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Hardy and Hardy-Sobolev Spaces on Strongly Lipschitz Domains and Some Applications

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Abstract: Let $\Omega \subset \mathbb{R}^n$ be a strongly Lipschitz domain. In this article, the authors study Hardy spaces, $H^p_r(\Omega)$ and $H^p_z(\Omega)$, and Hardy-Sobolev spaces, $H^{1,p}_r(\Omega)$ and $H^{1,p}_{z,0}(\Omega)$ on Ω, for $p \in (\frac{n}{n+1}, 1]$. The authors establish grand maximal function characterizations of these spaces. As applications, the authors obtain some div-curl lemmas in these settings and, when Ω is a bounded Lipschitz domain, the authors prove that the divergence equation div $\mathbf{u} = f$ for $f \in H^p_z(\Omega)$ is solvable in $H^{1,p}_{z,0}(\Omega)$ with suitable regularity estimates.

Keywords: Hardy space; Hardy-Sobolev space; grand maximal function; div-curl formula; divergence equation

MSC: Primary 42B30; Secondary 42B37, 46E35, 42B25, 35F15

1 Introduction

A domain Ω of $\mathbb{R}^n (n \geq 2)$ is said to be strongly Lipschitz if it is a Lipschitz domain and its boundary $\partial\Omega$ is a finite union of parts of rotated graphs of Lipschitz maps and, at most, one of these parts is possibly unbounded; for the definition of Lipschitz domains, we refer the reader to [12].

The real Hardy space $H^p(\mathbb{R}^n)$, $p \in (0, 1]$, is important since it is a good substitution of $L^p(\mathbb{R}^n)$ and has many applications in PDE and analysis; see Stein [26]. Hardy spaces on domains, with applications to the boundary value problems for the Laplace equation, have been studied, for instance, in [2, 6–8, 10, 15, 18–21, 23, 27, 28]. There are mainly two kinds of Hardy spaces on domains, one is the restriction of Hardy spaces to domains, the other is the collection of elements in Hardy spaces with supports in the considered domains. Let us recall the definition as follows (see, for instance, [3, 8, 21]).

For an open set $\Omega \subset \mathbb{R}^n$, let $\mathcal{D}(\Omega)$ be the collection of all smooth functions compactly supported in Ω and $\mathcal{D}'(\Omega)$ the space of all distributions on $\mathcal{D}(\Omega)$. For $p \in (\frac{n}{n+1}, 1]$, the *Hardy space* $H^p_z(\Omega)$ is defined as

$$H^p_z(\Omega) := \{f \in \mathcal{D}'(\mathbb{R}^n) : f \in H^p(\mathbb{R}^n), \ \mathrm{supp} f \subset \overline{\Omega}\}$$

equipped with the *quasi-norm* $\|f\|_{H^p_z(\Omega)} := \|f\|_{H^p(\mathbb{R}^n)}$, and the *Hardy space* $H^p_r(\Omega)$ as

$$H^p_r(\Omega) := \{f \in \mathcal{D}'(\Omega) : \exists F \in H^p(\mathbb{R}^n), \ \mathrm{s.t.}\ F|_\Omega = f\}$$

equipped with the *quasi-norm* $\|f\|_{H^p_r(\Omega)} := \inf_F \|F\|_{H^p(\mathbb{R}^n)}$, where the infimum is taken over the set $\{F \in H^p(\mathbb{R}^n) : F|_\Omega = f\}$. Since $\mathcal{D}'(\mathbb{R}^n) \subset \mathcal{D}'(\Omega)$, it follows that, every f in $H^p_z(\Omega)$ is actually in $H^p_r(\Omega)$ with $\|f\|_{H^p_z(\Omega)} \geq \|f\|_{H^p_r(\Omega)}$.

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Our first main result is the grand maximal function characterization of Hardy spaces on domains. To this end, we recall the following two grand maximal functions.

For any $x \in \mathbb{R}^n$ and $p \in (\frac{n}{n+1}, 1]$, denote by $F_x(\Omega)$ the collection of all $\phi \in \mathcal{D}(\mathbb{R}^n)$, for which there exists a cube Q such that $\operatorname{supp}\phi \subset Q$, $x \in Q$, $c_Q \in \Omega$ and

$$\|\phi\|_{L^\infty(\mathbb{R}^n)} + \ell_Q \|\nabla\phi\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{|Q|},$$

here and hereafter, c_Q denotes the center of the cube Q and ℓ_Q its side length. For each $x \in \Omega$, let

$$G_x(\Omega) := \{\phi \in F_x(\Omega) : \phi = 0 \text{ on } \partial\Omega\}.$$

For each $f \in \mathcal{D}'(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$, let

$$M_z f(x) := \sup_{\phi\in F_x(\Omega)} |\langle f, \phi\rangle|.$$

Let $p^\star := \frac{np}{n-p}$, $q := \frac{p^\star}{p^\star-1}$ and $W_0^{1,q}(\Omega)$ denote the Sobolev space with zero boundary values on Ω. For each bounded linear functional f on $W_0^{1,q}(\Omega)$, for any $x \in \Omega$, let

$$M_r f(x) := \sup_{\phi\in G_x(\Omega)} |\langle f, \phi\chi_\Omega\rangle|,$$

where χ_Ω denotes the characteristic function of Ω. From the fact that $\phi \in G_x(\Omega)$, it follows that $\phi\chi_\Omega \in W_0^{1,q}(\Omega)$, which implies that $M_r f$ is well defined.

Then we have the following characterizations concerning Hardy spaces. In what follows, for any $q \in [1, \infty]$, we denote by q' its *conjugate index*, namely, $\frac{1}{q} + \frac{1}{q'} = 1$.

Theorem 1.1. *Let Ω be a strongly Lipschitz domain and $p \in (\frac{n}{n+1}, 1]$.*

(i) If Ω is bounded, then $f \in H_z^p(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp} f \subset \overline{\Omega}$, $M_z f \in L^p(\Omega)$ and $\langle f, \phi\rangle = 0$ for each $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi \equiv 1$ on Ω. Moreover, $\|f\|_{H_z^p(\Omega)}$ is equivalent to $\|M_z f\|_{L^p(\Omega)}$ with the equivalent positive constants independent of f.

(ii) If Ω is unbounded, then $f \in H_z^p(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp} f \subset \overline{\Omega}$, $M_z f \in L^p(\Omega)$. Moreover, $\|f\|_{H_z^p(\Omega)}$ is equivalent to $\|M_z f\|_{L^p(\Omega)}$ with the equivalent positive constants independent of f.

In [15, Theorem 5.1], Jonsson et al. gave a result similar to Theorem 1.1. Notice that, in their Theorem 5.1, the considered set has to be unbounded (having the global Markov property) and, in the implication that $M_z f \in L^p(\Omega)$ implies $f \in H^{p,\infty,s}(\Omega)$, where $H^{p,\infty,s}(\Omega)$ is the atomic Hardy space on Ω, f is required to belong to $L^1_{\mathrm{loc}}(\Omega)$. To the best of our knowledge, Theorem 1.1 as above is new except the case $p = 1$ and Ω being unbounded.

We have the following characterization for the Hardy space of restricted type, whose proof uses some ideas from the proofs of Theorem 1.1 and Miyachi [21, Theorems 1 and 4].

Theorem 1.2. *Let Ω be a strongly Lipschitz domain, $p \in (\frac{n}{n+1}, 1]$ and $q = (p^\star)'$. Then $f \in H_r^p(\Omega)$ if and only if f is a bounded linear functional on $W_0^{1,q}(\Omega)$ and $M_r f \in L^p(\Omega)$. Moreover, $\|f\|_{H_r^p(\Omega)}$ is equivalent to $\|M_r f\|_{L^p(\Omega)}$ with the equivalent positive constants independent of f.*

We wish to emphasize that, whereas the proof of the characterization of $H_r^p(\Omega)$ (Theorem 1.2) is relatively simpler, the arguments for $H_z^p(\Omega)$ are not trivial, and highly depend on the geometric structure of the underlying domain. The proof of Theorem 1.1 is composed by three steps: we first deal with the case Ω being special Lipschitz, then the case Ω being bounded, and finally Ω being unbounded; see Section 2 below. Based on Theorems 1.1 and 1.2, we then establish grand maximal characterizations of vector-valued Hardy spaces; see Corollaries 2.13 and 2.14 below.

In the same manner as $H^p(\mathbb{R}^n)$ for $p \in (0, 1]$, Hardy-Sobolev spaces play important roles in partial differential equations and analysis at the borderline cases. We refer the reader to [3, 4, 13, 16, 17, 22, 24, 25] for relevant developments concerning Hardy-Sobolev spaces. Let us begin with recalling the definition of Hardy-Sobolev spaces; see Auscher et al. [3] for $p = 1$.

Definition 1.3. Let $\Omega \subset \mathbb{R}^n$ be an open set and $p \in (\frac{n}{n+1}, 1]$. The *Hardy-Sobolev space* $\dot{H}^{1,p}_r(\Omega)$ is defined as

$$\dot{H}^{1,p}_r(\Omega) := \{f \in \mathcal{D}'(\Omega) : \nabla f \in H^p_r(\Omega)\}$$

equipped with the *quasi-norm* defined by $\|f\|_{\dot{H}^{1,p}_r(\Omega)} := \|\nabla f\|_{H^p_r(\Omega)}$. The *Hardy-Sobolev space* $\dot{H}^{1,p}_z(\Omega)$ is also defined as

$$\dot{H}^{1,p}_z(\Omega) := \{f \in \mathcal{D}'(\mathbb{R}^n) : \nabla f \in H^p_z(\Omega)\}$$

equipped with the *quasi-norm* $\|f\|_{\dot{H}^{1,p}_z(\Omega)} := \|\nabla f\|_{H^p_z(\Omega)}$.

Above, ∇f denotes the distributional derivative of f. Precisely, if $U \subset \mathbb{R}^n$ is an arbitrary open set, then, for any $f \in \mathcal{D}'(U)$, $\nabla f \in \mathcal{D}'(U)$ is a *distributional derivative* of f if, for each $\phi \in \mathcal{D}(U)$ and $i \in \{1, \ldots, n\}$, it holds true that $\langle f, \partial_i \phi \rangle = -\langle \partial_i f, \phi \rangle$.

Our second main result is the following grand maximal characterization of Hardy-Sobolev spaces, which generalizes corresponding results by Auscher et al. [3] from $p = 1$ to $p \in (\frac{n}{n+1}, 1]$. Let us begin with recalling the definitions of grand maximal functions.

For any $x \in \mathbb{R}^n$, let $\mathbf{F}_x(\Omega)$ be the collection of all vector-valued functions Φ in $\mathcal{D}(\mathbb{R}^n; \mathbb{C}^n)$ satisfying $\operatorname{supp} \Phi \subset Q$, $x \in Q$, $c_Q \in \Omega$ and $\|\Phi\|_{L^\infty(\mathbb{R}^n)} + \ell_Q \|\operatorname{div} \Phi\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{|Q|}$. Moreover, let

$$\mathbf{G}_x(\Omega) := \{\Phi \in \mathbf{F}_x(\Omega) : \Phi \cdot \nu = 0 \text{ almost everywhere on } \partial\Omega\},$$

where the almost everywhere is in the sense of the $(n-1)$-Hausdorff measure and ν denotes the outer normal of $\partial\Omega$. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$\mathbf{M}^{(1)}_\Omega f(x) := \sup_{\Phi \in \mathbf{F}_x(\Omega)} \left| \int_{\mathbb{R}^n} f(y) \operatorname{div} \Phi(y)\, dy \right|$$

and, for any $f \in L^1_c(\Omega)$ and $x \in \Omega$, let

$$\mathbf{N}^{(1)}_\Omega f(x) := \sup_{\Phi \in \mathbf{G}_x(\Omega)} \left| \int_{\Omega} f(y) \operatorname{div} \Phi(y)\, dy \right|.$$

Here and hereafter, a measurable function f on Ω is said to belong to the *space* $L^q_c(\Omega)$ for some $q \in (0, \infty)$ if, for any compact $K \subset \mathbb{R}^n$, $\int_{K \cap \Omega} |f(x)|^q\, dx < \infty$.

Based on the grand maximal function characterization of Hardy spaces, we then establish the following grand maximal function characterization of homogeneous Hardy-Sobolev spaces $\dot{H}^{1,p}_r(\Omega)$ and $\dot{H}^{1,p}_z(\Omega)$.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a strongly Lipschitz domain and $p \in (\frac{n}{n+1}, 1]$. Then $\nabla f \in H^p_r(\Omega)$ if and only if $f \in \mathcal{D}'(\Omega)$, $f \in L^{p^*}_c(\Omega)$ and $\mathbf{N}^{(1)}_\Omega f \in L^p(\Omega)$. Moreover, $\|\nabla f\|_{H^p_r(\Omega)}$ is equivalent to $\|\mathbf{N}^{(1)}_\Omega f\|_{L^p(\Omega)}$ with the equivalent positive constants independent of f.*

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a strongly Lipschitz domain and $p \in (\frac{n}{n+1}, 1]$. Then $\nabla f \in H^p_z(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, $f \in L^{p^*}_{\mathrm{loc}}(\mathbb{R}^n)$ with $\operatorname{supp} \nabla f \subset \overline{\Omega}$, and $\mathbf{M}^{(1)}_\Omega f \in L^p(\Omega)$. Moreover, $\|\nabla f\|_{H^p_z(\Omega)}$ is equivalent to $\|\mathbf{M}^{(1)}_\Omega f\|_{L^p(\Omega)}$ with the equivalent positive constants independent of f.*

Notice that, in Theorem 1.5, we do not need to distinguish whether or not Ω is bounded as in Theorem 1.1. Indeed, in case of Ω being bounded, we have the implicit condition that, if $f \in \mathcal{D}'(\mathbb{R}^n)$, $f \in L^{p^*}_{\mathrm{loc}}(\mathbb{R}^n)$ with $\operatorname{supp} \nabla f \subset \overline{\Omega}$, then, for each $\Phi \in \mathcal{D}(\mathbb{R}^n; \mathbb{C}^n)$ with Φ being a constant vector on Ω, it holds true that $\langle \nabla f, \Phi \rangle = 0$; see the proof of Theorem 1.5 below.

As applications of the theories of Hardy and Hardy-Sobolev spaces, we establish some endpoint div-curl lemmas (see Section 4 below), and study the divergence equation

$$\operatorname{div} \mathbf{u} = f.$$

We show that the above equation is solvable in the Hardy-Sobolev space if $f \in H^p_z(\Omega)$. This type of equations has been studied intensively; see, for instance, [1, 3, 5, 11, 14]. Actually, this result even has an application in proving main result of this paper; see Lemma 3.3 in Section 3. For $p \in (1, \infty)$, let $L^p_0(\Omega)$ be the space of functions in $L^p(\Omega)$ that have zero integral, namely, $\int_\Omega f(x)\,dx = 0$. Then, for $f \in L^p_0(\Omega)$, there exists a vector-valued function $\mathbf{u} \in W^{1,p}_0(\Omega, \mathbb{C}^n)$ which satisfies $\operatorname{div}\mathbf{u} = f$ and a control condition $\|\mathbf{u}\|_{W^{1,p}(\Omega)} \le C\|f\|_{L^p(\Omega)}$, where C is a positive constant independent of f; see [1, 5, 14]. But the same estimate fails when p approaches the limit situation, namely, when $p = 1$ or $p = \infty$. Our intention is to find a substitution for the limit situation $p = 1$ and even $p \in (\frac{n}{n+1}, 1)$. In many circumstances, we know that H^p is a good substitution of L^p when $p \in (0, 1]$. Hence we are interested to know, for $f \in H^p_z(\Omega)$, whether or not there exists a $\mathbf{u}$ which satisfies both $\operatorname{div}\mathbf{u} = f$ and some kind of estimations. We show in Section 5 that, when $f \in H^p_z(\Omega)$, there exists a $\mathbf{u} \in H^{1,p}_{z,0}(\Omega)$ which satisfies $\operatorname{div}\mathbf{u} = f$ with $\|D\mathbf{u}\|_{[H^p_z(\Omega)]^{n\times n}} \le C\|f\|_{H^p_z(\Omega)}$, where C is a positive constant independent of f. It follows that, if $f \in H^p_z(\Omega)$, the derivatives of corresponding $\mathbf{u}$ are all in $H^p_z(\Omega)$.

In order to prove the result on the divergence equation, we need the atomic decomposition of $f \in H^p_z(\Omega)$ in bounded Lipschitz domains. But we did not find a complete proof for such an atomic decomposition in the existing literatures. In [15, Theorem 5.3], Jonsson et al. proved that a similar decomposition holds true when Ω is a closed convex bounded set with nonzero interior. Notice that, in [15, Theorem 5.3], the supporting balls of the atoms need not be contained in Ω. In [3], Auscher et al. stated the result when $p = 1$ without giving a proof. In [8], Chang et al. proved the atomic decomposition for local Hardy spaces on domains. However, it looks like that their method cannot be directly used to prove the global version, since their proof relies on the fact that, for $f \in h^p_z(\Omega)$ and η_i being one part of a partition of unity, $f\eta \in h^p_z(\Omega)$. Obviously, such a property is not true when replacing $h^p_z(\Omega)$ by $H^p_z(\Omega)$. Hence, for the completeness of the paper, we offer a proof of the atomic decomposition for $f \in H^p_z(\Omega)$ in Appendix A.

The paper is organized as follows. We prove the grand maximal function characterizations of Hardy spaces (Theorems 1.1 and 1.2) in Section 2, and the grand maximal function characterizations of Hardy-Sobolev spaces (Theorems 1.4 and 1.5) in Section 3. In Section 4, we deal with the div-curl lemmas and, in Section 5, we study the divergence equation. Moreover, in Appendix A, we provide a complete proof of the atomic decomposition of $H^p_z(\Omega)$ with $p \in (\frac{n}{n+1}, 1]$ and Ω being a bounded Lipschitz domain.

Finally, we make some conventions on notation. Throughout the paper, we denote by C or c a *positive constant* which is independent of the main parameters, but may vary from line to line. If $f \le Cg$, we then write $f \lesssim g$ or $g \gtrsim f$ and, if $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$. The symbol $B(x, R)$ denotes an open ball, with center x and radius R, and $CB(x, R) = B(x, CR)$. We denote by Q a cube with the center $c_Q \in \mathbb{R}^n$, the side-length ℓ_Q and all its sides parallel to axes of coordinates. Let $CQ := Q(c_Q, C\ell_Q)$. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We also use $W^{1,q}_0(\Omega)$ to denote the *collection of elements in Sobolev spaces* $W^{1,q}(\Omega)$ *with zero boundary values*. For $q \in (0, n)$, let $q^\star$ be its *Sobolev conjugate index* $\frac{nq}{n-q}$ and, for $p \in [1, \infty]$, let p' denote its *conjugate index* $\frac{p}{p-1}$. We let M_{HL} denote the *non-centered Hardy-Littlewood maximal operator* throughout the paper, namely, for any $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M_{HL}f(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\,dy,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ containing x.

2 Hardy spaces $H^p_r(\Omega)$ and $H^p_z(\Omega)$

In this section, we introduce basic notions of the Hardy spaces $H^p_r(\Omega)$ and $H^p_z(\Omega)$ on domains, and establish their grand maximal function characterizations. We first recall the definition of $H^p(\mathbb{R}^n)$, which can be found in [26].

Definition 2.1. Fix a $\phi \in \mathcal{D}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \phi(x)\,dx \ne 0$. For any Schwartz distribution f and $x \in \mathbb{R}^n$, let

$$M_\phi f(x) := \sup_{t \in (0,\infty)} |(f * \phi_t)(x)|,$$

where $\phi_t(\cdot) := t^{-n}\phi(\cdot/t)$ for any $t \in (0,\infty)$. The *Hardy space* $H^p(\mathbb{R}^n)$, $p \in (0,1]$, is defined as the collection of all Schwartz distributions f satisfying $M_\phi f \in L^p(\mathbb{R}^n)$, equipped with the *quasi-norm* as

$$\|f\|_{H^p(\mathbb{R}^n)} := \|M_\phi f\|_{L^p(\mathbb{R}^n)}, \ \forall f \in H^p(\mathbb{R}^n).$$

It is well known that the Hardy space $H^p(\mathbb{R}^n)$ is independent of the choice of function ϕ as long as $\phi \in \mathcal{D}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \phi(x)\,dx \neq 0$; see [26].

Let us recall the following notion of atoms; see [8, 21] for instance.

Definition 2.2. Let $p \in (\frac{n}{n+1}, 1]$, $q \in (1,\infty]$ and $\Omega \subset \mathbb{R}^n$. A function a is called a $(p,q,\Omega)_a$-*atom* if it satisfies:
(i) supp $a \subset Q \subset \Omega$, where Q is a cube;
(ii) $\|a\|_{L^q(\mathbb{R}^n)} \le |Q|^{\frac{1}{q}-\frac{1}{p}}$;
(iii) $\int_Q a(x)\,dx = 0$.
Furthermore, if $\Omega \subsetneqq \mathbb{R}^n$, then a function b is called a $(p,q,\Omega)_b$-*atom* if it satisfies (i), (ii) and
(iii)' $4Q \cap \partial\Omega \neq \emptyset$.
When $\Omega = \mathbb{R}^n$, a $(p,q,\Omega)_a$-atom is simply called a (p,q)-*atom*.

By using [21, Theorems 1 and 4], we know that the Hardy space $H^p_r(\Omega)$ admits the following atomic decomposition; see also [8].

Lemma 2.3. *Let Ω be a strongly Lipschitz domain, $p \in (\frac{n}{n+1}, 1]$ and $q \in (1,\infty]$. Then, for each $f \in H^p_r(\Omega)$, there exist $(p,q,\Omega)_a$-atoms $\{a_i\}_i$ and $(p,q,\Omega)_b$-atoms $\{b_j\}_j$ such that*

$$f = \sum_i \lambda_i a_i + \sum_j \mu_j b_j \quad in \quad \mathcal{D}'(\Omega)$$

and

$$\sum_i |\lambda_i|^p + \sum_j |\mu_j|^p \le C\|f\|^p_{H^p_r(\Omega)},$$

where the positive constant C is independent of f.

From Lemma 2.3, we deduce that each $f \in H^p_r(\Omega)$ can be considered as a bounded linear functional on $W^{1,q}_0(\Omega)$, where q denotes the conjugate index of $p^\star$.

Lemma 2.4. *Let Ω be a strongly Lipschitz domain, $p \in (\frac{n}{n+1}, 1]$ and $q = \frac{p^\star}{p^\star-1}$. Then each $f \in H^p_r(\Omega)$ induces a bounded linear functional on $W^{1,q}_0(\Omega)$ and there exists a positive constant C such that, for all $f \in H^p_r(\Omega)$ and $g \in W^{1,q}_0(\Omega)$,*

$$|\langle f, g\rangle| \le C\|f\|_{H^p_r(\Omega)}\|g\|_{W^{1,q}_0(\Omega)}. \tag{2.1}$$

Proof. Since $\mathcal{D}(\Omega)$ is dense in $W^{1,q}_0(\Omega)$, we only need to show that (2.1) holds true for each $\phi \in \mathcal{D}(\Omega)$.

If $p = 1$, then $q = (p^\star)' = n$. For each $f \in H^p_r(\Omega)$, there exists $F \in H^1(\mathbb{R}^n)$ such that $F|_\Omega = f$ and $\|F\|_{H^1(\mathbb{R}^n)} \le 2\|f\|_{H^1_r(\Omega)}$. By using the duality of $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$, and the embedding of $W^{1,n}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$, we conclude that

$$\begin{aligned}\left|\int_\Omega f(x)\phi(x)\,dx\right| &= \left|\int_{\mathbb{R}^n} F(x)\phi(x)\,dx\right| \le \|F\|_{H^1(\mathbb{R}^n)}\|\phi\|_{\mathrm{BMO}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{H^1_r(\Omega)}\|\nabla\phi\|_{L^n(\mathbb{R}^n)} \sim \|f\|_{H^1_r(\Omega)}\|\phi\|_{W^{1,n}(\Omega)}.\end{aligned} \tag{2.2}$$

If $p \in (\frac{n}{n+1}, 1)$, then $q = (p^\star)' > n$. For each $f \in H^p_r(\Omega)$, by Lemma 2.3, there exist $(p,q,\Omega)_a$-atoms $\{a_i\}_i$ and $(p,q,\Omega)_b$-atoms $\{b_j\}_j$ such that

$$f = \sum_i \lambda_i a_i + \sum_j \mu_j b_j \quad \text{in} \quad \mathcal{D}'(\Omega)$$

and

$$\sum_i |\lambda_i|^p + \sum_j |\mu_j|^p \lesssim \|f\|^p_{H^p_r(\Omega)}.$$

For each a_i, let $Q_i \subset \Omega$ be its supporting cube. Recall that $p \in (\frac{n}{n+1}, 1)$ and $q = (p^\star)' > n$. Then, for each $\phi \in \mathcal{D}(\Omega)$, by using the Hölder inequality, we obtain

$$\begin{aligned}
\left|\int_\Omega a_i(x)\phi(x)\,dx\right| &= \left|\int_\Omega a_i(x)[\phi(x) - \phi(c_{Q_i})]\,dx\right| \\
&\lesssim \int_\Omega |a_i(x)| \left[\int_0^1 |\nabla\phi(tx + (1-t)c_{Q_i})|\,dt\right] |x - c_{Q_i}|\,dx \\
&\lesssim \ell_{Q_i} \int_\Omega |a_i(x)| \left[\int_0^1 |\nabla\phi(tx + (1-t)c_{Q_i})|\,dt\right] dx \\
&\lesssim \ell_{Q_i} \int_0^1 \|a_i\|_{L^{p^\star}(\mathbb{R}^n)} \left[\int_\Omega |\nabla\phi(tx + (1-t)c_{Q_i})|^q\,dx\right]^{\frac{1}{q}} dt \\
&\lesssim \ell_{Q_i} \int_0^1 \|a_i\|_{L^{p^\star}(\mathbb{R}^n)} \left[\int_\Omega |\nabla\phi(x)|^q\,dx\right]^{\frac{1}{q}} t^{-\frac{n}{q}}\,dt \\
&\lesssim \ell_{Q_i}|Q_i|^{-\frac{1}{n}} \int_0^1 \frac{1}{t^{\frac{n}{q}}}\,dt \|\nabla\phi\|_{L^q(\Omega)} \lesssim \|\phi\|_{W^{1,q}(\Omega)}.
\end{aligned} \tag{2.3}$$

For each b_j, let $P_j \subset \Omega$ be its supporting cube. Then we choose $x_{P_j} \in \partial\Omega$ such that $|x - x_{P_j}| \lesssim \ell_{P_j}$ for each $x \in P_j$, which implies that

$$\begin{aligned}
\left|\int_\Omega b_j(x)\phi(x)\,dx\right| &= \left|\int_\Omega b_j(x)[\phi(x) - \phi(x_{P_j})]\,dx\right| \\
&\le \int_\Omega |b_j(x)| \left[\int_0^1 |\nabla\phi(tx + (1-t)x_{P_j})|\,dt\right] |x - x_{P_j}|\,dx \\
&\lesssim \ell_{P_j} \int_0^1 \int_\Omega |b_i(x)||\nabla\phi(tx + (1-t)x_{P_j})|\,dx\,dt \\
&\lesssim \ell_{P_j} \int_0^1 \|b_i\|_{L^{p^\star}(\mathbb{R}^n)} \|\nabla\phi\|_{L^q(\Omega)} \frac{dt}{t^{\frac{n}{q}}} \lesssim \|\phi\|_{W^{1,q}(\Omega)}.
\end{aligned} \tag{2.4}$$

Combining the inequalities (2.3) and (2.4), via the monotonicity of ℓ^p, we conclude that, when $p \in (\frac{n}{n+1}, 1)$,

$$\begin{aligned}
|\langle f, \phi\rangle| \le \sum_i |\langle\lambda_i a_i, \phi\rangle| + \sum_j |\langle\mu_j b_j, \phi\rangle| &\lesssim \left[\sum_i |\lambda_i| + \sum_j |\mu_j|\right] \|\phi\|_{W^{1,q}(\Omega)} \\
&\lesssim \left[\sum_{i=1}^\infty |\lambda_i|^p + \sum_{j=1}^\infty |\mu_j|^p\right]^{\frac{1}{p}} \|\phi\|_{W^{1,q}(\Omega)} \lesssim \|f\|_{H^p_r(\Omega)} \|\phi\|_{W^{1,q}(\Omega)}.
\end{aligned}$$

This, together with (2.2) and the density of $\mathcal{D}(\Omega)$ in $W^{1,q}_0(\Omega)$, finishes the proof of Lemma 2.4. □

Remark 2.5. Let p and q be the same as in Lemma 2.4. The argument same as that used in the proof of Lemma 2.4 also shows that Lemma 2.4 holds true with Ω replaced by $\mathbb{R}^n$; moreover, there exists a positive constant C such that, for any $F \in H^p(\mathbb{R}^n)$ and $\phi \in W^{1,q}(\mathbb{R}^n)$

$$|\langle F, \phi\rangle| \le C\|F\|_{H^p(\mathbb{R}^n)}\|\phi\|_{W^{1,q}(\mathbb{R}^n)}.$$

We first prove a weaker version of Theorem 1.1 as follows. In what follows, $\vec{0}_n$ denotes the origin of $\mathbb{R}^n$.

Proposition 2.6. *Let Ω be a strongly Lipschitz domain and $p \in (\frac{n}{n+1}, 1]$. Then $f \in H^p_z(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp} f \subset \overline{\Omega}$ and $M_z f \in L^p(\mathbb{R}^n)$. Moreover, $\|f\|_{H^p_z(\Omega)}$ is equivalent to $\|M_z f\|_{L^p(\mathbb{R}^n)}$ with the equivalent positive constants independent of f.*

Proof. Suppose $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp} f \subset \overline{\Omega}$ and $M_z f \in L^p(\mathbb{R}^n)$. Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ be a radial function such that $\operatorname{supp}\psi \subset Q := Q(\vec{0}_n, 1)$ and $\int_{\mathbb{R}^n}\psi(x)\,dx = 1$. Notice that, for all $x \in \mathbb{R}^n$,

$$f * \psi_t(x) = \langle f, \psi_t(x-\cdot)\rangle =: \langle f, \psi_{x,t}\rangle,$$

where $\operatorname{supp}\psi_{x,t} \subset Q_{x,t} := \{x\} - tQ$. If $Q_{x,t} \cap \Omega = \emptyset$, then $\langle f, \psi_{x,t}\rangle = 0$. If $Q_{x,t} \cap \Omega \neq \emptyset$, let $\widetilde{Q}_{x,t}$ be a cube such that $c_{\widetilde{Q}_{x,t}} \in Q_{x,t} \cap \Omega$, $Q_{x,t} \subset \widetilde{Q}_{x,t}$ and $\ell_{\widetilde{Q}_{x,t}} = 2\ell_{Q_{x,t}}$. Then there exists a positive constant $\widetilde{C}$ independent of x and t, but depending on ψ, such that

$$\left\|\frac{\psi_{x,t}}{\widetilde{C}}\right\|_{L^\infty(\mathbb{R}^n)} + \ell_{\widetilde{Q}_{x,t}}\left\|\nabla\left(\frac{\psi_{x,t}}{\widetilde{C}}\right)\right\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{|\widetilde{Q}_{x,t}|}.$$

Thus, $\widetilde{C}\psi_{x,t} \in F_x(\Omega)$ and

$$|f * \psi_t(x)| = |\langle f, \psi_{x,t}\rangle| \lesssim \sup_{\phi\in F_x(\Omega)}|\langle f, \phi\rangle| \lesssim M_z f(x),$$

which implies that $\sup_{t\in(0,\infty)}|f * \psi_t(x)| \lesssim M_z f(x)$ and hence $f \in H^p(\mathbb{R}^n)$. Since $\operatorname{supp} f \subset \overline{\Omega}$, we have $f \in H^p_z(\Omega)$ and $\|f\|_{H^p_z(\Omega)} \lesssim \|M_z f\|_{L^p(\mathbb{R}^n)}$.

Conversely, let $f = \sum_i \lambda_i a_i$ be an atomic decomposition in $H^p(\mathbb{R}^n)$, where $\{a_i\}_i$ are (p,∞)-atoms and $\{\lambda_i\}_i \subset \mathbb{C}$ satisfy $\sum_i |\lambda_i|^p \sim \|f\|^p_{H^p_z(\Omega)}$. If it holds true that $\|M_z a\|_{L^p(\mathbb{R}^n)} \lesssim 1$ for any (p,∞)-atom a, then, by the monotonicity of ℓ^p, we have

$$\int_{\mathbb{R}^n}[M_z f(y)]^p\,dy \le \int_{\mathbb{R}^n}\left[\sum_{i=1}^\infty |\lambda_i| M_z a_i(y)\right]^p dy \le \sum_{i=1}^\infty |\lambda_i|^p \int_{\mathbb{R}^n}[M_z a_i(y)]^p\,dy$$
$$\lesssim \sum_{i=1}^\infty |\lambda_i|^p \lesssim \|f\|^p_{H^p_z(\Omega)}.$$

This implies that $\|M_z f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{H^p_z(\Omega)}$, and hence $M_z f \in L^p(\mathbb{R}^n)$ follows automatically.

It remains to prove that $\|M_z a\|_{L^p(\mathbb{R}^n)} \lesssim 1$ for any (p,∞)-atom a. Let a be a (p,∞)-atom with $\operatorname{supp} a \subset I$. For $x \in 4I$, we have

$$M_z a(x) = \sup_{\phi\in F_x(\Omega)}\left|\int_{\mathbb{R}^n} a(y)\phi(y)\,dy\right| \le \sup_{\phi\in F_x(\Omega)}\|\phi\|_{L^\infty(\mathbb{R}^n)}\int_{\Omega\cap Q}|a(y)|\,dy$$
$$\le \sup_{\phi\in F_x(\Omega)}\frac{1}{|Q|}\int_{\Omega\cap Q}|a(y)|\,dy \le \|a\|_{L^\infty(\mathbb{R}^n)}.$$

By the property of (p,∞)-atoms, we know that

$$\int_{4I}[M_z a(x)]^p\,dx \le \|a\|^p_{L^\infty(\mathbb{R}^n)}|4I| \lesssim 1.$$

For any $x \notin 4I$ and $\phi \in F_x(\Omega)$ with $\operatorname{supp} \phi \subset Q$, by the fact $\int_{\mathbb{R}^n} a(y)\,dy = 0$, we find that

$$\left|\int_{\mathbb{R}^n} a(y)\phi(y)\,dy\right| = \left|\int_I a(y)[\phi(y) - \phi(c_I)]\,dy\right|$$
$$\lesssim \|a\|_{L^\infty(\mathbb{R}^n)}\ell_I^{1+n}\|\nabla\phi\|_{L^\infty(\mathbb{R}^n)} \lesssim \|\nabla\phi\|_{L^\infty(\mathbb{R}^n)}(\ell_I)^{1+n-n/p}. \tag{2.5}$$

If $\operatorname{supp} a \cap \operatorname{supp} \phi = \emptyset$, we have $\int_{\mathbb{R}^n} a(y)\phi(y)\,dy = 0$. If $\operatorname{supp} a \cap \operatorname{supp} \phi \neq \emptyset$, we then have $I \cap Q \neq \emptyset$. Since $x \notin 4I$ and $x \in Q$, it follows that $\frac{3}{2}\ell_I \le |x - y| < \ell_Q$ for each $y \in Q \cap I$, which implies that

$$|x - c_I| \le |x - y| + |y - c_I| \lesssim \ell_Q.$$

We then deduce from (2.5) that

$$\left|\int_{\mathbb{R}^n} a(y)\phi(y)\,dy\right| \lesssim \frac{(\ell_I)^{1+(1-\frac{1}{p})n}}{(\ell_Q)^{n+1}} \lesssim \frac{(\ell_I)^{1+(1-\frac{1}{p})n}}{|x - c_I|^{n+1}}$$

and hence

$$M_z a(x) \lesssim \frac{(\ell_I)^{1+(1-\frac{1}{p})n}}{|x - c_I|^{n+1}}, \quad \forall\, x \notin 4I.$$

Combining the above estimates, we know that

$$\int_{x \notin 4I} [M_z a(x)]^p\,dx \lesssim \int_{2\ell_I}^{\infty} \frac{(\ell_I)^{p+(p-1)n} r^{n-1}}{r^{p(n+1)}}\,dr \lesssim 1$$

and hence

$$\int_{\mathbb{R}^n} [M_z a(x)]^p\,dx = \int_{4I} [M_z a(x)]^p\,dx + \int_{(4I)^\complement} \cdots \lesssim 1.$$

This proves the claim and hence finishes the proof of Proposition 2.6. □

In view of Proposition 2.6, to prove Theorem 1.1, it remains to prove that $\|M_z f\|_{L^p(\mathbb{R}^n)} \sim \|M_z f\|_{L^p(\Omega)}$ with the equivalent positive constants independent of f. We do this by the following several lemmas.

Lemma 2.7. *Let $p \in (\frac{n}{n+1}, 1]$ and Ω be a special Lipschitz domain. Then $f \in H^p_z(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp} f \subset \overline{\Omega}$ and $M_z f \in L^p(\Omega)$. Moreover, $\|f\|_{H^p_z(\Omega)}$ is equivalent to $\|M_z f\|_{L^p(\Omega)}$ with the equivalent positive constants independent of f.*

Proof. If $f \in H^p_z(\Omega)$, then, from Proposition 2.6, it follows that $M_z f \in L^p(\Omega)$ and $\|f\|_{H^p_z(\Omega)} \gtrsim \|M_z f\|_{L^p(\Omega)}$.

Conversely, we assume $\Omega := \{x \in \mathbb{R}^n : \lambda(x_1, \ldots, x_{n-1}) < x_n\}$, where λ is a Lipschitz function from $\mathbb{R}^{n-1}$ to $\mathbb{R}$. Let $\Omega^- := \mathbb{R}^n \setminus \overline{\Omega}$. For $x := (x_1, \ldots, x_n) \in \Omega^-$, define a reflecting function

$$R(x) := (x_1, \ldots, x_{n-1}, 2\lambda(x_1, \ldots, x_{n-1}) - x_n) \tag{2.6}$$

and a cone

$$P(x) := \{y \in \mathbb{R}^n : x_n - y_n \ge L|(x_1, \ldots, x_{n-1}) - (y_1, \ldots, y_{n-1})|\},$$

where L is the Lipschitz constant of λ.

For each $x := (x_1, \ldots, x_n) \in \Omega^-$, let $\widetilde{x} := (x_1, \ldots, x_{n-1}, \lambda(x_1, \ldots, x_{n-1}))$. It follows that

$$B\left(x, \frac{1}{L^2+1}|x - \widetilde{x}|\right) \subset P(\widetilde{x}),$$

$B(x, \frac{1}{L^2+1}|x - \widetilde{x}|) \cap \Omega = \emptyset$ and

$$|x - R(x)| = 2|x - \widetilde{x}| \lesssim d(x, \partial\Omega). \tag{2.7}$$

Now, for each $x \in \Omega^-$, let $\phi \in F_x(\Omega)$ and $\operatorname{supp}\phi \subset Q$. Since $\operatorname{supp} f \subset \overline{\Omega}$, we only need to consider the case $Q \cap \Omega \neq \emptyset$, In this case, we have $d(x, \partial\Omega) \lesssim \ell_Q$ and hence $|x - R(x)| \lesssim \ell_Q$. Thus, we can choose a cube $\widetilde{Q}$ such that $\ell_{\widetilde{Q}} \lesssim \ell_Q$, $c_{\widetilde{Q}} = c_Q$, $Q \subset \widetilde{Q}$ and $R(x) \in \widetilde{Q}$. This implies that $\widetilde{C}\phi \in F_{R(x)}(\Omega)$ for some harmless positive constant $\widetilde{C}$ and hence, for each $x \in \Omega^-$,

$$M_z f(x) \lesssim M_z f(R(x)).$$

Since R is a bilipschitz map and $R(\Omega^-) = \Omega$, it follows that

$$\int_{\Omega^-} [M_z f(x)]^p\,dx \lesssim \int_{\Omega^-} [M_z f(R(x))]^p\,dx \lesssim \int_{\Omega} [M_z f(x)]^p\,dx$$

and hence $\|M_z f\|_{L^p(\mathbb{R}^n)} \lesssim \|M_z f\|_{L^p(\Omega)}$, which, combined with Proposition 2.6, further implies that $\|f\|_{H^p_z(\Omega)} \lesssim \|M_z f\|_{L^p(\Omega)}$, and hence completes the proof of Lemma 2.7. □

Let us now deal with the general case of Ω. In what follows, for each strongly Lipschitz domain Ω and $\epsilon \in (0,\infty)$, we let $\Omega_\epsilon := \{x \in \mathbb{R}^n :\ d(x,\Omega) < \epsilon\}$.

Lemma 2.8. *Let $p \in (\frac{n}{n+1}, 1]$ and Ω be a strongly Lipschitz domain. Assume $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp} f \subset \overline{\Omega}$ and $M_z f \in L^p(\Omega)$. Then there exist positive constants C and ϵ, independent of f, such that*

$$\int_{\Omega_\epsilon \setminus \overline{\Omega}} [M_z f(x)]^p\,dx \le C \int_{\Omega} [M_z f(x)]^p\,dx.$$

Proof. By the proof of [4, Lemma 8], we know that there exist open sets, $V_0, \ldots, V_m \subset \mathbb{R}^n$, $m \in \mathbb{N}$, and $\epsilon_0 \in (0,\infty)$ having the following properties:

(i) $\Omega_{\epsilon_0} \subset \bigcup_{j=0}^m V_j$;

(ii) $\overline{V}_0$ is compact and included in Ω;

(iii) for each $j \in \{1, \ldots, m\}$, there exists an open set $\Omega_j^+ \subset \mathbb{R}^n$, which is the image of a special Lipschitz domain under an orthogonal transformation, such that $\partial\Omega \cap V_j = \partial\Omega_j^+ \cap V_j$ and $\Omega \cap V_j = \Omega_j^+ \cap V_j$.

For each $j \in \{1, \ldots, m\}$, let $\Omega_j^- := \mathbb{R}^n \setminus \overline{\Omega_j^+}$ and $R_j : \mathbb{R}^n \to \mathbb{R}^n$ be a bilipschitz reflection across $\partial\Omega_j^+$, as in (2.6), such that $R_j(\Omega_j^-) = \Omega_j^+$. For all $\epsilon \in (0, \epsilon_0)$, one has $(\Omega_\epsilon \setminus \overline{\Omega}) \subset \bigcup_{j=1}^m \{(\Omega_\epsilon \setminus \overline{\Omega}) \cap V_j\}$. Moreover, there exists an ϵ_0' such that, for all $j \in \{1, \ldots, m\}$ and $y \in (\Omega_{\epsilon_0'} \setminus \overline{\Omega}) \cap V_j$, $R_j(y) \in \Omega$ and $|R_j(y) - y| \lesssim d(y, \Omega)$.

Denote ϵ_0' by ϵ for simplicity. For each $x \in \Omega_\epsilon \setminus \overline{\Omega}$, there exists V_j such that $x \in (\Omega_\epsilon \setminus \overline{\Omega}) \cap V_j$. For each $\phi \in F_x(\Omega)$ with $\operatorname{supp}\phi \subset Q$ and $Q \cap \Omega \neq \emptyset$, from the fact $|R_j(x) - x| \lesssim d(x,\Omega) \lesssim \ell_Q$, it follows that we can choose a $\widetilde{Q}$ such that $\ell_{\widetilde{Q}} \lesssim \ell_Q$, $Q \subset \widetilde{Q}$, $c_Q = c_{\widetilde{Q}}$ and $R_j(x) \in \widetilde{Q}$. We then have $\widetilde{C}\phi \in F_{R_j(x)}(\Omega)$ for some harmless positive constant $\widetilde{C}$ and hence

$$|\langle f, \phi\rangle| \lesssim M_z f(R_j(x)).$$

By taking the supremum over ϕ, we conclude that, for each $x \in (\Omega_\epsilon \setminus \overline{\Omega}) \cap V_j$,

$$M_z f(x) \lesssim M_z f(R_j(x)),$$

which implies that

$$\begin{aligned}\int_{\Omega_\epsilon \setminus \overline{\Omega}} [M_z f(x)]^p\,dx &\le \sum_{j=1}^m \int_{(\Omega_\epsilon \setminus \overline{\Omega}) \cap V_j} [M_z f(x)]^p\,dx \\ &\lesssim \sum_{j=1}^m \int_{R_j((\Omega_\epsilon \setminus \overline{\Omega}) \cap V_j)} [M_z f(x)]^p\,dx \lesssim \int_{\Omega} [M_z f(x)]^p\,dx. \qquad (2.8)\end{aligned}$$

This finishes the proof of Lemma 2.8. □

Lemma 2.9. *Let $p \in (\frac{n}{n+1}, 1]$ and Ω be a bounded Lipschitz domain. Then $f \in H^p_z(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp} f \subset \overline{\Omega}$, $M_z f \in L^p(\Omega)$ and $\langle f, \phi \rangle = 0$ for each $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi \equiv 1$ on Ω. Moreover, $\|f\|_{H^p_z(\Omega)}$ is equivalent to $\|M_z f\|_{L^p(\Omega)}$ with the equivalent positive constants independent of f.*

Proof. If $f \in H^p_z(\Omega)$, then it follows, from the fact $f \in H^p(\mathbb{R}^n)$ with $\operatorname{supp} f \subset \overline{\Omega}$, that, for each $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi \equiv 1$ on Ω, $\langle f, \phi \rangle = 0$. Moreover, by Proposition 2.6, we know that $\|f\|_{H^p_z(\Omega)} \gtrsim \|M_z f\|_{L^p(\Omega)}$.

Let us prove the converse side. By applying Lemma 2.8, we find that there exists an $\epsilon \in (0, \infty)$ such that

$$\int_{\Omega_\epsilon \setminus \overline{\Omega}} [M_z f(x)]^p \, dx \lesssim \int_\Omega [M_z f(x)]^p \, dx. \tag{2.9}$$

Let us estimate the integral of $M_z f(x)$ over $(\Omega_\epsilon)^\complement$. We choose a ball $B := B(\vec{0}_n, r_B)$ with r_B large enough such that $\Omega_\epsilon \subset B$. For any $x \in 2B \setminus \overline{\Omega}_\epsilon$ and any $\widetilde{x} \in \Omega$, we have $|x - \widetilde{x}| \sim d(x, \Omega)$, since Ω is bounded. For any $\phi \in F_x(\Omega)$ with $\operatorname{supp} \phi \subset Q$, by choosing a larger cube $\widetilde{Q}$ such that $\{x\} \subset Q \subset \widetilde{Q}$ and $\ell_{\widetilde{Q}} \lesssim \ell_Q$, we have $\widetilde{C}\phi \in F_{\widetilde{x}}(\Omega)$ for a harmless positive constant $\widetilde{C}$. This further implies that $M_z f(x) \lesssim M_z f(\widetilde{x})$ for each $x \in 2B \setminus \overline{\Omega}_\epsilon$, and hence

$$\begin{aligned}\frac{1}{|2B \setminus \overline{\Omega}_\epsilon|} \int_{2B \setminus \overline{\Omega}_\epsilon} [M_z f(x)]^p \, dx &\le \sup_{x \in 2B \setminus \overline{\Omega}_\epsilon} [M_z f(x)]^p \\ &\lesssim \inf_{x \in \Omega} [M_z f(x)]^p \lesssim \frac{1}{|\Omega|} \int_\Omega [M_z f(x)]^p \, dx.\end{aligned} \tag{2.10}$$

By this, we find that

$$\int_{2B \setminus \overline{\Omega}_\epsilon} [M_z f(x)]^p \, dx \lesssim \int_\Omega [M_z f(x)]^p \, dx. \tag{2.11}$$

Let $x \in (2B)^\complement$ and $\phi \in F_x(\Omega)$ with $\operatorname{supp} \phi \subset Q$ and $x \in Q$. If $Q \cap \Omega = \emptyset$, then we have $\langle f, \phi \rangle = 0$. Otherwise, we have $|x| \lesssim \ell_Q$ and hence $\frac{1}{\ell_Q^{n+1}} \lesssim \frac{1}{|x|^{n+1}}$. By choosing a function $\chi \in \mathcal{D}(\mathbb{R}^n)$ satisfying $\operatorname{supp} \chi \subset 2B$, $\chi \equiv 1$ on B and $|\nabla \chi| \lesssim \frac{1}{\operatorname{diam}(\Omega)}$, we conclude that $\langle f, \phi \rangle = \langle f, \phi\chi \rangle$, since $\chi \equiv 1$ on $B \supset \Omega_\epsilon \supset \Omega$. For each $\widetilde{x} \in \Omega$, by assumption, we have $\langle f, \chi \rangle = 0$ and hence

$$|\langle f, \phi \rangle| = \left| \langle f, [\phi - \phi(\widetilde{x})]\chi \rangle \right|.$$

Noticing that, for any $y \in \Omega$,

$$|\phi(y) - \phi(\widetilde{x})| \lesssim \|\nabla \phi\|_{L^\infty(\mathbb{R}^n)} |y - \widetilde{x}| \lesssim \frac{\operatorname{diam}(\Omega)}{|x|^{n+1}},$$

by choosing I to be a cube centered at origin such that $2B \subset I$ and $\ell_I \sim \operatorname{diam}(\Omega)$, we conclude that $\operatorname{supp}([\phi - \phi(\widetilde{x})]\chi) \subset I$ and

$$\|[\phi(\cdot) - \phi(\widetilde{x})]\chi(\cdot)\|_{L^\infty(\mathbb{R}^n)} + \ell_I \|\nabla\{[\phi(\cdot) - \phi(\widetilde{x})]\chi(\cdot)\}\|_{L^\infty(\mathbb{R}^n)} \lesssim \frac{\operatorname{diam}(\Omega)}{|x|^{n+1}}.$$

Thus, $\frac{|x|^{n+1}}{|I| \operatorname{diam}(\Omega)} \left[\phi - \phi(\widetilde{x})\right] \chi \in F_{\widetilde{x}}(\Omega)$ for each $\widetilde{x} \in \Omega$ up to some harmless positive constant and, therefore,

$$|\langle f, \phi \rangle| \lesssim \frac{|I| \operatorname{diam}(\Omega)}{|x|^{n+1}} \inf_{y \in \Omega} M_z f(y) \lesssim \frac{|I| \operatorname{diam}(\Omega)}{|x|^{n+1}} \left\{ \frac{1}{|\Omega|} \int_\Omega [M_z f(y)]^p \, dy \right\}^{\frac{1}{p}}.$$

By taking the supremum on ϕ, we know that, for each $x \in (2B)^\complement$,

$$[M_z f(x)]^p \lesssim \frac{|I|^p [\operatorname{diam}(\Omega)]^p}{|x|^{(n+1)p} |\Omega|} \int_\Omega [M_z f(y)]^p \, dy.$$

Since $p \in (\frac{n}{n+1}, 1]$, it follows that

$$\int_{(2B)^{\complement}} [M_z f(x)]^p \, dx \sim \int_{2r_B}^{\infty} \int_{|x|=\lambda} [M_z f(x)]^p \lambda^{n-1} \, dx \, d\lambda$$
$$\lesssim \int_{2r_B}^{\infty} \frac{|I|^p [\mathrm{diam}(\Omega)]^p \, d\lambda}{\lambda^{(n+1)p-n+1} |\Omega|} \int_{\Omega} [M_z f(y)]^p \, dy \lesssim \int_{\Omega} [M_z f(y)]^p \, dy. \tag{2.12}$$

Combining (2.9), (2.11) and (2.12), we finally conclude that

$$\int_{\mathbb{R}^n} [M_z f(x)]^p \, dx \lesssim \int_{\Omega} [M_z f(x)]^p \, dx,$$

which, together with Proposition 2.6, implies that $\|f\|_{H^p_z(\Omega)} \lesssim \|M_z f\|_{L^p(\Omega)}$. This finishes the proof of Lemma 2.9. □

Lemma 2.10. *Let $p \in (\frac{n}{n+1}, 1]$ and Ω be an unbounded strongly Lipschitz domain. Then $f \in H^p_z(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, supp $f \subset \overline{\Omega}$ and $M_z f \in L^p(\Omega)$. Moreover, $\|f\|_{H^p_z(\Omega)}$ is equivalent to $\|M_z f\|_{L^p(\Omega)}$ with the equivalent positive constants independent of f.*

Proof. If $f \in H^p_z(\Omega)$, by Proposition 2.6, we find that $\|f\|_{H^p_z(\Omega)} \gtrsim \|M_z f\|_{L^p(\Omega)}$.

Conversely, suppose first that $\Omega^{\complement}$ is bounded. By Lemma 2.8, we know that, for some $\epsilon \in (0, \infty)$,

$$\int_{\Omega_\epsilon \setminus \overline{\Omega}} [M_z f(x)]^p \, dx \lesssim \int_{\Omega} [M_z f(x)]^p \, dx. \tag{2.13}$$

Choose a ball $B := B(\vec{0}_n, r)$ with $r \in (0, \infty)$ sufficiently large such that $[B(\vec{0}_n, r)]^{\complement} \subset \Omega$. Since, for all $x \in \mathbb{R}^n \setminus \Omega_\epsilon$ and $\widetilde{x} \in B \cap \Omega$, we have $|x - \widetilde{x}| \lesssim d(x, \Omega)$ with the implicit positive constant depending also on ϵ. Similarly to the proof of (2.10), we have

$$\sup_{x \in \mathbb{R}^n \setminus \Omega_\epsilon} M_z f(x) \lesssim \inf_{x \in B \cap \Omega} M_z f(x).$$

Hence, noticing $|\mathbb{R}^n \setminus \Omega_\epsilon| < \infty$, we find that

$$\int_{\mathbb{R}^n \setminus \Omega_\epsilon} [M_z f(x)]^p \, dx \lesssim |\mathbb{R}^n \setminus \Omega_\epsilon| \inf_{x \in B \cap \Omega} M_z f(x) \lesssim \int_{\Omega} [M_z f(x)]^p \, dx. \tag{2.14}$$

Finally, by (2.13) and (2.14), we conclude that

$$\int_{\mathbb{R}^n} [M_z f(x)]^p \, dx \lesssim \int_{\Omega} [M_z f(x)]^p \, dx,$$

which, combined with Proposition 2.6, further implies that $\|f\|_{H^p_z(\Omega)} \lesssim \|M_z f\|_{L^p(\Omega)}$ and hence, in this case, $\|f\|_{H^p_z(\Omega)} \sim \|M_z f\|_{L^p(\Omega)}$.

It remains to prove the case when Ω and $\Omega^{\complement}$ are both unbounded. Since Ω is strongly Lipschitz, it follows that there exists a ball $B(\vec{0}_n, r_B)$ sufficiently large such that $[B(\vec{0}_n, r_B)]^{\complement} \cap \Omega$ is special Lipschitz. Define $\widetilde{\Omega} := [B(\vec{0}_n, r_B)]^{\complement} \cap \Omega$. By Lemma 2.8, we have

$$\int_{\Omega_\epsilon \setminus \overline{\Omega}} [M_z f(x)]^p \, dx \lesssim \int_{\Omega} [M_z f(x)]^p \, dx \tag{2.15}$$

for some $\epsilon \in (0, \infty)$. Moreover, similarly to the proof of (2.10), we find that

$$\sup_{x \in B(\vec{0}_n, r_B) \setminus \Omega_\epsilon} M_z f(x) \lesssim \inf_{x \in B(\vec{0}_n, r_B) \cap \Omega} M_z f(x).$$

This, together with $|B(\vec{0}_n, r_B) \setminus \Omega_\epsilon| < \infty$, implies that

$$\int_{B(\vec{0}_n, r_B)\setminus\Omega_\epsilon} [M_z f(x)]^p \, dx \lesssim |B(\vec{0}_n, r_B) \setminus \Omega_\epsilon| \inf_{x\in B(\vec{0}_n, r_B)\cap\Omega} M_z f(x) \lesssim \int_\Omega [M_z f(x)]^p \, dx. \tag{2.16}$$

Recall that $\widetilde{\Omega} := [B(\vec{0}_n, r_B)]^{\complement} \cap \Omega$ is a special Lipschitz domain. Let $\widetilde{\Omega}^-$ and $R : \widetilde{\Omega}^- \to \widetilde{\Omega}$ be defined as in the proof of Lemma 2.7, namely, when Ω is a special Lipschitz domain.

Since $[\Omega^{\complement} \setminus B(\vec{0}_n, r_B)] \subset \widetilde{\Omega}^-$, it follows that, for each $x \in [\Omega^{\complement} \setminus B(\vec{0}_n, r_B)]$, it holds true that $x \in \widetilde{\Omega}^-$ and $d(x, \Omega) \sim |x - R(x)|$. Similarly to the proof of Lemma 2.7, we conclude that, for each $x \in [\Omega^{\complement} \setminus B(\vec{0}_n, r_B)]$,

$$M_z f(x) \lesssim M_z f(R(x))$$

and hence

$$\int_{\Omega^{\complement}\setminus B(\vec{0}_n, r_B)} [M_z f(x)]^p \, dx \lesssim \int_{\widetilde{\Omega}} [M_z f(x)]^p \, dx \lesssim \int_\Omega [M_z f(x)]^p \, dx. \tag{2.17}$$

Combining (2.15), (2.16) and (2.17), we have

$$\int_{\mathbb{R}^n} [M_z f(x)]^p \, dx \lesssim \int_\Omega [M_z f(x)]^p \, dx,$$

which, together with Proposition 2.6, further implies that $\|f\|_{H^p_z(\Omega)} \lesssim \|M_z f\|_{L^p(\Omega)}$ and hence completes the proof of Lemma 2.10. □

Proof of Theorem 1.1. The desired conclusion follows from Lemmas 2.9 and 2.10. □

Let us now proceed to prove Theorem 1.2.

Proof of Theorem 1.2. Suppose first $f \in H^p_r(\Omega)$. By the definition of $H^p_r(\Omega)$, we choose an extension F of f on $\mathbb{R}^n$ such that $\|F\|_{H^p(\mathbb{R}^n)} \le 2\|f\|_{H^p_r(\Omega)}$. From the proof of Proposition 2.6, we deduce that

$$\|M_z F\|_{L^p(\mathbb{R}^n)} \lesssim \|F\|_{H^p(\mathbb{R}^n)} \lesssim \|f\|_{H^p_r(\Omega)}.$$

We now show $M_r f \lesssim M_z F$. To this end, let $x \in \Omega$. For each $\phi \in G_x(\Omega)$, since $\phi\chi_\Omega \in W^{1,q}_0(\Omega)$, it follows that there exists a sequence $\{\widetilde{\phi}_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ which converges to $\phi\chi_\Omega$ in $W^{1,q}(\Omega)$ as $k \to \infty$. Let $\eta_k(\cdot) := k^n\eta(k\cdot) \in \mathcal{D}(\mathbb{R}^n)$, $k \in \mathbb{N}$, be an approximation of identity, and let $\widetilde{\widetilde{\phi}}_k := \eta_k * (\phi\chi_\Omega)$ for all $k \in \mathbb{N}$.

Since $\operatorname{supp}(\phi\chi_\Omega)$ is contained in a cube Q, it follows that there exists a cube Q_k such that $Q \subset Q_k$ and $\operatorname{supp} \widetilde{\widetilde{\phi}}_k \subset Q_k$ for each $k \in \mathbb{N}$. Notice that $x \in Q_k$, $c_{Q_k} \in \Omega$ and, for large enough k, $|Q_k| \le 2|Q|$. By the triangle inequality and $\{\widetilde{\phi}_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\Omega)$, we know that, for all $k \in \mathbb{N}$,

$$\left|\langle f, \phi\chi_\Omega\rangle - \left\langle F, \widetilde{\widetilde{\phi}}_k\right\rangle\right| \le \left|\left\langle f, \phi\chi_\Omega - \widetilde{\phi}_k\right\rangle\right| + \left|\left\langle F, \widetilde{\phi}_k - \widetilde{\widetilde{\phi}}_k\right\rangle\right|.$$

From Lemma 2.4, it follows that, for all $k \in \mathbb{N}$,

$$\left|\left\langle f, \phi\chi_\Omega - \widetilde{\phi}_k\right\rangle\right| \le \|f\|_{H^p_r(\Omega)} \left\|\phi - \widetilde{\phi}_k\right\|_{W^{1,q}(\Omega)}$$

and, from Remark 2.5,

$$\left|\left\langle F, \widetilde{\phi}_k - \widetilde{\widetilde{\phi}}_k\right\rangle\right| \lesssim \|F\|_{H^p(\mathbb{R}^n)} \left\|\widetilde{\phi}_k - \widetilde{\widetilde{\phi}}_k\right\|_{W^{1,q}(\mathbb{R}^n)} \lesssim \|f\|_{H^p_r(\Omega)} \left\|\widetilde{\phi}_k - \widetilde{\widetilde{\phi}}_k\right\|_{W^{1,q}(\mathbb{R}^n)}.$$

By letting $k \to \infty$ and the choice of $\{\widetilde{\widetilde{\phi}}_k\}_{k\in\mathbb{N}}$, we conclude that, for all $x \in \Omega$,

$$|\langle f, \phi\chi_\Omega\rangle| \le \lim_{k\to\infty} \left|\left\langle F, \widetilde{\widetilde{\phi}}_k\right\rangle\right| = |\langle F, \phi\chi_\Omega\rangle| \lesssim M_z F(x).$$

Taking supremum over $\phi \in G_x(\Omega)$, we finally conclude that, for all $x \in \Omega$, $M_r f(x) \lesssim M_z F(x)$ and hence $\|M_r f\|_{L^p(\Omega)} \lesssim \|f\|_{H^p_r(\Omega)}$.

The converse follows from [21, Theorems 1 and 4], which completes the proof of Theorem 1.2. □

It then turns out that, in the definition of grand maximal functions, we only need to require ϕ to be in some $W^{1,q}(\Omega)$ instead of $W^{1,\infty}(\Omega)$, provided q is large enough.

For any $x \in \mathbb{R}^n$, $p \in (\frac{n}{n+1}, 1]$ and $q \in (\frac{p^*}{p^*-1}, \infty)$, we denote by $F_x^q(\Omega)$ the collection of all $\phi \in \mathcal{D}(\mathbb{R}^n)$, which satisfy $\operatorname{supp} \phi \subset Q$, $x \in Q$, $c_Q \in \Omega$ and

$$\|\phi\|_{L^q(\mathbb{R}^n)} + \ell_Q \|\nabla \phi\|_{L^q(\mathbb{R}^n)} \le |Q|^{-1/q'}.$$

Similarly, for each $x \in \Omega$, we let

$$G_x^q(\Omega) := \{\phi \in F_x^q(\Omega) : \ \phi = 0 \text{ on } \partial\Omega\}.$$

We then let, for $f \in \mathcal{D}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M_z^{(q)} f(x) := \sup_{\phi \in F_x^q(\Omega)} |\langle f, \phi \rangle|$$

and, for each bounded linear functional f on $W_0^{1,q}(\Omega)$ and $x \in \Omega$,

$$M_r^{(q)} f(x) := \sup_{\phi \in G_x^q(\Omega)} |\langle f, \phi\chi_\Omega \rangle|.$$

From the fact that $\phi \in G_x(\Omega)$, it follows that $\phi\chi_\Omega \in W_0^{1,q}(\Omega)$, which implies that $M_r^{(q)} f$ is well defined.

We then have the following grand maximal function characterizations, as generalizations of Theorems 1.1 and 1.2.

Theorem 2.11. *Let Ω be a strongly Lipschitz domain, $p \in (\frac{n}{n+1}, 1]$ and $q \in (\frac{p^*}{p^*-1}, \infty]$.*

(i) If Ω is bounded, then $f \in H_z^p(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp} f \subset \overline{\Omega}$, $M_z^{(q)} f \in L^p(\Omega)$ and $\langle f, \phi \rangle = 0$ for each $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi \equiv 1$ on Ω. Moreover, $\|f\|_{H_z^p(\Omega)}$ is equivalent to $\|M_z^{(q)} f\|_{L^p(\Omega)}$ with the equivalent positive constants independent of f.

(ii) If Ω is unbounded, then $f \in H_z^p(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp} f \subset \overline{\Omega}$ and $M_z^{(q)} f \in L^p(\Omega)$. Moreover, $\|f\|_{H_z^p(\Omega)}$ is equivalent to $\|M_z^{(q)} f\|_{L^p(\Omega)}$ with the equivalent positive constants independent of f.

Proof. Since $F_x(\Omega) \subset F_x^q(\Omega)$, it follows that $M_z f(x) \le M_z^{(q)} f(x)$ for all $x \in \mathbb{R}^n$, which, together with Theorem 1.1, further implies that

$$\|f\|_{H_z^p(\Omega)} \lesssim \|M_z^{(q)} f\|_{L^p(\Omega)}.$$

Let us now prove the converse side. We only prove (i), since (ii) can be proved in a similar way.

Suppose that Ω is bounded and $f \in H_z^p(\Omega)$. Then $f \in H^p(\mathbb{R}^n)$ with $\operatorname{supp} f \subset \overline{\Omega}$ and, for each $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi \equiv 1$ on Ω, it holds true that $\langle f, \phi \rangle = 0$. Moreover, there exists an atomic decomposition of f in $\mathcal{D}'(\mathbb{R}^n)$ such that $f = \sum_j \lambda_j a_j$ in $\mathcal{D}'(\mathbb{R}^n)$, where $\{a_j\}_j$ are (p, ∞)-atoms and $\{\lambda_j\}_j \subset \mathbb{C}$ satisfy

$$\sum_j |\lambda_j|^p \lesssim \|f\|_{H^p(\mathbb{R}^n)}^p \sim \|f\|_{H_z^p(\mathbb{R}^n)}^p.$$

To show $M_z^{(q)} f \in L^p(\Omega)$, it suffices to show that, for all j, $M_z^{(q)} a_j \in L^p(\Omega)$ and $\|M_z^{(q)} a_j\|_{L^p(\Omega)} \lesssim 1$. To this end, let I_j be the cube where a_j supports on. For each $x \in 4I_j$, from the Hölder inequality and the fact $\phi \in F_x^q(\Omega)$, with $\operatorname{supp} \phi \subset Q$, $x \in Q$ and $c_Q \subset \Omega$, it follows that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} a_j(y)\phi(y)\,dy \right| &\le \left[\int_Q |a_j(y)|^{q'}\,dy \right]^{\frac{1}{q'}} \|\phi\|_{L^q(\mathbb{R}^n)} \\ &\le \left[\frac{1}{|Q|} \int_{\mathbb{R}^n} |a_j(y)|^{q'}\,dy \right]^{\frac{1}{q'}} \le \|a_j\|_{L^\infty(\mathbb{R}^n)}, \end{aligned} \tag{2.18}$$

which implies that, for all $x \in 4I_j$,

$$M_z^{(q)} a_j(x) \leq \ell_{I_j}^{-n/p}$$

and hence

$$\int_{4I_j} [M_z^{(q)} a_j(x)]^p \, dx \lesssim 1. \tag{2.19}$$

For $x \notin 4I_j$ and $\phi \in F_x^q(\Omega)$, from the Hölder inequality and $\int_{\mathbb{R}^n} a_j(y)\, dy = 0$, it follows that

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} a_j(y)\phi(y)\, dy\right| &= \left|\int_{\mathbb{R}^n} a_j(y)[\phi(y) - \phi(c_{I_j})]\, dy\right| \\
&\lesssim \int_{I_j} |a_j(y)| \int_0^1 |\nabla\phi(ty + (1-t)c_{I_j})||y - c_{I_j}|\, dt\, dy \\
&\lesssim \ell_{I_j}^{1-n/p} \int_0^1 \int_{I_j} |\nabla\phi(ty + (1-t)c_{I_j})|\, dt\, dy \\
&\lesssim \ell_{I_j}^{1-n/p} \int_0^1 \ell_{I_j} \|\nabla\phi(t\cdot + (1-t)c_{I_j})\|_{L^q(\mathbb{R}^n)}\, dt \\
&\lesssim \ell_{I_j}^{1-n/p+n/q'} \int_0^1 \|\nabla\phi\|_{L^q(\mathbb{R}^n)} t^{-n/q}\, dt \\
&\lesssim \ell_{I_j}^{1-n/p+n/q'} (\ell_Q)^{-1-n/q'} \lesssim \ell_{I_j}^{n/p} \left(\frac{\ell_{I_j}}{|x - c_{I_j}|}\right)^{1+n/q'},
\end{aligned} \tag{2.20}$$

where c_{I_j} denotes the center of I_j. Taking the supremum over $\phi \in F_x^q(\Omega)$, we find that, for all $x \notin 4I_j$,

$$M_z^{(q)} a_j(x) \lesssim \frac{1}{|I_j|^{\frac{1}{p}}} \left(\frac{\ell_{I_j}}{|x - c_{I_j}|}\right)^{1+\frac{n}{q'}}$$

and hence

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus 4I_j} [M_z^{(q)} a_j(x)]^p \, dx &\lesssim \frac{1}{|I_j|} \int_{\mathbb{R}^n \setminus 4I_j} \left(\frac{\ell_{I_j}}{|x - c_{I_j}|}\right)^{(1+\frac{n}{q'})p} dx \\
&\lesssim \frac{1}{|I_j|} \int_{2\ell_{I_j}}^{\infty} r^{-(1+\frac{n}{q'})p+n-1} (\ell_{I_j})^{(1+\frac{n}{q'})p}\, dr \lesssim 1.
\end{aligned}$$

Again, the last inequality holds true since $(1 + \frac{n}{q'})p > n$. Thus, for all j, $\|M_z^{(q)} a_j\|_{L^p(\Omega)} \lesssim 1$, which completes the proof of Theorem 2.11. □

Theorem 2.12. *Let Ω be a strongly Lipschitz domain, $p \in (\frac{n}{n+1}, 1]$ and $q \in (\frac{p^*}{p^*-1}, \infty]$. Then $f \in H_r^p(\Omega)$ if and only if f is a bounded linear functional on $W_0^{1,q}(\Omega)$ and $M_r^{(q)} f \in L^p(\Omega)$. Moreover, $\|f\|_{H_r^p(\Omega)}$ is equivalent to $\|M_r^{(q)} f\|_{L^p(\Omega)}$ with the equivalent positive constants independent of f.*

Proof. Since $G_x(\Omega) \subset G_x^q(\Omega)$, it follows that $M_r f(x) \leq M_r^{(q)} f(x)$ for all $x \in \Omega$ and hence, from Theorem 1.2, it follows that $\|f\|_{H_r^p(\Omega)} \lesssim \|M_r^{(q)} f\|_{L^p(\Omega)}$.

Conversely, suppose $f \in H_r^p(\Omega)$. By Lemma 2.3, we can decompose f into

$$f = \sum_i \lambda_i a_i + \sum_j \mu_j b_j \quad \text{in} \quad \mathcal{D}'(\Omega),$$

where $\{a_i\}_i$ are $(p,\infty,\Omega)_a$-atoms and $\{b_j\}_j$ are $(p,\infty,\Omega)_b$-atoms, and$(\{\lambda_i\}_i\cup\{\mu_j\}_j)\subset\mathbb{C}$ satisfy

$$\sum_i|\lambda_i|^p+\sum_j|\mu_j|^p\lesssim\|f\|^p_{H^p_r(\Omega)}.$$

By the proof of Theorem 2.11, we know that $\|M_r^{(q)}a_i\|_{L^p(\mathbb{R}^n)}\leq\|M_z^{(q)}a_i\|_{L^p(\mathbb{R}^n)}\lesssim 1$ for each i. In order to finish the proof of Theorem 2.12, we still need to show $\|M_r^{(q)}b_j\|_{L^p(\mathbb{R}^n)}\lesssim 1$ for each j. Let the cube J_j be the support of b_j. For each $x\in 4J_j$ and $\phi\in G_x^q(\Omega)$, with supp $\phi\subset Q$, $x\in Q$ and $c_Q\in\Omega$, by the Hölder inequality and $\phi\in G^q_x(\Omega)$, we have

$$\left|\int_{\mathbb{R}^n}b_j(y)\phi(y)\,dy\right|\leq\left[\int_{\mathbb{R}^n}|b_j(y)|^{q'}\,dy\right]^{\frac{1}{q'}}\|\phi\|_{L^q(\mathbb{R}^n)}\leq\left[\frac{1}{|Q|}\int_{\mathbb{R}^n}|b_j(y)|^{q'}\,dy\right]^{\frac{1}{q'}}\leq\|b_j\|_{L^\infty(\mathbb{R}^n)},$$

which implies that, for each $x\in 4J_j$,

$$M_r^{(q)}b_j(x)\lesssim\ell_{J_j}^{-n/p}$$

and hence

$$\int_{4J_j}[M_r^{(q)}\,b_j(x)]^p\,dx\lesssim 1. \tag{2.21}$$

For $x\notin 4J_j$, let $\phi\in G_x^q(\Omega)$ and choose $x_0\in\partial\Omega\cap 4J_j$. Since $\phi(x_0)=0$, from $\int_{J_j}b_j(y)\,dy=0$, the Hölder inequality and an argument similar to that used in the proof of (2.20), we deduce that

$$\begin{aligned}\left|\int_{\mathbb{R}^n}b_j(y)\phi(y)\,dy\right|&\lesssim\left|\int_{J_j}b_j(y)[\phi(y)-\phi(x_0)]\,dy\right|\\&\lesssim\ell_{J_j}\int_{J_j}|b_j(y)|\int_0^1|\nabla\phi(ty+(1-t)c_{J_j})|\,dt\,dy\\&\lesssim\frac{1}{|J_j|^{\frac{1}{p}}}\left(\frac{\ell_{J_j}}{|x-c_{J_j}|}\right)^{1+\frac{n}{q'}}.\end{aligned}$$

By taking the supremum over $\phi\in G_x^q(\Omega)$, we conclude that, for each $x\in\mathbb{R}^n\setminus 4J_j$,

$$M_r^{(q)}b_j(x)\lesssim\frac{1}{|J_j|^{\frac{1}{p}}}\left(\frac{\ell_{J_j}}{|x-c_{J_j}|}\right)^{1+\frac{n}{q'}}$$

and hence

$$\begin{aligned}\int_{(4J_j)^{\complement}}[M_z^{(q)}b_j(x)]^p\,dx&\lesssim\frac{1}{|J_j|}\int_{(4J_j)^{\complement}}\left(\frac{\ell_{J_j}}{|x-c_{J_j}|}\right)^{(1+\frac{n}{q'})p}\,dx\\&\lesssim\frac{1}{|J_j|}\int_{\ell_{J_j}/2}^{\infty}r^{-(1+\frac{n}{q'})p+n-1}(\ell_{J_j})^{(1+\frac{n}{q'})p}\,dx\lesssim 1.\end{aligned}$$

This, combined with (2.21), implies that $\|M_r^{(q)}b_j\|_{L^p(\mathbb{R}^n)}\lesssim 1$.

Hence, we finally find that

$$\begin{aligned}\int_{\mathbb{R}^n}[M_r^{(q)}f(x)]^p\,dx&\leq\int_{\mathbb{R}^n}\sum_i|\lambda_i|^p[M_r^{(q)}a_i(x)]^p\,dx+\int_{\mathbb{R}^n}\sum_j|\mu_j|^p[M_r^{(q)}b_j(x)]^p\,dx\\&\lesssim\sum_i|\lambda_i|^p+\sum_j|\mu_j|^p\lesssim\|f\|^p_{H^p_r(\Omega)},\end{aligned}$$

which completes the proof of Theorem 2.12. □

Theorems 2.11 and 2.12 can be easily generalized to the vector-valued case. We first recall some notation.

For $q \in (\max\{\frac{p^*}{p^*-1}, 2n\}, \infty]$ and $x \in \mathbb{R}^n$, we define $\widetilde{\mathbf{F}}^q_x(\Omega)$ as the collection of all vectors $\Phi := (\Phi_1, \dots, \Phi_n)$ in $\mathcal{D}(\mathbb{R}^n, \mathbb{C}^n)$, for which there exists a cube Q such that supp $\Phi \subset Q$, $x \in Q$, $c_Q \in \Omega$ and

$$\|\Phi\|_{L^q(\mathbb{R}^n)} + \ell_Q \|D\Phi\|_{L^q(\mathbb{R}^n)} \le \frac{1}{|Q|^{\frac{1}{q'}}}.$$

Here and hereafter, for a vector function $\mathbf{u}$, we denote by $\mathbf{Du}$ its *gradient matrix*.

Moreover, for each $x \in \Omega$, let

$$\widetilde{\mathbf{G}}^q_x(\Omega) := \{\Phi \in \widetilde{\mathbf{F}}^q_x(\Omega) : \ \Phi = 0 \text{ on } \partial\Omega\}.$$

For each $\mathbf{F} \in \mathcal{D}'(\mathbb{R}^n; \mathbb{C}^n)$ and $x \in \mathbb{R}^n$, let

$$\widetilde{M}^{(q)}_z \mathbf{F}(x) := \sup_{\Phi \in \widetilde{\mathbf{F}}^q_x(\Omega)} |\langle \mathbf{F}, \Phi\rangle| = \sup_{\Phi \in \widetilde{\mathbf{F}}^q_x(\Omega)} \left| \sum_{i=1}^n \langle F_i, \Phi_i\rangle \right|$$

and, for each bounded linear functional $\mathbf{F}$ on $W^{1,q}_0(\mathbb{R}^n, C^n)$ and each $x \in \Omega$,

$$\widetilde{M}^{(q)}_r \mathbf{F}(x) := \sup_{\Phi \in \widetilde{\mathbf{G}}^q_x(\Omega)} |\langle \mathbf{F}, \Phi\chi_\Omega\rangle| = \sup_{\Phi \in \widetilde{\mathbf{G}}^q_x(\Omega)} \left| \sum_{i=1}^n \langle F_i, \Phi_i\chi_\Omega\rangle \right|.$$

In what follows, we write $\widetilde{M}^{(\infty)}_z \mathbf{F}$ and $\widetilde{M}^{\infty}_r \mathbf{F}$ simply as $\widetilde{M}_z \mathbf{F}$ and $\widetilde{M}_r \mathbf{F}$, respectively.

The following vector-valued versions follow directly from Theorems 2.11 and 2.12.

Corollary 2.13. *Let Ω be a strongly Lipschitz domain, $p \in (\frac{n}{n+1}, 1]$ and $q \in (\frac{p^*}{p^*-1}, \infty]$.*

(i) If Ω is bounded, then $\mathbf{F} \in H^p_z(\Omega)$ if and only if $\mathbf{F} \in \mathcal{D}'(\mathbb{R}^n; \mathbb{C}^n)$, supp $\mathbf{F} \subset \overline{\Omega}$, $\widetilde{M}^{(q)}_z \mathbf{F} \in L^p(\Omega)$ and $\langle \mathbf{F}, \Phi\rangle = 0$ for each $\Phi \in \mathcal{D}(\mathbb{R}^n; \mathbb{C}^n)$ with Φ being a constant vector on Ω. Moreover, $\|\mathbf{F}\|_{H^p_z(\Omega)}$ is equivalent to $\|\widetilde{M}^{(q)}_z \mathbf{F}\|_{L^p(\Omega)}$ with the equivalent positive constants independent of $\mathbf{F}$.

(ii) If Ω is unbounded, then $\mathbf{F} \in H^p_z(\Omega)$ if and only if $\mathbf{F} \in \mathcal{D}'(\mathbb{R}^n; \mathbb{C}^n)$, supp $\mathbf{F} \subset \overline{\Omega}$ and $\widetilde{M}^{(q)}_z \mathbf{F} \in L^p(\Omega)$. Moreover, $\|\mathbf{F}\|_{H^p_z(\Omega)}$ is equivalent to $\|\widetilde{M}^{(q)}_z \mathbf{F}\|_{L^p(\Omega)}$ with the equivalent positive constants independent of $\mathbf{F}$.

Corollary 2.14. *Let Ω be a strongly Lipschitz domain, $p \in (\frac{n}{n+1}, 1]$ and $q \in (\frac{p^*}{p^*-1}, \infty]$. Then $\mathbf{F} \in H^p_r(\Omega)$ if and only if $\mathbf{F} \in \mathcal{D}'(\Omega; \mathbb{C}^n)$ and $\widetilde{M}^{(q)}_r \mathbf{F} \in L^p(\Omega)$. Moreover, $\|\mathbf{F}\|_{H^p_r(\Omega)}$ is equivalent to $\|\widetilde{M}^{(q)}_r \mathbf{F}\|_{L^p(\Omega)}$ with the equivalent positive constants independent of* $\mathbf{F}$.

3 Hardy-Sobolev spaces $\dot{H}^{1,p}_r(\Omega)$ and $\dot{H}^{1,p}_z(\Omega)$

In this section, we study homogeneous Hardy-Sobolev spaces on domains. By the Poincaré inequality from [16, Proposition 3], we can improve the regularity of f as long as f lies in homogenous Hardy-Sobolev spaces as follows.

Proposition 3.1. *Let $p \in (\frac{n}{n+1}, 1]$ and p^* be the Sobolev conjugate index of p, namely, $p^* := \frac{np}{n-p}$. Then $\dot{H}^{1,p}_r(\Omega) \subset L^{p^*}_c(\Omega)$ and $\dot{H}^{1,p}_z(\Omega) \subset L^{p^*}_{\mathrm{loc}}(\mathbb{R}^n)$.*

Our main tool is to represent the norm of ∇f in $H^p_r(\Omega)$ or $H^p_z(\Omega)$ in terms of the L^p-norm of the appropriate maximal function, which can be considered as a generalization of [3, Theorem 6] to the case $p \in (\frac{n}{n+1}, 1)$.

In Lemma 2.4, we have shown that the pair $\langle f, \phi\rangle$ is well defined for $f \in H^p_r(\Omega)$ and $\phi \in W^{1,(p^*)'}_0(\Omega)$. Next, we prove that this extended definition is appropriate when we consider the Hardy-Sobolev space $\dot{H}^{1,p}_r(\Omega)$.

Lemma 3.2. *Let $p \in (\frac{n}{n+1}, 1]$ and $q \in (\frac{p^*}{p^*-1}, \infty]$. Then, for $f \in \dot{H}^{1,p}_r(\Omega)$, $\phi \in W^{1,q}_0(\Omega)$ and supp $\phi \subset Q$, where Q is a cube, it holds true that $\int_\Omega f(x)\partial_i\phi(x)\,dx = -\langle \partial_i f, \phi\rangle$ for each $i \in \{1, \dots, n\}$.*

Proof. For $\phi \in W_0^{1,q}(\Omega)$ and $\operatorname{supp}\phi \subset Q$, where Q is a cube, we have $\phi \in W_0^{1,(p^*)'}(\Omega \cap Q)$. Hence, for $g \in H_r^p(\Omega)$, $\langle g, \phi\rangle$ is well defined by Lemma 2.4.

Let us choose a sequence $\phi_k \in \mathcal{D}(\Omega \cap Q)$ such that ϕ_k converges to ϕ in $W_0^{1,q}(\Omega \cap Q)$ as $k \to \infty$. Then we know that ϕ_k converges to ϕ also in $W_0^{1,(p^*)'}(\Omega \cap Q)$ as $k \to \infty$. Hence, by Lemma 2.4, we obtain

$$\lim_{k\to\infty} \langle g, \phi_k\rangle = \langle g, \phi\rangle.$$

On the other hand, since, by Proposition 3.1, $f \in L_c^{p^*}(\Omega) \subset L_c^{q'}(\Omega)$, from this and the Hölder inequality, it follows that, for $i \in \{1, \dots, n\}$,

$$\begin{aligned}\left|\int_\Omega f(x)(\partial_i\phi - \partial_i\phi_k)(x)\,dx\right| &\le \|f\|_{L^{q'}(\Omega\cap Q)}\|\partial_i\phi - \partial_i\phi_k\|_{L^q(\Omega)}\\ &\le \|f\|_{L^{q'}(\Omega\cap Q)}\|\phi - \phi_k\|_{W^{1,q}(\Omega)} \to 0\end{aligned}$$

as $k \to \infty$. Finally, by $\partial_i f \in H_r^p(\Omega)$, we know that

$$\langle \partial_i f, \phi\rangle = \lim_{k\to\infty} \langle \partial_i f, \phi_k\rangle = -\lim_{k\to\infty}\int_\Omega f(x)\partial_i\phi_k(x)\,dx = -\int_\Omega f(x)\partial_i\phi(x)\,dx, \tag{3.1}$$

which completes the proof of Lemma 3.2. □

The following result on the divergence equation can be found, for instance, in [3, 5, 14].

Lemma 3.3. *Let U be a bounded Lipschitz domain of $\mathbb{R}^n$ and $p \in (1,\infty)$. If $f \in L^p(U)$ has zero integral, then there exists $\mathbf{F} \in W_0^{1,p}(U;\mathbb{C}^n)$ such that $f = \operatorname{div}\mathbf{F}$ and $\|D\mathbf{F}\|_{L^p(U)} \le C\|f\|_{L^p(U)}$, where the positive constant C depends only on p and the Lipschitz constant of U.*

Proof of Theorem 1.4. In the following, we choose q such that $q \in (\frac{p^*}{p^*-1}, \infty)$. For $\Phi \in \mathbf{G}_x(\Omega)$, it holds true that $\|\Phi\|_{L^\infty(\mathbb{R}^n)} + \ell_Q\|\operatorname{div}\Phi\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{|Q|}$. Since

$$\int_{Q\cap\Omega} \operatorname{div}\Phi(x)\,dx = \int_{\partial(Q\cap\Omega)} \Phi(x)\cdot\nu(x)\,d\sigma(x) = 0,$$

where ν denotes the outer normal of $\partial(Q\cap\Omega)$, from Lemma 3.3, it follows that there exists a $\widetilde{\Phi} \in W_0^{1,q}(Q \cap \Omega;\mathbb{C}^n)$ such that $\operatorname{div}\Phi = \operatorname{div}\widetilde{\Phi}$ and $\|D\widetilde{\Phi}\|_{L^q(Q\cap\Omega)} \lesssim \|\operatorname{div}\Phi\|_{L^q(Q\cap\Omega)}$. By the fact that $Q\cap\Omega$ is a Lipschitz domain, together with the Sobolev inequality, we conclude that

$$\left\|\widetilde{\Phi}\right\|_{L^q(Q\cap\Omega)} \lesssim \left\|D\widetilde{\Phi}\right\|_{L^{q_\star}(Q\cap\Omega)} \lesssim \left\|D\widetilde{\Phi}\right\|_{L^q(Q\cap\Omega)}\ell_Q,$$

where $q_\star = \frac{nq}{n+q}$. This implies that

$$\left\|\widetilde{\Phi}\right\|_{L^q(\Omega\cap Q)} + \ell_Q\left\|D\widetilde{\Phi}\right\|_{L^q(\Omega\cap Q)} \lesssim \ell_Q\left\|D\widetilde{\Phi}\right\|_{L^q(\Omega\cap Q)} \lesssim \ell_Q\|\operatorname{div}\Phi\|_{L^q(\Omega\cap Q)} \lesssim |Q|^{-\frac{1}{q'}}$$

and hence $\frac{\widetilde{\Phi}}{C} \in \widetilde{\mathbf{G}}_x^q(\Omega)$ for some harmless positive constant C. By applying Lemma 3.2, we obtain

$$\left|\int_\Omega f(x)\operatorname{div}\Phi(x)\,dx\right| = \left|\int_\Omega f(x)\operatorname{div}\widetilde{\Phi}(x)\,dx\right| = |\langle\nabla f, \widetilde{\Phi}\rangle|,$$

which implies that $\mathbf{N}_\Omega^{(1)}f(x) \lesssim \widetilde{M}_r^{(q)}(\nabla f)(x)$ for each $x \in \Omega$. From Corollary 2.14, it follows that

$$\|\mathbf{N}_\Omega^{(1)}f\|_{L^p(\Omega)} \lesssim \|\widetilde{M}_r^{(q)}(\nabla f)\|_{L^p(\Omega)} \sim \|\nabla f\|_{H_r^p(\Omega)}.$$

Conversely, let $f \in L_c^{p^\star}(\Omega)$ and $\mathbf{N}_\Omega^{(1)} f \in L^p(\Omega)$. Choose a $C_c^\infty(\mathbb{R}^n)$ function ϕ such that $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$ and $\operatorname{supp} \phi \subset B(0,1)$. For each $t \in (0,\infty)$, let $\phi_t(x) := t^{-n}\phi(x/t)$ for all $x \in \mathbb{R}^n$. For each $g \in \mathcal{D}'(\Omega)$, following [21, Section 1] (see also [16]), for each $x \in \Omega$, we define

$$g^+_{\phi,\Omega}(x) := \sup_{0<t<d(x,\partial\Omega)/2} |g \star \phi_t(x)|.$$

Since, for each $x \in \Omega$ and $0 < t < d(x,\partial\Omega)/2$, $\phi_t(x - \cdot) \in \mathcal{D}(\Omega)$, it follows that

$$\left| \int_\Omega f(y)\partial_i\phi_t(x-y)\,dy \right| = \left|(\partial_i f) \star \phi_t(x)\right|$$

for each $i \in \{1,\dots,n\}$. Moreover, by the choice of ϕ, we know that, for each $t \in (0, d(x,\partial\Omega)/2)$, the vector $\overbrace{(0,\dots,0,\ \phi_t, 0,\dots,0)}^{n}$ belongs to $\mathbf{G}_x(\Omega)$ up to a harmless positive constant. By taking supremum over t, we conclude that, for all $x \in \Omega$,

$$(\nabla f)^+_{\phi,\Omega}(x) \lesssim \mathbf{N}_\Omega^{(1)} f(x)$$

and hence $(\nabla f)^+_{\phi,\Omega} \in L^p(\Omega)$. This, together with [21, Theorem 4] (see also [16]), implies that $\nabla f \in H_r^p(\Omega)$ and $\|\nabla f\|_{H_r^p(\Omega)} \lesssim \|\mathbf{N}_\Omega^{(1)} f\|_{L^p(\Omega)}$. The proof of Theorem 1.4 is therefore completed. □

We next proceed to prove Theorem 1.5. The following conclusion follows from an application of Theorem 1.4.

Proposition 3.4. *Let $p \in (\frac{n}{n+1}, 1]$ and $p^\star = \frac{np}{n-p}$. Then $\nabla f \in H_z^p(\Omega)$ if and only if $f \in L_{\mathrm{loc}}^{p^\star}(\mathbb{R}^n)$, $\operatorname{supp} \nabla f \subset \overline{\Omega}$ and $\mathbf{M}_\Omega^{(1)} f \in L^p(\mathbb{R}^n)$. Moreover, $\|\nabla f\|_{H_z^p(\Omega)}$ is equivalent to $\|\mathbf{M}_\Omega^{(1)} f\|_{L^p(\mathbb{R}^n)}$ with the equivalent positive constants independent of f.*

Proof. Suppose first $\nabla f \in H_z^p(\Omega)$. It follows, from Proposition 3.1, that $f \in L_{\mathrm{loc}}^{p^\star}(\mathbb{R}^n)$. By applying Theorem 1.4 to the case $\Omega = \mathbb{R}^n$, we know that

$$\|\mathbf{N}_{\mathbb{R}^n}^{(1)} f\|_{L^p(\mathbb{R}^n)} \sim \|\nabla f\|_{H^p(\mathbb{R}^n)} \sim \|\nabla f\|_{H_z^p(\Omega)}.$$

Since $\mathbf{F}_x(\Omega) \subset \mathbf{G}_x(\mathbb{R}^n)$, it follows that, for each $x \in \mathbb{R}^n$,

$$\mathbf{M}_\Omega^{(1)} f(x) = \sup_{\Phi \in \mathbf{F}_x(\Omega)} \left| \int_{\mathbb{R}^n} f(y) \operatorname{div} \Phi(y)\,dy \right| \le \sup_{\Phi \in \mathbf{G}_x(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} f(y) \operatorname{div} \Phi(y)\,dy \right| \le \mathbf{N}_{\mathbb{R}^n}^{(1)} f(x).$$

This implies that

$$\|\mathbf{M}_\Omega^{(1)} f\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathbf{N}_{\mathbb{R}^n}^{(1)} f\|_{L^p(\mathbb{R}^n)} \sim \|\nabla f\|_{H_z^p(\Omega)}.$$

Conversely, for each $x \in \mathbb{R}^n$ and $\Phi \in \mathbf{G}_x(\mathbb{R}^n)$ with $\operatorname{supp} \Phi \subset Q$ and $x \in Q$, we just need to consider those Φ satisfying $Q \cap \Omega \neq \emptyset$, since $\operatorname{supp} \nabla f \subset \overline{\Omega}$ and

$$\langle f, \operatorname{div} \Phi\rangle = -\langle \nabla f, \Phi\rangle.$$

Notice that, in this case, we have $d(x,\Omega) \lesssim \ell_Q$ and hence we can find a cube $\widetilde{Q}$ such that $Q \subset \widetilde{Q}$, $c_{\widetilde{Q}} \in \Omega$ and $\ell_{\widetilde{Q}} \lesssim \ell_Q$. We then have $\widetilde{C}\Phi \in \mathbf{F}_x(\Omega)$ for some harmless positive constant $\widetilde{C}$, which implies that, for each $x \in \mathbb{R}^n$,

$$\mathbf{N}_{\mathbb{R}^n}^{(1)} f(x) = \sup_{\Phi \in \mathbf{G}_x(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} f(y) \operatorname{div} \Phi(y)\,dy \right| \sim \sup_{\Phi \in \mathbf{F}_x(\Omega)} \left| \int_{\mathbb{R}^n} f(y) \operatorname{div} \Phi(y)\,dy \right| \lesssim \mathbf{M}_\Omega^{(1)} f(x).$$

From this, we finally deduce that

$$\|\nabla f\|_{H_z^p(\Omega)} = \|\nabla f\|_{H^p(\mathbb{R}^n)} \sim \|\mathbf{N}_{\mathbb{R}^n}^{(1)} f\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathbf{M}_\Omega^{(1)} f\|_{L^p(\mathbb{R}^n)},$$

which completes the proof of Proposition 3.4. □

In view of Proposition 3.4, to prove Theorem 1.5, we need to replace $\|\mathbf{M}_{\Omega}^{(1)} f\|_{L^p(\mathbb{R}^n)}$ by a relatively small quantity $\|\mathbf{M}_{\Omega}^{(1)} f\|_{L^p(\Omega)}$, which we do next.

We first introduce an auxiliary maximal function. Recall that, for each $x \in \mathbb{R}^n$, the class $\widetilde{\mathbf{F}}_x^{\infty}(\Omega)$ consists of all vector-valued functions in $\mathcal{D}(\mathbb{R}^n;\mathbb{C}^n)$, which satisfy $\operatorname{supp} \Phi \subset Q$, $x \in Q$, $c_Q \in \Omega$ and $\|\Phi\|_{L^\infty(\mathbb{R}^n)} + \ell_Q \|D\Phi\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{|Q|}$.

For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we then let

$$\widetilde{\mathbf{M}}_{\Omega}^{(1)} f(x) := \sup_{\Phi \in \widetilde{\mathbf{F}}_x^{\infty}(\Omega)} \left| \int_{\mathbb{R}^n} f(y) \operatorname{div} \Phi(y)\, dy \right|.$$

Notice that, for each $f \in L^{p^*}_{\mathrm{loc}}(\mathbb{R}^n)$ with $\nabla f \in \mathcal{D}'(\mathbb{R}^n)$ and $\operatorname{supp} \nabla f \subset \overline{\Omega}$, it holds true that, for each $x \in \mathbb{R}^n$,

$$\widetilde{\mathbf{M}}_{\Omega}^{(1)} f(x) = \sup_{\Phi \in \widetilde{\mathbf{F}}_x^{\infty}(\Omega)} \left| \int_{\mathbb{R}^n} f(y) \operatorname{div} \Phi(y)\, dy \right| = \sup_{\Phi \in \widetilde{\mathbf{F}}_x^{\infty}(\Omega)} |\langle \nabla f, \Phi \rangle| = \widetilde{M}_z(\nabla f)(x). \tag{3.2}$$

With the new maximal function, we now finish the proof of Theorem 1.5.

Proof of Theorem 1.5. Let Ω be a bounded Lipschitz domain. From Propositions 3.4 and 3.1, and the definition of $H^p_z(\Omega)$, we deduce that, if $\nabla f \in H^p_z(\Omega)$, then $\mathbf{M}_{\Omega}^{(1)} f \in L^p(\mathbb{R}^n)$ and $\operatorname{supp} \nabla f \subset \overline{\Omega}$. Moreover, if Ω is bounded, then $\langle \nabla f, \Phi \rangle = 0$ for each $\Phi \in \mathcal{D}(\mathbb{R}^n;\mathbb{C}^n)$ with Φ being a constant vector on Ω. Hence, if $\nabla f \in H^p_z(\Omega)$, we have $\mathbf{M}_{\Omega}^{(1)} f \in L^p(\Omega)$ and, by Proposition 3.4 again,

$$\|\mathbf{M}_{\Omega}^{(1)} f\|_{L^p(\Omega)} \le \|\mathbf{M}_{\Omega}^{(1)} f\|_{L^p(\mathbb{R}^n)} \sim \|\nabla f\|_{H^p_z(\Omega)}. \tag{3.3}$$

Conversely, since, for every $\Phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{C}^n)$, it holds true that

$$\|\operatorname{div} \Phi\|_{L^\infty(\mathbb{R}^n)} \lesssim \|D\Phi\|_{L^\infty(\mathbb{R}^n)},$$

it follows that, for $\Phi \in \widetilde{\mathbf{F}}_x^{\infty}(\Omega)$, we have $\widetilde{C}\Phi \in \mathbf{F}_x(\Omega)$ for a harmless positive constant $\widetilde{C}$. This, combined with (3.2), implies that, for every $x \in \Omega$,

$$\widetilde{M}_z(\nabla f)(x) = \widetilde{\mathbf{M}}_{\Omega}^{(1)} f(x) \lesssim \mathbf{M}_{\Omega}^{(1)} f(x).$$

Hence, if $\mathbf{M}_{\Omega}^{(1)} f \in L^p(\Omega)$, then $\widetilde{M}_z(\nabla f) \in L^p(\Omega)$. It follows, from this and Corollary 2.13(ii), that, if Ω is unbounded, then $\nabla f \in H^p_z(\Omega)$.

It remains to consider the case Ω being bounded. Take r_B large enough such that $\Omega \subset B(\vec{0}_n, r_B)$. Since $f \in L^{p^*}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\operatorname{supp} \nabla f \subset \overline{\Omega}$, for each $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} \phi \subset \mathbb{R}^n \setminus \overline{\Omega}$, it follows that

$$\int_{\mathbb{R}^n} f(x) \partial_i \phi(x)\, dx = -\langle \partial_i f, \phi \rangle = 0.$$

By the arbitrariness of ϕ and the fact $[\mathbb{R}^n \setminus B(\vec{0}_n, r_B)] \subset [\mathbb{R}^n \setminus \Omega]$, we conclude that f is a constant $\widetilde{C}$ on $\mathbb{R}^n \setminus B(\vec{0}_n, r_B)$.

For each $\Phi \in \mathcal{D}(\mathbb{R}^n;\mathbb{C}^n)$ with Φ being a constant vector on Ω, we choose a $\widetilde{\Phi} \in \mathcal{D}(\mathbb{R}^n;\mathbb{C}^n)$ such that $\widetilde{\Phi}$ being a constant vector on $B(\vec{0}_n, r_B)$ and $\widetilde{\Phi} \equiv \Phi$ on Ω. The choice of $\widetilde{\Phi}$ and the fact $\operatorname{supp} \nabla f \subset \overline{\Omega}$ imply that

$$\langle \nabla f, \Phi \rangle = \langle \nabla f, \widetilde{\Phi} \rangle = \langle f, \operatorname{div} \widetilde{\Phi} \rangle = \langle f - \widetilde{C}, \operatorname{div} \widetilde{\Phi} \rangle = 0,$$

since $\operatorname{supp} (f - \widetilde{C}) \subset \overline{B(\vec{0}_n, r_B)}$ and $\operatorname{div} \widetilde{\Phi} \equiv 0$ on $\overline{B(\vec{0}_n, R)}$. This, together with $\widetilde{M}_z(\nabla f) \in L^p(\Omega)$ and Corollary 2.13 (i), implies that $\nabla f \in H^p_z(\Omega)$, which completes the proof of Theorem 1.5. □

4 The div-curl lemma

The characterizations of Hardy-Sobolev spaces have some immediate applications. In this section, we deal with the div-curl lemma, which generalizes [3, Theorem 19] to the case $p \in (\frac{n}{n+1}, 1]$.

Theorem 4.1. *Let Ω be a strongly Lipschitz domain and $\mathbf{e} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{C}^n)$ be a vector field with divergence zero such that $\mathbf{e} = \vec{0}_n$ on $\partial\Omega$. If $f \in \dot{H}^{1,p}_r(\Omega)$, then $\mathbf{e} \cdot \nabla f \in \mathcal{D}'(\mathbb{R}^n)$ in the sense that, for each $\phi \in \mathcal{D}(\mathbb{R}^n)$,*

$$\langle \mathbf{e} \cdot \nabla f, \phi \rangle := \langle \nabla f, \mathbf{e}\phi \rangle,$$

$\mathbf{e} \cdot \nabla f \in H^p_z(\Omega)$ and

$$\|\mathbf{e} \cdot \nabla f\|_{H^p_z(\Omega)} \le C\|\mathbf{e}\|_{L^\infty(\mathbb{R}^n)}\|\nabla f\|_{H^p_r(\Omega)},$$

where C is a positive constant independent of $\mathbf{e}$ and f.

Proof. For each $x \in \Omega$ and $\phi \in F_x(\Omega)$, since $\operatorname{div} \mathbf{e} = 0$, it follows that $\operatorname{div}(\mathbf{e}\phi) = \nabla\phi \cdot \mathbf{e}$ and hence

$$\|\mathbf{e} \cdot \phi\|_{L^\infty(\mathbb{R}^n)} \le \|\mathbf{e}\|_{L^\infty(\mathbb{R}^n)}\|\phi\|_{L^\infty(\mathbb{R}^n)}$$

and

$$\|\operatorname{div}(\mathbf{e}\phi)\|_{L^\infty(\mathbb{R}^n)} \le \|\mathbf{e}\|_{L^\infty(\mathbb{R}^n)}\|\nabla\phi\|_{L^\infty(\mathbb{R}^n)}.$$

Hence $\frac{\mathbf{e}\phi}{\|\mathbf{e}\|_{L^\infty(\mathbb{R}^n)}} \in \mathbf{F}_x(\Omega)$. By $\mathbf{e} = 0$ on $\partial\Omega$, we have $\frac{\mathbf{e}\phi}{\|\mathbf{e}\|_{L^\infty(\mathbb{R}^n)}} \in \mathbf{G}_x(\Omega)$, which implies that, for all $x \in \Omega$,

$$M_z(\mathbf{e} \cdot \nabla f)(x) \lesssim \|\mathbf{e}\|_{L^\infty(\mathbb{R}^n)}\mathbf{N}^{(1)}_\Omega f(x).$$

Hence, by Theorem 1.4, we know that $M_z(\mathbf{e} \cdot \nabla f) \in L^p(\Omega)$ and

$$\|M_z(\mathbf{e} \cdot \nabla f)\|_{L^p(\Omega)} \lesssim \|\mathbf{e}\|_{L^\infty(\mathbb{R}^n)}\|\mathbf{N}^{(1)}_\Omega f\|_{L^p(\Omega)} \sim \|\mathbf{e}\|_{L^\infty(\mathbb{R}^\infty)}\|\nabla f\|_{H^p_r(\Omega)}. \tag{4.1}$$

If Ω is bounded, then, for each $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi \equiv 1$ on Ω, we have $\phi\mathbf{e} \in W^{1,\infty}_0(\Omega, \mathbb{C}^n)$. By Lemma 3.2, we find that

$$\int_{\mathbb{R}^n} \mathbf{e}(x) \cdot \nabla f(x)\phi(x)\,dx = -\int_\Omega f(x)\operatorname{div}(\mathbf{e}\phi)(x)\,dx = 0.$$

From this, Theorem 1.1 and (4.1), it follows that $\mathbf{e} \cdot \nabla f \in H^p_z(\Omega)$ and

$$\|\mathbf{e} \cdot \nabla f\|_{H^p_z(\Omega)} \sim \|M_z(\mathbf{e} \cdot \nabla f)\|_{L^p(\Omega)} \lesssim \|\mathbf{e}\|_{L^\infty(\mathbb{R}^n)}\|\nabla f\|_{H^p_r(\Omega)},$$

which completes the proof of Theorem 4.1. □

Remark 4.2. In the case $p = 1$ of Theorem 4.1, we can relax the assumption on $\mathbf{e}$ as $\mathbf{e} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{C}^n)$ and $\mathbf{e} \cdot \nu = 0$ on $\partial\Omega$, where ν denotes the outer normal of Ω; see [3].

If we drop the condition $\mathbf{e} = 0$ on $\partial\Omega$ in Theorem 4.1, we then have the following conclusions. The proofs are similar to that of Theorem 4.1, the details being omitted.

Theorem 4.3. *If $\nabla f \in H^p_r(\Omega)$ and $\mathbf{e} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{C}^n)$ is a vector field with divergence zero in $\mathbb{R}^n$, then $\mathbf{e} \cdot \nabla f \in \mathcal{D}'(\Omega)$ in the sense that, for each $\phi \in \mathcal{D}(\Omega)$,*

$$\langle \mathbf{e} \cdot \nabla f, \phi \rangle := \langle \nabla f, \mathbf{e}\phi \rangle,$$

$\mathbf{e} \cdot \nabla f \in H^p_r(\Omega)$ and

$$\|\mathbf{e} \cdot \nabla f\|_{H^p_r(\Omega)} \le C\|\mathbf{e}\|_{L^\infty(\mathbb{R}^n)}\|\nabla f\|_{H^p_r(\Omega)},$$

where C is a positive constant independent of $\mathbf{e}$ and f.

Theorem 4.4. *If $f \in \dot{H}^{1,p}_z(\Omega)$ and $\mathbf{e} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{C}^n)$ is a vector field with divergence zero in $\mathbb{R}^n$, then $\mathbf{e} \cdot \nabla f \in H^p_z(\Omega)$ and*

$$\|\mathbf{e} \cdot \nabla f\|_{H^p_z(\Omega)} \le C\|\mathbf{e}\|_{L^\infty(\mathbb{R}^n)}\|\nabla f\|_{H^p_z(\Omega)},$$

where C is a positive constant independent of $\mathbf{e}$ and f.

5 Solvability of the divergence equation

In this section, we apply the previous results on Hardy and Hardy-Sobolev spaces to study the divergence equation.

Definition 5.1. Let $\Omega \subset \mathbb{R}^n$ be an open set and $p \in (\frac{n}{n+1}, 1]$. The *Hardy-Sobolev space* $H^{1,p}_{z,0}(\Omega)$ is then defined as

$$H^{1,p}_{z,0}(\Omega) := \left\{ f \in L^1_{\rm loc}(\mathbb{R}^n) : \ \operatorname{supp} f \subset \overline{\Omega},\ \nabla f \in H^p_z(\Omega) \right\}$$

equipped with the *quasi-norm*

$$\|f\|_{H^{1,p}_{z,0}(\Omega)} := \left[\|f\|^p_{L^p(\Omega)} + \|\nabla f\|^p_{H^p_z(\Omega)} \right]^{1/p}.$$

Remark 5.2. (i) In Definition 5.1, it is not a restriction to require $f \in L^1_{\rm loc}(\mathbb{R}^n)$, since Proposition 3.1 indicates that $\dot{H}^{1,p}_z(\Omega) \subset L^{p^*}_{\rm loc}(\mathbb{R}^n)$.

(ii) When $p = 1$, $H^{1,1}_{z,0}(\Omega)$ can also be defined as

$$H^{1,1}_{z,0}(\Omega) := \left\{ f \in L^1(\Omega) : \ \operatorname{Tr} f = 0 \text{ on } \partial\Omega \text{ and } \nabla f \in H^p_z(\Omega) \right\};$$

see [3]. We choose Definition 5.1 since, when $p \in (0, 1)$, this is more convenient to us.

A measurable vector function $\mathbf{u}$ is called a solution of the divergence equation $\operatorname{div} \mathbf{u} = f$ for some $f \in H^p_z(\Omega)$ with $p \in (\frac{n}{n+1}, 1]$ if, for every $\phi \in \mathcal{D}(\mathbb{R}^n)$, it holds true that

$$\int_\Omega \mathbf{u}(x) \cdot \nabla\phi(x)\, dx = -\langle f, \phi \rangle.$$

A vector-valued distribution $\mathbf{u} := (u_1, \dots, u_i, \dots, u_n)$ is in $[H^{1,p}_{z,0}(\Omega)]^n$ if each component of $\mathbf{u}$ is in $H^{1,p}_{z,0}(\Omega)$ and we let

$$\|\mathbf{u}\|_{[H^{1,p}_{z,0}(\Omega)]^n} := \sum_{i=1}^n \|u_i\|_{H^{1,p}_{z,0}(\Omega)}.$$

Lemma 5.3. *Let $p \in (\frac{n}{n+1}, 1]$ and Ω be a bounded Lipschitz domain. Then, for each $(p, 2, \Omega)_a$-atom a, with $\operatorname{supp} a \subset Q \subset \Omega$, there exists $\mathbf{u} \in [H^{1,p}_{z,0}(\Omega)]^n$ such that $\operatorname{div} \mathbf{u} = a$ and*

$$\|\mathbf{u}\|_{[H^{1,p}_{z,0}(\Omega)]^n} \le C(\max\{\ell_Q, 1\}),$$

where C is a positive constant independent of a.

Proof. From [5, Theorem 2] (see also [1, Theorem 4.1]), it follows that, for each $(p, 2, \Omega)_a$-atom a, there exists $\mathbf{u} \in W^{1,2}_0(Q)$ such that $\operatorname{div} \mathbf{u} = a$ and

$$\|D\mathbf{u}\|_{L^2(Q)} \lesssim \|a\|_{L^2(Q)}. \tag{5.1}$$

Write $\mathbf{u}$ as $(u_1, \dots, u_n)$. Next, we proceed to estimate the size of $\|u_i\|_{H^{1,p}_{z,0}(\Omega)}$ for each $i \in \{1, \dots, n\}$. By Theorem 1.5, we find that, for each $i \in \{1, \dots, n\}$, it holds true that

$$\|\nabla u_i\|_{H^p_z(\Omega)} \sim \|\mathbf{M}^{(1)}_\Omega(u_i)\|_{L^p(\Omega)}.$$

Since $u_i \in W^{1,2}_0(Q)$, from the definition of $\mathbf{M}^{(1)}_\Omega(u_i)$, we deduce that, for each $x \in \Omega$,

$$\mathbf{M}^{(1)}_\Omega(u_i)(x) \lesssim M_{HL}(\nabla u_i)(x).$$

This, together with the Hölder inequality, the boundedness on $L^2(\mathbb{R}^n)$ of M_{HL} and (5.1), further implies that

$$\int\limits_{2Q\cap\Omega} \left[\mathbf{M}_\Omega^{(1)}(u_i)(x)\right]^p\, dx \lesssim |Q|^{1-\frac{p}{2}} \|M_{HL}(\nabla u_i)\|_{L^2(2Q)}^p \lesssim 1. \tag{5.2}$$

For $x \in \Omega \setminus (2Q)$, let $\Phi \in \mathbf{F}_x(\Omega)$ be such that $x \in I$, $\operatorname{supp}\,\Phi \subset I$ and

$$\|\Phi\|_{L^\infty(\mathbb{R}^n)} + \ell_I \|\operatorname{div}\Phi\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{|I|}.$$

If $(\operatorname{supp}\,\Phi) \cap Q = \emptyset$, then $\int_{\mathbb{R}^n} u_i(y) \operatorname{div}\Phi(y)\, dy = 0$ for $i \in \{1, \dots, n\}$.

Suppose $(\operatorname{supp}\,\Phi)\cap Q \neq \emptyset$. Then there exists $y \in [(\operatorname{supp}\,\Phi)\cap Q] \subset [I\cap Q]$, which implies $\ell_Q \le |y-x| \le \sqrt{n}\ell_I$ and hence $|x - c_Q| \lesssim \ell_I$. Therefore, we can choose a cube $\widetilde{I}$ such that $c_{\widetilde{I}} \in Q$, $I \subset \widetilde{I}$ and $\ell_{\widetilde{I}} \lesssim \ell_I$. The choice of $\widetilde{I}$ implies that $\widetilde{C}\Phi \in \mathbf{F}_x(Q)$ for some positive constant $\widetilde{C}$ independent of Q and Ω and hence, for each $x \in \Omega \setminus (2Q)$,

$$\mathbf{M}_\Omega^{(1)}(u_i)(x) \lesssim \mathbf{M}_Q^{(1)}(u_i)(x).$$

Notice that it follows, from Theorem 1.5 and Proposition 3.4, that

$$\|\mathbf{M}_Q^{(1)}(u_i)\|_{L^p(Q)} \sim \|\mathbf{M}_Q^{(1)}(u_i)\|_{L^p(\mathbb{R}^n)}.$$

Hence, by this, the Hölder inequality, the boundedness on $L^2(\mathbb{R}^n)$ of M_{HL} and (5.1), we find that

$$\begin{aligned}\int\limits_{\Omega\setminus(2Q)} \left[\mathbf{M}_\Omega^{(1)}(u_i)(x)\right]^p\, dx &\lesssim \|\mathbf{M}_Q^{(1)}(u_i)\|_{L^p(Q)}^p \lesssim \|M_{HL}(\nabla u_i)\|_{L^p(Q)}^p \\ &\lesssim |Q|^{1-\frac{p}{2}} \|M_{HL}(\nabla u_i)\|_{L^2(2Q)}^p \lesssim 1.\end{aligned} \tag{5.3}$$

It follows, from (5.2), (5.3) and Theorem 1.5, that

$$\|\nabla u_i\|_{H_z^p(\Omega)} \sim \|\mathbf{M}_\Omega^{(1)}(u_i)\|_{L^p(\Omega)} \lesssim 1. \tag{5.4}$$

On the other hand, by the Hölder inequality, the Sobolev inequality and (5.1), we know that, for each $i \in \{1, \dots, n\}$,

$$\|u_i\|_{L^p(\Omega)} \le |Q|^{\frac{1}{p}-\frac{1}{2}} \|u_i\|_{L^2(Q)} \lesssim |Q|^{\frac{1}{p}-\frac{1}{2}} |Q|^{\frac{1}{n}} \|\nabla u_i\|_{L^2(Q)} \lesssim \ell_Q, \tag{5.5}$$

which, combined with (5.4), implies that $\|\mathbf{u}\|_{H_{z,0}^{1,p}(\Omega)} \lesssim \max\{\ell_Q, 1\}$ and hence completes the proof of Lemma 5.3. □

Theorem 5.4. *Let $p \in (\frac{n}{n+1}, 1]$ and Ω be a bounded Lipschitz domain. Then, for each $f \in H_z^p(\Omega)$, there exists $\mathbf{u} \in [H_{z,0}^{1,p}(\Omega)]^n$ such that $\operatorname{div}\mathbf{u} = f$. Moreover, there exists a positive constant C, independent of f, such that $\|\mathbf{u}\|_{[H_{z,0}^{1,p}(\Omega)]^n} \le C\|f\|_{H_z^p(\Omega)}$.*

Proof. By Theorem A.8, for each $f \in H_z^p(\Omega)$, we have its atomic decomposition as $f = \sum_{i=1}^\infty \lambda_i a_i$ in $\mathcal{D}'(\mathbb{R}^n)$, with $\{a_i\}_{i=1}^\infty$ being $(p, 2, \Omega)_a$-atoms and $\{\lambda_i\}_{i=1}^\infty \subset \mathbb{C}$ satisfying $\sum_{i=1}^\infty |\lambda_i|^p \lesssim \|f\|_{H_z^p(\Omega)}^p$.

By Lemma 5.3, we find that, for every $i \in \mathbb{N}$ and a_i with $\operatorname{supp}\, a_i \subset Q_i \subset \Omega$, there exists $\mathbf{u}_i \in [H_{z,0}^{1,p}(\Omega)]^n$ such that $\operatorname{div}\mathbf{u}_i = a_i$, $\operatorname{supp}\,\mathbf{u}_i \subset Q_i$ and $\|\mathbf{u}_i\|_{[H_{z,0}^{1,p}(\Omega)]^n} \lesssim \operatorname{diam}(\Omega)$.

By using the Sobolev-Poincaré inequality from [16], we know that, for each i,

$$\|u_i\|_{L^1(\Omega)} \le |Q_i|^{1-1/p^*} \|u_i\|_{L^{p^*}(Q_i)} \le |Q_i|^{1-1/p^*} \|\nabla u_i\|_{H_z^p(\Omega)} \lesssim C(\operatorname{diam}(\Omega)),$$

where $C(\operatorname{diam}(\Omega))$ is an absolute positive constant which only depends on the diameter of Ω. From

$$\sum_{i=1}^\infty |\lambda_i| \le \left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{1/p} \lesssim \|f\|_{H_z^p(\Omega)},$$

it follows that $\{\sum_{i=1}^{N}\lambda_i\mathbf{u}_i\}_{N\in\mathbb{N}}$ converges in $L^1(\Omega)$ and hence in $\mathcal{D}'(\Omega)$. Let $\mathbf{u} := \sum_{i=1}^{\infty}\lambda_i\mathbf{u}_i$ in $\mathcal{D}'(\Omega)$. By (5.5), we find that $\mathbf{u}_i \in L^p(\Omega)$ with $\|\mathbf{u}_i\|_{L^p(\Omega)} \le 1$ for each $i \in \mathbb{N}$. This, together with $\sum_{i=1}^{\infty}|\lambda_i|^p \lesssim \|f\|^p_{H^p_z(\Omega)}$, implies that $\mathbf{u} \in L^p(\Omega)$ and

$$\|\mathbf{u}\|_{L^p(\Omega)} \lesssim \|f\|_{H^p_z(\Omega)}. \tag{5.6}$$

From Theorem 1.5 and (5.4), we deduce that

$$\|D\mathbf{u}\|^p_{H^p_z(\Omega)} \sim \|\mathbf{M}^{(1)}_\Omega(\mathbf{u})\|^p_{L^p(\Omega)} \lesssim \sum_{i=1}^{\infty}|\lambda_i|^p\|\mathbf{M}^{(1)}_\Omega(\mathbf{u}_i)\|^p_{L^p(\Omega)} \lesssim \sum_{i=1}^{\infty}|\lambda_i|^p \lesssim \|f\|^p_{H^p_z(\Omega)}.$$

This, combined with (5.6) and the fact $\operatorname{supp}\mathbf{u} \subset \overline{\Omega}$, implies that $\mathbf{u} \in H^{1,p}_{z,0}(\Omega)$ and $\|\mathbf{u}\|_{H^{1,p}_{z,0}(\Omega)} \lesssim \|f\|_{H^p_z(\Omega)}$.

Finally, by $\mathbf{u} = \sum_{i=1}^{\infty}\lambda_i\mathbf{u}_i$ in $\mathcal{D}'(\Omega)$ and the divergence theorem, we further conclude that, for every $\phi \in \mathcal{D}(\mathbb{R}^n)$,

$$\int_\Omega \mathbf{u}(x)\cdot\nabla\phi(x)\,dx = \sum_{i=1}^{\infty}\lambda_i\int_\Omega \mathbf{u}_i(x)\cdot\nabla\phi(x)\,dx = -\sum_{i=1}^{\infty}\lambda_i\int_\Omega a_i(x)\phi(x)\,dx = -\langle f,\phi\rangle,$$

which means that $\mathbf{u}$ is a solution to the divergence equation $\operatorname{div}\mathbf{u} = f$. This finishes the proof of Theorem 5.4. □

A Appendix

In this appendix, we wish to provide a complete proof of the atomic decomposition of $H^p_z(\Omega)$, where $p \in (\frac{n}{n+1}, 1]$ and Ω is a bounded Lipschitz domain.

First, we recall the local version of the space $H^p_z(\Omega)$; see, for instance, [8, 19, 20].

Definition A.1. Let $p \in (\frac{n}{n+1}, 1]$ and Ω be a bounded Lipschitz domain. A distribution f is said to be in the space $h^p_z(\Omega)$ if it is a Schwarz distribution in $h^p(\mathbb{R}^n)$ such that $\operatorname{supp} f \subset \overline{\Omega}$, namely,

$$h^p_z(\Omega) = \{f \in h^p(\mathbb{R}^n) : \ \operatorname{supp} f \subset \overline{\Omega}\}.$$

Let us recall the notion of the atoms for the local Hardy space $h^p_z(\Omega)$.

Definition A.2. Let $p \in (\frac{n}{n+1}, 1]$ and Ω be a bounded Lipschitz domain. Let $\delta \in (0,\infty)$ be fixed. A measurable function a, with $\operatorname{supp} a \subset Q \subset \Omega$, is called an *$h^p_z(\Omega)$-atom* if it satisfies

(i) $\|a\|_{L^2(Q)} \le |Q|^{\frac{1}{2}-\frac{1}{p}}$;

(ii) $\ell_Q \le \delta$ and $\int_Q a(x)\,dx = 0$, or $\ell_Q > \delta$.

By [8], we know that each element in $h^p_z(\Omega)$ has an atomic decomposition as follows.

Theorem A.3. *Let $p \in (\frac{n}{n+1}, 1]$ and Ω be a bounded Lipschitz domain. Then, for each $f \in h^p_z(\Omega)$, there exist $h^p_z(\Omega)$-atoms $\{a_i\}_i$ and $\{\lambda_i\}_i \subset \mathbb{C}$ such that $f = \sum_i \lambda_i a_i$ in $\mathcal{D}'(\mathbb{R}^n)$ and $\sum_i|\lambda_i|^p \le C\|f\|^p_{h^p_z(\Omega)}$, where C is a positive constant independent of f.*

Next, we introduce a notion to be used latter on.

Definition A.4. Let $p \in (\frac{n}{n+1}, 1]$ and Ω be a bounded Lipschitz domain. A bounded and measurable function a support on $Q \subset \Omega$ is called a *pseudo $H^p_z(\Omega)$* atom if $\|a\|_{L^2(Q)} \le |Q|^{\frac{1}{2}-\frac{1}{p}}$.

In order to obtain atomic decompositions for $H^p_z(\Omega)$, we recall two technical lemmas. The following definition on emanating conditions was introduced in [9, Definition 3.5].

Definition A.5. Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain and let $\sigma_1, \sigma_2 \in [1, \infty)$. Then Ω is said to satisfy the *emanating chain condition* with constants σ_1 and σ_2 if there exists a covering $\mathcal{W} := \{W_i : i \in \mathbb{Z}_+\}$ of Ω consisting of open cubes (or balls) such that

(C1) $\sigma_1 W \subset \Omega$ for all $W \in \mathcal{W}$ and $\sum_{W\in\mathcal{W}} \chi_{\sigma_1 W} \leq \sigma_2 \chi_\Omega$ on $\mathbb{R}^n$.

(C2) For every $i \in \mathbb{Z}_+$ and $W_i \in \mathcal{W}$, there exist $\{W_{i,0}, W_{i,1}, \ldots, W_{i,m_i}\} \subset \mathcal{W}$ such that $W_{i,0} = W_i$, $W_{i,m_i} = W_0$ and $W_{i,k_1} \subset \sigma_2 W_{i,k_2}$ for $0 \leq k_1 \leq k_2 \leq m_i$. Moreover, $W_{i,k} \cap W_{i,k+1}$, $0 \leq k < m_i$, contains a ball $B_{i,k}$ such that $(W_{i,k} \cup W_{i,k+1}) \subset \sigma_2 B_{i,k}$. The chain $\{W_{i,0}, \ldots, W_{i,m_i}\}$ is called the *chain emanating* from W_i. The number $m_i \in \mathbb{Z}_+$ is called the *length* of this chain.

(C3) The set $\{i \in \mathbb{Z}_+ : W_i \cap K \neq \emptyset\}$ is finite for every compact subset $K \subset \Omega$.

The next lemma replies on [9, Theorem 3.8] and the fact that a bounded Lipschitz domain is a John domain.

Lemma A.6. *Let Ω be a bounded Lipschitz domain and $\epsilon \in (0, \infty)$. Then there exists a fixed point $x_0 \in \Omega$ such that, for each x in $\Omega^\epsilon := \{y \in \Omega : d(y, \Omega^\complement) > \epsilon\}$, there exists a finite chain of cubes, $\{I_1, \ldots, I_m\}$, with $m \in \mathbb{N}$, satisfying $x \in I_1$ and $c_{I_m} = x_0$, and another finite chain of cubes, $\{J_1, \ldots, J_{m-1}\}$, satisfying $J_k \subset (I_{k+1} \cap I_k)$ and $\ell_{J_k} \sim \ell_{I_k}$. Moreover, the quantity $\ell_x := \min\{\ell_{I_1}, \ldots, \ell_{I_m}\}$ is bounded from below by a positive constant l independent of x, and m is bounded from above by a positive constant M independent of x.*

Proof. Since a bounded Lipschitz domain is a John domain, from [9, Theorem 3.8], we deduce that Ω satisfies an emanating chain condition with constants $\sigma_1, \sigma_2 \in [1, \infty)$. Let $x_0 \in \Omega$ be fixed. Since Ω is bounded and $\overline{\Omega}^\epsilon$ compact, from (C3), it follows that the set $\{i \in \mathbb{Z}_+ : W_i \cap \overline{\Omega}^\epsilon \neq \emptyset\}$ is finite. Thus,

$$\min\{\ell_{W_i} : W_i \cap \overline{\Omega}^\epsilon \neq \emptyset\} = \alpha > 0.$$

For each $x \in \Omega^\epsilon$, there exists a W_i such that $x \in W_i$, $W_i \cap \overline{\Omega}^\epsilon \neq \emptyset$ and $\ell_{W_i} \geq \alpha$. From (C2), it follows that there exists a chain $\{W_{i,1}, \ldots, W_{i,m_i}\}$ such that $c_{W_{i,m_i}} = c_{W_0} = x_0$ and $W_{i,1} = W_i$. Since it follows from (C2) that $W_{i,k_1} \subset \sigma_2 W_{i,k_2}$ for each $1 \leq k_1 \leq k_2 \leq m_i$, we have $\min_{0\leq k\leq m_i}\{\ell_{W_{i,k}}\} = \beta \in (0, \infty)$, which is independent of x. Let $m := m_k$ and, for each $k \in \{1, \ldots, m\}$, let $I_k := W_{i,k}$. Since, for each $k \in \{1, \ldots, m-1\}$, there exists a ball $B_{i,k}$ such that $B_{i,k} \subset (W_{i,k} \cap W_{i,k+1})$ and $(W_{i,k} \cup W_{i,k+1}) \subset \sigma_2 B_{i,k}$, it follows that there exists a cube $J_k \subset (W_{i,k} \cap W_{i,k+1})$ such that $\ell_{J_k} \sim \ell_{W_{i,k}}$. Thus, $\ell_x \gtrsim \beta$, where β is independent of x.

Moreover, by (C1), we know $\sum_{W\in\mathcal{W}} \chi_{\sigma_1 W} \leq \sigma_2 \chi_\Omega$ on $\mathbb{R}^n$. Thus, for any positive constant c, the set $\{W \in \mathcal{W} : \ell_W \geq c\}$ is finite, which implies that m is bounded from above by a positive constant M independent of x. The proof of Lemma A.6 is then completed. □

Lemma A.7. *Let $p \in (\frac{n}{n+1}, 1]$, $\delta \in (0, \infty)$ and Ω be a bounded Lipschitz domain. Then there exists a cube $W \subset\subset \Omega$, which means $\overline{W} \subset \Omega$, such that, for each pseudo $H^p_z(\Omega)$ atom a with $\operatorname{supp} a \subset Q$ and $\ell_Q > \delta$, $a = C(\sum_{i=0}^{m-1} h_i + g)$, where $\{h_i\}_{i=0}^{m-1}$ are $(p, 2, \Omega)_a$-atoms, g is a pseudo $H^p_z(\Omega)$ atom satisfying $\operatorname{supp} g \subset W$, C and M are positive constants independent of a, but depending on δ, such that $m \leq M$.*

Proof. Let $\delta \in (0, \infty)$. For each pseudo $H^p_z(\Omega)$ atom a with $\operatorname{supp} a \subset Q \subset \Omega$ and $l_Q > \delta$, it follows that there exists $\epsilon \in (0, \infty)$ such that $c_Q \in \Omega^\epsilon := \{y \in \Omega : d(y, \Omega^\complement) > \epsilon\}$. By Lemma A.6, we know that there exists a fixed point $x_0 \in \Omega$ such that, for each $x \in \Omega^\epsilon$, there exist two finite chains $\{I_1, \ldots, I_m\}$ and $\{J_1, \ldots, J_{m-1}\}$ with $m \in \mathbb{N}$, satisfying the conclusions of Lemma A.6.

Let $W := I_m$. Since $c_Q \in I_1$ and $\ell_{I_1} \sim \operatorname{diam}(\Omega) \sim \ell_Q$, it follows that there exists a cube J_0 such that $J_0 \subset (I_1 \cap Q)$ and $\ell_{J_0} \gtrsim \ell_Q$. We then have

$$\operatorname{supp}\left(a - \frac{\int_{\mathbb{R}^n} a(y)\,dy}{|J_0|}\chi_{J_0}\right) \subset I_1 \subset \Omega,$$

$$\left\|a - \frac{\int_{\mathbb{R}^n} a(y)\,dy}{|J_0|}\chi_{J_0}\right\|_{L^2(\mathbb{R}^n)} \leq \|a\|_{L^2(\mathbb{R}^n)} + \|a\|_{L^2(\mathbb{R}^n)} \frac{|Q|^{\frac12}}{|J_0|}|J_0|^{\frac12} \lesssim |I_1|^{\frac12-\frac1p}$$

and

$$\int_{\mathbb{R}^n} \left[a(x) - \frac{\int_{\mathbb{R}^n} a(y)\,dy}{|J_0|} \chi_{J_0}(x) \right] dx = 0.$$

Thus, $a - \frac{\int_{\mathbb{R}^n} a(y)\,dy}{|J_0|}\chi_{J_0} =: \widetilde{C}h_0$, where h_0 is a $(p, 2, \Omega)_a$-atom and $\widetilde{C}$ a positive constant independent of the atom a.

Similarly, for each $i \in \{1, \dots, m-1\}$, we choose a cube J_i such that $J_i \subset (I_{i+1} \cap I_i)$ and $\ell_{J_i} \gtrsim \ell_{I_i}$. By this, we have

$$\operatorname{supp}\left(\frac{\int_{\mathbb{R}^n} a(y)\,dy}{|J_{i-1}|}\chi_{J_{i-1}} - \frac{\int_{\mathbb{R}^n} a(y)\,dy}{|J_i|}\chi_{J_i} \right) \subset I_i \subset \Omega,$$

$$\left\| \frac{\int_{\mathbb{R}^n} a(y)\,dy}{|J_{i-1}|}\chi_{J_{i-1}} - \frac{\int_{\mathbb{R}^n} a(y)\,dy}{|J_i|}\chi_{J_i} \right\|_{L^2(\mathbb{R}^n)} \le \frac{\|a\|_{L^2(\mathbb{R}^n)}|Q|^{\frac12}}{|J_{i-1}|}|J_{i-1}|^{\frac12} + \frac{\|a\|_{L^2(\mathbb{R}^n)}|Q|^{\frac12}}{|J_i|}|J_i|^{\frac12} \lesssim |I_i|^{\frac12-\frac1p}$$

and

$$\int_{\mathbb{R}^n} \left[\frac{\int_{\mathbb{R}^n} a(y)\,dy}{|J_{i-1}|}\chi_{J_{i-1}}(x) - \frac{\int_{\mathbb{R}^n} a(y)\,dy}{|J_i|}\chi_{J_i}(x) \right] dx = 0.$$

Thus,

$$\frac{\int_{\mathbb{R}^n} a(y)\,dy}{|J_{i-1}|}\chi_{J_{i-1}} - \frac{\int_{\mathbb{R}^n} a(y)\,dy}{|J_i|}\chi_{J_i} =: \widetilde{C}h_i,$$

where h_i is a $(2, p, \Omega)_a$–atom and $\widetilde{C}$ a positive constant independent of a. Finally, by choosing a suitable constant $\widetilde{C}$ and defining $g := \frac{\int_{\mathbb{R}^n} a(y)\,dy}{\widetilde{C}|J_{m-1}|}\chi_{J_{m-1}}$, we find that $\|g\|_{L^2(\mathbb{R}^n)} \le |W|^{\frac12-\frac1p}$, which completes the proof of Lemma A.7. ☐

The main result of this appendix is to prove the following global version of the above atomic decomposition.

Theorem A.8. *Let $p \in (\frac{n}{n+1}, 1]$ and Ω be a bounded Lipschitz domain. Then, for each $f \in H^p_z(\Omega)$, there exist $(p, 2, \Omega)_a$-atoms $\{a_i\}_i$ and $\{\lambda_i\}_i \subset \mathbb{C}$ such that $f = \sum_i \lambda_i a_i$ in $\mathcal{D}'(\mathbb{R}^n)$ and $\sum_i |\lambda_i|^p \le C\|f\|^p_{H^p_z(\Omega)}$, where the positive constant C is independent of f.*

Proof. Since $H^p_z(\Omega) \subset h^p_z(\Omega)$, from Theorem A.3, it follows that $f = \sum_{i=1}^\infty \lambda_i a_i + \sum_{j=1}^\infty \mu_j b_j$ in $\mathcal{D}'(\mathbb{R}^n)$, where $\{a_i\}_{i=1}^\infty$ and $\{b_j\}_{j=1}^\infty$ are $h^p_z(\Omega)$ atoms, with a_i satisfying $\int_{\mathbb{R}^n} a_i(x)\,dx = 0$ for all $i \in \mathbb{N}$ and $\operatorname{supp} b_j \subset Q_j$ with $\ell_{Q_j} > \delta$ for all $j \in \mathbb{N}$. Moreover, we have

$$\sum_{i=1}^\infty |\lambda_i|^p + \sum_{j=1}^\infty |\mu_j|^p \lesssim \|f\|^p_{h^p_z(\Omega)} \lesssim \|f\|^p_{H^p_z(\Omega)}.$$

Thus, by Lemma A.7, we conclude that

$$\begin{aligned} f &= \sum_{i=1}^\infty \lambda_i a_i + \sum_{j=1}^\infty \mu_j b_j \\ &\sim \sum_{i=1}^\infty \lambda_i a_i + \sum_{j=1}^\infty \mu_j (h_{j,0} + h_{j,1} + \cdots + h_{j,m_j-1} + g) \\ &\sim \sum_{i=1}^\infty \lambda_i a_i + \sum_{j=1}^\infty \mu_j (h_{j,0} + h_{j,1} + \cdots + h_{j,m_j-1}) + \sum_{j=1}^\infty \mu_j g_j, \end{aligned}$$

where $\{a_i\}_{i=1}^\infty$ and $\{h_{i,j}\}_{i\in\mathbb{N}, j\in\{0,\dots,m_j-1\}}$ are all $(p, 2, \Omega)_a$-atoms, and $m_j \le M$ for all $j \in \mathbb{N}$ is uniformly bounded. Therefore, we have

$$\sum_{i=1}^\infty |\lambda_i|^p + \sum_{j=1}^\infty m_j|\mu_j|^p \lesssim \|f\|^p_{h^p_z(\Omega)} \lesssim \|f\|^p_{H^p_z(\Omega)}.$$

Observing that $\left(\sum_{i=1}^{\infty} |\lambda_i|\right)^p \le \sum_{i=1}^{\infty} |\lambda_i|^p$ for all $p \in (\frac{n}{n+1}, 1]$, to complete the proof of Theorem A.8, we only need to show that $g := \frac{\sum_{j=1}^{\infty} \mu_j g_j}{\sum_{j=1}^{\infty} |\mu_j|}$ is a $(p, 2, \Omega)_a$-atom.

By Lemma A.7, we have supp $g \subset W \subset \Omega$, where $\mathcal{W}$ is as in Lemma A.7. Since $f \in H_z^p(\Omega)$, it follows that, for any $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi \equiv 1$ on Ω,

$$0 = \int_{\mathbb{R}^n} f(x)\phi(x)\,dx = \sum_{i=1}^{\infty} \int_{\Omega} \lambda_i a_i(x)\,dx + \sum_{j=1}^{\infty} \sum_{k=0}^{m_j-1} \int_{\Omega} C_j \mu_j h_{j,k}(x)\,dx + \int_{\Omega} g(x)\,dx,$$

where C_j is a positive constant depending only on $j \in \mathbb{N}$, and hence $\int_\Omega g(x)\,dx = 0$. By the fact

$$\left\| \sum_{j=1}^{\infty} \mu_j g_j \right\|_{L^2(\mathbb{R}^n)} \le \sum_{j=1}^{\infty} \left\|\mu_j g_j\right\|_{L^2(\mathbb{R}^n)} = \sum_{j=1}^{\infty} |\mu_j| \left\|g_j\right\|_{L^2(\mathbb{R}^n)} \lesssim |W|^{\frac{1}{2}-\frac{1}{p}} \sum_{j=1}^{\infty} |\mu_j|,$$

we know that $\frac{\sum_{j=1}^{\infty} \mu_j g_j}{\sum_{j=1}^{\infty} |\mu_j|}$ is a $(p, 2, \Omega)_a$-atom up to some harmless positive constant. The proof of Theorem A.8 is therefore completed. □

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