diam } \mathcal{S}_E = \frac{1}{4} \text{diam } E$. Let \mathcal{P} be a $(\frac{1}{4} \text{diam } E)$ -partition of \mathcal{S}_E and $\mathcal{P}' = \{\Gamma' \cap \Gamma : \Gamma' \in \mathcal{P}\}$. The weak chord-arc property of \mathcal{S}_E and Lemma 3.2 imply that there exists $N_0 > 0$ such that $|\mathcal{P}'| \leq |\mathcal{P}| \leq N_0(\text{diam } \Gamma)^{-1}$. Since $\text{diam } \Gamma' \leq (\text{diam } \Gamma)^2$ for each $\Gamma' \in \mathcal{P}'$ it follows that Γ has the weak chord-arc property for some M^* depending on N_0 .

Suppose now that Γ contains none of the \mathcal{S}_{E_i} , $i = 1, \dots, 4$. Then, there exist $i \in \{1, 2, 3\}$ and maximal integers $r, q \geq 0$ such that $\Gamma \subset \mathcal{S}_{E_{i4q}} \cup \mathcal{S}_{E_{(i+1)r}}$. In this case, apply the arguments above for $\Gamma \cap \mathcal{S}_{E_{i4q}}$ and $\Gamma \cap \mathcal{S}_{E_{(i+1)r}}$.

If $n_0 = 0$ then apply the arguments above for $\Gamma \cap \mathcal{S}_{E_1}, \dots, \Gamma \cap \mathcal{S}_{E_N}$ where E_1, \dots, E_N are the edges of S_0 . □

5.1 A non-rectifiable Jordan curve that satisfies the weak chord-arc property

Let \mathcal{S} be the homogeneous (N, p) -snowflake where the first polygonal arc in Figure 1 is used only at the 10^n -th steps, $n \in \mathbb{N}$. We also require that $\text{diam } \mathcal{S} < (4p)^{-1}$. Note that at the 10^k step of construction, the length of the polygonal curve S_{10^k} is equal to $(4p)^k$. Thus, \mathcal{S} is not rectifiable. We claim that \mathcal{S} has the weak chord-arc property.

By Lemma 5.1, it suffices to check that all subarcs \mathcal{S}_E are weak chord-arc curves. Fix an edge E built at step n . Then $\text{diam } E \geq 4^{-n}$. Let k_0 be the smallest $k \in \mathbb{N}$ such that $\text{diam } E' \leq (\text{diam } E)^2$ for all $E' \in S_{n+k}$. We claim that $k_0 \leq 9n$. Indeed, the construction of \mathcal{S} implies that at step $10n$, each edge has diameter equal to $(4p)4^{-9n} \text{diam } E \leq (4p)(\text{diam } E)^{10}$ since the first polygonal arc in Figure 1 has been used only once. The claim follows from our assumption that $\text{diam } E \leq \text{diam } \mathcal{S} < (4p)^{-1}$.

Let \mathcal{P} be the set of all subarcs $\mathcal{S}_{E'}$ where E' are constructed at step $n + k_0$ and have E as their common parent. Then, $\text{diam } E' = 4^{-k_0} C \text{diam } E$ with $C = 4p$ if the first polygonal arc has been used in the k_0 steps or $C = 1$ otherwise. Since k_0 is minimal,

$$\frac{1}{4}(\text{diam } \mathcal{S}_E)^2 \leq \text{diam } \mathcal{S}_{E'} \leq (\text{diam } \mathcal{S}_E)^2.$$

Therefore, $|\mathcal{P}| = 4^{k_0} \leq 4C(\text{diam } E)^{-1} = 4C(\text{diam } \mathcal{S}_E)^{-1}$. By Lemma 3.1, there exists a $\text{diam } \mathcal{S}_E$ -partition of \mathcal{S}_E that has at most $C'(\text{diam } \mathcal{S}_E)^{-1}$ elements, for some $C' > 1$ depending on N, p . Hence, \mathcal{S}_E has the weak chord-arc property.

Corollary 5.2. *There exists a Jordan domain Ω with nonrectifiable boundary such that $\Sigma(\Omega, t^\alpha)$ is a quasisymmetric sphere for all $\alpha \in (0, 1]$.*

Proof. It follows from the discussion in Section 7 of [11] that there exists $p_0 \in (\frac{1}{4}, \frac{1}{2})$ and an integer $N_0 \geq 4$ such that every homogeneous (N, p) -snowflake with $p \leq p_0$, $N \geq N_0$ bounds a domain that satisfies the LQC property. Let Ω be the domain bounded by the snowflake constructed above with $p \leq p_0$, $N \geq N_0$. Since \mathcal{S} is a quasicircle, $\Sigma(\Omega, t)$ is a quasisymmetric sphere. Moreover, the weak chord-arc property of \mathcal{S} and Theorem 1.2 imply that $\Sigma(\Omega, t^\alpha)$ is a quasisymmetric sphere for all $\alpha \in (0, 1)$. □

5.2 A quasicircle of Assouad dimension 1 that does not satisfy the weak chord-arc property

Let \mathcal{S} be the homogeneous (N, p) -snowflake where the first polygonal arc in Figure 1 is used only at the n^2 -th steps for $n \in \mathbb{N}$. For convenience we also assume that each edge of S_0 has length equal to 1. Then, if E is an edge of S_n , $\text{diam } E = 4^{-n}(4p)^{\lfloor \sqrt{n} \rfloor}$ where, $\lfloor x \rfloor$ denotes the greatest integer which is smaller than x .

We show that \mathcal{S} has Assouad dimension equal to 1 but does not have the weak chord-arc property.

Fix $\epsilon > 0$; we claim that \mathcal{S} has Assouad dimension less than $1 + \epsilon$. Similarly to Lemma 5.1, it is easy to show that it is enough to verify the Assouad condition only for the subarcs \mathcal{S}_E . Take $\delta \in (0, 1)$ and an edge E of the n -th step polygon S_n , for some $n \in \mathbb{N}$. Let m be the largest integer such that $\text{diam } E' \geq \delta \text{diam } E$ for all edges E' of S_m and \mathcal{P} be the set of all subarcs $\mathcal{S}_{E'} \subset \mathcal{S}_E$ where E' is an edge of S_m . Then, $4^{n-m}(4p)^{\lfloor \sqrt{m} \rfloor - \lfloor \sqrt{n} \rfloor} \geq \delta$ and

$$(m - n) \log 4 - (\sqrt{m} - \sqrt{n}) \log 4p \leq -\log \delta + \log 4p \tag{5.1}$$

By elementary calculus, there exists $M > 0$ depending on ϵ, p such that $\sqrt{x} - \sqrt{y} \leq \frac{\epsilon}{1+\epsilon} \frac{\log 4}{\log 4p} (x-y)$ for all $x > M$ and $0 < y < x$. If $m < M$ then clearly $|\mathcal{P}| = 4^{m-n} \leq 4^M < 4^M \delta^{-1-\epsilon}$. If $m \geq M$ then $(m-n) \log 4 - (\sqrt{m} - \sqrt{n}) \log 4p \geq \frac{\log 4}{1+\epsilon} (m-n)$ and by (5.1)

$$(m-n) \log 4 \leq \log \delta^{-1-\epsilon} + 2 \log 4p.$$

Therefore, $|\mathcal{P}| = 4^{m-n} \leq 4^M (4p)^2 \delta^{-1-\epsilon}$ and the claim follows. Since ϵ was chosen arbitrarily, \mathcal{S} has Assouad dimension equal to 1.

We show now that \mathcal{S} does not have the weak chord-arc property. Let $n \in \mathbb{N}$, E be an edge of the n -th step polygon S_n and $m \geq n$ be the greatest integer such that $\text{diam } E' \geq (\text{diam } E)^2$ for each edge E' of S_m . Let \mathcal{P} be the set of all subarcs $S_{E'} \subset S_E$ where E' are edges of S_m . Then,

$$\left(4^{-n}(4p)^{\sqrt{n}-1}\right)^2 \leq (\text{diam } E)^2 \leq \text{diam } E' \leq 4^{-m}(4p)^{\sqrt{m}}$$

which yields

$$\frac{1}{2} - \frac{n}{m} \leq \frac{\log(4p)}{\sqrt{m} \log 4} \left(\frac{1}{2} - \sqrt{\frac{n}{m}}\right) + \frac{\log 4p}{m \log 4}.$$

Note that as n goes to infinity, m goes to infinity and n/m goes arbitrarily close to $1/2$. Hence, $m > \frac{25}{16}n$ for all sufficiently large n . Therefore,

$$M(S_E, \mathcal{P}) \simeq |\mathcal{P}| \frac{\text{diam } S_{E'}}{\text{diam } S_E} \geq (4p)^{\sqrt{m}-\sqrt{n}-1} \gtrsim (4p)^{\sqrt{m}-\sqrt{n}} \geq (4p)^{\sqrt{n}/4}$$

which goes to infinity as n goes to infinity. Thus \mathcal{S} is not a weak chord-arc curve.

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