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Applications of Integral Geometry to Geometric Properties of Sets in the 3D-Heisenberg Group

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Abstract: By studying the group of rigid motions, $PSH(1)$, in the 3D-Heisenberg group H_1, we define a density and a measure in the set of horizontal lines. We show that the volume of a convex domain $D \subset H_1$ is equal to the integral of the length of chords of all horizontal lines intersecting D. As in classical integral geometry, we also define the kinematic density for $PSH(1)$ and show that the measure of all segments with length ℓ intersecting a convex domain $D \subset H_1$ can be represented by the p-area of the boundary ∂D, the volume of D, and 2ℓ. Both results show the relationship between geometric probability and the natural geometric quantity in [10] derived by using variational methods. The probability that a line segment be contained in a convex domain is obtained as an application of our results.

Keywords: Heisenberg Group; kinematic formula; Integral Geometry; CR-manifolds; method of moving frames

MSC: 020, 030

1 Introduction

We adapt the methods of integral geometry to study the 3D-Heisenberg group, which is a non-compact CR manifold with zero Tanaka-Webster torsion and zero Tanaka-Webster curvature. After introducing a brief account of integral geometry and CR geometry, we will explain how our formulas connect both fields.

The roots of integral geometry date back to geometric probability and the study of invariant measures by integration techniques, which consider the probability of random geometric objects interacting with each other under a group of transformations as, for example, Buffon's needle problem and Bertrand's paradox. In the late nineteenth and early twentieth century, a variety of problems in geometric probability arose and led to systematic studies in this field. Works of Crofton, Poincaré, Sylvester and others built up the foundation of integral geometry. A series of articles related to the developments of geometric probability in this period was elaborated by Maran [21, 22], Little [19], and Baddeley [1]. When the concept of invariant measure became clear, Wilhelm Blaschke [3] and his school initiated integral geometry. Santaló's book [26] has been one of the most important monographs on the subject; Howard's book [17] deals with the case of Riemannian geometry; Zhou [34, 35] derived several integral formulas for submanifolds in Riemannian homogeneous spaces; The book of Schneider and Weil [27] included the fundamental knowledge of integral geometry and recent development of integral and stochastic geometry. Due to page restriction, we refer to the surveys [32] [33].

CR geometry initially studied the geometry of the boundary of a smooth strictly pseudo-convex domain in $\mathbb{C}^n$, and then evolved to the study of abstract CR manifolds. The foundational work was produced by Chern-Moser [9] in 1974, and closely connected work was given by Webster [29] and Tanaka [28] independently; they

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introduced the pseudohermitian geometry, where a connection is given associated to a choice of a contact form and curvature invariants. We point out that the relationship between CR geometry and pseudohermitian geometry have the strong analogy to that between conformal geometry and Riemannian geometry [2]. Introductory surveys, emphasizing recent development of three dimensional pseudohermitian geometry, are [30, 31].

Next we give the background of our studying target. For more details, we refer to [10, Appendix], [24], our previous work [15], and also to [11, 30, 31]. Some additional works with a sub-Riemannian approach are, for example, [4–6, 16, 18, 24, 25].

The 3D-Heisenberg group H_1 is the Euclidean space $\mathbb{R}^3$, as a set, with the group multiplication (left-invariant translation)

$$L_{(a,b,c)} \circ (x, y, t) = (a + x, b + y, c + t + bx - ay).$$

H_1 is also a 3-dimensional Lie group. Any left-invariant vector field at the point (x, y, t) is a linear combination of the following standard vector fields:

$$\mathring{e}_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, \quad \mathring{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial t}, \tag{1.1}$$

$$T = \frac{\partial}{\partial t}, \tag{1.2}$$

where T is called the Reeb vector field. The standard contact structure $\xi = span\{\mathring{e}_1, \mathring{e}_2\}$ on H_1 is a subbundle of the tangent bundle TH; equivalently, we can define ξ to be the kernel of the standard contact form

$$\Theta = dt + xdy - ydx.$$

Note that $\Theta(T) = 1$ and $d\Theta(\cdot, T) = 0$. The standard CR structure on H_1 is the almost complex structure J defined on ξ such that

$$J^2 = -I, J(\mathring{e}_1) = \mathring{e}_2, \ J(\mathring{e}_2) = -\mathring{e}_1.$$

On ξ there exists a natural metric

$$L_\Theta(X, Y) = \frac{1}{2}d\Theta(X, JY) = \frac{1}{2}\left((dx)^2 + (dy)^2\right) := \langle X, X\rangle,$$

called the *Levi metric* and the associated length is $|X| = \sqrt{\langle X, X\rangle}$.

A rigid motion (called a *pseudohermitian transformation*) in H_1 is a diffeomorphism Φ defined on H_1 preserving the CR structure J and the contact form Θ, namely,

$$\Phi_\star J = J\Phi_\star \text{ on the contact plane } \xi, \qquad \Phi^\star\Theta = \Theta \text{ in } H_1.$$

Denote the group of pseudohermitian transformations by $PSH(1)$. Similarly to the group of rigid motions in $\mathbb{R}^3$, in the previous work [15] we showed that any pseudohermitian transformation $\Phi_{Q,\alpha} \in PSH(1)$ can be represented by a left-invariant translation L_Q for $Q = (a, b, c) \in \mathbb{R}^3$ and a rotation $R_\alpha \in SO(2)$. Actually there exists the following one-to-one correspondence between the group actions and the matrix multiplications

$$\Phi_{Q,\alpha}(x, y, t) := L_{(a,b,c)} \circ R_\alpha(x, y, t) \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & \cos\alpha & -\sin\alpha & 0 \\ b & \sin\alpha & \cos\alpha & 0 \\ c & b & -a & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \\ t \end{pmatrix}. \tag{1.3}$$

Notice that understanding of the structure of $PSH(1)$ plays an important role when using the method of moving frames (see (2.4)).

Let Σ be a smooth hypersurface in H_1. Recall that a point $p \in \Sigma$ is called *singular* if the contact plane conincides with the tangent plane at p, namely, $\xi_p = T_p\Sigma$; otherwise the point p is called *regular*. Let S_Σ denote the set of singular points. It is easy to see that S_Σ is a closed set. At each regular point p, there exists a vector $e_1 \in \xi_p \cap T_p\Sigma$, unique up to a sign, that defines a one-dimensional foliation consisting of integral curves of e_1, called *characteristic curves*. The vector $e_2 = Je_1$ perpendicular to e_1, in the sense of Levi metric, is called the *Legendrian normal* or the *Gauss map* [10].

Let $D \subset H_1$ be a smooth domain with boundary $\partial D = \Sigma$ and $(\omega^1, \omega^2, \Theta)$ be the dual basis of (e_1, e_2, T). Cheng-Hwang-Malchiodi-Yang [10] studied the minimal surface in H_1 via a variational approach and defined the volume and the p-area respectively by

$$V(D) = \frac{1}{2} \int_D \Theta \wedge d\Theta, \tag{1.4}$$

$$p\text{-area}(\Sigma) = \int_\Sigma \Theta \wedge w^1. \tag{1.5}$$

We point out that $\frac{1}{2}$ is a normalization constant and this volume is just the usual Euclidean volume. While the p-area comes from a variation of the surface Σ in the normal direction fe_2 for some suitable function with compact support on the regular points of Σ. Note that we can continuously extend $\Theta \wedge \omega^1$ over the singular set S_Σ in such a way that it vanishes on S_Σ. Thus the p-area is globally defined on Σ. In the same paper, the authors also defined the p-mean curvature and the associated p-minimal surfaces; Malchiodi [20] summarized other two equivalent definitions for the p-mean curvature. Moreover, similar notions of volume and area were also studied in [5, 16, 24, 25], when considering a C^1-surface Σ enclosing a bounded set D; in this case the area of Σ coincides with the H_1-perimeter of of D. The authors adapt the usual normal vector perpendicular to the surface, while we consider the Legendrian normal, later used on the study of umbilic hypersurfaces in the higher dimensional Heisenberg groups H_n, see [12].

A *horizontal line* in H_1 is a line in $\mathbb{R}^3$ such that its velocity vector is always tangent to the contact plane. In the next section, we shall show that every horizontal line L can be uniquely determined by three parameters $(p, \theta, t) \in \mathbb{R} \times [0, 2\pi) \times \mathbb{R}$ (equivalently, by the base point $B \in L$). We frequently use the notation $L_{p,\theta,t}$ to emphasize the parameters. We make use of the convention that all lines considered here are oriented. The integral formulas for the non-oriented lines will be one-half of our cases. Following classical integral geometry [8, 26], we show, see (2.5), that the three-form $dp \wedge d\theta \wedge dt$ is invariant under $PSH(1)$. We have the following definition of a measure for a set of horizontal lines.

Definition 1.1. The *measure* of a set X of horizontal lines $L_{p,\theta,t}$ is defined by the integral over the set X,

$$m(X) = \int_X dG,$$

where the differential form $dG = dp \wedge d\theta \wedge dt$ is called the *density* for sets of horizontal lines.

Notice that the density dG is the only one that is invariant under motions of $PSH(1)$. By convention, the density will always be taken at absolute value.

Next we state some classical results and our corresponding results. Recall the classical Crofton's formula: for a convex domain $D \subset \mathbb{R}^2$ with boundary length ℓ, we have ([26], Chapter 4.2)

$$\int_{\{L: L \cap D \neq \emptyset\}} dG = 2\ell, \tag{1.6}$$

where $L = L_{p,\theta}$ denotes the lines and $dG = dp \wedge d\theta$ is the density of lines in $\mathbb{R}^2$. In our previous work ([15] Theorem 1.11), by using the method of moving frame for the convex domain $D \subset H_1$ with boundary $\partial D = \Sigma$, we show the Crofton-type formula

$$\int_{\{L: L \cap D \neq \emptyset\}} dG = 2 \cdot p\text{-area}(\Sigma), \tag{1.7}$$

where L denotes a horizontal line and dG is the density of set of horizontal lines in H_1.

Going back to the Euclidean plane, in Chapter 3 of [26] it is showed that the area of a convex domain $D \subset \mathbb{R}^2$ is equal to the integral of the length of chords σ over all lines L intersecting D, namely,

$$\int_{\{L:L\cap D\neq\emptyset\}} \sigma\, dG = \pi \cdot \text{area}(D).$$

In a similar way, we have the following result in H_1:

Theorem 1.2. *Given a convex domain $D \subset H_1$, denote by $\mathcal{G}$ the set of horizontal lines intersecting D and let σ be the length of the intersection w.r.t. the Levi metric. Then*

$$\int_{\mathcal{G}} \sigma dG = 2\pi V(D),$$

where dG is the density in the space of horizontal lines and $V(D)$ is the Lebesgue volume D.

Remark 1.3. To simplify our argument, the domain is assumed to be convex. However, in general the convexity is not necessary. In the general case, we will need the additional assumption that there exists finitely many components of intersections of lines with the domain.

Once we have defined the measure of sets of lines, the *probability* ([26] Chapter 2) that a random line L is in the set X when it is known to be in the set Y containing X can be defined by the quotient of the measures

$$P(L \in X | L \in Y) = \frac{m(L \cap X \neq \emptyset)}{m(L \cap Y \neq \emptyset)}, \tag{1.8}$$

where $m(L \cap Z) = \int_{L\cap Z\neq\emptyset} dG$ is the measure of the set $Z \subset \mathbb{R}^2$ with respect to the density dG. Thus, by (1.7) and the conditional probability (1.8), we immediately have the corollary:

Corollary 1.4. *Given a convex 3-domain D in H_1 with boundary $\partial D = \Sigma$, and randomly throw an oriented horizontal line L intersecting D once a time with chord length σ. The average chord length of the lines intersecting D is*

$$\frac{m(\sigma; L\cap D \neq \emptyset)}{m(L; L\cap D\neq\emptyset)} = \frac{\int_{L\cap D\neq\emptyset} \sigma dG}{\int_{L\cap D\neq\emptyset} dG} = \frac{2\pi V(D)}{2\cdot p\text{-}area(\Sigma)}.$$

For our second theorem, recall that any element in the group of rigid motions in $\mathbb{R}^2$ can be associated to an orthogonal frame by the Lie group action. Similarly, we can have the association to $PSH(1)$ and the frames in H_1 as follows: using the Lie group structure of H_1, we define the the moving frame $(Q; e_1(Q), e_2(Q), T)$ at the point Q by moving the standard frame $(O; \mathring{e}_1(O), \mathring{e}_2(O), T)$ (defined in (1.1)) at the original O under translations in $PSH(1)$. Note that the contact plane $\xi_Q = span\{e_1(Q), e_2(Q)\}$ for any $Q \in H_1$. There exists a one-to-one correspondence between $PSH(1)$ and the moving frames, and thus any element in $PSH(1)$ can be uniquely parameterized by four variables (a, b, c, ϕ), where the point $Q = (a, b, c)$ determines the position of the frame, ϕ the rotation from $\mathring{e}_1$ to e_1 at Q. In the next section, we shall show that the following 4-form is invariant under $PSH(1)$, and hence one defines it as the kinematic density.

Definition 1.5. The invariant 4-form

$$dK := da \wedge db \wedge dc \wedge d\phi \tag{1.9}$$

is called the *kinematic density* for the group of motions $PSH(1)$ in H_1, where $Q = (a, b, c)$, and ϕ is the angle from the standard vector $\mathring{e}_1(Q)$ to the frame vector $e_1(Q)$.

When restricting the frame to a horizontal line L, and denoting by h the oriented distance from the base point $B \in L$ to Q w.r.t. the Levi metric, we shall show, see (2.7), that the invariant volume element of $PSH(1)$ has the alternative expression

$$dK = dG \wedge dh, \tag{1.10}$$

where dG is the density defined in Definition 1.1.

By integrating dK over a domain in $PSH(1)$, one gets the measure m of the corresponding set of motions (as in [26] called the *kinematic measure*). This kinematic measure is related to an unexpected geometric quantity, the p-area, so it is worth to believe that this approach can help us in understanding some aspects of CR (pseudohermitian) geometry using the viewpoint of integral geometry. The following result gives an evidence of the connection between these two fields.

Theorem 1.6. *Given a convex domain $D \subset H_1$ with boundary Σ, let $\mathcal{G}_\ell$ be the set of points $(Q, \phi) = (a, b, c, \phi)$ such that the segment starting at Q with direction ϕ and length $|v| = \ell$ (w.r.t. the Levi metric) meets D. Then*

$$\int_{\mathcal{G}_\ell} dK = 2\pi V(D) + 2\ell \cdot p\text{-}area(\Sigma), \tag{1.11}$$

where dK is the kinematic density for the group of motions $PSH(1)$, p-area(Σ) is the (sub-Riemannian) p-area of the boundary Σ, and $V(D)$ the Lebesgue volume of D.

A geometric interpretation of the integral in Theorem 1.6, which will be used later in the proof of the theorem, is as follows: For any point (Q, ϕ), there exists a unique horizontal line L and a vector v such that v starts at Q with direction ϕ and length $|v| = \ell$, see Figure 1. Then $\int_{\mathcal{G}_\ell} dK$ represents the measure of the set of vectors v such that the intersection of v and D is not empty.

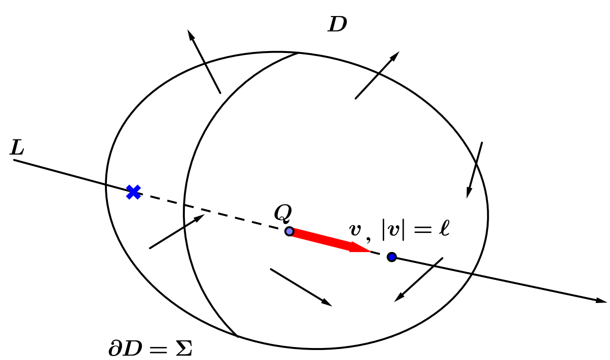

Figure 1: Random intersections of vector v

Remark 1.7. We take for granted that (Q, ϕ) meets D (equivalently, $v \cap D \neq \emptyset$) in the sense of $v \cap D \neq \emptyset$ or $v \cap \Sigma \neq \emptyset$. When we restrict our attention to $v \subsetneq D$ only, the equation (1.11) becomes

$$\int_{L \cap D \neq \emptyset, v \subset D} (\sigma - \ell) dG$$

and the measure of $v \subset D$ is

$$m(v; v \subset D) = 2\pi V(D) - 2\ell \cdot p\text{-area}(\Sigma).$$

With the measure, we immediately have the probability of containment problem.

Corollary 1.8. *Let $D_i \subset H_1$ be convex domains with boundary Σ_i for $i = 1, 2$, such that $D_1 \subset D_2$. The probability of randomly throwing a vector v with length ℓ w.r.t. the Levi metric in D_2 intersecting D_1 is*

$$P(v \cap D_1 | v \cap D_2) = \frac{2\pi V(D_1) + 2\ell \cdot \ p\text{-}area(\Sigma_1)}{2\pi V(D_2) + 2\ell \cdot p\text{-}area(\Sigma_2)}.$$

2 Invariants for sets of horizontal lines

In the section, we will derive the invariants for the set of horizontal lines as shown in Definition 2.5 and the other expression of the kinematic density for $PSH(1)$.

Given a regular curve $\gamma : t \in I \mapsto H_1$. Its velocity can always be decomposed into the part tangent to the contact plane ξ and the other orthogonal to ξ w.r.t. the Levi metric, namely,

$$\gamma'(t) = \underbrace{\gamma'_\xi(t)}_{\in \xi} + \underbrace{\gamma'_T(t)}_{\in T}.$$

A *horizontally regular curve* is a regular curve with non-zero contact part

$$\gamma'_\xi(t) \neq 0 \text{ for all } t \in I.$$

In Proposition 4.1 in [15], the authors showed that any horizontally regular curve can be parametrized by the horizontal arc-length s such that $|\gamma'_\xi(s)| = 1$. Throughout the article, we always assume that the curve (or line) is parametrized under this condition. Moreover, if $\gamma(s)$ is a curve joining points $A = \gamma(s_0)$ and $B = \gamma(s_1)$, we have the length

$$\text{length}(\gamma) = \int_{s_0}^{s_1} |\gamma'_\xi(s)| ds. \tag{2.1}$$

In particular, denote by $|AB|$ the length of the line segment joining the points A and B.

Now we characterize the horizontal lines. Any *horizontal line* L in H_1 can be uniquely determined by three parameters $(p, \theta, t) \in \mathbb{R} \times [0, 2\pi) \times \mathbb{R}$, and therefore we also use the notation $L = L_{p,\theta,t}$. Denote the projection $\pi(L)$ of L onto the xy-plane. It is known that any line on the plane can be determined by its distance p from the origin and the angle $\theta (0 \le \theta < 2\pi)$ of the normal with the x-axis (Figure 2). We also denote

- the *footpoint* $b = (p\cos\theta, p\sin\theta, 0) \in \pi(L)$,
- the *base point* B: the lift of b on L,
- t: the t-coordinate of the base point B.

Since L is horizontal, by (1.1) we also have the unit vector U along L

- $U = \sin\theta\, \mathring{e}_1(B) - \cos\theta\, \mathring{e}_2(B)$.

Using this, we immediately observe that the horizontal line L can be parametrized by the parameter $s \in \mathbb{R}$ with B and U, namely,

$$\begin{aligned} L : B + sU &= (p\cos\theta, p\sin\theta, t) + s(\sin\theta \mathring{e}_1(B) - \cos\theta \mathring{e}_2(B)) \\ &= (p\cos\theta + s\cdot\sin\theta, p\sin\theta - s\cdot\cos\theta, t + s(y\sin\theta + x\cos\theta)). \end{aligned} \tag{2.2}$$

Moreover, (2.2) implies that the points $(x, y, z) \in L$ satisfy the conditions

$$\begin{aligned} p &= x\cos\theta + y\sin\theta, \\ z &= (x\sin\theta - y\cos\theta)p + t. \end{aligned}$$

Therefore, we have the following expression for any horizontal line

$$L_{p,\theta,t} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{aligned} p &= x\cos\theta + y\sin\theta, \\ z &= (x\sin\theta - y\cos\theta)p + t \end{aligned} \right\}. \tag{2.3}$$

Now we show that the 3-form $dp \wedge d\theta \wedge dt$ is invariant under $PHS(1)$ and introduce the invariant measure. Suppose the horizonal line $L'_{p',\theta',t'}$ is obtained by $L_{p,\theta,t}$ transformed under a pseudohermition transformation

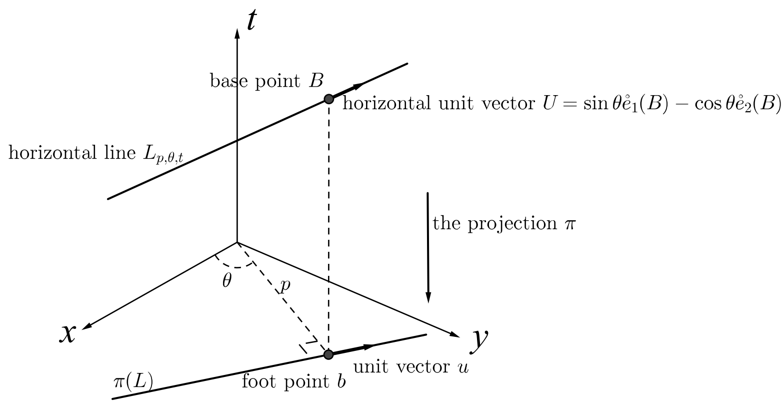

Figure 2: horizontal line $L_{p,\theta,t}$

$L_Q \circ \Phi_\alpha$. By the matrix multiplication (1.3), it is easy to calculate the transformed line $L'_{p',\theta',t'}$ satisfying

$$L'_{p',\theta',t'}: \begin{cases} \theta' = \theta - \alpha \\ p' = p + a\cos(\theta-\alpha) + b\sin(\theta-\alpha) \\ t' = 2p(a\sin(\theta-\alpha) + b\cos(\theta-\alpha)) + c + t, \end{cases} \tag{2.4}$$

and therefore

$$dp \wedge d\theta \wedge dt = dp' \wedge d\theta' \wedge dt'. \tag{2.5}$$

To the end, the 3-form $dp \wedge d\theta \wedge dt$ is invariant under the rigid motions in H_1. In addition, suppose that the measure of a set X of horizontal lines is defined by any integral of the form

$$m(X) = \int_X f(p,\theta,t)dp \wedge d\theta \wedge dt \tag{2.6}$$

for some function f. Following the spirit of classical integral geometry and geometric probability, the most natural measure should be *invariant under the group of rigid motions* $PSH(1)$ in H_1. If we wish that the measure $m(X)$ be equal to the measure of transformed set $m(X') = m(L_Q \circ \Phi_\alpha X)$ for any set X and any motion, by (2.4) and (2.5), the function f must be a constant. Choosing the constant equal to one, Definition 1.1 is obtained.

Next we derive an alternative expression of the kinematic density (1.10). Let $(Q; e_1(Q), e_2(Q), T)$ be the moving frame obtained from the standard frame $(O; \mathring{e}_1, \mathring{e}_2, T)$ at the origin O by the left-invariant translation to the point Q and the angle ϕ that makes $e_1(Q)$ with the standard vector $\mathring{e}_1(Q)$, as in Figure 3.

We observe that $e_2(Q) = Je_1(Q) \in \xi_Q$ and that the angle ϕ indicates the rotation of the frame vector $e_1(Q)$ from the vector $\mathring{e}_1(Q)$ on the contact plane ξ_Q. Denote the oriented distance

$$h = \pm|\vec{BQ}|,$$

w.r.t. the Levi metric, the sign depending on the direction $\vec{BQ}$ and the orientation of L. Since

$$Q = (a,b,c) = B + h(\sin\theta, -\cos\theta, p) = (p\cos\theta + h\sin\theta, p\sin\theta - h\cos\theta, t + hp),$$

we have

$$\begin{cases} a = p\cos\theta + h\sin\theta, \\ b = p\sin\theta - h\cos\theta, \\ c = t + hp, \\ \phi = \pi/2 \pm \theta \text{ (the sign needs the orientations of line } L). \end{cases}$$

Taking derivatives and making the wedge product, we reach the expression (1.10) of the kinematic density for the group of motions in H_1

$$da \wedge db \wedge dc \wedge d\phi = dp \wedge d\theta \wedge dt \wedge dh = dG \wedge dh, \tag{2.7}$$

which is indeed invariant under $PSH(1)$ by (2.5).

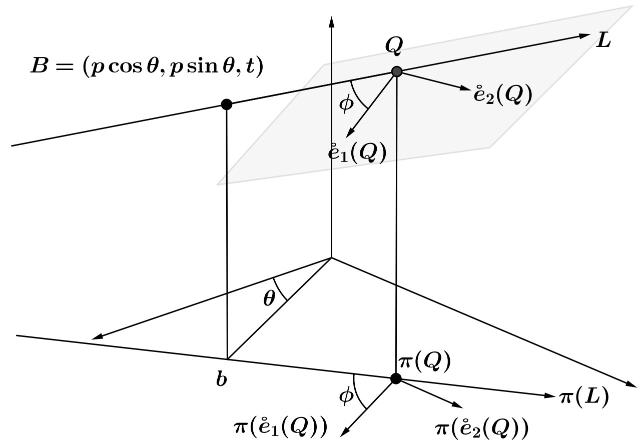

Figure 3: PSH(1)-action

3 Proof of Theorem 1.2

We shall prove Lemma 3.1 first. Take a horizontal line $\gamma(s)$ and two points $A = \gamma(s_0)$, $B = \gamma(s_1)$ on L. Since L is horizontal, $B \in \xi_A \cap \xi_B$. If we consider the distance joining A and B defined as (2.1), the infinitesimal lengths $|\gamma'(s)|$ vary depending on the Levi-metric, defined on different contact planes $\xi_{\gamma(s)}$ for all points $\gamma(s)$ between A and B. Considering $B \in \xi_A$, we define the distance

$$|AB|_A := \int_{s_0}^{s_1} |\gamma'(s)|_{\xi_A} ds, \tag{3.1}$$

where the infinitesimal length $|\gamma'(s)|_{\xi_A}$ depends only on the contact plane ξ_A of the initial point A. However, both distances are exactly same.

Lemma 3.1. *Given an oriented horizontal line $\gamma(s)$, parametrized by horizontal arc-length s, passing through the points $A = \gamma(s_0)$ to $B = \gamma(s_1)$. Then the two distances defined by* (2.1) *and* (3.1) *coincide:* $|AB|_A = |AB|$.

Proof. Note that, since $\gamma(s)$ is horizontal, $A \in \xi_B$ and $B \in \xi_B$. Thus, $|AB|_A = |AB|_B$. Therefore, we can always consider any point between A and B being parameterized as the end of the vector ejecting from A by

$$\gamma(s) := A + s(\sin\theta \mathring{e}_1(A) - \cos\theta \mathring{e}_2(A)).$$

Clearly, $\gamma'(s) = \sin\theta\mathring{e}_1(A) - \cos\theta\mathring{e}_2(A)$ is a unit vector on $\xi_A \cap \xi_{\gamma(s)}$ for any $s \in [s_0, s_1]$, so $|\gamma'(s)|_{\xi_A} = |\gamma'(s)|_{\xi_{\gamma(s)}} = 1$ and the result follows immediately. □

Now we prove Theorem 1.2.

Proof. First we observe that the slope of projection $\pi(L)$ of L on xy-plane is $-\cot\theta$, which is independent of the orientation of L. Now fixed a pair of (p, θ) and consider the cross-section of domain D and the vertical plane along the projection $\pi(L)$

$$S_{p,\theta} = \left\{(x, y, t) \in \mathbb{R}^3;\ p = x\cos\theta + y\sin\theta, (x, y) \in \pi(L \cap D),\ t \in I_2 \text{ for some interval } I_2\right\}.$$

Since the projection $\pi(S_{p,\theta})$ onto the xy-plane is again $\pi(L)$, we may set the first two coordinates of points on $S_{p,\theta}$ satisfying

$$y = y_{p,\theta}(x) = p - x\cot\theta.$$

Thus, for $\theta \neq 0$ or π, the plane $S_{p,\theta}$ can be parameterized by

$$X : (u, v) \in I_1 \times I_2 \mapsto (x(u, v), y(u, v), t(u, v)),$$

for some interval I_1, I_2 depending the range of domain D, where

$$\begin{aligned} x(u, v) &= u, \\ y(u, v) &= y_{p,\theta}(u) = p - u \cot\theta, \\ t(u, v) &= v. \end{aligned} \tag{3.2}$$

Now we use the following Lemma.

Lemma 3.2 ([15] Lemma 8.7). *Let $E = \alpha\mathbb{X}_u + \beta\mathbb{X}_v$ be the tangent vector field defined on the regular surface $X(u, v)$ in H_1. Then the vector E is also on the contact bundle ξ (and hence in $TH_1 \cap \xi$) if and only if pointwisely the coefficients α and β satisfy*

$$\alpha(t_u + xy_u - yx_u) + \beta(t_v + xy_v - yx_v) = 0. \tag{3.3}$$

Since $X(u, v) \cap \xi_{X(u,v)}$ is an one-dimensional foliation (a horizontal line in this case) E restricted on $S_{p,\theta}$, E is a linear combination of X_u and X_v. By Lemma 3.2, we choose $\alpha := -(t_v + xy_v - yx_v)$ and $\beta := (t_u + xy_u - yx_u)$ which satisfy (3.3). Use (3.2), we have

$$\begin{aligned} E := E(u, v) &:= -(t_v + xy_v - yx_v)X_u + (t_u + xy_u - yx_u)X_v \\ &= -(1 + x \cdot 0 - y \cdot 0)(1, y', 0) + (0 + xy' - y)(0, 0, 1) \\ &= (-1, -y', xy' - y) \\ &= (-1)\mathring{e}_1(x, y, z) + (-y')\mathring{e}_2(x, y, z). \end{aligned} \tag{3.4}$$

By Lemma 3.1 and (3.4),

$$\begin{aligned} \sigma = |E|_A &= |E| \\ &= \int\limits_{q \in E} \sqrt{\langle E(q), E(q)\rangle_{\xi_q}} \\ &= \int\limits_{u \in I_1} \sqrt{1 + (y')^2}\, du \\ &= \int\limits_{u \in I_1} |\csc\theta| du. \end{aligned} \tag{3.5}$$

When $\theta = 0$ or π, the set $\{\theta = 0\} \cap \{\theta = \pi\}$ has measure 0, which implies the density

$$dG = 0. \tag{3.6}$$

Finally, combining both cases, (3.5) and (3.6), we have

$$\begin{aligned} \int\limits_{L_{p,\theta,t} \cap D \neq \emptyset} \sigma dG &= \int\limits_{\theta \neq 0,\pi} \sigma dG + \int\limits_{\theta = 0,\pi} \sigma dG \\ &= \int \Big(\int\limits_{x \in I_1} |\csc\theta| dx \Big) dt \wedge dp \wedge d\theta \\ &= \int \int\limits_{x \in I_1} |\csc\theta| dx \wedge dt \wedge (dx\cos\theta + dy\sin\theta) \wedge d\theta \\ &= \int \int\limits_{x \in I_1} dx \wedge dt \wedge dy \wedge d\theta \\ &= 2\pi V(D), \end{aligned}$$

where we have used the fact that $dx \wedge dy \wedge dt$ is the Lebesgue volume form in $\mathbb{R}^3$ in the last identity. This completes the proof. □

Remark 3.3. If D consists of finitely many simply-connected subsets, then the right-hand-side of (1.2) becomes the sum of the volumes of each subset.

4 Proof of Theorem 1.6

Proof. For any element in $\mathcal{G}_\ell$, there exists the corresponding vector v and the horizontal line $L = L_{p,\theta,t}$ such that L intersects D at two points and $v \in L$ starts from Q with direction ϕ and length $|v| = \ell$. Notice that when v moves along L such that $v \cap D \neq \emptyset$, the point Q also travels over the distance $\sigma + \ell$ on L. Therefore by Theorem 1.2, (1.7), and (2.7) we have

$$\begin{aligned}\int_{\mathcal{G}_\ell} dK &= \int_{v\cap D\neq\emptyset} dp \wedge d\theta \wedge dt \wedge dh \\ &= \int_{L\cap D\neq\emptyset} (\sigma + \ell) dG \\ &= 2\pi V(D) + 2\ell \cdot p\text{-area}(\Sigma).\end{aligned}$$

□

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