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Applications of the 'Ham Sandwich Theorem' to Eigenvalues of the Laplacian

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Abstract: We apply Gromov's ham sandwich method to get: (1) domain monotonicity (up to a multiplicative constant factor); (2) reverse domain monotonicity (up to a multiplicative constant factor); and (3) universal inequalities for Neumann eigenvalues of the Laplacian on bounded convex domains in Euclidean space.

Keywords: Eigenvalues of the Laplacian; convexity; Ham Sandwich Theorem

MSC: 58J50, 94B75, 53C23, 46M20

1 Introduction and the statement of main results

Let Ω be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary. We denote by $\lambda_0^D(\Omega) \leq \lambda_1^D(\Omega) \leq \cdots \leq \lambda_k^D(\Omega) \leq \cdots$ the Dirichlet eigenvalues of the Laplacian on Ω and by $0 = \lambda_0^N(\Omega) < \lambda_1^N(\Omega) \leq \lambda_2^N(\Omega) \leq \cdots \leq \lambda_k^N(\Omega) \leq \cdots$ the Neumann eigenvalues of the Laplacian on Ω. It is known that the following two properties for these eigenvalues hold:

1. (Domain monotonicity for Dirichlet eigenvalues) If $\Omega \subseteq \Omega'$ are bounded domains, then $\lambda_k^D(\Omega') \leq \lambda_k^D(\Omega)$ for any k.
2. (Restricted reverse domain monotonicity for Neumann eigenvalues) If in addition $\Omega' \setminus \Omega$ has measure zero then $\lambda_k^N(\Omega) \leq \lambda_k^N(\Omega')$ for any k.

These two properties are direct consequence of Courant's minimax principle (see [5]). The following two examples suggest that domain monotonicity does not hold for Neumann eigenvalues in general.

Example 1.1. Let Ω' be the n-dimensional unit cube $[0, 1]^n$. Then $\lambda_1^N(\Omega') = 1$. However, if Ω is a convex domain in $[0, 1]^n$ that approximates the segment connecting the origin and the point $(1, 1, \cdots, 1)$ then $\lambda_1^N(\Omega) \sim 1/n$.

Example 1.2. Let $p \in [1, 2]$ and B_p^n be the n-dimensional ℓ_p-ball centered at the origin. Suppose that $r_{n,p}$ is the positive number such that $vol(r_{n,p}B_p^n) = 1$ and set $\Omega' := r_{n,p}B_p^n$. Then $r_{n,p} \sim n^{1/p}$ and $\lambda_1^N(\Omega') \geq c$ for some absolute constant $c > 0$ ([27, Section 4 (2)]). If the segment in Ω' connecting the origin and $(r_{n,p}, 0, 0, \cdots, 0)$ is approximated by a convex domain Ω in Ω' then $\lambda_1^N(\Omega) \sim r_{n,p}^{-2} \sim n^{-2/p}$.

In this paper we study the above two properties for Neumann eigenvalues of the Laplacian on convex domains in a Euclidean space. For two real numbers α, β we denote $\alpha \lesssim \beta$ if $\alpha \leq c\beta$ for some absolute constant $c > 0$.

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One of our main results is the following:

Theorem 1.3. *For any natural number $k \geq 2$ and any two bounded convex domains Ω, Ω' in $\mathbb{R}^n$ with piecewise smooth boundaries such that $\Omega \subseteq \Omega'$ we have*

$$\lambda_k^N(\Omega') \lesssim (n \log k)^2 \lambda_{k-1}^N(\Omega).$$

As a corollary we get the following inner radius estimate:

Corollary 1.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain with piecewise smooth boundary. For any $k \geq 2$ we have*

$$\mathrm{inrad}(\Omega) \lesssim \frac{n \log k \sqrt{\lambda_{k-1}^N(B_1)}}{\sqrt{\lambda_k^N(\Omega)}},$$

where B_1 is a unit ball in $\mathbb{R}^n$.

We also obtain the opposite inequality to the one in Theorem 1.3:

Theorem 1.5. *Let Ω, Ω' be bounded convex domains in $\mathbb{R}^n$ having piecewise smooth boundaries. Assume that Ω is symmetric with respect to the origin (i.e., $\Omega = -\Omega$) and $\Omega \subseteq \Omega'$. Set $v := vol\,\Omega / vol\,\Omega' \in [0, 1]$. Then for any natural number $k \geq 3$ we have*

$$\lambda_{k-2}^N(\Omega') \gtrsim \min\left\{\frac{(\log(1-v))^2}{n^8(\log k)^6}, \frac{1}{n^6(\log k)^4}\right\}\lambda_k^N(\Omega). \tag{1.1}$$

For general (not necessarily symmetric) Ω we have

$$\lambda_{k-2}^N(\Omega') \gtrsim \min\left\{\frac{(\log(1-2^{-n}v))^2}{n^8(\log k)^6}, \frac{1}{n^6(\log k)^4}\right\}\lambda_k^N(\Omega). \tag{1.2}$$

As a corollary of Theorems 1.3 and 1.5 we obtain

$$\lambda_k^N(\Omega') \lesssim (n \log k)^2 \lambda_k^N(\Omega) \text{ and } \lambda_k^N(\Omega') \gtrsim \min\left\{\frac{(\log(1-2^{-n}v))^2}{n^8(\log k)^6}, \frac{1}{n^6(\log k)^4}\right\}\lambda_k^N(\Omega)$$

for all $k \geq 2$, which corresponds to the above properties (1) and (2) up to multiplicative constant factors. In [22] E. Milman obtained the corresponding inequality for $k = 1$ (see (5.1)). Despite the fact that his inequality is independent of dimension, our two inequalities above involve dimensional terms. However $\log k$ bounds in the two inequalities are nontrivial (Compare with (5.2)). The case where $p = 1$ in Example 1.2 shows that the n^2 order in Theorem 1.3 cannot be improved. Probably there is a chance to express the multiplicative constant factor in Theorem 1.3 in terms of the volume ratio $v = vol\,\Omega / vol\,\Omega'$ to avoid the dependence of dimension (see Question 5.3).

In the special case where $\Omega = \Omega'$ in Theorem 1.3 we obtain the following universal inequalities among Neumann eigenvalues :

$$\lambda_k^N(\Omega) \lesssim (n \log k)^2 \lambda_{k-1}^N(\Omega). \tag{1.3}$$

By 'universal' we mean it does not depend on the underlying domain Ω itself. Payne, Pólya, and Weinberger studied universal inequalities among Dirichlet eigenvalues ([24, 25]). Since then many universal inequalities for Dirichlet eigenvalues were studied (see [1]). For Neumann eigenvalues, Liu ([19]) showed the sharp inequalities

$$\lambda_k^N(\Omega) \lesssim k^2 \lambda_1^N(\Omega) \tag{1.4}$$

for any bounded convex domain Ω, which improves author's exponential bounds in k in [11]. On the other hand, one can get

$$\lambda_k^N(\Omega) \gtrsim k^{2/n} \lambda_1^N(\Omega) \tag{1.5}$$

for any bounded convex domain $\Omega \subseteq \mathbb{R}^n$. This inequality follows from the combination of E. Milman's result [22, Remark 2.11] and Cheng-Li's result [7] (see [26, Chapter III §5]). In fact E. Milman described the Sobolev inequality in terms of $\lambda_1^N(\Omega)$ and Cheng-Li showed lower bounds of $\lambda_k^N(\Omega)$ in terms of the Sobolev constant. The Weyl asymptotic formula says that the inequality (1.5) is sharp. In particular combining (1.4) with (1.5) we can obtain

$$\lambda_k^N(\Omega) \lesssim k^{2-2/n}\lambda_{k-1}^N(\Omega).$$

Compared with this inequality our inequality (1.3) includes the dimensional term. However the dependence on k is the best possible to author's knowledge. It should be mentioned that the author's conjecture in [11, 12] is $\lambda_k^N(\Omega) \lesssim \lambda_{k-1}^N(\Omega)$ for any bounded convex domain Ω with piecewise smooth boundary.

In the proof of Theorems 1.3 and 1.5 we will use Gromov's method on bisections of finite subsets by the zero set of a finite combination of eigenfunctions. It enables us to get lower bounds for eigenvalues of the Laplacian in terms of the Cheeger constants and the maximal multiplicity of a covering of a domain (Proposition 3.1). We will try to find 'nice' convex partitions in order to get 'nice' lower bounds for the Cheeger constants of pieces of the partition.

2 Preliminaries

2.1 Separation distance

Let Ω be a bounded domain in Euclidean space. For two subsets $A, B \subseteq \Omega$ we set $d_\Omega(A,B) := \inf\{|x-y| \mid x \in A, y \in B\}$. We denote by μ the Lebesgue measure on Ω normalized as $\mu(\Omega) = 1$.

Definition 2.1 (Separation distance, [13]). For any $\kappa_0, \kappa_1, \cdots, \kappa_k \geq 0$ with $k \geq 1$, we define the (k-)*separation distance* $\mathrm{Sep}(\Omega; \kappa_0, \kappa_1, \cdots, \kappa_k)$ of Ω as the supremum of $\min_{i\neq j} d_\Omega(A_i, A_j)$, where $A_0, A_1, \cdots, A_k$ are any Borel subsets of Ω satisfying $\mu(A_i) \geq \kappa_i$ for all $i = 0, 1, \cdots, k$.

Theorem 2.2 ([12, Theorem 1]). *There exists an absolute constant $c > 0$ satisfying the following property. Let Ω be a bounded convex domain in Euclidean space with piecewise smooth boundary and k, l be two natural numbers with $l \leq k$. Then we have*

$$\mathrm{Sep}(\Omega; \kappa_0, \cdots, \kappa_l) \leq \frac{c^{k-l+1}}{\sqrt{\lambda_k^N(\Omega)}} \max_{i\neq j} \log \frac{1}{\kappa_i\kappa_j}.$$

The case where $k = l = 1$ was first proved by Gromov and V. Milman without the convexity assumption ([14]). Chung, Grigor'yan, and Yau then extended the result to the case where $k = l$ ([8, 9]). To reduce the number l of subsets in Ω in a dimension-free way we need the convexity of Ω (see [12]).

2.2 Cheeger constant and eigenvalues of the Laplacian

For a Borel subset $A \subseteq \Omega$ and $r > 0$ we denote by $U_r(A)$ the r-neighborhood of A in Ω. We define the *Minkowski boundary measure* of A as

$$\mu_+(A) := \liminf_{r\to 0} \frac{\mu(U_r(A)\setminus A)}{r}.$$

Definition 2.3 (Cheeger constant). For a bounded domain Ω in a Euclidean space we define the *Cheeger constant* of Ω as

$$h(\Omega) := \inf_{A_0, A_1} \max\left\{\frac{\mu_+(A_0)}{\mu(A_0)}, \frac{\mu_+(A_1)}{\mu(A_1)}\right\},$$

where the infimum runs over all non-empty disjoint two Borel subsets A_0, A_1 of Ω.

Let μ be a finite Borel measure on a bounded domain $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a Borel measurable function. A real number m_f is called a *median* of f if it satisfies

$$\mu(\{x \in \Omega \mid f(x) \geq m_f\}) \geq \mu(\Omega)/2 \text{ and } \mu(\{x \in \Omega \mid f(x) \leq m_f\}) \geq \mu(\Omega)/2.$$

The following characterization of the Cheeger constant is due to Maz'ya and Federer-Fleming. See [22, Lemma 2.2] for example.

Theorem 2.4 ([10], [21]). *The Cheeger constant $h(\Omega)$ is the best constant for the following $(1,1)$-Poincaré inequality:*

$$h(\Omega)\|f - m_f\|_{L^1(\Omega,\mu)} \leq \||\nabla f|\|_{L^1(\Omega,\mu)} \text{ for any } f \in C^\infty(\Omega).$$

Theorem 2.5 (E. Milman [23, Theorem 2.1]). *Let Ω be a bounded convex domain in Euclidean space and assume that Ω satisfies the following concentration inequality for some $r > 0$ and $\kappa \in (0, 1/2) : \mu(\Omega \setminus U_r(A)) \leq \kappa$ for any Borel subset $A \subseteq \Omega$ such that $\mu(A) \geq 1/2$. Then $h(\Omega) \geq (1 - 2\kappa)/r$.*

One can easily check that Theorem 2.5 has the following equivalent interpretation in terms of separation distance.

Proposition 2.6. *Let Ω be a convex domain in a Euclidean space. Then for any $\kappa \in (0, 1/2)$ we have*

$$\mathrm{Sep}(\Omega; \kappa, 1/2) \geq (1 - 2\kappa)/h(\Omega).$$

In particular we have

$$\operatorname{diam} \Omega \geq 1/h(\Omega).$$

The latter statement can be found in [16, Theorem 5.1] and [22, Theorem 5.12] up to some absolute constant.

Theorem 2.7 ([17, Theorem 1.1], [6]). *Let Ω be a bounded convex domain in $\mathbb{R}^n$ with piecewise smooth boundary. For any natural number k we have*

$$\operatorname{diam} \Omega \lesssim nk/\sqrt{\lambda_k^N(\Omega)}.$$

The Buser-Ledoux inequality asserts that $\sqrt{\lambda_1^N(\Omega)} \gtrsim h(\Omega)$ for any bounded convex domain $\Omega \subseteq \mathbb{R}^n$ with piecewise smooth boundary ([4], [18]). As a corollary of Theorem 2.7 we obtain

$$\operatorname{diam} \Omega \lesssim n/h(\Omega). \tag{2.1}$$

2.3 Voronoi partition

Let X be a metric space and $\{x_i\}_{i \in I}$ be a subset of X. For each $i \in I$ we define the *Voronoi cell* C_i associated with the point x_i as

$$C_i := \{x \in X \mid d(x, x_i) \leq d(x, x_j) \text{ for all } j \neq i\}.$$

Note that if X is a bounded convex domain Ω in a Euclidean space then $\{C_i\}_{i \in I}$ is a convex partition of Ω (the boundaries ∂C_i may overlap each other). Observe also that if the balls $\{B(x_i, r)\}_{i \in I}$ of radius r cover Ω then $C_i \subseteq B(x_i, r)$, and thus $\operatorname{diam} C_i \leq 2r$ for any $i \in I$.

3 Gromov's ham sandwich method

In this section we explain Gromov's ham sandwich method to estimate eigenvalues of the Laplacian from below. Recall that the classical ham sandwich theorem in algebraic topology asserts that given three subsets in $\mathbb{R}^3$ of finite volume, there is a plane that bisects all these subsets ([20]). Instead of bisecting by a plane, bisecting by the zero set of a finite combination of eigenfunctions of the Laplacian is considered in Gromov's ham sandwich method.

Let Ω be a bounded domain in a Euclidean space with piecewise smooth boundary and $\{A_i\}_{i=1}^l$ be a finite covering of Ω; $\Omega = \bigcup_i A_i$. We denote by $M(\{A_i\})$ the maximal multiplicity of the covering $\{A_i\}$ and by $h(\{A_i\})$ the minimum of the Cheeger constants of A_i, $i = 1, 2, \cdots, l$.

Although the following argument essentially appears in [13, Appendix C_+] we include the proof for completeness.

Proposition 3.1 (Compare with [13, Appendix C_+]). *Under the above situation, we have*

$$\lambda_l^N(\Omega) \geq \frac{h(\{A_i\})^2}{4M(\{A_i\})^2}.$$

Sketch of Proof. We abbreviate $M := M(\{A_i\})$ and $h := h(\{A_i\})$. Take orthonormal eigenfunctions $f_1, f_2, \cdots, f_l$, with eigenvalues $\lambda_1^N(\Omega), \lambda_2^N(\Omega), \cdots, \lambda_l^N(\Omega)$, respectively.

Step 1. Use the Borsuk-Ulam theorem to get constants $c_0, c_1, \cdots, c_l$ such that $f := c_0 + \sum_{i=1}^l c_i f_i$ bisects each $A_1, A_2, \cdots A_l$, i.e.,

$$\mu(A_i \cap f^{-1}[0, \infty)) \geq \mu(A_i)/2 \text{ and } \mu(A_i \cap f^{-1}(-\infty, 0]) \geq \mu(A_i)/2.$$

In fact, according to the Corollary in [29], in order to bisect l subsets by a finite combination of $f_0 \equiv 1, f_1, \cdots, f_l$, it suffices to check that $f_0, f_1, \cdots, f_l$ are linearly independent modulo sets of measure zero (i.e., whenever $a_0 f_0 + a_1 f_1 + \cdots + a_l f_l = 0$ over a Borel subset of positive measure, we have $a_0 = a_1 = \cdots = a_l = 0$). This is possible since the zero set of any finite combination of $f_0, f_1, \cdots, f_l$ has finite codimension 1 Hausdorff measure ([2, Subsection 1.1.1]).

Step 2. Put $f_+(x) := \max\{f(x), 0\}$ and $f_-(x) := \max\{-f(x), 0\}$. Then we set $g_\pm := f_\pm^2$. Note that 0 is the median of the restriction of $g_\pm$ to each A_i by Step 1. Apply Theorem 2.4 to get $h\|g_\pm\|_{L^1(A_i, \mu|_{A_i})} \leq \||\nabla g_\pm|\|_{L^1(A_i, \mu|_{A_i})}$ for each i.

Step 3. Use Step 2 to get

$$\int_\Omega g_\pm d\mu \leq \sum_{i=1}^l \int_{A_i} g_\pm d\mu \leq \frac{1}{h} \sum \int_{A_i} |\nabla g_\pm| d\mu \leq \frac{M}{h} \int_\Omega |\nabla g_\pm| d\mu.$$

Recalling that $g_\pm = f_\pm^2$ and using the Cauchy-Schwarz inequality we have

$$\int_\Omega f_\pm^2 d\mu \leq \frac{4M^2}{h^2} \int_\Omega |\nabla f_\pm|^2 d\mu.$$

Since the zero set $f^{-1}(0)$ has measure zero we get

$$\int_\Omega f^2 d\mu \leq \frac{4M^2}{h^2} \int_\Omega |\nabla f|^2 d\mu.$$

We therefore obtain $\sum_{i=0}^l c_i^2 \leq (4M^2/h^2) \sum_{i=1}^l c_i^2 \lambda_i^N(\Omega)$ and thus the conclusion of the proposition. □

Remark 3.2. 1. In [13] Gromov treated the case where Ω is a closed Riemannian manifold of Ricci curvature $\geq -(n-1)$ and the covering consists of some balls B_i of radius ε in Ω. Instead of considering the $(1, 1)$-Poincaré inequality in terms of Cheeger constants in Step 2 he proved that $\|g\|_{L^1(B_i, \mu|_{B_i})} \leq c(n, \varepsilon) \||\nabla g|\|_{L^1(\widetilde{B}_i, \mu|_{\widetilde{B}_i})}$, where $g = f^2$, $c(n, \varepsilon)$ is a constant depending only on dimension n and ε, and $\widetilde{B}_i$ is the ball of radius 2ε with the same center of B_i.

2. The above proposition is also valid for the case where Ω is a closed Riemannian manifold or a compact Riemannian manifold with boundary. In the latter case we impose the Neumann boundary condition.

As an application of Proposition 3.1 we can obtain estimates of eigenvalues of the Laplacian of closed hyperbolic manifolds due to Buser ([3, Theorems 3.1, 3.12, 3.14]). In fact, Buser gave a partition of a closed hyperbolic manifold and lower bound estimates of Cheeger constants of each piece of the partition.

4 Proof of main theorems

Let Ω, Ω' be bounded convex domains in a Euclidean space. Throughout this section μ will be the Lebesgue measure on Ω' normalized by $\mu(\Omega') = 1$.

Proof of Theorem 1.3. We apply Gromov's ham sandwich method (Proposition 3.1) to bound $\lambda^N_{k-1}(\Omega)$ from below in terms of $\lambda^N_k(\Omega')$. To apply the proposition we want to find a finite partition $\{\Omega_i\}_{i=1}^l$ of Ω with $l \le k-1$ such that the Cheeger constant of each Ω_i can be comparable with $\sqrt{\lambda^N_k(\Omega')}$.

According to Theorem 2.2 we have

$$\mathrm{Sep}\Big(\Omega'; \underbrace{\frac{1}{k^n}, \frac{1}{k^n}, \cdots, \frac{1}{k^n}}_{k \text{ times}}\Big) \le \frac{cn \log k}{\sqrt{\lambda^N_k(\Omega')}} \tag{4.1}$$

for some absolute constant $c > 0$. We set $R := (cn \log k)/\sqrt{\lambda^N_k(\Omega')}$.

Suppose that Ω' includes k $(4R)$-separated points $x_1, x_2, \cdots, x_k$. By Theorem 2.7 we have $\operatorname{diam}\Omega' \le c'nk/\sqrt{\lambda^N_k(\Omega')}$ for some absolute constant $c' > 0$. Applying the Bishop-Gromov inequality we have

$$\mu(B(x_i, R)) \ge (R/\operatorname{diam}\Omega')^n \ge (c \log k)^n/(c'k)^n$$

for each i. If we take a larger c in (4.1) so that $(c \log k)/c' \ge 1$ we get $\mu(B(x_i, R)) \ge 1/k^n$. Since $B(x_i, R)$'s are $2R$-separated this contradicts (4.1).

Let $y_1, y_2, \cdots, y_l$ be maximal $4R$-separated points in Ω', where $l \le k-1$. Since $\Omega' \subseteq \bigcup_{i=1}^l B(y_i, 4R)$, if $\{\Omega_i'\}_{i=1}^l$ is the Voronoi partition associated with $\{y_i\}$ then we have $\operatorname{diam}\Omega_i' \le 8R$. Setting $\Omega_i := \Omega_i' \cap \Omega$ we get $\Omega = \bigcup_{i=1}^l \Omega_i$ and $\operatorname{diam}\Omega_i \le 8R$. Since each Ω_i is convex, Proposition 2.6 gives $h(\Omega_i) \ge 1/(8R)$. Applying Proposition 3.1 to the covering $\{\Omega_i\}$ we obtain

$$\lambda^N_{k-1}(\Omega) \ge \lambda^N_l(\Omega) \ge 1/\{4(8R)^2\} \ge \lambda^N_k(\Omega')/(16cn\log k)^2,$$

which yields the conclusion of the theorem. This completes the proof. □

In order to prove Theorem 1.5 we prove several lemmas.

Lemma 4.1 ([22, Lemma 5.2]). *Let Ω, Ω' be bounded convex domains in $\mathbb{R}^n$ such that $\Omega \subseteq \Omega'$. Assume that $\operatorname{vol}\Omega \ge v \operatorname{vol}\Omega'$. Then we have $h(\Omega') \ge v^2 h(\Omega)$.*

Lemma 4.2. *Let Ω, Ω' be bounded convex domains in $\mathbb{R}^n$ with piecewise smooth boundaries such that $\Omega \subseteq \Omega'$. Assume that $\operatorname{vol}\Omega \ge (1-k^{-n}) \operatorname{vol}\Omega'$ for some natural number $k \ge 2$. Then we have*

$$(n^2 \log k)^2 \lambda^N_{k-1}(\Omega') \gtrsim \lambda^N_k(\Omega).$$

Proof. Due to Theorem 2.2 we have

$$\mathrm{Sep}\Big(\Omega; \underbrace{\frac{1}{k^n}, \frac{1}{k^n}, \cdots, \frac{1}{k^n}}_{k \text{ times}}\Big) \le \frac{cn \log k}{\sqrt{\lambda^N_k(\Omega)}}. \tag{4.2}$$

We set $R := (cn^2 \log k)/\sqrt{\lambda^N_k(\Omega)}$. As in the proof of Theorem 1.3 we have maximal $4R$-separated points $x_1, x_2, \cdots, x_l \in \Omega$ such that $l \le k-1$. We get $\Omega \subseteq \bigcup_{i=1}^l B(x_i, 4R)$.

Claim 4.3. $U_R(\Omega) = \Omega'$.

Let us admit the above claim for a while. Then we have $\Omega' \subseteq \bigcup_{i=1}^{l} B(x_i, 5R)$. Let $\{\Omega_i'\}_{i=1}^{l}$ be the Voronoi partition associated with $\{x_i\}$ then we have $\operatorname{diam} \Omega_i' \le 10R$. Proposition 2.6 gives $h(\Omega_i') \ge 1/(10R)$. According to Proposition 3.1 we obtain

$$\lambda_{k-1}^N(\Omega') \ge 1/(20R)^2 = \lambda_k^N(\Omega)/(20cn^2 \log k)^2,$$

which implies the lemma.

Suppose that $U_R(\Omega) \neq \Omega'$. There exists $x \in \partial\Omega'$ such that $B(x, R) \cap \Omega = \emptyset$. Lemma 4.1 together with Proposition 2.6 and (2.1) show that

$$\operatorname{diam} \Omega' \lesssim n/h(\Omega') \lesssim n/h(\Omega) \le n \operatorname{diam} \Omega,$$

which gives the existence of an absolute constant $c_1 > 0$ such that $\operatorname{diam} \Omega' \le c_1 n \operatorname{diam} \Omega$. The Bishop-Gromov inequality yields

$$\mu(B(x, R)) \ge (R/\operatorname{diam} \Omega')^n \ge R^n/(c_1 n \operatorname{diam} \Omega)^n.$$

Since $\operatorname{diam} \Omega \le c_2 nk/\sqrt{\lambda_k^N(\Omega)}$ for some absolute constant $c_2 > 0$ (Theorem 2.7) we have

$$\mu(B(x, R)) \ge (c \log k)^n/(c_1 c_2 k)^n > 1/k^n,$$

provided that c is a large enough absolute constant such that $(c \log k)/(c_1 c_2) > 1$. We thereby obtain

$$\mu(B(x, R) \cup \Omega) = \mu(B(x, R)) + \mu(\Omega) > 1/k^n + (1 - 1/k^n) = 1,$$

which is a contradiction. □

In order to adapt to the hypothesis of Lemma 4.2 we use the following improvement of Borell's lemma.

Theorem 4.4 ([15, Section 1 Remark]). *Let Ω, Ω' be bounded convex domains such that $\Omega \subseteq \Omega'$. Assume that Ω is symmetric. Then for any $r \ge 1$ we have*

$$\mu(\Omega' \setminus r\Omega) \le (1 - \mu(\Omega))^{\frac{r+1}{2}},$$

where $r\Omega := \{rx \mid x \in \Omega\}$.

Proof of Theorem 1.5. We first consider the case where Ω is symmetric. According to Theorem 4.4, setting

$$r := 2 \max\left\{\frac{n \log k}{-\log(1 - \nu)}, 1\right\}$$

we have $\mu(\Omega' \setminus r\Omega) < 1/k^n$. Take a bounded convex domain $\widetilde{\Omega} \subseteq r\Omega \cap \Omega'$ with piecewise smooth boundary such that $\Omega \subseteq \widetilde{\Omega}$ and $\mu(\Omega' \setminus \widetilde{\Omega}) < 1/k^n$. Since $\widetilde{\Omega} \subseteq r\Omega$ Theorem 1.3 implies

$$(nr \log k)^2 \lambda_{k-1}^N(\widetilde{\Omega}) \gtrsim r^2 \lambda_k^N(r\Omega) = \lambda_k^N(\Omega). \tag{4.3}$$

Using Lemma 4.2 we also obtain

$$(n^2 \log k)^2 \lambda_{k-2}^N(\Omega') \gtrsim \lambda_{k-1}^N(\widetilde{\Omega}). \tag{4.4}$$

Combining the above two inequalities (4.3) and (4.4) we obtain (1.1).

For general (not necessarily symmetric) Ω, there exists a choice of a center (we may assume here that the center is the origin without loss of generality) such that *vol*$(\Omega \cap -\Omega) \ge 2^{-n}$ *vol* Ω ([28, Corollary]). By virtue of Theorem 4.4, setting

$$r := 2 \max\left\{\frac{n \log k}{-\log(1 - 2^{-n}\nu)}, 1\right\}$$

we get $\mu(\Omega' \setminus r\Omega) \le \mu(\Omega' \setminus r(\Omega \cap -\Omega)) < 1/k^n$. Thus applying the same proof of the symmetric case we obtain (1.2). This completes the proof. □

5 Questions

In this section we raise several questions concerning this paper.

Question 5.1. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ with piecewise smooth boundary. Then, for any natural number k and $\kappa_0, \kappa_1, \cdots, \kappa_k > 0$, can we get*

$$\mathrm{Sep}(\Omega; \kappa_0, \kappa_1, \cdots, \kappa_k) \lesssim \frac{1}{(\log k)\sqrt{\lambda_k^N(\Omega)}} \max_{i \neq j} \log \frac{1}{\kappa_i \kappa_j} \ ?$$

We can subtract $\log k$ terms in Theorems 1.3 and 1.5 once we get an affirmative answer to Question 5.1 since [12, Theorem 3.4] gives

$$\mathrm{Sep}(\Omega; \kappa_0, \kappa_1, \cdots, \kappa_l) \leq \frac{c^{k-l+1}}{(\log k)\sqrt{\lambda_k^N(\Omega)}} \max_{i \neq j} \log \frac{1}{\kappa_1 \kappa_j}$$

for any two natural numbers $l \leq k$ and any $\kappa_0, \kappa_1, \cdots, \kappa_l > 0$, where $c > 0$ is an absolute constant.

Question 5.2. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ with piecewise smooth boundary and assume that Ω satisfies the following $(k-1)$-separation inequality for some k:*

$$\mathrm{Sep}((\Omega, \mu); \kappa_0, \kappa_1, \cdots, \kappa_{k-1}) \leq \frac{1}{D} \max_{i \neq j} \log \frac{1}{\kappa_i \kappa_j} \quad (\forall \kappa_0, \kappa_1, \cdots, \kappa_{k-1} > 0).$$

Then, does there exist an absolute constant $c > 0$ and a convex partition $\Omega = \bigcup_{i=1}^{l} \Omega_i$ with $l \leq k-1$ such that

$$\mu(\Omega_i) \geq \frac{1}{ck} \text{ and } \mathrm{Sep}((\Omega_i, \mu|_{\Omega_i}); \kappa, \kappa) \leq \frac{c}{D} \log \frac{1}{\kappa}$$

for any κ ?

An affirmative answer to Question 5.2 would imply the universal inequality $\lambda_k^N(\Omega) \lesssim (\log k)^2 \lambda_{k-1}^N(\Omega)$ via Theorem 2.5 and Proposition 3.1. If both Questions 5.1 and 5.2 is affirmative then we can obtain $\lambda_k^N(\Omega) \lesssim \lambda_{k-1}^N(\Omega)$.

Question 5.3. *Let Ω, Ω' be bounded convex domains with piecewise smooth boundaries such that $\Omega \subseteq \Omega'$. Set $v := \mathrm{vol}\,\Omega / \mathrm{vol}\,\Omega' \in [0, 1]$. Can we prove $\lambda_k^N(\Omega) \leq f_1(v) g_1(\log k) \lambda_k^N(\Omega')$ and $\lambda_k(\Omega') \leq f_2(v) g_2(\log k) \lambda_k^N(\Omega)$, where f_1 and f_2 are any functions and g_1 and g_2 are some rational functions ?*

When $k = 1$ E. Milman obtained

$$\lambda_1^N(\Omega') \geq v^4 \lambda_1^N(\Omega) \text{ and } \lambda_1^N(\Omega) \gtrsim (1/\log(1 + 1/v))^2 \lambda_1^N(\Omega') \tag{5.1}$$

(see [22, Lemmas 5.1, 5.2]). Combining this inequality with (1.4) and (1.5) we can get

$$\lambda_k^N(\Omega') \gtrsim v^4 k^{\frac{2}{n}-2} \lambda_k^N(\Omega) \text{ and } \lambda_k^N(\Omega) \gtrsim (k^{\frac{1}{n}-1}/\log(1 + 1/v))^2 \lambda_k^N(\Omega'), \tag{5.2}$$

but this does not imply an answer to Question 5.3.

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