Anal. Geom. Metr. Spaces 2016; 4:314–316 **DE GRUYTER** OPEN

Research Article **Open Access**

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Isoperimetric Symmetry Breaking: a Counterexample to a Generalized Form of the Log-Convex Density Conjecture

DOI 10.1515/agms-2016-0014
Received October 12, 2016; accepted November 16, 2016

Abstract: We give an example of a smooth surface of revolution for which all circles about the origin are strictly stable for fixed area but small isoperimetric regions are nearly round discs away from the origin.

Keywords: symmetry breaking; isoperimetric

MSC: 53A10, 49Q10

1 Introduction

Our Theorem 2.1 provides an example of a smooth, complete, radial metric on the plane, as suggested by Figure 1, for which all circles about the origin are stable (indeed have positive second variation of length for fixed area), but small isoperimetric regions are not rotationally symmetric (discs or annuli about the origin), but rather approximately round discs located away from the origin, because the Gauss curvature has its maximum away from the origin.

The basic idea of our new example is to find a surface of revolution for which all circles about the origin are stable but the Gauss curvature has a maximum away from the origin, since it is known that small isoperimetric regions are nearly round discs at a maximum of the Gauss curvature. The difficulty is that the function $Q(r)$ of the stability condition $Q(r) \leq 1$ and the Gauss curvature $G(r)$ increase or decrease together, and since $Q(0) = 1$

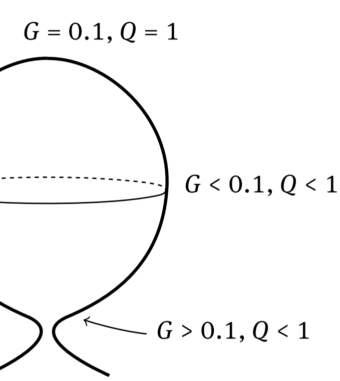

Figure 1: A surface of revolution can have all circles about the origin stable but some other circles isoperimetric, where the Gauss curvature is larger. The trick is to have the Gauss curvature decrease when the circumference is large and increase when the circumference is small

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it can never increase above its initial value. The ratio Q'/G' of their derivatives equals f^2, where $2\pi f$ is the circumference of the circle about the origin. So the trick is to have them decrease where f is big and then increase where f is small, leaving Q below $Q(0) = 1$ but raising G above $G(0)$, as suggested by Figure 1.

This example runs in the face of a number of earlier indications to the contrary:

1. In planes of revolution with strictly decreasing Gauss curvature, isoperimetric regions are composed of discs and/or annuli about the origin (Morgan, Hutchings, and Howards, [7]). The key ingredient is the Gauss-Bonnet formula. All circles about the origin have positive second variation of length for fixed area (Ritoré, [9, §2.1]).
2. In $\mathbb{R}^n$ with smooth, radial volume and perimeter densities such that the perimeter of a ball about the origin is a strictly convex function of its volume, spheres about the origin are uniquely isoperimetric (Howe, [5]). The strong convexity hypothesis means that variations from constant radius are costly.
3. The Log-convex Density Theorem proved by Chambers [1] says that in $\mathbb{R}^n$ with smooth, radial density, all spheres about the origin are isoperimetric if and only if they all have nonnegative second variation for fixed volume (if and only if the log of the density is convex), uniquely so unless the density is constant in a neighborhood of the origin. Chambers' proof carefully analyzes the planar generating curve, which satisfies a nice differential equation.
4. In $\mathbb{R}^n$ with a smooth, radial metric such that spheres about the origin have positive second variation of area for fixed volume, spheres about the origin are uniquely isoperimetric among polar graphs over the unit sphere about the origin (Guan, Li, and Wang, [4]). The proof uses a volume-preserving mean curvature flow.
5. In $\mathbb{R}^n$ with radial volume and perimeter densities r^m and r^k ($m, k > 0$), all spheres about the origin are uniquely isoperimetric if and only if they all have nonnegative second variation for fixed volume (Di Giosia et al., [2]). The proof follows the proof of Chambers [1]. They hope for a similar result for all smooth, radial volume and perimeter densities.

2 The Example

The following theorem provides our symmetry-breaking example:

Theorem 2.1 ([6]). *There are smooth surfaces of revolution for which every circle about the origin has positive second variation of length for fixed area but small isoperimetric regions are approximately round discs away from the origin.*

Proof. We consider a smooth surface of revolution with polar coordinates r, θ, and metric

$$ds^2 = dr^2 + f(r)^2 d\theta^2.$$

After Ritoré ([9, Lemma 1.6]), as elegantly explained by Engelstein et al. ([3, Sect. 6]), the circle about the origin of radius r has positive second variation for fixed area if and only if

$$Q(r) = (f')^2 - ff'' < 1,$$

the Gauss curvature is given by

$$G(r) = -\frac{f''}{f},$$

and the derivatives satisfy

$$Q' = f^2 G'.$$

In particular, Q and G increase or decrease together, G more slowly when f is large.

Now from the origin start like a sphere of Gauss curvature 0.1, initially decreasing a tiny bit so that Q is slightly less than 1. When f is about 10 (approximately a hemisphere of radius 10), let G decrease by a larger but still small amount δ. Since $f \gg 1$ for most of the decrease, Q decreases much more than G. Next, when f is about say $1/4$ (approximately a large spherical cap of radius 10), increase G to $0.1 + \delta$. Since $f < 1$, Q increases more slowly than G, so stays below its original value of 1. Finally, to make sure that f does not go to 0, let G and Q decrease rapidly to -100. So even though Q stays less than 1 and all circles about the origin have positive second variation of length for fixed area, small circles where the Gauss curvature is $0.1+\delta$ have less perimeter than ones about the origin of the same area. Existence of isoperimetric regions is not an issue, because curves far from the origin, in a region of constant Gauss curvature -100, have lots of perimeter and the enclosed area well inside this region in a minimizing sequence is vanishing. The arguments of Morgan and Johnson ([8, §§2.2, 3.1]) apply and say that small isoperimetric regions are approximately round discs where the Gauss curvature is maximum. □

Remark 2.2. If instead of a plane you want a sphere, just reflect across the minimum of f for a C^2 example, which you can smooth to a C^∞ example.

Remark 2.3. The strongest remaining general conjecture is that in $\mathbb{R}^n$ with smooth, radial metric and (volume and perimeter) densities, if all spheres about the origin have nonnegative second variation of area for fixed volume, then they are isoperimetric among polar graphs over the unit sphere.

Acknowledgement: The author thanks the referees for their kind and helpful comments.

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