DE GRUYTER OPEN Anal. Geom. Metr. Spaces 2017; 5:69–77

Research Article **Open Access**

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Some Invariant Properties of Quasi-Möbius Maps

https://doi.org/10.1515/agms-2017-0004
Received July 4, 2017; accepted August 3, 2017

Abstract: We investigate properties which remain invariant under the action of quasi-Möbius maps of quasi-metric spaces. A metric space is called doubling with constant D if every ball of finite radius can be covered by at most D balls of half the radius. It is shown that the doubling property is an invariant property for (quasi-)Möbius maps. Additionally it is shown that the property of uniform disconnectedness is an invariant for (quasi-)Möbius maps as well.

Keywords: Möbius structures, doubling property, quasi-Möbius maps, uniform disconnectedness

MCS: 30C65, 53C23, 54F45

1 Introduction

Let (X, d) be a metric space. X is *doubling* if there exists a constant $D > 0$, such that every ball of finite radius can be covered by at most D balls of half the radius. X is *uniformly disconnected* if there exists a constant $\theta < 1$, such that X contains no θ-chain, i.e. a sequence of (at least 3 distinct) points $(x_0, x_1, \dots, x_n)$ such that

$$d(x_i, x_{i+1}) \leq \theta d(x_0, x_n).$$

A map $f : (X, d) \to (Y, d')$ is *quasi-Möbius* if it is a homeomorphism and there exists a homeomorphism $\nu : [0, \infty[\to [0, \infty[$, such that for all quadruples $Q = (x_1, x_2, x_3, x_4)$ of distinct points of X and $Q' := (f(x_1), f(x_2), f(x_3), f(x_4))$,

$$\mathrm{cr}(Q', d') \leq \nu(\mathrm{cr}(Q, d))$$

holds. Here the *cross-ratio* cr is given by

$$\mathrm{cr}(Q, d) := \frac{d(x_1, x_3)d(x_2, x_4)}{d(x_1, x_4)d(x_2, x_3)}.$$

The aim of this paper is to prove the following two theorems:

Theorem 1.1 (Invariance of doubling under quasi-Möbius maps)**.** *Let (X, d) be a doubling space. Let $f : (X, d) \to (Y, d')$ be a quasi-Möbius homeomorphism. Then (Y, d') is doubling.*

Theorem 1.2 (Invariance of uniform disconnectedness under quasi-Möbius maps)**.** *Let (X, d) be a metric uniformly disconnected space and let $f : (X, d) \to (Y, d')$ be a quasi-Möbius homeomorphism. Then (Y, d') is uniformly disconnected.*

The results are related to results of Lang-Schlichenmaier [5] and Xie [12] who proved that quasi-symmetric maps respectively quasi-Möbius maps preserve the Nagata dimension of metric spaces. The present work has

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been inspired by the article of Xie [12] and the work of Väisälä [11]. We note that a space is doubling if and only if it has finite Assouad dimension [8]. However the Assouad dimension is not a quasi-symmetric (and therefore also not a quasi-Möbius) invariant [10].

We would like to note that we have been informed that Theorem 1.1 is a direct consequence of a published result of Li-Shanmugalingam [6].

It is well known that uniform disconnectedness is invariant under quasi-symmetric maps [4, 8]. However its behaviour under quasi-Möbius maps has not been studied before.

The related property of uniform perfectness has been shown to be invariant under the metric inversion in [9]. It is therefore also invariant under quasi-Möbius maps.

In Appendix A we prove a slight generalization of Theorem 1.1 and Theorem 1.2 for K-quasi-metric spaces.

2 Basic Definitions and Preparations

We introduce the necessary definitions which we will require later.

2.1 Extended Metrics

Let X be a set with cardinality at least 3. We call a map $d : X \times X \to [0, \infty]$ an *extended metric* on X if there exists a set $\Omega(d) \subset X$ with cardinality 0 or 1 and furthermore all of the following requirements are satisfied:

1. $d_{|X\setminus\Omega(d)\times X\setminus\Omega(d)} : X \setminus \Omega(d) \times X \setminus \Omega(d) \to [0, \infty[$ is a metric;
2. $d(x, \omega) = d(\omega, x) = \infty$ for all $x \in X \setminus \Omega(d)$ and $\omega \in \Omega(d)$;
3. $d(\omega, \omega) = 0$ for $\omega \in \Omega(d)$.

If $\Omega(d)$ is non empty we call $\omega \in \Omega(d)$ the *infinitely remote point* of X. By abuse of notation we may write ∞ for the point ω.

2.2 Doubling Property

We call a metric space *doubling with constant D* if every ball of finite radius can be covered by at most D balls of half the radius.

2.3 Uniform Disconnectedness

For $\theta < 1$ we call a sequence of (at least 3 distinct) points $(x_0, x_1, \dots, x_n)$ in a metric space (X, d) a θ*-chain* if

$$d(x_i, x_{i+1}) \leq \theta d(x_0, x_n)$$

holds for all $i \in \{0, 1, \dots, n-1\}$. A metric space is called *uniformly disconnected with constant* θ if it contains no θ-chains.[1]

2.4 Quasi-Symmetric Maps

We call a homeomorphism $f : (X, d) \to (Y, d')$ ν*-quasi-symmetric* if for all pairwise distinct $x_1, x_2, x_3 \in X$ we have

$$\frac{d'(f(x_1), f(x_2))}{d'(f(x_1), f(x_3))} \leq \nu\left(\frac{d(x_1, x_2)}{d(x_1, x_3)}\right).$$

1 And therefore also no θ'-chains for any $\theta' \leq \theta$.

A homeomorphism $f : (X, d) \to (Y, d')$ is called *quasi-symmetric* if it is ν-quasi-symmetric for some homeomorphism $\nu : [0, \infty[\to [0, \infty[$. It is called *symmetric* if for all pairwise distinct $x_1, x_2, x_3 \in X$ we have

$$\frac{d'(f(x_1), f(x_2))}{d'(f(x_1), f(x_3))} = \frac{d(x_1, x_2)}{d(x_1, x_3)}.$$

3 Invariance of Doubling Property

3.1 Preparations for the Proof

For the proof we need the following proposition of Xie and a result of Väisälä which we cite verbatim

Proposition 3.1 (Proposition 3.6 in [12]). *Let $f : (X_1, d_1) \to (X_2, d_2)$ be a quasi-Möbius homeomorphism. Then f can be written as $f = f_2^{-1} \circ f' \circ f_1$, where f' is a quasi-symmetric map, and f_i for $i \in \{1, 2\}$ is either a metric inversion or the identity map on the metric space (X_i, d_i).*

Proposition 3.2 (Theorem 3.10 in [11]). *Let (X, d) be an unbounded metric space and let $f : X \to Y$ be a quasi-Möbius map. Then f is quasi-symmetric if and only if $f(x) \to \infty$ as $x \to \infty$. If X is any metric space and if $f : X \cup \{\infty\} \to Y \cup \{\infty\}$ is quasi-Möbius with $f(\infty) = \infty$, then $f_{|X}$ is quasi-symmetric.*

Remark 3.3. *Let (X, d) be an unbounded space. Then we can build the completed space with respect to the infinitely remote point $\bar{X} := X \cup \{\infty\}$ together with an extended metric $\bar{d}$. Let $\bar{d}(x, y) := d(x, y)$ and $\bar{d}(\infty, x) := \bar{d}(x, \infty) = \infty$ for all $x, y \in X$. Furthermore let $\bar{d}(\infty, \infty) = 0$. Then clearly (X, d) is doubling if and only if $(\bar{X}, \bar{d})$ is doubling. It then follows from [7] that the completion also has a doubling measure. Then Proposition 3.2 and Proposition 4.2 from [6] directly imply the following theorem:*

Theorem 3.4. *Let (X, d) be an metric doubling space with doubling constant D, where d is an extended metric [3] and denote by $\infty \in X$ the infinitely remote point in (X, d). Furthermore let $p \in X$ with $p \neq \infty$ and let i_p be given by $i_p(x, y) := \frac{d(x,y)}{d(p,x)\, d(p,y)}$ for all $x, y \in X \setminus \{\infty\}$ and $i_p(\infty, x) := i_p(x, \infty) := \frac{1}{d(p,x)}$. Define $d_p(x, y) := \inf\{\sum_{i=1}^{k} i_p(x_i, x_{i-1}) \,|\, x = x_0, \dots, x_k = y \in X \setminus \{p\}\}$. Then (X, d_p) is doubling with constant at most $D^{10} + 1$.*

Remark 3.5. *Note that if in addition $d \in \mathcal{M}$ where $(X, \mathcal{M})$ is Ptolemy Möbius, then $i_p = d_p$ and in particular (X, d_p) is doubling with constant at most $D^8 + 1$.*

3.2 Proof of Theorem 1.1

Proof of Theorem 1.1. It remains to show the theorem for (X, d) being a doubling metric space, $f : (X, d) \to (X, d')$ a metric inversion and we have the following cases to check:

1. (X, d) unbounded, (X, d') bounded;
2. (X, d) and (X, d') both unbounded but with different points at infinity.

Case 2 follows directly from Theorem 3.4. In the situation of 1, d' is a metric inversion d_p where p is an isolated point in X. That is there exists a $\epsilon > 0$ such that $d(p, x) > \epsilon$ for all $x \in X \setminus \{p\}$. The proof of Theorem 3.4 still holds. □

4 Invariance of Uniform Disconnectedness

The proof of Theorem 1.2 will again make use of some of the propositions from the previous sections.

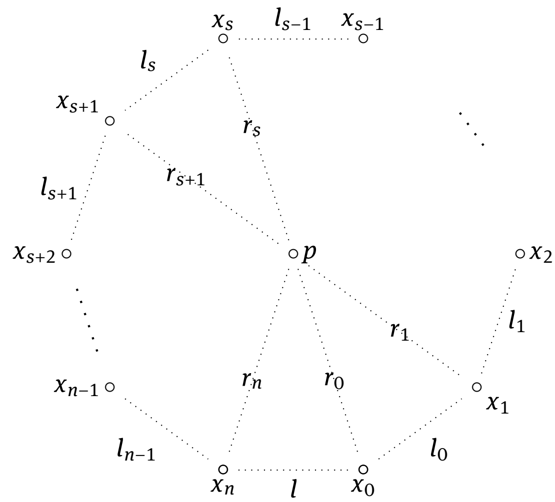

Figure 1: The view of the θ-chain in (X, d)

In the following let (X, d) be a metric space, $p \in X$ and $\theta \leq \frac{1}{32}$. We assume that (X, d_p) is not θ-uniformly disconnected, in particular there is some θ-chain $(x_0, x_1, \dots, x_n)$ in $(X \setminus \{p\}, d_p)$. We keep this notation for the rest of this section. In addition we introduce the following notation for convenience: Let $r_i := d(p, x_i)$, $l := d(x_0, x_n)$ and $l_i := d(x_i, x_{i+1})$. This is illustrated in Figure 1. Without loss of generality we can assume $r_n \geq r_0$.

Remark 4.1. *The condition for $(x_0, x_1, \dots, x_n)$ being a θ-chain in (X, d_p) implies that*

$$\frac{l_i}{r_i r_{i+1}} \leq \frac{4\theta l}{r_n r_0} \; \forall i \in \{0, \dots, n-1\}.$$

On the other hand if

$$\frac{l_i}{r_i r_{i+1}} \leq \frac{\theta l}{4 r_n r_0} \; \forall i \in \{0, \dots, n-1\}$$

holds, then $(x_0, x_1, \dots, x_n)$ is a θ-chain in (X, d_p).

Lemma 4.2. *Assume that (X, d) contains no $\sqrt[3]{4\theta}$-chains. Then there is an index $s \in \{0, \dots, n-1\}$ such that*

$$l_s > l\sqrt[3]{4\theta}$$

and

$$\max\{r_s, r_{s+1}\}\sqrt[3]{4\theta} \geq r_0.$$

Proof. Assume for a contradiction that $r_s\sqrt[3]{4\theta} < r_0$ and $r_{s+1}\sqrt[3]{4\theta} < r_0$. Then from the condition in the remark above it follows

$$\frac{l_s}{r_s r_{s+1}} \leq \frac{4\theta l}{r_n r_0} < \frac{4\theta l_s}{\sqrt[3]{4\theta} r_n r_0} < \frac{4\theta l_s}{\sqrt[3]{4\theta}^3 r_s r_{s+1}} = \frac{l_s}{r_s r_{s+1}} \tag{4.1}$$

which is a contradiction. □

Proposition 4.3. *(X, d) contains a $\sqrt[3]{4\theta}$-chain.*

Proof. By the previous lemma we know that there must be some index q such that $r_q\sqrt[3]{4\theta} \geq r_0$ and for all $i \in \{0, \dots, q-1\}$ we have that $r_i\sqrt[3]{4\theta} < r_0$.

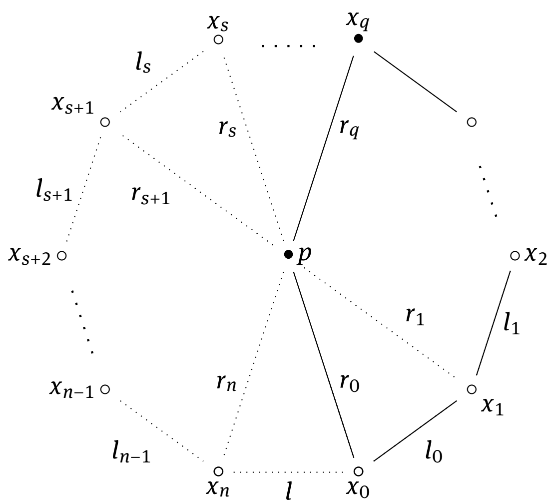

Figure 2: The constructed $\sqrt[3]{4\theta}$-chain in (X, d)

We claim that $(x_q, x_{q-1}, \dots, x_1, x_0, p)$ is a $\sqrt[3]{4\theta}$-chain in (X, d). If this were not so, there would be some $i \in \{0, \dots, q-1\}$ for which $r_q\sqrt[3]{4\theta} < l_i$. But then

$$\frac{r_q\sqrt[3]{4\theta}^2}{r_0 r_q} < \frac{r_q\sqrt[3]{4\theta}}{r_i r_q} \le \frac{r_q\sqrt[3]{4\theta}}{r_i r_{i+1}} < \frac{l_i}{r_i r_{i+1}} \le \frac{4\theta l}{r_n r_0} \tag{4.2}$$

implies

$$r_n < \sqrt[3]{4\theta} l \le \frac{1}{2} l \tag{4.3}$$

which is a contradiction to the triangle inequality of the metric space (X, d). ☐

Proof of Theorem 1.2. The proof of the theorem now follows directly from Theorem 3.1. ☐

5 Applications of the Theorems

For the following we need a short definition [4]: Let F be a finite set with $k \ge 2$ elements and let F^∞ denote the set of sequences $\{x_i\}_{i=1}^\infty$ with $x_i \in F$. For two elements $x = \{x_i\}, y = \{y_i\} \in F^\infty$ let

$$L(x, y) = \sup\{I \in \mathbb{N} \mid \forall 1 \le i \le I : x_i = y_i\}.$$

In particular we have $L(x, x) = \infty$ and $L(x, y) = 0$ if $x_1 \neq y_1$. Given $0 < a < 1$ set $\rho_a(x, y) = a^{L(x,y)}$. This defines an ultrametric on F^∞. We call (F^∞, ρ_a) the *symbolic k-Cantor set with parameter a*.

As an application of the theorems we provide a generalization of the following result by David and Semmes:

Proposition 5.1 (Proposition 15.11 (Uniformization) in [4]). *Suppose that (M, d) is a compact metric space which is bounded, complete, doubling, uniformly disconnected, and uniformly perfect. Then M is quasi-symmetrically equivalent to the symbolic Cantor set F^∞, where we take $F = \{0, 1\}$ and we use the metric ρ_a on F^∞ with parameter $a = \frac{1}{2}$.*

We can generalize this result as follows:

Theorem 5.2. *Suppose that (M, d) is a complete, doubling, uniformly perfect and uniformly disconnected metric space. Then M is quasi-Möbius equivalent to the symbolic Cantor set as given above.*

Proof. Let $p \in M$ be some point and let $s_p(x,y) = \frac{d(x,y)}{(d(x,p)+1)(d(y,p)+1)}$. Let $\hat{d}_p(x,y) = \inf\{\sum_{i=1}^k s_p(x_i, x_{i-1}) : x = x_0, \dots, x_k = y \in X\}$. We have [2]

$$\frac{1}{4}s_p(x,y) \leq \hat{d}_p(x,y) \leq s_p(x,y) \leq \frac{1}{1+d(x,p)} + \frac{1}{1+d(y,p)}.$$

Then the space $(M, \hat{d}_p)$ is bounded and satisfies all the properties of the above proposition: The map $f : (X,d) \to (X, \hat{d}_p)$ given by $d \mapsto \hat{d}_p$ is Möbius. By Theorem 1.2 and Theorem 1.1, doubling and uniformly disconnectedness are invariant under Möbius maps. The invariance of uniformly perfectness follows from [9], and the invariance of completeness follows from [1]. Totally boundedness follows from the doubling property and therefore the space $(X, \hat{d}_p)$ is compact. □

We can apply the same idea to Proposition 16.9 in [4] and we get:

Corollary 5.3. *Let (M,d) be a complete Ahlfors regular metric space of dimension γ which is uniformly disconnected. Then there exists a doubling measure μ on F^∞, and (M,d) is quasi-Möbius equivalent to (F^∞, D), where D is given by*

$$D(x,y) = \left(\mu(\bar{B}(x, d_a(x,y))) + \mu(\bar{B}(y, d_a(x,y)))\right)^{\frac{1}{\gamma}},$$

and $0 < a < 1$.

This follows from the above remarks and the invariance of Ahlfors regularity under $d \mapsto \bar{d}_p$ as shown in [6].

A Appendix

Proposition A.1. *Let (X,d) be a K-quasi-metric space [3]. Let X_∞ denote the infinite remote set and let $\infty \in X_\infty$, i.e. the space satisfies the relations*

1. $d(x,y) = 0 \iff x = y$,
2. $d(x,y) = d(y,x)$,
3. $d(x,y) \leq K \max\{d(x,z), d(z,y)\}$ *for all $x, y, z \in X$ for which all distances are defined,*
4. $d(x,y) < \infty \iff x, y \in X \setminus X_\infty$.

Let $\lambda : X \to [0,\infty]$, $L > 0$ and $K' \geq K$ be such that $X_\infty = \lambda^{-1}(\infty)$ and

1. $d(x,y) \leq K' \max\{L\lambda(x), L\lambda(y)\}$,
2. $L\lambda(x) \leq K' \max\{d(x,y), L\lambda(y)\}$.

Denote by $X'_\infty := \{\lambda^{-1}(0)\}$. Define a new metric $d_\lambda : (X \times X) \setminus (X'_\infty \times X'_\infty) \to [0,\infty]$ by

1. $d_\lambda(x,y) := \frac{d(x,y)}{\lambda(x)\lambda(y)}$ *for $x, y \in X \setminus X'_\infty$,*
2. $d_\lambda(x,\infty) := d_\lambda(\infty, x) := \frac{L}{\lambda(x)}$ *for $\infty \in X_\infty$,*
3. $d_\lambda(\infty,\infty) = 0$ *for $\infty \in X_\infty$,*
4. $d_\lambda(x,p) := d_\lambda(p,x) := \infty$ *for $p \in X'_\infty$.*

If (X,d) is doubling with constant D then (X, d_λ) is doubling with constant at most $D^{\lceil \log_2(8K'^{10}K) \rceil} + 1$.

Proof. By Prop 5.3.6 in [3], d_λ is a K'^2-quasi-metric. In particular we have for all $x, y, z \in X$ for which all distances are defined, that:

$$d_\lambda(x,y) \leq K'^2 \max\{d(d,z), d(z,y)\}.$$

Let $x_0 \in X$, $x_0 \neq p \in X'_\infty$ and $r > 0$ and let $B' := B'_r(x_0) := \{x \in X \mid d_\lambda(x_0, x) \leq r\}$. Consider the following cases

1. If $B' \cap B'_{\frac{1}{2}r}(\infty) \neq \emptyset$, then let $A' := B' \setminus B'_{\frac{1}{2}r}(\infty)$. For all $x, y \in B'$ we have

$$d_\lambda(x,y) = \frac{d(x,y)}{\lambda(x)\lambda(y)} \leq K'^2 r,$$

from which it follows that

$$d(x, y) \le K'^2 r\lambda(x)\lambda(y).$$

Furthermore we have for all $x \in A'$ that $d_\lambda(\infty, x) = \frac{L}{\lambda(x)} > \frac{1}{2}r$ and therefore also $\lambda(x) < \frac{2L}{r}$. Combining both equations we get that for all $x, y \in A'$ we have

$$d(x, y) \le K'^2 r \frac{2L}{r}\frac{2L}{r} = \frac{K'^2 4L^2}{r}.$$

Without loss of generality assume $x_0 \in A'$. By the doubling property of (X, d) we can cover $B_{\frac{K'^2 4L^2}{r}}(x_0)$ by at most D^N balls b_i of radius $\frac{K'^2 4L^2}{r}2^{-N}$. Let $\tilde{b}_i := b_i \cap A'$ then we have for all $x, y \in \tilde{b}_i$:

$$d_\lambda(x, y) \le \frac{\frac{K'^2 4L^2}{r2^N}K}{\lambda(x)\lambda(y)}.$$

By the assumption there is a $\bar{x} \in B' \cap B'_{\frac{1}{2}r}(\infty)$ and we have for $x \in B'$ that $d_\lambda(x, \bar{x}) \le K'^2 r$, therefore we also have $\frac{L}{\lambda(x)} = d_\lambda(x, \infty) \le K'^4 r$ and $\lambda(x) \ge \frac{L}{K'^4 r}$. In conclusion we get for all $x, y \in \tilde{b}_i$:

$$d_\lambda(x, y) \le \frac{\frac{K'^2 4L^2}{r2^N}K}{\lambda(x)\lambda(y)} \le \frac{\frac{K'^2 4L^2}{r2^N}K}{\frac{L}{K'^4 r}\frac{L}{K'^4 r}} = \frac{K'^{10}K4r}{2^N}.$$

In particular for $N := \lceil \log_2(8K'^{10}K) \rceil$ we get a cover of B' by at most $D^N + 1$ balls of half the radius.

2. If $B' \cap B'_{\frac{1}{2}r}(\infty) = \emptyset$, then we have $d_\lambda(x_0, \infty) > r$ and $d_\lambda(B', \infty) > \frac{1}{2}r$. For all $y \in B'$ we have $d_\lambda(x_0, y) = \frac{d(x_0, y)}{\lambda(x_0)\lambda(y)} \le r$ and therefore also

$$d(x_0, y) \le r\lambda(x_0)\lambda(y) \le \frac{rL^2}{d_\lambda(\infty, x_0)d_\lambda(\infty, y)} = \frac{rL^2}{d_\lambda(B', \infty)^2}.$$

By the doubling property of (X, d) we can find D^N balls b_i of radius $\frac{rL^2}{d_\lambda(B', \infty)^2}2^{-N}$ covering B'. Let $\tilde{b}_i := b_i \cap B'$, then we have for any $x, y \in \tilde{b}_i$:

$$d_\lambda(x, y) = \frac{d(x, y)}{\lambda(x)\lambda(y)} \le \frac{K\frac{rL^2 2^{-N}}{d_\lambda(B', \infty)^2}}{\lambda(x)\lambda(y)} = \frac{Kr2^{-N}d_\lambda(\infty, x)d_\lambda(\infty, y)}{d_\lambda(B', \infty)^2}.$$

Furthermore for any $x \in B'$ we have

$$d_\lambda(x, \infty) \le K'^2 \max\{d_\lambda(x_0, x), d_\lambda(x_0, \infty)\} \le K'^2 r \le K'^2 2d_\lambda(B', \infty).$$

We can combine the estimates to get

$$d_\lambda(x, y) \le \frac{Kr2^{-N}K'^4 4d_\lambda(B', \infty)^2}{d_\lambda(B', \infty)^2} = Kr2^{-N}K'^4 4.$$

In particular for $N := \lceil \log_2(8KK'^4) \rceil$ we have constructed a covering by D^N balls of radius at most $\frac{1}{2}r$.

□

Proposition A.2. *Let (X, d) be a K-quasi-metric space [3]. Let X_∞ denote the infinite remote set and let $\infty \in X_\infty$, i.e. the space satisfies the relations*

1. $d(x, y) = 0 \iff x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \le K \max\{d(x, z), d(z, y)\}$ *for all $x, y, z \in X$ for which all distances are defined,*
4. $d(x, y) < \infty \iff x, y \in X \setminus X_\infty$.

Let $\lambda : X \to [0, \infty]$, $L > 0$ and $K' \ge K$ be such that $X_\infty = \lambda^{-1}(\infty)$ and

1. $d(x, y) \leq K' \max\{L\lambda(x), L\lambda(y)\}$,
2. $L\lambda(x) \leq K' \max\{d(x, y), L\lambda(y)\}$.

Denote by $X'_\infty := \{\lambda^{-1}(0)\}$. *Define a new metric* $d_\lambda : (X \times X) \setminus (X'_\infty \times X'_\infty) \to [0, \infty]$ *by*

1. $d_\lambda(x, y) := \frac{d(x,y)}{\lambda(x)\lambda(y)}$ *for* $x, y \in X \setminus X'_\infty$,
2. $d_\lambda(x, \infty) := d_\lambda(\infty, x) := \frac{L}{\lambda(x)}$ *for* $\infty \in X_\infty$,
3. $d_\lambda(\infty, \infty) = 0$ *for* $\infty \in X_\infty$,
4. $d_\lambda(x, p) := d_\lambda(p, x) := \infty$ *for* $p \in X'_\infty$.

Let $\theta \leq \frac{1}{K^{19}}$. *If* (X, d_λ) *has a* θ*-chain, then* (X, d) *has a* $\sqrt[3]{\theta K'^4}$*-chain.*

Proof. Using the same notation as before in section 4 we note that for all $i \in \{0, \dots, n-1\}$ the following relation holds:

$$\frac{l_i}{\frac{K'^2}{L^2} r_i r_{i+1}} \leq \frac{l_i}{\lambda(x_i)\lambda(x_{i+1})} \leq \frac{l\theta}{\lambda(x_0)\lambda(x_n)} \leq \frac{l\theta}{\frac{1}{K'L} r_0 r_n}.$$

We can apply a similar argument as in Theorem 4.2 to get an index q for which

$$r_0 \leq \sqrt[3]{\theta K'^4} r_q,$$

and such that for all $i \in \{0, \dots, q-1\}$ we have

$$r_0 > \sqrt[3]{\theta K'^4} r_i.$$

Assume again for a contradiction that $(x_q, x_{x-1}, \dots, x_0, p)$ is not a $\sqrt[3]{\theta K'^4}$-chain. Then for some $i \in \{0, \dots, q-1\}$:

$$\frac{\sqrt[3]{\theta K'^4}^2 r_q}{\frac{K'^2}{L^2} r_0 r_q} \leq \frac{\sqrt[3]{\theta K'^4} r_q}{\frac{K'^2}{L^2} r_i r_q} \leq \frac{\sqrt[3]{\theta K'^4} r_q}{\frac{K'^2}{L^2} r_i r_{i+1}} \leq \frac{\sqrt[3]{\theta K'^4} r_q}{\lambda(x_i)\lambda(x_{i+1})} < \frac{l_i}{\lambda(x_i)\lambda(x_{i+1})} \tag{A.1}$$

$$\leq \frac{\theta l}{\lambda(x_0)\lambda(x_n)} \leq \frac{\theta l}{\frac{1}{K'^2 L^2} r_0 r_n} \tag{A.2}$$

From this it follows that

$$r_n < \sqrt[3]{\theta K'^4} K'^4 l \leq K^{-1} l.$$

□

Acknowledgement: The author would like to thank Viktor Schroeder and Urs Lang for several helpful discussions.

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