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# An Analog of the Neumann Problem for the 1-Laplace Equation in the Metric Setting: Existence, Boundary Regularity, and Stability

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**Abstract:** We study an inhomogeneous Neumann boundary value problem for functions of least gradient on bounded domains in metric spaces that are equipped with a doubling measure and support a Poincaré inequality. We show that solutions exist under certain regularity assumptions on the domain, but are generally nonunique. We also show that solutions can be taken to be differences of two characteristic functions, and that they are regular up to the boundary when the boundary is of *positive mean curvature*. By regular up to the boundary we mean that if the boundary data is 1 in a neighborhood of a point on the boundary of the domain, then the solution is  $-1$  in the intersection of the domain with a possibly smaller neighborhood of that point. Finally, we consider the stability of solutions with respect to boundary data.

**Keywords:** bounded variation; metric measure space; Neumann problem; positive mean curvature; stability

**MSC:** 30L99, 26B30, 43A85

## 1 Introduction

The goal of the Neumann boundary value problem for  $\Delta_p$  in a smooth Euclidean domain  $\Omega \subset \mathbb{R}^n$  is to find a function  $u \in W^{1,p}(\Omega)$  such that

$$\begin{aligned} \Delta_p u &= -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad \text{in } \Omega, \text{ and} \\ |\nabla u|^{p-2} \partial_\eta u &= f \quad \text{on } \partial\Omega, \end{aligned}$$

where  $\partial_\eta u$  is the derivative of  $u$  in the direction of outer normal to  $\partial\Omega$  and  $f \in L^\infty(\partial\Omega, \mathcal{H}^{n-1})$  such that  $\int_{\partial\Omega} f \, d\mathcal{H}^{n-1} = 0$ . In the case  $p = 1$  the problem degenerates to finding  $u \in \operatorname{BV}(\Omega)$  such that

$$\begin{aligned} \Delta_1 u &:= -\operatorname{div} \left( \frac{Du}{\|Du\|} \right) = 0 \quad \text{in } \Omega, \text{ and} \\ \frac{\partial_\eta u}{\|Du\|} &= f \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

and the boundary condition also means that we must have  $|f| \leq 1$  on  $\partial\Omega$  in order for the problem to make sense. Here  $\frac{\partial_\eta u}{\|Du\|}$  denotes the Radon-Nikodym derivative of the vector-valued Radon measure  $\partial_\eta u$  with respect to the Radon measure  $\|Du\|$ . The goal of this paper is to study this problem in the setting of metric measure spaces equipped with a doubling measure supporting a  $(1, 1)$ -Poincaré inequality.

When  $\Omega$  is a smooth Euclidean domain, an application of the divergence theorem, together with tools from the calculus of variations, tell us that  $u$  is a solution to the Neumann problem for  $\Delta_p$  if and only if  $u$

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minimizes the energy

$$I_p(v) = \int_{\Omega} |\nabla v(x)|^p dx + \int_{\partial\Omega} Tv(y) f(y) d\mathcal{H}^{n-1}(y)$$

within the class of all Sobolev functions  $v \in W^{1,p}(\Omega)$ , where  $Tv$  is the trace of  $v$  at the boundary of the domain  $\Omega \subset \mathbb{R}^n$ . When  $p = 1$ , the above energy is replaced with

$$I_1(v) = \|Dv\|(\Omega) + \int_{\partial\Omega} Tv(y) f(y) d\mathcal{H}^{n-1}(y),$$

with  $v \in BV(\Omega)$ . However, unlike in the case of  $p > 1$ , the functional  $I_1$  in general has no minimizer in the class  $BV(\Omega)$  (even if  $|f| \leq 1$ ), see the discussion below, the discussion around Definition 3.4, and the discussion in Section 8. But by restricting the minimization problem by also requiring the test functions  $v$  to satisfy  $|v| \leq 1$ , we obtain the existence of minimizers. Note that the boundary value problem (1.1) is scale invariant, that is, if  $u$  solves the problem, then so does  $\lambda u$  for each  $\lambda > 0$ . However, the energy  $I_1$  is not scale invariant as  $I_1(\lambda u) = \lambda I_1(u)$  whenever  $\lambda \geq 0$ . Requiring the test functions  $v$  to satisfy  $|v| \leq 1$  takes care of this issue. In fact, we demonstrate in the proof of Theorem 4.15 that we can take such a solution to be the difference of two characteristic functions (and hence, this solution takes on at most three values, namely  $-1, 0, 1$ ).

The Neumann problem can be formulated as a minimization problem even if  $\Omega$  is not smooth, as long as  $\Omega$  is regular enough for us to talk about an “outer normal vector field” to  $\partial\Omega$ . One case where this is possible is when  $\Omega$  is of finite perimeter (and thus a divergence-type theorem holds for  $\Omega$ ). In the non-smooth setting of metric measure spaces where a notion of first-order calculus has been developed, the above energy minimization problem makes sense provided  $\Omega$  is regular enough, meaning that it is of finite perimeter, the codimension 1 Hausdorff measure of  $\partial\Omega$  is finite (so that the boundary integral term in  $I_1$  makes sense), and the trace  $Tv$  exists. In such a setting, there are ways of making sense of  $\partial_{\eta}u$ , see for example [34].

As a consequence of the study undertaken in this paper, we know that when  $\Omega$  is a bounded domain with boundary of positive mean curvature, and if  $f$  takes on just three values,  $-1, 0, 1$ , there are solutions to the Neumann problem such that the zero extension of the solution function  $u$  to the complement of  $\Omega$  jumps from 1 to 0 in the relative interior of the set  $\{y \in \partial\Omega : f(y) = -1\}$ , and jumps from  $-1$  to 0 in the relative interior of the set  $\{y \in \partial\Omega : f(y) = 1\}$ , see Theorem 5.13. In the Euclidean smooth domain setting, this means that  $\partial_{\eta}u/\|Du\| = f$  in these parts of  $\partial\Omega$ . We also give examples to show that such a phenomenon cannot be guaranteed if  $\partial\Omega$  is not of positive mean curvature. Thus as in the case of the Dirichlet problem, here too the mean curvature of  $\partial\Omega$  plays a key role in interpreting the boundary condition

$$\frac{\partial_{\eta}u}{\|Du\|} = f \text{ on } \partial\Omega,$$

both in the Euclidean and in the non-Euclidean metric setting. In the metric setting, the notion of positive mean curvature of  $\partial\Omega$  was first proposed in [29]; we utilize this notion in the present paper.

Especially in metric spaces, more attention has generally been given to Dirichlet problems than to Neumann problems. In this setting, nonlinear potential theory for Dirichlet problems when  $p > 1$  is now well developed, see the monograph [4] as well as e.g. [5, 8, 9, 42]. See also e.g. [35, 37, 41, 44, 46] for previous studies of the Dirichlet problem when  $p = 1$  in the Euclidean setting, and [18, 25, 29] in the metric setting. By contrast, Neumann problems have been studied very little. The paper [10] dealt mostly with the homogeneous Neumann boundary value problem, while in the paper [32], a Neumann problem was formulated as the minimization of the functional

$$I_p(u) = \int_{\Omega} g_u^p d\mu + \int_{\partial\Omega} Tu f dP(\Omega, \cdot),$$

where  $g_u$  is an upper gradient of  $u$  and  $p > 1$ , see Section 2 for notation. The functional  $I_1$  is obtained by replacing the first term with  $\|Du\|(\Omega)$ . In the Euclidean setting, with  $\Omega$  a smooth domain, a variant of this boundary value problem was studied in [36], and a connection between the problem for  $p > 1$  and the problem for  $p = 1$  was established through a study of the behavior of solutions  $u_p$  for  $p > 1$  as  $p \rightarrow 1^+$ . For functions  $f \in L^{\infty}(\partial\Omega)$ , the following norm was associated in [36, 37]:

$$\|f\|_* = \sup \left\{ \frac{\int_{\partial\Omega} fw dP(\Omega, \cdot)}{\|Dw\|(\Omega)} : w \in BV(\Omega) \text{ with } w \neq 0, \int_{\partial\Omega} w dP(\Omega, \cdot) = 0 \right\}.$$

The Neumann problem for  $p = 1$  was studied in [37] and then in [36] for Euclidean domains with Lipschitz boundary, and with  $\|f\|_* \leq 1$ . The paper [36] also gave an application of this problem to the study of electrical conductivity. We point out here that the condition  $\|f\|_* \leq 1$  gives the minimal energy  $I_1(u) = 0$ , and hence constant functions will certainly minimize the energy. Our focus in the present paper is to study the situation corresponding to  $\|f\|_* > 1$ , in which case there are no minimizers for the energy  $I_1$  if one seeks to minimize  $I_1(u)$  within the class of *all* functions  $u \in BV(\Omega)$ , see the discussion in the proof of [36, Proposition 3.1]. Thus we are compelled to add a further natural constraint on the competitor functions  $u$ , namely that  $-1 \leq u \leq 1$ . This constraint is not as restrictive as it might seem, and instead for any  $\beta > 0$  we can also consider constraints of the form that all competitor functions satisfy  $-\beta \leq u \leq \beta$ . Then  $u_\beta$  is a minimizer for the constraint that all competitors  $v$  should satisfy  $-\beta \leq v \leq \beta$  if and only if  $\beta^{-1}u_\beta$  is a minimizer for the constraint that all competitors  $v$  satisfy  $-1 \leq v \leq 1$ . Thus the study undertaken here complements the results in [36, 37] in the smooth Euclidean domains setting. For instance, if  $\partial\Omega$  is of positive mean curvature, then  $\|f\|_* > 1$  whenever the boundary data  $f$  is not  $\mathcal{H}$ -a.e. zero on  $\partial\Omega$  and takes on only three values,  $-1, 0, 1$ ; this interesting case, excluded in the studies in [36, 37], is covered in Section 5 of the present paper.

For an alternate (but equivalent) framing of the Neumann boundary value problem for  $p = 1$ , see [39]. The paper [39] also gives an application of the problem to the study of conductivity, see [39, Section 1.1]. The problem as framed in [39] is not tractable in the metric setting as it relies heavily on the theory of divergence free  $L^\infty$ -vector fields, a tool that is lacking in the non-smooth setting. In this paper, following the formulation given in [32], we consider minimization of the functional

$$I(u) = \|Du\|(\Omega) + \int_{\partial^*\Omega} Tuf \, dP(\Omega, \cdot).$$

We study the existence, uniqueness, regularity, and stability properties of solutions. In Section 3 we consider basic properties of solutions and note that they are generally nonunique. However, in Proposition 3.8 we show that if a solution exists, it can be taken to be of the form  $\chi_{E_1} - \chi_{E_2}$  for disjoint sets  $E_1, E_2 \subset \Omega$ . In most of the rest of the paper, we consider only such solutions. It is clear that these solutions cannot exhibit much interior regularity, but in Proposition 3.14 we show that  $\chi_{E_1}$  and  $\chi_{E_2}$  are functions of least gradient.

In Section 4 we show that under some regularity assumptions on  $\Omega$ , and additionally with  $-1 \leq f \leq 1$ , the functional  $I(\cdot)$  is lower semicontinuous with respect to convergence in  $L^1(\Omega)$ , and we use this fact to establish the existence of solutions; this is Theorem 4.15. In Section 5 we study the boundary regularity of solutions when  $\Omega$  has boundary of positive mean curvature and  $f$  only takes the values  $-1, 0, 1$ . While solutions are generally nonunique, in Theorem 6.5 we show that so-called minimal solutions are unique. In Section 7 we study stability properties of solutions with respect to boundary data, and show that a convergent sequence of boundary data yields a sequence of solutions that converges up to a subsequence; this is Theorem 7.4. In Theorem 7.9 we present one method of explicitly constructing a solution for limit boundary data. Finally, in Section 8 we consider the Neumann problem without the constraint  $-1 \leq u \leq 1$ .

## 2 Notation and definitions

In this section we introduce the necessary notation and assumptions.

In this paper,  $(X, d, \mu)$  is a complete metric space equipped with a Borel regular outer measure  $\mu$  satisfying a doubling property, that is, there is a constant  $C_d \geq 1$  such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball  $B = B(x, r)$  with center  $x \in X$  and radius  $r > 0$ . If a property holds outside a set of  $\mu$ -measure zero, we say that it holds almost everywhere, or a.e. We assume that  $X$  consists of at least two points. When we want to specify that a constant  $C$  depends on the parameters  $a, b, \dots$ , we write  $C = C(a, b, \dots)$ .

A complete metric space with a doubling measure is proper, that is, closed and bounded subsets are compact. Since  $X$  is proper, for any open set  $\Omega \subset X$  we define  $\text{Lip}_{\text{loc}}(\Omega)$  to be the space of functions that are

Lipschitz in every open  $\Omega' \Subset \Omega$ . Here  $\Omega' \Subset \Omega$  means that  $\overline{\Omega'}$  is a compact subset of  $\Omega$ . Other local spaces of functions are defined analogously.

For any set  $A \subset X$  and  $0 < R < \infty$ , the restricted spherical Hausdorff content of codimension 1 is defined by

$$\mathcal{H}_R(A) = \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i \leq R \right\}.$$

The codimension 1 Hausdorff measure of a set  $A \subset X$  is given by

$$\mathcal{H}(A) = \lim_{R \rightarrow 0} \mathcal{H}_R(A).$$

The measure theoretic boundary  $\partial^* E$  of a set  $E \subset X$  is the set of points  $x \in X$  at which both  $E$  and its complement have positive upper density, i.e.

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \rightarrow 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.$$

The measure theoretic interior and exterior of  $E$  are defined respectively by

$$I_E = \left\{ x \in X : \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} = 0 \right\} \quad (2.1)$$

and

$$O_E = \left\{ x \in X : \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 0 \right\}. \quad (2.2)$$

A curve  $\gamma$  is a nonconstant rectifiable continuous mapping from a compact interval into  $X$ . The length of a curve  $\gamma$  is denoted by  $\ell_\gamma$ . We will assume every curve to be parametrized by arc-length, which can always be done (see e.g. [15, Theorem 3.2]). A nonnegative Borel function  $g$  on  $X$  is an upper gradient of an extended real-valued function  $u$  on  $X$  if for all curves  $\gamma$  on  $X$ , we have

$$|u(x) - u(y)| \leq \int_0^{\ell_\gamma} g(\gamma(s)) ds, \quad (2.3)$$

where  $x$  and  $y$  are the end points of  $\gamma$ . We interpret  $|u(x) - u(y)| = \infty$  whenever at least one of  $|u(x)|, |u(y)|$  is infinite. Upper gradients were originally introduced in [20].

If  $g$  is a nonnegative  $\mu$ -measurable function on  $X$  and (2.3) holds for 1-a.e. curve, we say that  $g$  is a 1-weak upper gradient of  $u$ . A property holds for 1-a.e. curve if it fails only for a curve family with zero 1-modulus. A family  $\Gamma$  of curves is of zero 1-modulus if there is a nonnegative Borel function  $\rho \in L^1(X)$  such that for all curves  $\gamma \in \Gamma$ , the curve integral  $\int_\gamma \rho ds$  is infinite.

Let  $\Omega \subset X$  be open. By only considering curves in  $\Omega$ , we can say that  $g$  is an upper gradient of  $u$  in  $\Omega$ . We let

$$\|u\|_{N^{1,1}(\Omega)} = \|u\|_{L^1(\Omega)} + \inf \|g\|_{L^1(\Omega)},$$

where the infimum is taken over all upper gradients  $g$  of  $u$  in  $\Omega$ . The substitute for the Sobolev space  $W^{1,1}(\Omega)$  in the metric setting is the Newton-Sobolev space

$$N^{1,1}(\Omega) := \{u : \|u\|_{N^{1,1}(\Omega)} < \infty\}.$$

We understand Newton-Sobolev functions to be defined everywhere (even though  $\|\cdot\|_{N^{1,1}(\Omega)}$  is then only a seminorm). For more on Newton-Sobolev spaces, we refer to [4, 21, 43].

The 1-capacity of a set  $A \subset X$  is given by

$$\text{Cap}_1(A) = \inf \|u\|_{N^{1,1}(X)}, \quad (2.4)$$

where the infimum is taken over all functions  $u \in N^{1,1}(X)$  such that  $u \geq 1$  in  $A$ . We know that when  $X$  supports a (1, 1)-Poincaré inequality (see below),  $\text{Cap}_1$  is an outer capacity, meaning that

$$\text{Cap}_1(A) = \inf \{ \text{Cap}_1(U) : U \supset A \text{ is open} \}$$

for any  $A \subset X$ , see e.g. [4, Theorem 5.31]. If a property holds outside a set  $A \subset X$  with  $\text{Cap}_1(A) = 0$ , we say that it holds 1-quasieverywhere, or 1-q.e.

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, following [38]. See also e.g. [2, 11, 12, 14, 45] for the classical theory in the Euclidean setting. For  $u \in L^1_{\text{loc}}(X)$ , we define the total variation of  $u$  in  $X$  to be

$$\|Du\|(X) = \inf \left\{ \liminf_{i \rightarrow \infty} \int_X g_{u_i} d\mu : u_i \in \text{Lip}_{\text{loc}}(X), u_i \rightarrow u \text{ in } L^1_{\text{loc}}(X) \right\},$$

where each  $g_{u_i}$  is an upper gradient of  $u_i$ . We say that a function  $u \in L^1(X)$  is of bounded variation, denoted by  $u \in \text{BV}(X)$ , if  $\|Du\|(X) < \infty$ . By replacing  $X$  with an open set  $\Omega \subset X$  in the definition of the total variation, we can define  $\|Du\|(\Omega)$ . For an arbitrary set  $A \subset X$ , we define

$$\|Du\|(A) = \inf \{ \|Du\|(\Omega) : A \subset \Omega, \Omega \subset X \text{ is open} \}.$$

If  $u \in \text{BV}(X)$ ,  $\|Du\|(\cdot)$  is a finite Radon measure on  $X$  by [38, Theorem 3.4]. A  $\mu$ -measurable set  $E \subset X$  is said to be of finite perimeter in  $\Omega$  if  $\|D\chi_E\|(\Omega) < \infty$ , where  $\chi_E$  is the characteristic function of  $E$ . The perimeter of  $E$  in  $\Omega$  is also denoted by

$$P(E, \Omega) := \|D\chi_E\|(\Omega).$$

We have the following coarea formula from [38, Proposition 4.2]: if  $\Omega \subset X$  is an open set and  $u \in \text{BV}(\Omega)$ , then for any Borel set  $A \subset \Omega$ ,

$$\|Du\|(A) = \int_{-\infty}^{\infty} P(\{u > t\}, A) dt. \quad (2.5)$$

We will assume throughout that  $X$  supports a  $(1, 1)$ -Poincaré inequality, meaning that there exist constants  $C_P > 0$  and  $\lambda \geq 1$  such that for every ball  $B(x, r)$ , every locally integrable function  $u$  on  $X$ , and every upper gradient  $g$  of  $u$ , we have

$$\int_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq C_P r \int_{B(x,\lambda r)} g d\mu,$$

where

$$u_{B(x,r)} := \int_{B(x,r)} u d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u d\mu.$$

By applying the Poincaré inequality to approximating locally Lipschitz functions in the definition of the total variation, we get the following for  $\mu$ -measurable sets  $E \subset X$ :

$$\min\{\mu(B(x, r) \cap E), \mu(B(x, r) \setminus E)\} \leq 2C_P r P(E, B(x, \lambda r)). \quad (2.6)$$

For an open set  $\Omega \subset X$  and a  $\mu$ -measurable set  $E \subset X$  with  $P(E, \Omega) < \infty$ , we know that for any Borel set  $A \subset \Omega$ ,

$$P(E, A) = \int_{\partial^* E \cap A} \theta_E d\mathcal{H}, \quad (2.7)$$

where  $\theta_E: X \rightarrow [\alpha, C_d]$  with  $\alpha = \alpha(C_d, C_P, \lambda) > 0$ , see [1, Theorem 5.3] and [3, Theorem 4.6].

The lower and upper approximate limits of a function  $u$  on  $X$  are defined respectively by

$$u^\wedge(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{u < t\})}{\mu(B(x, r))} = 0 \right\}$$

and

$$u^\vee(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{u > t\})}{\mu(B(x, r))} = 0 \right\}.$$

The jump set of a function  $u$  is the set

$$S_u := \{x \in X : u^\wedge(x) < u^\vee(x)\}.$$

By [3, Theorem 5.3], the variation measure of a BV function can be decomposed into the absolutely continuous and singular part, and the latter into the Cantor and jump part, as follows. Given an open set  $\Omega \subset X$  and  $u \in BV(\Omega)$ , we have for any Borel set  $A \subset X$

$$\begin{aligned} \|Du\|(A) &= \|Du\|^a(A) + \|Du\|^s(A) \\ &= \|Du\|^a(A) + \|Du\|^c(A) + \|Du\|^j(A) \\ &= \int_A a \, d\mu + \|Du\|^c(A) + \int_{A \cap S_u} \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x) \, dt \, d\mathcal{H}(x), \end{aligned} \quad (2.8)$$

where  $a \in L^1(\Omega)$  is the density of the absolutely continuous part and the functions  $\theta_{\{u>t\}}$  are as in (2.7).

**Definition 2.9.** Let  $\Omega \subset X$  be an open set and let  $u$  be a  $\mu$ -measurable function on  $\Omega$ . For  $x \in \partial\Omega$ , the number  $Tu(x)$  is the *trace* of  $u$  if

$$\lim_{r \rightarrow 0} \int_{B(x,r) \cap \Omega} |u - Tu(x)| \, d\mu = 0.$$

It is straightforward to check that the trace is always a Borel function on the set where it exists.

**Definition 2.10.** Let  $\Omega \subset X$  be an open set. A function  $u \in BV_{\text{loc}}(\Omega)$  is said to be of *least gradient* in  $\Omega$  if

$$\|Du\|(\Omega) \leq \|D(u + \varphi)\|(\Omega)$$

for every  $\varphi \in BV(\Omega)$  with compact support in  $\Omega$ .

### 3 Preliminary results

In this section we define the Neumann problem and consider various basic properties of solutions.

In this section, we always assume that  $\Omega \subset X$  is a nonempty bounded open set with  $P(\Omega, X) < \infty$ , such that for any  $u \in BV(\Omega)$ , the trace  $Tu(x)$  exists for  $\mathcal{H}$ -a.e.  $x \in \partial^* \Omega$  and thus also for  $P(\Omega, \cdot)$ -a.e.  $x \in \partial^* \Omega$ , by (2.7). See [31, Theorem 3.4] for conditions on  $\Omega$  that guarantee that this holds.

For some of our results, we will also assume that the following exterior measure density condition holds:

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, r) \setminus \Omega)}{\mu(B(x, r))} > 0 \quad \text{for } \mathcal{H}\text{-a.e. } x \in \partial\Omega. \quad (3.1)$$

Moreover, in this section we always assume that  $f \in L^1(\partial^* \Omega, P(\Omega, \cdot))$  such that

$$\int_{\partial^* \Omega} f \, dP(\Omega, \cdot) = 0. \quad (3.2)$$

Throughout this paper we will consider the following functional: for  $u \in BV(\Omega)$ , let

$$I(u) = \|Du\|(\Omega) + \int_{\partial^* \Omega} Tu f \, dP(\Omega, \cdot).$$

First we note the following basic property of the functional. We denote  $u_+ = \max\{u, 0\}$  and  $u_- = \max\{-u, 0\}$ .

**Lemma 3.3.** For any  $u \in BV(\Omega)$ , we have  $I(u) = I(u_+) + I(u_-)$ .

*Proof.* Note that for any  $\mu$ -measurable  $E \subset X$ , we have  $P(E, \Omega) = P(\Omega \setminus E, \Omega)$ . Since  $\mu$  is  $\sigma$ -finite on  $X$ , it follows that for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}$  we have  $\mu(\{u = t\}) = 0$  and thus  $P(\{u < t\}, \Omega) = P(\{u \leq t\}, \Omega)$ , where  $\mathcal{L}^1$  is the Lebesgue

measure. Thus by the BV coarea formula (2.5), we have

$$\begin{aligned}
I(u) &= \int_{-\infty}^{\infty} P(\{u > t\}, \Omega) dt + \int_{\partial^* \Omega} T u f dP(\Omega, \cdot) \\
&= \int_0^{\infty} P(\{u > t\}, \Omega) dt + \int_{-\infty}^0 P(\{u < t\}, \Omega) dt + \int_{\partial^* \Omega} T u f dP(\Omega, \cdot) \\
&= \int_{-\infty}^{\infty} P(\{u_+ > t\}, \Omega) dt + \int_{-\infty}^{\infty} P(\{u_- > t\}, \Omega) dt + \int_{\partial^* \Omega} T u f dP(\Omega, \cdot) \\
&= \|Du_+\|(\Omega) + \int_{\partial^* \Omega} T u_+ f dP(\Omega, \cdot) + \|Du_-\|(\Omega) - \int_{\partial^* \Omega} T u_- f dP(\Omega, \cdot) \\
&= I(u_+) + I(-u_-). \quad \square
\end{aligned}$$

Note that for  $u \equiv 0$ ,  $I(u) = 0$ . Thus, if  $I(u) \geq 0$  for all  $u \in BV(\Omega)$ , then we find a minimizer simply by taking the zero function. Hence, we are more interested in the case where  $I(u) < 0$  for some  $u \in BV(\Omega)$ . But then

$$\lim_{\beta \rightarrow \infty} I(\beta u) = \lim_{\beta \rightarrow \infty} \beta I(u) = -\infty.$$

Thus, we consider the following restricted minimization problem.

**Definition 3.4.** We say that a function  $u \in BV(\Omega)$  solves the restricted Neumann boundary value problem with boundary data  $f$  if  $-1 \leq u \leq 1$  and  $I(u) \leq I(v)$  for all  $v \in BV(\Omega)$  with  $-1 \leq v \leq 1$ .

The restricted problem does not always have a solution. It may also have only trivial, i.e., constant, solutions even though the boundary data are non-trivial. Moreover, non-trivial solutions need not be unique. In the Euclidean setting these issues were observed in [36].

**Example 3.5.** In the unweighted plane (endowed with the Euclidean distance), consider the unit square, i.e.,  $\Omega = (0, 1)^2$ . Fix a constant  $a > 0$  and let  $f = -a$  on the middle third portion of the bottom side,  $f = a$  on the middle third portion of the top side, and  $f = 0$  elsewhere on the boundary.

If  $a > 1$ , then

$$\inf_{u \in BV(\Omega), \|u\|_{L^\infty(\Omega)} \leq 1} I(u) = \frac{2(1-a)}{3},$$

but no admissible function gives this infimum.

**Example 3.6.** Consider again the unit square  $\Omega = (0, 1)^2$  in the Euclidean plane. Fix a constant  $a > 0$  and let  $f = -a$  on the bottom side,  $f = a$  on the top side, and  $f = 0$  on the vertical sides.

- (a) If  $a \in (0, 1)$ , then  $\inf_u I(u) = 0$ , which is attained only by  $u \equiv c$  for any constant  $c \in [-1, 1]$ . The fact that no other solutions exist can be proven using Proposition 3.8 and Proposition 3.14 below.
- (b) If  $a = 1$ , then  $\inf_u I(u) = 0$ , which is attained by any constant function  $u \equiv c$  with  $c \in [-1, 1]$  as well as by any function  $u(x, y) = v(y)$ ,  $(x, y) \in \Omega$ , where  $v$  is an arbitrary decreasing function with  $v(0^+) = 1$ ,  $v(1^-) = -1$ .

See also Example 7.5 for an example of nonuniqueness with  $I(u) < 0$ .

Next we will show that it suffices to consider only a special subclass of BV functions as candidates for a solution to the restricted Neumann problem. First we note that we have the following version of Cavalieri's principle, which can be obtained from the usual Cavalieri's principle by decomposing  $v$  into its positive and negative parts.

**Lemma 3.7.** Let  $\nu$  be a signed Radon measure on  $X$ . Then, for any nonnegative  $h \in L^1(X, |\nu|)$ ,

$$\int_X h d\nu = \int_0^{\infty} \nu(\{h > t\}) dt.$$

**Proposition 3.8.** *Let  $u \in BV(\Omega)$  with  $-1 \leq u \leq 1$ . Then, there exist disjoint  $\mu$ -measurable sets  $E_1, E_2 \subset \Omega$  such that*

$$I(\chi_{E_1} - \chi_{E_2}) \leq I(u).$$

Furthermore, if  $u$  is a solution to the restricted Neumann problem with boundary data  $f$ , then for  $\mathcal{L}^1$ -a.e.  $t_1, t_2 \in (0, 1)$ , the sets

$$E_1 := \{x \in \Omega : u(x) > t_1\} \quad \text{and} \quad E_2 := \{x \in \Omega : u(x) < -t_2\}$$

give a solution  $\chi_{E_1} - \chi_{E_2}$  to the same restricted Neumann problem.

*Proof.* By Lemma 3.3 we have  $I(u) = I(u_+) + I(-u_-)$ . By using the BV coarea formula (2.5), and applying the above Cavalieri's principle with  $dv = f dP(\Omega, \cdot)$ ,

$$I(u_+) = \int_0^1 \left( P(\{u_+ > t\}, \Omega) + \int_{\{Tu_+ > t\}} f dP(\Omega, \cdot) \right) dt. \quad (3.9)$$

If  $t \in (0, 1)$  and  $Tu_+(x) < t$  for some  $x \in \partial^* \Omega$ , then

$$\begin{aligned} \limsup_{r \rightarrow 0} \int_{B(x,r) \cap \Omega} |\chi_{\{u_+ > t\}} - 0| d\mu &= \limsup_{r \rightarrow 0} \frac{\mu(B(x,r) \cap \{u_+ > t\})}{\mu(B(x,r) \cap \Omega)} \\ &\leq \frac{1}{t - Tu_+(x)} \limsup_{r \rightarrow 0} \int_{B(x,r) \cap \Omega} |u_+ - Tu_+(x)| d\mu = 0. \end{aligned}$$

Thus,  $T\chi_{\{u_+ > t\}}(x) > 0$  yields that  $Tu_+(x) \geq t$ . Conversely, we see that if  $Tu_+(x) > t$ , then  $T\chi_{\{u_+ > t\}}(x) = 1$ . In conclusion,

$$\chi_{\{Tu_+ > t\}} \leq T\chi_{\{u_+ > t\}} \leq \chi_{\{Tu_+ \geq t\}}.$$

However,  $P(\Omega, \{Tu_+ = t\}) = 0$  for  $\mathcal{L}^1$ -a.e.  $t \in (0, 1)$ , since  $P(\Omega, \cdot)$  is a finite measure. Thus (3.9) becomes

$$I(u_+) = \int_0^1 \left( P(\{u_+ > t\}, \Omega) + \int_{\partial^* \Omega} T\chi_{\{u_+ > t\}} f dP(\Omega, \cdot) \right) dt,$$

that is,

$$I(u_+) = \int_0^1 I(\chi_{\{u_+ > t\}}) dt. \quad (3.10)$$

Thus there is  $t_1 \in (0, 1)$  such that  $I(\chi_{\{u_+ > t_1\}}) \leq I(u_+)$ , which is the same as  $I(\chi_{\{u > t_1\}}) \leq I(u)$ .

Denoting  $I(\cdot) = I_f(\cdot)$  to make the dependence on  $f$  explicit, with the substitutions of  $f$  by  $-f$  and  $u_+$  by  $u_-$ , inequality (3.10) becomes

$$I_f(-u_-) = I_{-f}(u_-) = \int_0^1 I_{-f}(\chi_{\{u_- > t\}}) dt = \int_0^1 I_f(-\chi_{\{u_- > t\}}) dt. \quad (3.11)$$

Therefore, there is  $t_2 \in (0, 1)$  such that  $I_f(-\chi_{\{u_- > t_2\}}) \leq I_f(-u_-)$ , i.e.,  $I_f(-\chi_{\{u < -t_2\}}) \leq I_f(-u_-)$ . Letting  $E_1 = \{u > t_1\}$  and  $E_2 = \{u < -t_2\}$ , we now have by Lemma 3.3

$$I(\chi_{E_1} - \chi_{E_2}) = I(\chi_{E_1}) + I(-\chi_{E_2}) \leq I(u_+) + I(-u_-) = I(u),$$

proving the first claim.

Now let  $u$  be a solution, and  $t_1$  and  $t_2$  as above. If we had  $I(\chi_{\{u > s\}}) < I(u_+)$  for some  $s \in (0, 1)$ , then

$$I(\chi_{\{u > s\}}) + I(-\chi_{\{u < -t_2\}}) < I(u),$$

which contradicts  $u$  being a solution. Thus from (3.10) it follows that  $I(\chi_{\{u > s\}}) = I(u_+)$  for  $\mathcal{L}^1$ -a.e.  $s \in (0, 1)$ . Analogously, using (3.11) we find that  $I(-\chi_{\{u < -s\}}) = I(-u_-)$  for  $\mathcal{L}^1$ -a.e.  $s \in (0, 1)$ , and this proves the second claim.  $\square$

**Lemma 3.12.** *If  $E \subset \Omega$  is of finite perimeter in  $\Omega$ , then  $I(-\chi_{\Omega \setminus E}) = I(\chi_E)$ . Thus, if  $E_1, E_2 \subset \Omega$  are disjoint sets such that  $\chi_{E_1} - \chi_{E_2}$  solves the restricted Neumann problem, then necessarily  $I(\chi_{E_1}) = I(-\chi_{E_2})$ , and  $\chi_{E_1} - \chi_{\Omega \setminus E_1}$  is also a solution.*



*Proof.* If  $P(E, \Omega) < \infty$ , note that  $P(\Omega \setminus E, \Omega) = P(E, \Omega)$ , and that for  $\mathcal{H}$ -a.e.  $x \in \partial^* \Omega$ ,

$$T\chi_E(x) + T\chi_{\Omega \setminus E}(x) = T(\chi_E(x) + \chi_{\Omega \setminus E}(x)) = T\chi_\Omega(x) = 1.$$

Thus,

$$\begin{aligned} I(-\chi_{\Omega \setminus E}) &= P(\Omega \setminus E, \Omega) - \int_{\partial^* \Omega} T\chi_{\Omega \setminus E} f \, dP(\Omega, \cdot) \\ &= P(E, \Omega) - \int_{\partial^* \Omega} T\chi_{\Omega \setminus E} f \, dP(\Omega, \cdot) \\ &= P(E, \Omega) - \int_{\partial^* \Omega} T\chi_{\Omega \setminus E} f \, dP(\Omega, \cdot) + \int_{\partial^* \Omega} f \, dP(\Omega, \cdot) \quad \text{by (3.2)} \\ &= P(E, \Omega) + \int_{\partial^* \Omega} T\chi_E f \, dP(\Omega, \cdot) \\ &= I(\chi_E). \end{aligned}$$

Next, let  $E_1, E_2 \subset \Omega$  be disjoint sets such that  $\chi_{E_1} - \chi_{E_2}$  solves the restricted Neumann problem. If  $I(\chi_{E_1}) < I(-\chi_{E_2})$ , then by the above, we also have  $I(-\chi_{\Omega \setminus E_1}) < I(-\chi_{E_2})$ . Then, by Lemma 3.3,

$$I(\chi_{E_1} - \chi_{\Omega \setminus E_1}) = I(\chi_{E_1}) + I(-\chi_{\Omega \setminus E_1}) < I(\chi_{E_1}) + I(-\chi_{E_2}) = I(\chi_{E_1} - \chi_{E_2}),$$

a contradiction. Similarly,  $I(\chi_{E_1}) > I(-\chi_{E_2})$  is impossible. Moreover, now

$$\begin{aligned} I(\chi_{E_1} - \chi_{\Omega \setminus E_1}) &= I(\chi_{E_1}) + I(-\chi_{\Omega \setminus E_1}) \\ &= 2I(\chi_{E_1}) = I(\chi_{E_1}) + I(-\chi_{E_2}) = I(\chi_{E_1} - \chi_{E_2}), \end{aligned}$$

so that  $\chi_{E_1} - \chi_{\Omega \setminus E_1}$  is also a solution.  $\square$

**Lemma 3.13.** *Let  $E_1, E_2 \subset \Omega$  be disjoint  $\mu$ -measurable sets. Then,  $\chi_{E_1} - \chi_{E_2}$  solves the restricted Neumann problem if and only if*

$$I(\chi_{E_1}) \leq I(\chi_F) \quad \text{and} \quad I(-\chi_{E_2}) \leq I(-\chi_F)$$

for all  $\mu$ -measurable sets  $F \subset \Omega$ .

*Proof.* Suppose that  $\chi_{E_1} - \chi_{E_2}$  is a solution. If there is a  $\mu$ -measurable set  $F \subset \Omega$  with  $I(\chi_F) < I(\chi_{E_1})$ , then by Lemma 3.3 and Lemma 3.12

$$\begin{aligned} I(\chi_F - \chi_{\Omega \setminus F}) &= I(\chi_F) + I(-\chi_{\Omega \setminus F}) \\ &= 2I(\chi_F) < 2I(\chi_{E_1}) = I(\chi_{E_1}) + I(-\chi_{E_2}) = I(\chi_{E_1} - \chi_{E_2}), \end{aligned}$$

a contradiction. Similarly,  $I(-\chi_F) < I(-\chi_{E_2})$  is impossible.

If  $E_1, E_2 \subset \Omega$  are such that  $I(\chi_{E_1}) \leq I(\chi_F)$  and  $I(-\chi_{E_2}) \leq I(-\chi_F)$  for all  $\mu$ -measurable sets  $F \subset \Omega$ , then

$$I(\chi_{E_1} - \chi_{E_2}) = I(\chi_{E_1}) + I(-\chi_{E_2}) \leq I(\chi_{F_1}) + I(-\chi_{F_2}) = I(\chi_{F_1} - \chi_{F_2})$$

for any two disjoint  $\mu$ -measurable sets  $F_1, F_2 \subset \Omega$ . In view of Proposition 3.8,  $\chi_{E_1} - \chi_{E_2}$  must be a solution.  $\square$

Recall that a function  $u \in BV(\Omega)$  is of least gradient in  $\Omega$  if

$$\|Du\|(\Omega) \leq \|D(u + \varphi)\|(\Omega)$$

for every  $\varphi \in BV(\Omega)$  with compact support in  $\Omega$ .

**Proposition 3.14.** *Let  $E_1, E_2 \subset \Omega$  be disjoint sets such that  $\chi_{E_1} - \chi_{E_2}$  solves the restricted Neumann problem. Then,  $\chi_{E_1}$  and  $\chi_{E_2}$  are functions of least gradient in  $\Omega$ .*

*Proof.* To show that  $\chi_{E_1}$  is a function of least gradient, it suffices to show that  $\|D\chi_{E_1}\|(\Omega) \leq \|D\chi_F\|(\Omega)$  whenever  $F \subset \Omega$  is a  $\mu$ -measurable set with  $F \triangle E_1 \Subset \Omega$ , see [22, Lemma 3.2]. Let  $F$  be such a set. By Lemma 3.13,  $I(\chi_{E_1}) \leq I(\chi_F)$ . On the other hand,  $\chi_{E_1} = \chi_F$  in a neighborhood of  $\partial\Omega$ . It follows that

$$\|D\chi_{E_1}\|(\Omega) \leq \|D\chi_F\|(\Omega),$$

so that  $\chi_{E_1}$  is of least gradient. The proof for  $E_2$  is analogous.  $\square$

The above is our main result on the interior regularity of solutions; from the proposition it follows that the sets  $E_1, E_2$  and their complements are porous in  $\Omega$ , see [22, Theorem 5.2].

Since solutions can be constructed from sets  $E$  of finite perimeter in  $\Omega$  and since  $\Omega$  is itself of finite perimeter in  $X$ , it is useful to know that the sets  $E$  are also of finite perimeter in  $X$ .

**Theorem 3.15** ([25, Corollary 6.13]). *Assume that  $\widehat{\Omega} \subset X$  is a bounded open set with  $P(\widehat{\Omega}, X) < \infty$ , and suppose that there exists  $N \subset \partial\widehat{\Omega}$  with  $\mathcal{H}(N) < \infty$  such that*

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, r) \setminus \widehat{\Omega})}{\mu(B(x, r))} > 0$$

for every  $x \in \partial\widehat{\Omega} \setminus N$ . Let  $E \subset \widehat{\Omega}$  such that  $P(E, \widehat{\Omega}) < \infty$ . Then  $E$  is of finite perimeter in  $X$ .

Note that if  $\widehat{\Omega}$  satisfies the condition listed in (3.1), then  $\mathcal{H}(N) = 0$  above.

**Lemma 3.16.** *Assume that  $\Omega$  satisfies the exterior measure density condition (3.1). Let  $E \subset \Omega$  be a  $\mu$ -measurable set with  $P(E, X) < \infty$ . Then, for any Borel set  $A \subset \partial\Omega$ , we have*

$$P(E, A) = P(\Omega, A \cap \{T\chi_E = 1\}).$$

By Theorem 3.15, we can equally well only assume that  $P(E, \Omega) < \infty$ .

*Proof.* Note that the trace  $T\chi_E(x)$  is defined for  $\mathcal{H}$ -a.e.  $x \in \partial^*\Omega$  and can only take the values 0 and 1. Also,  $P(E, \cdot)$  is concentrated on  $\partial^*E$ , and  $(\partial\Omega \setminus \partial^*\Omega) \cup \{T\chi_E = 0\} \subset O_E$ , i.e., the measure theoretic exterior of  $E$  as defined by (2.2). Thus, we have

$$P(E, \partial\Omega \setminus \{T\chi_E = 1\}) \leq P(E, \partial\Omega \setminus \partial^*\Omega) + P(E, \partial^*\Omega \cap \{T\chi_E = 0\}) = 0. \quad (3.17)$$

Consider a point  $x \in \partial\Omega$  where  $T\chi_E(x) = 1$ . We have

$$\frac{\mu(B(x, r) \cap (\Omega \triangle E))}{\mu(B(x, r))} \rightarrow 0 \quad \text{as } r \rightarrow 0,$$

so that  $x \in \partial^*E$  if and only if  $x \in \partial^*\Omega$ , and according to [2, Proposition 6.2], for  $\mathcal{H}$ -a.e. such point we have  $\theta_E(x) = \theta_\Omega(x)$ ; recall (2.7). In total, by (3.17) and by applying (2.7) twice,

$$\begin{aligned} P(E, A) &= P(E, A \cap \{T\chi_E = 1\}) = \int_{A \cap \{T\chi_E = 1\} \cap \partial^*E} \theta_E d\mathcal{H} \\ &= \int_{A \cap \{T\chi_E = 1\} \cap \partial^*\Omega} \theta_\Omega d\mathcal{H} = P(\Omega, A \cap \{T\chi_E = 1\}). \end{aligned} \quad \square$$

**Lemma 3.18.** *Suppose that  $\Omega$  satisfies the exterior measure density condition (3.1), and that  $-1 \leq f \leq 1$ . Let  $E \subset \Omega$  be of finite perimeter in  $\Omega$ . Then*

$$P(E, X) \leq I(\chi_E) + 2P(\Omega, X).$$

*Proof.* By Theorem 3.15,  $P(E, X) < \infty$ . By the definition of the functional and the fact that  $-1 \leq f \leq 1$ ,

$$P(E, \Omega) \leq I(\chi_E) + P(\Omega, X),$$

whereas by Lemma 3.16,  $P(E, \partial\Omega) \leq P(\Omega, X)$ . Thus we get

$$P(E, X) = P(E, \Omega) + P(E, \partial\Omega) \leq I(\chi_E) + 2P(\Omega, X). \quad \square$$

## 4 Existence of solutions

In this section, we prove that under fairly mild assumptions on  $\Omega$ , solutions to the restricted Neumann problem given on page 7 exist. This is Theorem 4.15.

We say that a set  $A \subset X$  is 1-*quasiopen* if for every  $\varepsilon > 0$  there is an open set  $G \subset X$  with  $\text{Cap}_1(G) < \varepsilon$  such that  $A \cup G$  is open. Note that 1-quasiopen sets do not in general form a topology: as is noted in [6], all singletons in unweighted  $\mathbb{R}^n$ ,  $n \geq 2$ , are 1-quasiopen, but not all sets are 1-quasiopen. Nonetheless, countable unions as well as finite intersections of 1-quasiopen sets are 1-quasiopen by [13, Lemma 2.3].

The following lemma is well known in the Euclidean setting, and has been proved in the metric setting in [28, Lemma 3.8].

**Lemma 4.1.** *Let  $\Omega \subset X$  be an open set, and let  $u \in L^1_{\text{loc}}(\Omega)$  with  $\|Du\|(\Omega) < \infty$ . Then for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $A \subset \Omega$  with  $\text{Cap}_1(A) < \delta$ , then  $\|Du\|(A) < \varepsilon$ .*

From this lemma it easily follows that 1-quasiopen sets are always  $\|Du\|$ -measurable, and we will use this fact without further notice.

The total variation is easily seen to be lower semicontinuous with respect to  $L^1$ -convergence in any open set. We will need the following more general semicontinuity result that follows from [27, Theorem 4.5].

**Proposition 4.2.** *Let  $u \in L^1_{\text{loc}}(X)$  such that  $\|Du\|(X) < \infty$ , and suppose that  $u_i \rightarrow u$  in  $L^1_{\text{loc}}(X)$ . Then, for every 1-quasiopen set  $U \subset X$ , we have*

$$\|Du\|(U) \leq \liminf_{i \rightarrow \infty} \|Du_i\|(U).$$

To deal with boundary values given by a function  $f$  defined only on  $\partial\Omega$ , we first need to extend  $f$  to the whole space in a suitable way. We will consider open sets  $\Omega$  whose boundary is codimension 1 Ahlfors regular in the following sense: for every  $x \in \partial\Omega$ , every  $0 < r \leq \text{diam}(\Omega)$ , and some constant  $C_A \geq 1$ ,

$$\frac{1}{C_A} \frac{\mu(B(x, r))}{r} \leq \mathcal{H}(B(x, r) \cap \partial\Omega) \leq C_A \frac{\mu(B(x, r))}{r}. \quad (4.3)$$

**Theorem 4.4.** *Let  $\Omega \subset X$  be a bounded open set whose boundary is codimension 1 Ahlfors regular as given in (4.3). Let  $f \in L^1(\partial\Omega, \mathcal{H})$  with  $-1 \leq f \leq 1$ . Then, there exists  $Ef \in N^{1,1}(X \setminus \partial\Omega)$  with  $-1 \leq Ef \leq 1$  and*

$$\lim_{r \rightarrow 0} \int_{B(x, r)} |Ef - f(x)| d\mu = 0$$

for  $\mathcal{H}$ -a.e.  $x \in \partial\Omega$ .

*Proof.* This follows from [33] and the proofs therein. Note that the argument of the extension theorem for Besov boundary data [33, Theorem 1.1] needs to be slightly modified to produce a Newtonian extension not only inside  $\Omega$  but in the whole set  $X \setminus \partial\Omega$ . Namely, when constructing a Whitney-type decomposition  $\mathcal{W}_{X \setminus \partial\Omega}$ , we consider only balls whose distance from  $\partial\Omega$  is at most  $2 \text{diam}(\Omega)$ . Such a collection of balls covers  $\Omega$  as well as the  $2 \text{diam}(\Omega)$ -neighborhood of  $\partial\Omega$ , leaving out  $\partial\Omega$ . Then, we relax the requirements on the partition of unity  $\{\phi_{j,i}\}_{j,i}$  subordinate to  $\mathcal{W}_{X \setminus \partial\Omega}$  by demanding that

$$\begin{aligned} \sum_{j,i} \phi_{j,i}(x) &= 1 \quad \text{for } x \in X \setminus \partial\Omega, \text{ dist}(x, \partial\Omega) \leq \text{diam}(\Omega), \quad \text{and} \\ \sum_{j,i} \phi_{j,i}(x) &\leq 1 \quad \text{for } x \in X \setminus \partial\Omega, \text{ dist}(x, \partial\Omega) > \text{diam}(\Omega). \end{aligned}$$

Using such a ‘‘partition of unity’’ gives us an extension of  $f$  in the class  $N^{1,1}(X \setminus \partial\Omega) \cap \text{Lip}_{\text{loc}}(X \setminus \partial\Omega)$  such that this extension vanishes outside of the  $3 \text{diam}(\Omega)$ -neighborhood of  $\partial\Omega$ .

The extension theorem for Besov boundary data modified as described above can then be used directly in [33, Theorem 1.2] to find the desired extension  $Ef \in N^{1,1}(X \setminus \partial\Omega) \cap \text{Lip}_{\text{loc}}(X \setminus \partial\Omega)$  for any  $L^1$ -boundary data  $f: \partial\Omega \rightarrow [-1, 1]$ , truncating  $Ef$  at levels  $\pm 1$  if needed.  $\square$

Note that for any  $A \subset X$ , by [16, Theorem 4.3, Theorem 5.1] (see also [19, Corollary 5.3]) we have

$$\mathcal{H}(A) = 0 \quad \text{if and only if} \quad \text{Cap}_1(A) = 0. \quad (4.5)$$

In the following, given a ball  $B = B(x, r)$  we sometimes abbreviate  $2B := B(x, 2r)$ .

**Proposition 4.6.** *Let  $\Omega \subset X$  be a bounded open set with  $\mathcal{H}(\partial\Omega) < \infty$ , and let  $f$  be a function on  $X$  such that  $-1 \leq f \leq 1$ ,  $f \in N^{1,1}(X \setminus \partial\Omega)$ , and*

$$\lim_{r \rightarrow 0} \int_{B(x,r)} |f - f(x)| d\mu = 0 \quad (4.7)$$

for  $\mathcal{H}$ -a.e.  $x \in \partial\Omega$ . Then,  $f \in N^{1,1}(X)$ .

*Proof.* Fix  $i \in \mathbb{N}$ . By the compactness of  $\partial\Omega$ , we find a covering  $\{B_j = B(x_j, r_j)\}_{j=1}^M$  such that  $r_j \leq 1/i$  for all  $j$ , and

$$\sum_{j=1}^M \frac{\mu(B(x_j, r_j))}{r_j} < \mathcal{H}(\partial\Omega) + 1/i. \quad (4.8)$$

Then, pick  $2/r_j$ -Lipschitz functions  $\eta_j \in \text{Lip}_c(B(x_j, 2r_j))$  such that  $0 \leq \eta_j \leq 1$  and  $\eta_j = 1$  on  $B(x_j, r_j)$ . Define  $v_i = \max_{j \in \{1, \dots, M\}} \eta_j$ . Consider the function

$$f_i := (1 - v_i)f.$$

Let  $g \in L^1(X)$  be an upper gradient of  $f$  in  $X \setminus \partial\Omega$ . Clearly  $2\chi_{2B_j}/r_j$  is an upper gradient of  $\eta_j$ , and then  $2 \sum_{j=1}^M \frac{\chi_{2B_j}}{r_j}$  is an upper gradient of  $v_i$ . We show that

$$g_i := g + 2 \sum_{j=1}^M \frac{\chi_{2B_j}}{r_j}$$

is a 1-weak upper gradient of  $f_i$  in  $X$ . By the Leibniz rule, see e.g. [4, Theorem 2.15],  $g_i$  is a 1-weak upper gradient of  $f_i$  in  $X \setminus \partial\Omega$ . Take a curve  $\gamma$  such that the upper gradient inequality is satisfied by  $f_i$  and  $g_i$  on all subcurves of  $\gamma$  in  $X \setminus \partial\Omega$ ; this is true for 1-a.e.  $\gamma$ , by [4, Lemma 1.34].

Note that

$$\text{dist} \left( \partial\Omega, X \setminus \bigcup_{j=1}^M B_j \right) > 0.$$

Thus,  $\gamma$  can be split into a finite number of subcurves each of which lies either entirely in  $\bigcup_{j=1}^M B_j$ , or entirely in  $X \setminus \partial\Omega$ . If  $\gamma_1$  is a subcurve lying entirely in  $\bigcup_{j=1}^M B_j$ ,

$$|f_i(\gamma_1(0)) - f_i(\gamma_1(\ell_{\gamma_1}))| = |0 - 0| = 0,$$

so the upper gradient inequality is satisfied. If  $\gamma_2$  is a subcurve lying entirely in  $X \setminus \partial\Omega$ , then

$$|f_i(\gamma_2(0)) - f_i(\gamma_2(\ell_{\gamma_2}))| \leq \int_{\gamma_2} g_i ds$$

by our choice of  $\gamma$ . Summing over the subcurves, we obtain

$$|f_i(\gamma(0)) - f_i(\gamma(\ell_\gamma))| \leq \int_\gamma g_i ds.$$

Thus,  $g_i$  is a 1-weak upper gradient of  $f_i$  in  $X$ . By (4.8) we have

$$\|g_i\|_{L^1(X)} \leq \|g\|_{L^1(X)} + 2C_d(\mathcal{H}(\partial\Omega) + 1/i),$$

and thus  $f_i \in N^{1,1}(X)$ . Since Lipschitz functions are dense in  $N^{1,1}(X)$ , see e.g. [4, Theorem 5.1], we have also  $f_i \in \text{BV}(X)$ , with

$$\|Df_i\|(X) \leq \|g_i\|_{L^1(X)}.$$

Clearly  $f_i \rightarrow f$  in  $L^1(X)$  as  $i \rightarrow \infty$ . By lower semicontinuity,

$$\|Df\|(X) \leq \liminf_{i \rightarrow \infty} \|Df_i\|(X) \leq \|g\|_{L^1(X)} + 2C_d \mathcal{H}(\partial\Omega).$$

Thus,  $f \in \text{BV}(X)$ . Recall the decomposition of the variation measure from (2.8). Since  $f \in N^{1,1}(X \setminus \partial\Omega)$ ,  $\|Df\|^s(X \setminus \partial\Omega) = 0$ . Since  $\mathcal{H}(\partial\Omega) < \infty$ , also  $\|Df\|^c(\partial\Omega) = 0$  by [3, Theorem 5.3]. Finally, by (2.8),

$$\|Df\|^j(\partial\Omega) \leq C_d \int_{\partial\Omega \cap S_f} (f^\vee - f^\wedge) d\mathcal{H} = 0,$$

since by the Lebesgue point condition (4.7),  $f^\wedge(x) = f^\vee(x) \in \mathbb{R}$  for  $\mathcal{H}$ -a.e.  $x \in \partial\Omega$ . Thus,  $\|Df\|^s(X) = 0$ , so that by [17, Theorem 4.6, Remark 4.7], there exists a function  $h \in N^{1,1}(X)$  with  $\mu(\{h \neq f\}) = 0$ . By the Lebesgue point theorem [24, Theorem 4.1, Remark 4.2] (note that this result assumes that  $\mu(X) = \infty$ , but this assumption can be avoided by using [40, Lemma 3.1] instead of [24, Theorem 3.1] in the proof of Lebesgue point theorem found in [24]),

$$\lim_{r \rightarrow 0} \int_{B(x,r)} |f - h(x)| d\mu = \lim_{r \rightarrow 0} \int_{B(x,r)} |h - h(x)| d\mu = 0$$

for 1-q.e. or equivalently  $\mathcal{H}$ -a.e.  $x \in X$ , by (4.5). By the same Lebesgue point theorem, and (4.7), we have

$$\lim_{r \rightarrow 0} \int_{B(x,r)} |f - f(x)| d\mu = 0$$

for 1-q.e.  $x \in X$ . Thus, necessarily  $h(x) = f(x)$  for 1-q.e.  $x \in X$ . Thus by [4, Proposition 1.61],  $f \in N^{1,1}(X)$ .  $\square$

**Corollary 4.9.** *Let  $\Omega \subset X$  be a bounded open set whose boundary is codimension 1 Ahlfors regular as given in (4.3). Let  $f \in L^1(\partial\Omega, \mathcal{H})$  with  $-1 \leq f \leq 1$ . Then, there exists  $E_f \in N^{1,1}(X)$  with  $E_f(x) = f(x)$  for every  $x \in \partial\Omega$ .*

*Proof.* Combine Theorem 4.4 and Proposition 4.6.  $\square$

In Example 3.5, it is crucial that  $a > 1$ . If  $f$  is restricted in the same way as  $u$ , solutions exist at least if  $\Omega$  is sufficiently regular. The proof relies on the following lower semicontinuity result, which will also be used later in other contexts. Such a restriction is necessary even in Euclidean setting with smooth domains, see [37] and Example 3.5 (which, while is not a smooth domain, can be modified to be one).

**Lemma 4.10.** *Let  $\Omega \subset X$  be a nonempty bounded open set satisfying the exterior measure density condition (3.1), and such that for any  $u \in \text{BV}(\Omega)$ , the trace  $Tu(x)$  exists for  $\mathcal{H}$ -a.e.  $x \in \partial^*\Omega$ . Assume also that  $\partial\Omega$  is codimension 1 Ahlfors regular as given in (4.3). Let  $f \in L^1(\partial^*\Omega, P(\Omega, \cdot))$  with  $-1 \leq f \leq 1$ , satisfying (3.2). Then if  $E_i \subset \Omega$ ,  $i \in \mathbb{N}$ , are such that  $P(E_i, \Omega) < \infty$  and  $\chi_{E_i} \rightarrow \chi_E$  in  $L^1(\Omega)$ , it follows that*

$$I(\chi_E) \leq \liminf_{i \rightarrow \infty} I(\chi_{E_i}) \quad \text{and} \quad I(-\chi_E) \leq \liminf_{i \rightarrow \infty} I(-\chi_{E_i}).$$

Note that since  $\partial\Omega$  is compact, we have in particular  $\mathcal{H}(\partial\Omega) < \infty$ , and then  $P(\Omega, X) < \infty$  (see e.g. [23, Proposition 6.3]).

*Proof.* By (2.7), we have  $f \in L^1(\partial^*\Omega, \mathcal{H})$  and so we can extend  $f$  to a function  $f \in L^1(\partial\Omega, \mathcal{H})$ , e.g. by zero extension. By Corollary 4.9, there is an extension of  $f$ , still denoted simply by  $f$ , such that  $f \in N^{1,1}(X)$ . We know that every function in the class  $N^{1,1}(X)$  is 1-quasicontinuous, see [7, Theorem 1.1] or [4, Theorem 5.29]. Therefore by [6, Proposition 3.4] we know that for every  $t \in \mathbb{R}$ ,  $\{f > t\}$  and  $\{f < t\}$  are 1-quasiopen. Then by [13, Lemma 2.3],  $\{t_1 < f < t_2\}$  is also 1-quasiopen for any  $t_1, t_2 \in \mathbb{R}$ .

Let  $E \subset \Omega$  such that  $P(E, \Omega) < \infty$ . By Cavalieri's principle,

$$\begin{aligned} I(\chi_E) &= P(E, \Omega) + \int_0^1 \int_{\{f > t\}} T\chi_E dP(\Omega, \cdot) dt - \int_0^1 \int_{\{f < -t\}} T\chi_E dP(\Omega, \cdot) dt \\ &= \int_0^1 \left[ P(E, \Omega) + P(\Omega, \{T\chi_E = 1\} \cap \{f > t\}) - P(\Omega, \{T\chi_E = 1\} \cap \{f < -t\}) \right] dt. \end{aligned} \quad (4.11)$$

Fix  $t \in (0, 1)$ . Suppose  $E_i \subset \Omega$ ,  $i \in \mathbb{N}$ , are such that  $P(E_i, \Omega) < \infty$  and  $\chi_{E_i} \rightarrow \chi_E$  in  $L^1(\Omega)$  (and thus in fact in  $L^1(X)$ ). By Theorem 3.15, also  $P(E_i, X) < \infty$ . By lower semicontinuity and Lemma 3.18, we have

$$P(E, X) \leq \liminf_{i \rightarrow \infty} P(E_i, X) \leq \liminf_{i \rightarrow \infty} I(\chi_{E_i}) + 2P(\Omega, X),$$

where we can assume the limit on the right-hand side to be finite. Thus  $P(E, X) < \infty$ . By Proposition 4.2, we now have

$$\begin{aligned} & P(E, \Omega \cap \{f > t\}) + P(E, \partial\Omega \cap \{f > t\}) \\ &= P(E, \{f > t\}) \\ &\leq \liminf_{i \rightarrow \infty} P(E_i, \{f > t\}) \\ &= \liminf_{i \rightarrow \infty} \left( P(E_i, \Omega \cap \{f > t\}) + P(E_i, \partial\Omega \cap \{f > t\}) \right). \end{aligned}$$

Thus, by Lemma 3.16,

$$\begin{aligned} & P(E, \Omega \cap \{f > t\}) + P(\Omega, \{TX_E = 1\} \cap \{f > t\}) \\ &\leq \liminf_{i \rightarrow \infty} \left( P(E_i, \Omega \cap \{f > t\}) + P(\Omega, \{TX_{E_i} = 1\} \cap \{f > t\}) \right). \end{aligned} \quad (4.12)$$

Since also  $\chi_{\Omega \setminus E_i} \rightarrow \chi_{\Omega \setminus E}$  in  $L^1(X)$ , by the lower semicontinuity of perimeter we also get

$$\begin{aligned} & P(\Omega \setminus E, \Omega \cap \{f < -t\}) + P(\Omega \setminus E, \partial\Omega \cap \{f < -t\}) \\ &= P(\Omega \setminus E, \{f < -t\}) \\ &\leq \liminf_{i \rightarrow \infty} P(\Omega \setminus E_i, \{f < -t\}) \\ &= \liminf_{i \rightarrow \infty} \left( P(\Omega \setminus E_i, \Omega \cap \{f < -t\}) + P(\Omega \setminus E_i, \partial\Omega \cap \{f < -t\}) \right). \end{aligned}$$

Note that  $TX_{\Omega \setminus E}(x) = 1$  if and only if  $TX_E(x) = 0$ . Thus by Lemma 3.16,

$$\begin{aligned} & P(\Omega \setminus E, \Omega \cap \{f < -t\}) + P(\Omega, \{TX_E = 0\} \cap \{f < -t\}) \\ &\leq \liminf_{i \rightarrow \infty} \left( P(\Omega \setminus E_i, \Omega \cap \{f < -t\}) + P(\Omega, \{TX_{E_i} = 0\} \cap \{f < -t\}) \right). \end{aligned}$$

By subtracting  $P(\Omega, \{f < -t\})$  from both sides and noting that  $P(F, A) = P(\Omega \setminus F, A)$  for any  $\mu$ -measurable  $F \subset X$  and any set  $A \subset \Omega$ , we obtain

$$\begin{aligned} & P(E, \Omega \cap \{f < -t\}) - P(\Omega, \{TX_E = 1\} \cap \{f < -t\}) \\ &\leq \liminf_{i \rightarrow \infty} \left( P(E_i, \Omega \cap \{f < -t\}) - P(\Omega, \{TX_{E_i} = 1\} \cap \{f < -t\}) \right). \end{aligned} \quad (4.13)$$

By the fact that  $P(E, \cdot)$  is a finite measure, for  $\mathcal{L}^1$ -a.e.  $t \in (0, 1)$  we have

$$P(E, \Omega \cap (\{f = t\} \cup \{f = -t\})) = 0.$$

For such  $t$ , by (4.12), and (4.13) and using lower semicontinuity once more, in the 1-quasiopen set  $\{-t < f < t\}$ ,

$$\begin{aligned} & P(E, \Omega) + P(\Omega, \{TX_E = 1\} \cap \{f > t\}) - P(\Omega, \{TX_E = 1\} \cap \{f < -t\}) \\ &= P(E, \Omega \cap \{f > t\}) + P(E, \Omega \cap \{f < -t\}) + P(E, \Omega \cap \{-t < f < t\}) \\ &\quad + P(\Omega, \{TX_E = 1\} \cap \{f > t\}) - P(\Omega, \{TX_E = 1\} \cap \{f < -t\}) \\ &\leq \liminf_{i \rightarrow \infty} \left( P(E_i, \Omega \cap \{f > t\}) + P(\Omega, \{TX_{E_i} = 1\} \cap \{f > t\}) \right) \\ &\quad + \liminf_{i \rightarrow \infty} \left( P(E_i, \Omega \cap \{f < -t\}) - P(\Omega, \{TX_{E_i} = 1\} \cap \{f < -t\}) \right) \\ &\quad + \liminf_{i \rightarrow \infty} P(E_i, \Omega \cap \{-t \leq f \leq t\}) \\ &\leq \liminf_{i \rightarrow \infty} \left( P(E_i, \Omega) \right. \\ &\quad \left. + P(\Omega, \{TX_{E_i} = 1\} \cap \{f > t\}) \right) - P(\Omega, \{TX_{E_i} = 1\} \cap \{f < -t\}). \end{aligned} \quad (4.14)$$

By combining (4.11) and (4.14) and using Fatou's lemma, we obtain

$$I(\chi_E) \leq \liminf_{i \rightarrow \infty} I(\chi_{E_i}).$$

Denoting  $I(\cdot) = I_f(\cdot)$  to make the dependence on  $f$  explicit, we have also

$$I_f(-\chi_E) = I_{-f}(\chi_E) \leq \liminf_{i \rightarrow \infty} I_{-f}(\chi_{E_i}) = \liminf_{i \rightarrow \infty} I_f(-\chi_{E_i}),$$

and thus the claim is proved.  $\square$

**Theorem 4.15.** *Let  $\Omega$  and  $f$  be as in Lemma 4.10. Then the restricted Neumann problem given in Definition 3.4 has a solution.*

*Proof.* Take a sequence  $(u_i) \subset BV(\Omega)$  with  $-1 \leq u_i \leq 1$  and

$$I(u_i) < \inf_{v \in BV(\Omega), \|v\|_{L^\infty(\Omega)} \leq 1} I(v) + 1/i \quad \text{for all } i \in \mathbb{N}.$$

By Proposition 3.8 we can assume that  $u_i = \chi_{E_1^i} - \chi_{E_2^i}$  for disjoint sets  $E_1^i, E_2^i \subset \Omega$ ,  $i \in \mathbb{N}$ . By Lemma 3.18 and Lemma 3.3,

$$\begin{aligned} P(E_1^i, X) &\leq I(\chi_{E_1^i}) + 2P(\Omega, X) = I(u_i) - I(-\chi_{E_2^i}) + 2P(\Omega, X) \\ &\leq I(u_i) + 3P(\Omega, X), \end{aligned}$$

and similarly for the sets  $E_2^i$ . We conclude that the sequences  $P(E_1^i, X)$  and  $P(E_2^i, X)$  are bounded, and so by [38, Theorem 3.7] there are sets  $E_1, E_2 \subset \Omega$  such that  $\chi_{E_1^i} \rightarrow \chi_{E_1}$  in  $L^1(X)$  and  $\chi_{E_2^i} \rightarrow \chi_{E_2}$  in  $L^1(X)$ , passing to a subsequence if needed (without relabeling the sequences). Then, clearly also  $\mu(E_1 \cap E_2) = 0$ . By lower semicontinuity,  $P(E_1, X) < \infty$  and  $P(E_2, X) < \infty$ . Thus by Lemma 3.3 and Lemma 4.10,

$$\begin{aligned} I(\chi_{E_1} - \chi_{E_2}) &= I(\chi_{E_1}) + I(-\chi_{E_2}) \\ &\leq \liminf_{i \rightarrow \infty} I(\chi_{E_1^i}) + \liminf_{i \rightarrow \infty} I(-\chi_{E_2^i}) \\ &\leq \liminf_{i \rightarrow \infty} \left( I(\chi_{E_1^i}) + I(-\chi_{E_2^i}) \right) \\ &= \liminf_{i \rightarrow \infty} \left( I(\chi_{E_1^i} - \chi_{E_2^i}) \right) = \liminf_{i \rightarrow \infty} I(u_i). \end{aligned}$$

Thus,  $\chi_{E_1} - \chi_{E_2}$  is a solution.  $\square$

## 5 When $f: \partial\Omega \rightarrow \{-1, 0, 1\}$

In this section, we always assume that  $\Omega \subset X$  is a nonempty bounded domain with  $P(\Omega, X) < \infty$ , such that for any  $u \in BV(\Omega)$ , the trace  $Tu(x)$  exists for  $\mathcal{H}$ -a.e.  $x \in \partial^*\Omega$ . We also assume that the boundary data  $f \in L^1(\partial^*\Omega, P(\Omega, \cdot))$  satisfies (3.2), that is,  $\int_{\partial^*\Omega} f dP(\Omega, \cdot) = 0$ . To know that minimizers exist, we need  $-1 \leq f \leq 1$  as in the previous section, see also [37].

In light of the result from the previous sections that  $\chi_{E_1} - \chi_{E_2}$  is a solution for some choice of  $E_1, E_2 \subset \Omega$ , we see that the “relative outer normal derivative” of the solution (in relation to the total variation of the function) is directed either entirely outward (i.e.,  $\partial_\eta u / \|Du\| = \pm 1$  in the Euclidean setting) or has vanishing derivative. Thus, in the Euclidean setting, if one is to make sense of  $f$  as the relative outer normal derivative of the solution, then the only permissible values one has for  $f$  are 0, 1, and  $-1$ . This section is dedicated to the study of boundary behavior of solutions  $\chi_{E_1} - \chi_{E_2}$  for such  $f$ .

**Definition 5.1.** For  $E \subset \Omega$  of finite perimeter in  $\Omega$ , let  $\partial_E \Omega$  denote the collection of all points  $x \in \partial^*\Omega$  for which  $TX_E(x) = 1$ .

Suppose  $E_1, E_2 \subset \Omega$  are disjoint sets such that  $\chi_{E_1} - \chi_{E_2}$  solves the restricted Neumann problem. Note that

$$0 \geq I(\chi_{E_1}) = P(E_1, \Omega) - P(\Omega, \partial_{E_1}\Omega \cap \{f = -1\}) + P(\Omega, \partial_{E_1}\Omega \cap \{f = 1\}).$$

Therefore

$$P(E_1, \Omega) + P(\Omega, \partial_{E_1}\Omega \cap \{f = 1\}) \leq P(\Omega, \partial_{E_1}\Omega \cap \{f = -1\}). \quad (5.2)$$

From Lemma 3.12, we can conclude that  $I(\chi_{E_1} - \chi_{E_2}) = 2I(\chi_{E_1}) = 2I(-\chi_{E_2})$ . If  $I(\chi_{E_1} - \chi_{E_2}) \neq 0$ , then  $I(\chi_{E_1} - \chi_{E_2}) < 0$ , and hence  $I(\chi_{E_1}) < 0$ . If  $P(E_1, \Omega) = 0$ , then by the facts that  $X$  supports the relative isoperimetric inequality (2.6) and  $\Omega$  is connected, we must have either that  $\mu(\Omega \setminus E_1) = 0$  or  $\mu(E_1) = 0$ , from either of which we would have that  $I(\chi_{E_1}) = 0$ . Thus we must have  $P(E_1, \Omega) > 0$ . However, from this and (5.2) we can only infer that

$$P(\Omega, \partial_{E_1}\Omega \cap \{f = 1\}) < P(\Omega, \partial_{E_1}\Omega \cap \{f = -1\}). \quad (5.3)$$

On the set  $\partial_{E_1}\Omega$  one should understand that the relative outer normal derivative of  $\chi_{E_1} - \chi_{E_2}$  must be  $-1$ ; thus on the set  $\partial_{E_1}\Omega \cap \{f = 1\}$  the relative outer normal derivative of  $\chi_{E_1} - \chi_{E_2}$  does not agree with the boundary data  $f = 1$ . The above inequality therefore implies that the relative outer normal derivative of  $\chi_{E_1} - \chi_{E_2}$  agrees more often than not with the boundary data  $f$  where  $f \neq 0$ . We would prefer to obtain a better quantitative version of this statement.

**Proposition 5.4.** *Suppose that  $\Omega$ , as a metric measure space equipped with the measure  $\mu|_{\Omega}$ , supports a  $(1, 1)$ -Poincaré inequality and a measure density condition: there is some  $C \geq 1$  and  $r_0 > 0$  such that*

$$\mu(B(x, r) \cap \Omega) \geq \frac{\mu(B(x, r))}{C} \quad (5.5)$$

for every  $x \in \partial\Omega$  and  $0 < r < r_0$ . Suppose also that  $\partial\Omega$  is codimension 1 Ahlfors regular as defined in (4.3). Assume that  $\emptyset \neq E_1 \subsetneq \Omega$  is such that  $\chi_{E_1} - \chi_{\Omega \setminus E_1}$  solves the restricted Neumann problem with boundary data  $f: \partial^*\Omega \rightarrow \{-1, 0, 1\}$ . If  $\mu(E_1) \leq \mu(\Omega \setminus E_1)$ , then

$$P(\Omega, \partial_{E_1}\Omega \cap \{f = 1\}) \leq \frac{C_{\Omega} - 1}{C_{\Omega} + 1} P(\Omega, \partial_{E_1}\Omega \cap \{f = -1\}), \quad (5.6)$$

where the constant  $C_{\Omega} > 1$  is independent of  $f$  and  $E_1$ . Otherwise,

$$P(\Omega, \partial_{\Omega \setminus E_1}\Omega \cap \{f = -1\}) \leq \frac{C_{\Omega} - 1}{C_{\Omega} + 1} P(\Omega, \partial_{\Omega \setminus E_1}\Omega \cap \{f = 1\}).$$

It is straightforward to check that (5.5) can equivalently be required for every  $x \in \overline{\Omega}$ , possibly with different constants  $C, r_0$ . Moreover, we will see that one can express  $C_{\Omega} = \|T\|(1 + 2C_{P_{\Omega}} \text{diam}(\Omega))$ , where  $C_{P_{\Omega}} > 0$  is the constant associated with the Poincaré inequality on  $\Omega$  and  $\|T\|$  is the norm of the trace operator  $T: BV(\Omega) \rightarrow L^1(\partial^*\Omega, P(\Omega, \cdot))$ .

*Proof.* We will focus only on the situation when  $\mu(E_1) \leq \mu(\Omega \setminus E_1)$ . The other case can be proven analogously. According to [31, Theorem 5.5], the trace operator  $T: BV(\Omega) \rightarrow L^1(\partial^*\Omega, P(\Omega, \cdot))$  is bounded, that is,

$$\begin{aligned} P(\Omega, \partial_{E_1}\Omega \cap \{f = 1\}) + P(\Omega, \partial_{E_1}\Omega \cap \{f = -1\}) &\leq \int_{\partial^*\Omega} T\chi_{E_1} dP(\Omega, \cdot) \\ &\leq \|T\|(\mu(E_1) + P(E_1, \Omega)). \end{aligned}$$

The  $(1, 1)$ -Poincaré inequality on  $\Omega$  yields that

$$\frac{\mu(E_1)\mu(\Omega \setminus E_1)}{\mu(\Omega)} \leq C_{P_{\Omega}} \text{diam}(\Omega)P(E_1, \Omega), \quad (5.7)$$

where  $C_{P_{\Omega}} > 0$ . As  $\mu(\Omega \setminus E_1) \geq \mu(\Omega)/2$  due to the assumed relation  $\mu(E_1) \leq \mu(\Omega \setminus E_1)$ , we obtain that  $\mu(E_1) \leq 2C_{P_{\Omega}} \text{diam}(\Omega)P(E_1, \Omega)$ . Therefore

$$\begin{aligned} P(\Omega, \partial_{E_1}\Omega \cap \{f = 1\}) + P(\Omega, \partial_{E_1}\Omega \cap \{f = -1\}) \\ \leq \|T\|(1 + 2C_{P_{\Omega}} \text{diam}(\Omega))P(E_1, \Omega) =: C_{\Omega} P(E_1, \Omega). \end{aligned}$$



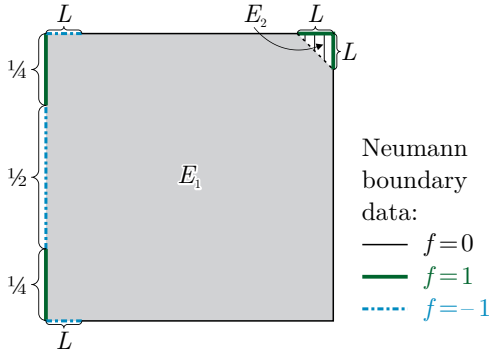


Figure 1: See Example 5.9

Consequently, we obtain from (5.2) that

$$\begin{aligned} & P(\Omega, \partial_{E_1}\Omega \cap \{f = 1\}) + P(\Omega, \partial_{E_1}\Omega \cap \{f = -1\}) \\ & \leq C_\Omega (P(\Omega, \partial_{E_1}\Omega \cap \{f = -1\}) - P(\Omega, \partial_{E_1}\Omega \cap \{f = 1\})), \end{aligned}$$

which immediately implies the validity of (5.6).  $\square$

The inequality  $\mu(E_1) \leq \mu(\Omega \setminus E_1)$  turns out to be crucial when applying the estimate (5.7) to compare  $\mu(E_1)$  with  $P(E_1, \Omega)$ . Otherwise, we cannot obtain (5.6) with a constant  $C_\Omega$  independent of  $E_1$ , see Example 5.9.

Nevertheless, we can define  $C(E_1) = C_{P_\Omega} \text{diam}(\Omega)\mu(\Omega)/\mu(\Omega \setminus E_1)$ . Then, (5.7) leads to  $\mu(E_1) \leq C(E_1)P(E_1, \Omega)$  and hence to the quantitative estimate

$$P(\Omega, \partial_{E_1}\Omega \cap \{f = 1\}) < \frac{\|T\|(1 + C(E_1)) - 1}{\|T\|(1 + C(E_1)) + 1} P(\Omega, \partial_{E_1}\Omega \cap \{f = -1\}). \quad (5.8)$$

If the  $L^1$ -boundedness of the trace operator is established by other means, we can remove the assumptions of a  $(1, 1)$ -Poincaré inequality for  $\Omega$  and the measure density condition (5.5). Then, we can bypass (5.7) by setting  $C(E_1) = \mu(E_1)/P(E_1, \Omega)$  to get (5.8).

The following example shows that it is in general impossible to obtain an estimate better than (5.3) in case we wish the constants to be independent of  $E_1$ . On the other hand, the situation is different if  $\partial\Omega$  is of positive mean curvature in the sense of [29], see Definition 5.10 below.

**Example 5.9.** Fix  $0 < L < 1/8$ . Let  $\Omega = (0, 1)^2$  be the unit square in  $\mathbb{R}^2$  (unweighted), and let  $F_1 \subset \partial\Omega$  be given by the union of the four line segments: one connecting  $(1 - L, 1)$  to  $(1, 1)$ , one connecting  $(1, 1 - L)$  to  $(1, 1)$ , one connecting  $(0, 0)$  to  $(0, \frac{1}{4})$ , and one connecting  $(0, \frac{3}{4})$  to  $(0, 1)$ , the first two of which are each of length  $L$  and the latter two of which are each of length  $\frac{1}{4}$ . Let  $F_2 \subset \partial\Omega$  be the union of three line segments, one connecting  $(0, \frac{1}{4})$  to  $(0, \frac{3}{4})$  of length  $\frac{1}{2}$ , and the other two, each of length  $L$ , one connecting  $(0, 1)$  to  $(L, 1)$  and the other connecting  $(0, 0)$  to  $(L, 0)$ . Let  $f = \chi_{F_1} - \chi_{F_2}$ . Now the restricted Neumann problem has exactly one solution, given by  $u = \chi_{E_1} - \chi_{E_2}$ , where  $E_1 = \Omega \setminus E_2$  and  $E_2$  is the triangular region in  $\Omega$  with vertices  $(1 - L, 1)$ ,  $(1, 1)$ , and  $(1, 1 - L)$ .

Using the above solution to the given Neumann problem, let us now show that it is in general impossible to obtain an estimate of the form (5.6) in case  $\mu(E_2) < \mu(E_1)$ , with a constant  $C_\Omega$  independent of  $E_1$ . Indeed, we have  $P(\Omega, \partial_{E_1}\Omega \cap \{f = -1\}) = 2L + \frac{1}{2}$  and  $P(\Omega, \partial_{E_1}\Omega \cap \{f = 1\}) = \frac{1}{2}$ . Therefore,

$$\lim_{L \rightarrow 0} \frac{P(\Omega, \partial_{E_1}\Omega \cap \{f = -1\})}{P(\Omega, \partial_{E_1}\Omega \cap \{f = 1\})} = 1.$$

The example above heavily relies on the fact that the boundary data are non-zero on flat parts of  $\partial\Omega$ . In the remaining part of this section, we will discuss the case when  $\partial\Omega$  is of positive mean curvature in the sense of [29]; see also [44].

**Definition 5.10.** Let  $h \in BV_{\text{loc}}(X)$ . We say that  $u \in BV_{\text{loc}}(X)$  is a *weak solution to the Dirichlet problem for least gradients in  $\Omega$  with boundary data  $h$*  if  $u = h$  on  $X \setminus \Omega$  and

$$\|Du\|(\overline{\Omega}) \leq \|Dv\|(\overline{\Omega})$$

whenever  $v \in BV_{\text{loc}}(X)$  with  $v = h$  on  $X \setminus \Omega$ .

A weak solution exists whenever  $h \in BV_{\text{loc}}(X)$  with  $\|Dh\|(X) < \infty$ , see [29, Lemma 3.1].

**Definition 5.11.** We say that the boundary  $\partial\Omega$  has *positive mean curvature* if there exists a non-decreasing function  $\varphi: (0, \infty) \rightarrow (0, \infty)$  and a constant  $r_0 > 0$  such that for all  $z \in \partial\Omega$  and all  $0 < r < r_0$  with  $P(B(z, r), X) < \infty$ , we have that  $u^\vee \geq 1$  everywhere on  $B(z, \varphi(r))$  for any weak solution  $u$  to the Dirichlet problem for least gradients in  $\Omega$  with boundary data  $\chi_{B(z, r)}$ .

Recall that the perimeter measure  $P(E, \cdot)$  relates to  $\mathcal{H}|_{\partial E}$  via the function  $\theta_E: X \rightarrow [\alpha, C_d]$  as stated in (2.7).

**Definition 5.12** ([3, Definition 6.1]). We say that  $X$  is a *local space* if, given any two sets of locally finite perimeter  $E_1 \subset E_2 \subset X$ , we have  $\theta_{E_1}(x) = \theta_{E_2}(x)$  for  $\mathcal{H}$ -a.e.  $x \in \partial^*E_1 \cap \partial^*E_2$ .

The assumption  $E_1 \subset E_2$  can in fact be dropped as shown in the discussion after [17, Definition 5.9]. See [3] and [26] for some examples of local spaces. See also [30, Example 5.2] for an example of a space that fails to be local, despite being equipped with a doubling measure that supports a Poincaré inequality.

**Theorem 5.13.** *Suppose  $X$  is a local space. Assume that  $\Omega$  satisfies the exterior measure density condition (3.1), that  $\mathcal{H}(\partial\Omega) < \infty$ , and that  $\partial\Omega$  has positive mean curvature. Suppose that  $\chi_{E_1} - \chi_{E_2}$  solves the restricted Neumann problem with boundary data  $f: \partial^*\Omega \rightarrow \{-1, 0, 1\}$ . If  $z \in \partial\Omega$  such that  $f = -1$  in a neighborhood of  $z$ , then  $T\chi_{E_1}(z) = 1$ .*

*Moreover, if  $u \in BV(\Omega)$  is any solution to the restricted Neumann problem with boundary data  $f$  and  $f = -1$  on  $B(z, r) \cap \partial\Omega$  for some  $r > 0$ , then  $u = 1$  on  $B(z, \varphi(r)) \cap \Omega$  and hence  $Tu(z) = 1$ .*

In the above,  $r \mapsto \varphi(r)$  is the function associated with positive mean curvature of  $\partial\Omega$  as in Definition 5.11.

*Proof.* If  $z \in \partial\Omega$  such that  $f = -1$  in a neighborhood of  $z$ , we find  $r > 0$  such that  $f = -1$  on  $B(z, r) \cap \partial^*\Omega$ , and  $P(B(z, r), X) < \infty$  and  $\mathcal{H}(\partial B(z, r) \cap \partial\Omega) = 0$ ; the latter two facts hold for  $\mathcal{L}^1$ -a.e.  $r > 0$  by the BV coarea formula (2.5) and the fact that  $\mathcal{H}(\partial\Omega) < \infty$ . Take  $K \subset X$  such that  $\chi_K$  is a weak solution to the Dirichlet problem for least gradients in  $\Omega$  with boundary data  $\chi_{B(z, r)}$ ; in particular,  $\chi_K = \chi_{B(z, r)}$  on  $X \setminus \Omega$ . We let  $E = E_1 \cup (B(z, r) \setminus \Omega)$  and claim that  $K \cap E$  is another weak solution to the Dirichlet problem. Suppose it is not. Then

$$P(K, \overline{\Omega}) < P(K \cap E, \overline{\Omega}).$$

By [29, Corollary 4.6], we have  $T\chi_K(x) = \chi_{B(z, r)}(x)$  for  $\mathcal{H}$ -a.e.  $x \in \partial\Omega$ , and thus  $\mathcal{H}(\partial^*K \cap \partial\Omega) = 0$ , whence  $P(K, \partial\Omega) = 0$  by (2.7). Thus

$$P(K, \overline{\Omega}) = P(K, \Omega).$$

Now we also have  $T\chi_{K \cap E} \leq \chi_{B(z,r)}$   $\mathcal{H}$ -a.e. on  $\partial\Omega$ , and so  $\mathcal{H}(\partial^*(K \cap E) \cap \partial\Omega \setminus B(z,r)) = 0$ . Note that  $P(E_1, X) < \infty$  by Theorem 3.15, and then  $P(K \cap E, X) < \infty$  by [38, Proposition 4.7]. Thus by the fact that  $P(K \cap E, \cdot)$  is a Borel outer measure and (2.7),

$$\begin{aligned} P(K \cap E, \overline{\Omega}) &= P(K \cap E, \Omega) + P(K \cap E, \partial\Omega) \\ &= P(K \cap E, \Omega) + P(K \cap E, \partial\Omega \cap B(z,r)) \\ &= P(K \cap E, \Omega) + P(B(z,r) \setminus (K \cap E), \partial\Omega \cap B(z,r)) \\ &= P(K \cap E, \Omega) + P(\Omega, B(z,r) \cap \{T\chi_{K \cap E} = 0\}) \quad \text{by Lemma 3.16} \\ &= P(K \cap E, \Omega) + P(\Omega, B(z,r) \setminus \partial_E \Omega), \end{aligned}$$

since  $T\chi_K = 1$   $\mathcal{H}$ -a.e. on  $B(z,r)$ . See Definition 5.1 for the definition of  $\partial_E \Omega$ . Combining these,

$$P(K, \Omega) < P(K \cap E, \Omega) + P(\Omega, B(z,r) \setminus \partial_E \Omega). \quad (5.14)$$

It is straightforward to verify that

$$\partial^*(K \cap E) \subset (\partial^*K \setminus O_E) \cup (\partial^*E \cap I_K),$$

where  $I_K$  and  $O_E$  stand for the measure theoretic interior and exterior, respectively, as defined by (2.1) and (2.2). By (2.7) and by  $X$  being local, we obtain that

$$\begin{aligned} P(K \cap E, \Omega) &= \int_{\Omega \cap \partial^*(K \cap E)} \theta_{K \cap E} d\mathcal{H} \\ &\leq \int_{\Omega \cap \partial^*K \setminus O_E} \theta_K d\mathcal{H} + \int_{\Omega \cap \partial^*E \cap I_K} \theta_E d\mathcal{H} \\ &= P(K, \Omega \setminus O_E) + P(E, \Omega \cap I_K). \end{aligned}$$

Combining this with (5.14), we get

$$P(K, \Omega \cap O_E) < P(E, \Omega \cap I_K) + P(\Omega, B(z,r) \setminus \partial_E \Omega). \quad (5.15)$$

On the other hand, comparing  $E$  against  $E \cup K$  in the Neumann problem (note that also  $P(E \cup K, X) < \infty$  by [38, Proposition 4.7]), by Lemma 3.13 we obtain

$$\begin{aligned} P(E, \Omega) + \int_{\partial_E \Omega} f dP(\Omega, \cdot) &\leq P(E \cup K, \Omega) + \int_{\partial_{E \cup K} \Omega} f dP(\Omega, \cdot) \\ &= P(E \cup K, \Omega) + \int_{\partial_K \Omega \setminus \partial_E \Omega} f dP(\Omega, \cdot) + \int_{\partial_E \Omega} f dP(\Omega, \cdot) \\ &= P(E \cup K, \Omega) + \int_{B(z,r) \setminus \partial_E \Omega} f dP(\Omega, \cdot) + \int_{\partial_E \Omega} f dP(\Omega, \cdot), \end{aligned} \quad (5.16)$$

since we had  $T\chi_K(x) = \chi_{B(z,r)}(x)$  for  $\mathcal{H}$ -a.e.  $x \in \partial\Omega$ . Similarly as before, it is straightforward to verify that

$$\partial^*(E \cup K) \subset (\partial^*E \setminus I_K) \cup (\partial^*K \cap O_E).$$

By (2.7) and the fact that  $X$  is local, we now see that

$$\begin{aligned} P(E \cup K, \Omega) &= \int_{\Omega \cap \partial^*(E \cup K)} \theta_{E \cup K} d\mathcal{H} \\ &\leq \int_{\Omega \cap \partial^*E \setminus I_K} \theta_E d\mathcal{H} + \int_{\Omega \cap \partial^*K \cap O_E} \theta_K d\mathcal{H} \\ &= P(E, \Omega \setminus I_K) + P(K, \Omega \cap O_E). \end{aligned} \quad (5.17)$$

Combining (5.17) with (5.16) yields that

$$\begin{aligned} P(E, \Omega) - P(\Omega, \partial_E \Omega \cap \{f = -1\}) + P(\Omega, \partial_E \Omega \cap \{f = 1\}) \\ \leq P(E, \Omega \setminus I_K) + P(K, \Omega \cap O_E) - P(\Omega, B(z,r) \setminus \partial_E \Omega) \\ + P(\Omega, \partial_E \Omega \cap \{f = 1\}) - P(\Omega, \partial_E \Omega \cap \{f = -1\}). \end{aligned}$$

It follows that

$$P(E, \Omega \cap I_K) \leq P(K, \Omega \cap O_E) - P(\Omega, B(z, r) \setminus \partial_E \Omega). \quad (5.18)$$

Since (5.15) is in contradiction with (5.18), we have established the claim that  $K \cap E$  is a weak solution to the Dirichlet problem for least gradients in  $\Omega$  with boundary data  $\chi_{B(z,r)}$ . Therefore, by the definition of positive mean curvature,  $B(z, \varphi(r)) \subset K \cap E \subset E$  (up to a  $\mu$ -negligible set) and in particular,  $T\chi_{E_1}(z) = T\chi_E(z) = 1$ .

We complete the proof of this theorem by considering a solution  $u$  for boundary data  $f$  with  $f = -1$  on  $B(z, r) \cap \partial^* \Omega$ . By the last part of Proposition 3.8, we can find two sequences  $t_{1,k}, t_{2,k} \in (0, 1)$  with  $\lim_{k \rightarrow \infty} t_{1,k} = 1$  and  $\lim_{k \rightarrow \infty} t_{2,k} = 1$  such that each  $\chi_{\{u > t_{1,k}\}} - \chi_{\{u < -t_{2,k}\}}$  is a solution to the same Neumann problem. Thus, by the above argument, we have that  $u \geq t_{1,k}$  in  $B(z, \varphi(r)) \cap \Omega$  for each  $k \in \mathbb{N}$ , and thus the desired conclusion follows by letting  $k \rightarrow \infty$ .  $\square$

In particular, it follows from the above result that *every*  $z$  in the interior of the set  $\{x \in \partial \Omega : f(x) = -1\}$  satisfies  $z \in \partial_{E_1} \Omega$ . Conversely,  $z \notin \partial_{E_1} \Omega$  whenever  $z$  lies in the interior of the set  $\{x \in \partial \Omega : f(x) = 1\}$ .

**Remark 5.19.** Note that if  $f = \chi_{F_1} - \chi_{F_2}$  with  $F_1, F_2 \subset \partial \Omega$  disjoint, the above theorem gives us good control over the solutions to the restricted Neumann problem with boundary data  $f$  when both  $F_1$  and  $F_2$  are relatively open subsets of  $\partial \Omega$ . However, if  $F_1$  and  $F_2$  have empty interior, the above theorem gives us no control over the solutions near the boundary.

Compare this to the situation regarding the Dirichlet problem on domains whose boundary has positive mean curvature. It is known that if the Dirichlet boundary data are continuous, then the solution to the least gradient problem on the domain has trace on the boundary that agrees with the boundary data, see [29]. However, if the boundary data are not continuous, no such control over the trace of the solution is known except in special circumstances such as characteristic functions of relatively open subsets  $F \subset \partial \Omega$  for which  $\mathcal{H}(\partial \Omega \cap \partial F) = 0$ . Indeed, in the Euclidean setting, with a Euclidean ball playing the role of the domain, there are known to be boundary data, taken from the class  $L^1$  of the boundary sphere, for which solutions to the Dirichlet problem fail to have the correct trace, see [35].

A natural question is whether we have any control over the solution near the part of the boundary where  $f = 0$ .

**Example 5.20.** Consider the simple example of  $\Omega = B(0, 1) \subset \mathbb{R}^2$  (unweighted) with the boundary data

$$f(x, y) := \begin{cases} \operatorname{sgn} x & \text{for } (x, y) \in \partial \Omega \text{ with } |x| \geq \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

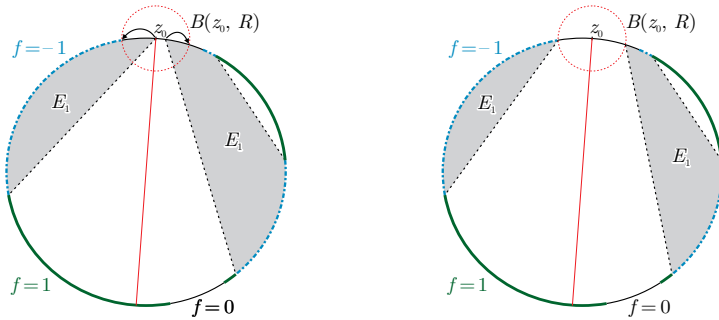
We can easily see that it is impossible to determine what value a solution  $u$  will have near the boundary points where  $f = 0$ . Indeed, the problem is solved by each of the following three functions:

$$\begin{aligned} u_1(x, y) &= \chi_{(-1, 1/2)}(x) - \chi_{(1/2, 1)}(x), & (x, y) \in \Omega, \\ u_2(x, y) &= \chi_{(-1, -1/2)}(x) - \chi_{(-1/2, 1)}(x), & (x, y) \in \Omega, \quad \text{and} \\ u_3(x, y) &= \chi_{(-1, -1/2)}(x) - \chi_{(1/2, 1)}(x), & (x, y) \in \Omega. \end{aligned}$$

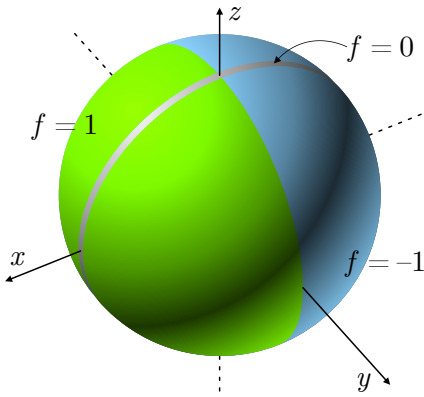
Then,  $Tu_1 \equiv 1$ ,  $Tu_2 \equiv -1$ , and  $Tu_3 \equiv 0$  on the set  $\{f = 0\}$ .

One might therefore wonder whether the zero Neumann data in a neighborhood of a boundary point guarantee that the solution is constant in a neighborhood of this point. In the following example, where a disk in the unweighted plane is discussed, we will see that such a conclusion indeed holds true. However, the subsequent two examples will prove the unweighted planar domain to be highly misleading.

**Example 5.21.** Let  $\Omega = B(0, 1) \subset \mathbb{R}^2$  (unweighted) and let  $f : \partial \Omega \rightarrow \{0, \pm 1\}$ . Let  $u = \chi_{E_1} - \chi_{\Omega \setminus E_1} \in \operatorname{BV}(\Omega)$  be a solution to the restricted Neumann problem with boundary data  $f$ . We will now show that if  $z_0 \in \partial \Omega$  lies in the interior of the set  $\{f = 0\}$ , then there is  $r > 0$  such that  $u$  is constant in  $B(z_0, r) \cap \Omega$ .



**Figure 2:** The perimeter of  $E_1$  inside  $\Omega$  is decreased by moving the endpoints of  $\partial E_1$  from  $\partial\Omega \cap B(z_0, R)$  to  $\partial\Omega \cap \partial B(z_0, R)$ . (See Example 5.21)



**Figure 3:** See Example 5.22

Suppose for the sake of contradiction that  $u$  is not constant on  $B(z_0, r)$  for any  $r > 0$ . Fix  $R > 0$  such that  $f(z) = 0$  for all  $z \in B(z_0, R) \cap \partial\Omega$ . Since  $\chi_{E_1}$  is a function of least gradient by Proposition 3.14, we can assume that  $\partial E_1 \cap \Omega$  consists of straight line segments that connect points in  $\partial\Omega$  and do not intersect each other. Consider the two closed half-disks whose union is  $\bar{\Omega}$  and whose intersection contains  $z_0$ . Take all the line segments of  $\partial E_1$  that reach  $B(z_0, R) \cap \partial\Omega$  and lie within one of these half-disks. Then, move their end-points that lie within  $B(z_0, R) \cap \partial\Omega$  to  $\partial B(z_0, R) \cap \partial\Omega$  within the respective half-disk. Such a modification of  $E_1$  will decrease the perimeter of  $E_1$  inside  $\Omega$  but the boundary integral will remain unchanged (since  $f = 0$  at all points where the trace of  $\chi_{E_1}$  changed). In other words, such a modification will decrease the value of the functional  $I(\cdot)$  and hence  $u$  could not have been a solution.

Let us now consider a domain in 3-dimensional Euclidean space, where the situation turns out to be very different from the plane.

**Example 5.22.** Let  $\Omega = B(0, 1) \subset \mathbb{R}^3$  (unweighted) and let

$$f(x, y, z) = \begin{cases} \operatorname{sgn} x & \text{when } |y| > \frac{1}{100}; \\ 0 & \text{otherwise.} \end{cases}$$

Based on Theorem 5.13, the trace of a solution to the restricted minimization problem  $u = \chi_{E_1} - \chi_{\Omega \setminus E_1}$  necessarily attains the values of  $-f$  in the region where  $f \neq 0$ . Therefore, the set  $E_1$  has to cover the surface of a unit half-ball with  $x < 0$ , perhaps apart from the thin slit  $|y| < \frac{1}{100}$ . However, if  $E_1$  consisted of at least two connected components, one for each component of the set  $\{f = -1\}$ , then the perimeter of  $E_1$  inside  $\Omega$  would be greater than the perimeter of the half-ball  $B(0, 1) \cap \{x < 0\}$ , which equals the area of a unit disk  $\{(0, y, z) \in \Omega\}$ . Hence,  $E_1$  consists of a single connected component.

Then,  $\partial E_1$  connects the two half-circles on  $\partial\Omega$  with  $x < 0$  and  $y = \pm \frac{1}{100}$ . If the set  $\tilde{E}_1 := \{(x, y, z) \in \partial E_1 : x < 0, |y| < \frac{1}{100}\}$  lies entirely inside  $\Omega$ , then the perimeter of this portion of  $\partial E_1$  can be bounded below by a half of the surface area of a cylinder of height  $\frac{2}{100}$  and radius  $(1 - (\frac{1}{100})^2)^{1/2}$ . Thus, the perimeter of  $E_1$  inside  $\Omega$  will decrease if a sufficiently large part of  $\tilde{E}_1$  lies on  $\partial\Omega$ .

Therefore, the jump set of the trace of the solution  $u$  has a nonempty intersection with the interior of the set  $\{f = 0\}$  and so the solution is nonconstant near the said intersection.

Next we show that the case of  $\mathbb{R}^2$  equipped with an Ahlfors 2-regular measure also differs from the unweighted plane.

**Example 5.23.** Consider  $X = \mathbb{R}^2$  endowed with the Euclidean distance and weighted Lebesgue measure  $d\mu(z) := w(z) dz$ , where

$$w(z) = \begin{cases} \frac{1}{2} & \text{for } z \in [-\frac{1}{10}, \frac{1}{10}] \times [-\frac{9}{10}, \frac{9}{10}], \\ 1 & \text{otherwise.} \end{cases}$$

Let  $\Omega = B(0, 1)$  and define  $f(x, y) = \operatorname{sgn} x$  for  $(x, y) \in \partial\Omega$  if  $|x| > 1/\sqrt{2}$  and  $f(x, y) = 0$  otherwise. Considering  $v(x, y) = -\operatorname{sgn} x$ ,  $(x, y) \in \Omega$ , we obtain that

$$\inf_u I(u) \leq I(v) = 2(P(B_+(0, 1), \Omega) - P(\Omega, \{f = 1\})) = 2\left(\frac{11}{10} - \frac{\pi}{2}\right) < 0,$$

where  $B_+(0, 1)$  denotes the right half-disk  $\{(x, y) \in B(0, 1) : x > 0\}$ . Observe also that the function  $v$  is not actually a solution.

Let us now consider only candidates for solutions that are of least gradient in  $\Omega$  and of the form  $w = \chi_{E_1} - \chi_{E_2}$  such that the jump set of  $w$  does not reach to the interior of the set  $\{f = 0\}$ . It is easy to verify for all  $\alpha, \beta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$  (and similarly for all  $\alpha, \beta \in [\frac{3\pi}{4}, \frac{5\pi}{4}]$ ) that the path of least weighted length that connects the boundary point  $(\cos \alpha, \sin \alpha)$  with  $(\cos \beta, \sin \beta)$  is a straight line segment. Thus, letting  $w_0(x, y) = \chi_{(-1, -1/\sqrt{2})}(x) - \chi_{(1/\sqrt{2}, 1)}(x)$  for  $(x, y) \in \Omega$ , we have  $I(w_0) \leq I(w)$ , while

$$I(w_0) = 2\left(P\left(\left\{(x, y) \in \Omega : x > \frac{1}{\sqrt{2}}\right\}, \Omega\right) - P(\Omega, \{f = 1\})\right) = 2\left(\sqrt{2} - \frac{\pi}{2}\right).$$

In particular,  $I(w) > I(v)$ .

Thus, the jump set of a solution  $u = \chi_{E_1} - \chi_{E_2}$  does reach to the interior of the set  $\{f = 0\}$ , i.e., there is  $z_0 \in \partial\Omega$  and  $r_0 > 0$  such that  $f \equiv 0$  in  $\partial\Omega \cap B(z_0, r_0)$ , but  $u$  is not constant in  $B(z_0, r)$  for any  $r < r_0$ . It can be verified that

$$\begin{aligned} E_1 &= \{(x, y) \in \Omega : x < -\max\{0.1, |y|/9\}\}, \\ E_2 &= \{(x, y) \in \Omega : x > \max\{0.1, |y|/9\}\}. \end{aligned}$$

## 6 Minimal solutions and their uniqueness

In this section, we assume that  $\Omega \subset X$  is a nonempty bounded open set with  $P(\Omega, X) < \infty$ , such that for any  $u \in BV(\Omega)$ , the trace  $Tu(x)$  exists for  $\mathcal{H}$ -a.e.  $x \in \partial^* \Omega$ . We also assume that the boundary data  $f \in L^1(\partial^* \Omega, P(\Omega, \cdot))$  satisfies (3.2).

We saw in Example 3.6 that solutions to the restricted Neumann problem need not be unique. However, we will see in this section that *minimal solutions* exist and are unique.

**Lemma 6.1.** *Let  $E, K \subset \Omega$  be of finite perimeter in  $\Omega$ . Then*

$$I(\chi_{E \cap K}) + I(\chi_{E \cup K}) \leq I(\chi_E) + I(\chi_K).$$

*Proof.* We have  $P(E \cap K, \Omega) + P(E \cup K, \Omega) \leq P(E, \Omega) + P(K, \Omega)$  by [38, Proposition 4.7]. Then by linearity of traces,  $\mathcal{H}$ -a.e. on  $\partial\Omega$  we have

$$T\chi_{E \cap K} + T\chi_{E \cup K} = T(\chi_{E \cap K} + \chi_{E \cup K}) = T(\chi_E + \chi_K) = T\chi_E + T\chi_K.$$

The claim follows.  $\square$

**Definition 6.2.** A solution  $u = \chi_{E_1} - \chi_{E_2}$  to the restricted Neumann problem is said to be *minimal* if whenever  $\tilde{E}_1, \tilde{E}_2 \subset \Omega$  are disjoint sets such that  $v = \chi_{\tilde{E}_1} - \chi_{\tilde{E}_2}$  is a solution, it follows that  $\mu(E_1 \setminus \tilde{E}_1) = 0$  and  $\mu(E_2 \setminus \tilde{E}_2) = 0$ .

By Lemma 3.12, it is enough to compare with solutions of the form  $\chi_{\tilde{E}} - \chi_{\Omega \setminus \tilde{E}}$ .

**Lemma 6.3.** *Suppose that  $u_a = \chi_{E_a} - \chi_{\Omega \setminus E_a}$  and  $u_b = \chi_{E_b} - \chi_{\Omega \setminus E_b}$  are both solutions to the restricted Neumann problem. Then, so are*

$$u := \chi_{E_a \cap E_b} - \chi_{\Omega \setminus (E_a \cap E_b)} \quad \text{and} \quad v := \chi_{E_a \cup E_b} - \chi_{\Omega \setminus (E_a \cup E_b)}.$$

*Proof.* By Lemma 6.1 we know that  $I(\chi_{E_a \cap E_b}) + I(\chi_{E_a \cup E_b}) \leq I(\chi_{E_a}) + I(\chi_{E_b})$ . By Lemma 3.12 we obtain that  $I(u) = 2I(\chi_{E_a \cap E_b})$ ,  $I(v) = 2I(\chi_{E_a \cup E_b})$ , and analogously for  $I(u_a)$  and  $I(u_b)$  as well. Then,

$$\frac{I(u) + I(v)}{2} = I(\chi_{E_a \cap E_b}) + I(\chi_{E_a \cup E_b}) \leq I(\chi_{E_a}) + I(\chi_{E_b}) = \frac{I(u_a) + I(u_b)}{2}. \quad (6.4)$$

As  $u_a$  and  $u_b$  are solutions, we can estimate  $I(u_a) = I(u_b) \leq I(u)$  and  $I(u_a) \leq I(v)$ , which together with (6.4) yields that  $I(u) = I(v) = I(u_a)$  and hence both  $u$  and  $v$  are solutions.  $\square$

**Theorem 6.5.** *Assume that  $\Omega$  satisfies the exterior measure density condition (3.1), that  $\partial\Omega$  is codimension 1 Ahlfors regular as given in (4.3), and that  $-1 \leq f \leq 1$ . Then there exists a unique (up to sets of  $\mu$ -measure zero) minimal solution to the restricted Neumann problem.*

*Proof.* By Theorem 4.15 we know that a solution exists. Let  $\beta = \inf_E \mu(E)$ , where the infimum is taken over all sets  $E$  such that  $u = \chi_E - \chi_{\Omega \setminus E}$  is a solution. By Proposition 3.8 and the fact that  $\Omega$  is bounded,  $\beta < \infty$ . Let  $\{E_k\}_{k=1}^\infty$  be a sequence of subsets of  $\Omega$  such that  $u_k = \chi_{E_k} - \chi_{\Omega \setminus E_k}$  are solutions and  $\mu(E_k) \rightarrow \beta$ . Let  $\tilde{E}_k = \bigcap_{j=1}^k E_j$ . Then, all functions

$$v_k := \min_{1 \leq j \leq k} u_j = \chi_{\tilde{E}_k} - \chi_{\Omega \setminus \tilde{E}_k}, \quad k = 1, 2, \dots$$

are also solutions by Lemma 6.3.

Let  $E_a = \bigcap_{k=1}^\infty E_k$ . Then  $v_k \rightarrow \chi_{E_a} - \chi_{\Omega \setminus E_a}$  in  $L^1(\Omega)$  and  $\mu(E_a) = \beta$ . By Lemma 4.10 we obtain that  $I(\chi_{E_a} - \chi_{\Omega \setminus E_a}) \leq \liminf_{k \rightarrow \infty} I(v_k)$ , and so  $\chi_{E_a} - \chi_{\Omega \setminus E_a}$  is also a solution.

Now if  $E \subset \Omega$  is such that  $\chi_E - \chi_{\Omega \setminus E}$  is a solution, by Lemma 6.3 we know that  $\chi_{E_a \cap E} - \chi_{\Omega \setminus (E_a \cap E)}$  is also a solution, and so  $\mu(E_a \cap E) \geq \beta$ . Since  $\mu(E_a) = \beta$ , necessarily  $\mu(E_a \setminus E) = 0$ . By the same argument, we obtain that  $E_a$  is the unique set with these properties, up to sets of  $\mu$ -measure zero.

By an entirely analogous argument, we find a unique (up to sets of  $\mu$ -measure zero) set  $E_b \subset \Omega$  such that  $\chi_{\Omega \setminus E_b} - \chi_{E_b}$  is a solution, and whenever  $\chi_{\Omega \setminus E} - \chi_E$  is another solution, then  $\mu(E_b \setminus E) = 0$ . By Lemma 3.13,  $\chi_{E_a} - \chi_{E_b}$  is the desired unique minimal solution.  $\square$

## 7 Stability

In this section, we always assume that  $\Omega \subset X$  is a nonempty bounded open set with  $P(\Omega, X) < \infty$ , such that for any  $u \in BV(\Omega)$ , the trace  $Tu(x)$  exists for  $\mathcal{H}$ -a.e.  $x \in \partial^* \Omega$ .

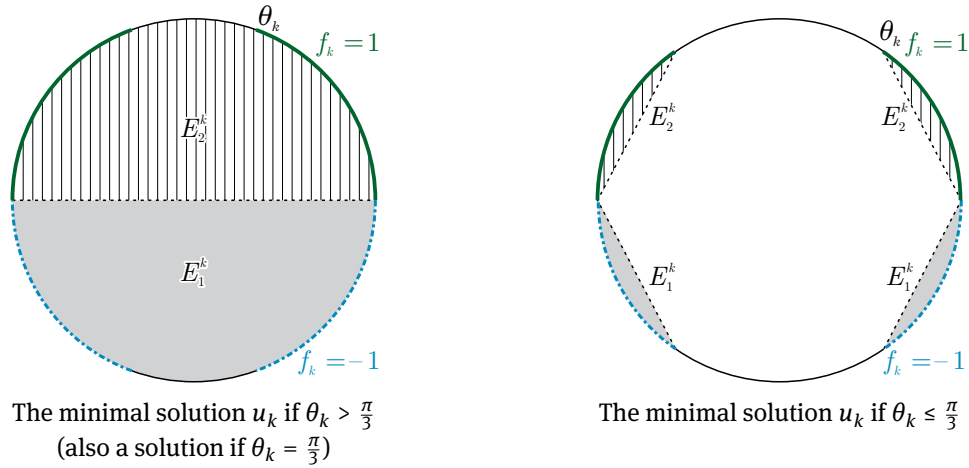


Figure 4: See Example 7.1

The goal here is to investigate stability of solutions to the restricted Neumann problem. By stability we mean that if a sequence of Neumann boundary data converges in  $L^1(\partial^* \Omega)$  to a function, then the corresponding sequence of solutions converges (perhaps up to a subsequence) to a solution to the Neumann problem with the limit boundary data. Stability properties give us a method by which we can, by hand, construct a solution to the Neumann problem for complicated boundary data by using solutions to simpler boundary data.

**Example 7.1.** Let  $\Omega = B(0, 1) \subset \mathbb{R}^2 \cong \mathbb{C}$  (unweighted). For each  $k \in \mathbb{N}$ , let  $\theta_k = \frac{\pi}{3} + \frac{(-1)^k}{k}$  and consider the sequence of boundary data functions

$$f_k(e^{i\theta}) := \begin{cases} 1 & \text{when } \theta \in (0, \theta_k] \cup [\pi - \theta_k, \pi), \\ -1 & \text{when } \theta \in [-\theta_k, 0) \cup (\pi, \pi + \theta_k], \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that there are two types of minimal solutions based on the value of  $\theta_k$ . If  $\theta_k \in [\frac{\pi}{3}, \frac{\pi}{2}]$ , then a solution can be expressed as  $u(x, y) = -\text{sgn}(y)$ , which is also minimal in case  $\theta_k > \frac{\pi}{3}$ . However, if  $\theta_k \in (0, \frac{\pi}{3}]$ , then the minimal solution  $u_k = \chi_{E_1^k} - \chi_{E_2^k}$  is determined by four disk segments whose arcs cover the connected components of  $\{f_k \neq 0\}$ , i.e.,

$$\begin{aligned} E_1^k &= \{(x, y) \in \Omega : (1 - \cos \theta_k)y \leq (|x| - 1) \sin \theta_k\}, \\ E_2^k &= \{(x, y) \in \Omega : (1 - \cos \theta_k)y \geq (1 - |x|) \sin \theta_k\}. \end{aligned} \tag{7.2}$$



Thus,  $u_{2k} = u$  for all  $k = 1, 2, \dots$ , and trivially  $u_{2k} \rightarrow u$  as  $k \rightarrow \infty$ . On the other hand  $u_{2k+1} \rightarrow u_\infty = \chi_{E_1^\infty} - \chi_{E_2^\infty} \neq u$ , where  $E_1^\infty$  and  $E_2^\infty$  are the sets as in (7.2) for  $\theta_\infty = \frac{\pi}{3}$ .

Consequently, the sequence of solutions  $\{u_k\}_{k=1}^\infty$  does not have any limit even though the sequence of boundary data functions converges in  $L^1(\partial^* \Omega, P(\Omega, \cdot))$ .

Note however that both functions  $u$  and  $u_\infty$  are solutions to the restricted Neumann problem with boundary data given by  $f = \lim_k f_k$ . This observation suggests that a weaker notion of stability might apply here. Indeed, Theorem 7.4 below will show that stability can be recovered if we allow for passing to a subsequence of the sequence of solutions.

In this section, we use the abbreviation  $L^1(\partial^* \Omega) := L^1(\partial^* \Omega, P(\Omega, \cdot))$ .

**Lemma 7.3.** *If  $u$  is a solution to the restricted Neumann problem with  $L^1(\partial^* \Omega)$ -boundary data  $f$  and  $v$  is a solution with  $L^1(\partial^* \Omega)$ -boundary data  $h$ , then*

$$|I_f(u) - I_h(v)| \leq \|f - h\|_{L^1(\partial^* \Omega)}.$$

*Proof.* Note that  $-1 \leq v \leq 1$  and  $-1 \leq u \leq 1$ . Therefore,

$$I_h(u) - I_f(u) \leq |I_h(u) - I_f(u)| = \left| \int_{\partial^* \Omega} (f - h) Tu \, dP(\Omega, \cdot) \right| \leq \|f - h\|_{L^1(\partial^* \Omega)}$$

and

$$I_f(v) - I_h(v) \leq |I_f(v) - I_h(v)| = \left| \int_{\partial^* \Omega} (f - h) Tv \, dP(\Omega, \cdot) \right| \leq \|f - h\|_{L^1(\partial^* \Omega)}.$$

It follows that

$$I_f(u) \geq I_h(u) - \|f - h\|_{L^1(\partial^* \Omega)} \geq I_h(v) - \|f - h\|_{L^1(\partial^* \Omega)}$$

and

$$I_h(v) \geq I_f(v) - \|f - h\|_{L^1(\partial^* \Omega)} \geq I_f(u) - \|f - h\|_{L^1(\partial^* \Omega)}.$$

In the above, we used the facts that  $v$  is a solution for  $I_h$  and that  $u$  is a solution for  $I_f$ . The desired conclusion now follows.  $\square$

**Theorem 7.4.** *Assume that  $\Omega$  satisfies the exterior measure density condition (3.1) and that  $\partial \Omega$  is codimension 1 Ahlfors regular as given in (4.3). Assume that  $f_k: \partial^* \Omega \rightarrow [-1, 1]$  satisfy (3.2), that  $f_k \rightarrow f$  in  $L^1(\partial^* \Omega)$  as  $k \rightarrow \infty$ , and that  $u_k = \chi_{E_1^k} - \chi_{E_2^k}$  are solutions to the restricted Neumann problem with boundary data  $f_k$ , for disjoint sets  $E_1^k, E_2^k \subset \Omega$ . Then, there is a subsequence  $\{u_{k_j}\}_{j=1}^\infty$  and a function  $u = \chi_{E_1} - \chi_{E_2}$  such that  $u_{k_j} \rightarrow u$  in  $L^1(\Omega)$  and  $u$  is a solution to the restricted Neumann problem with boundary data  $f$ .*

*Proof.* Clearly  $f$  also satisfies (3.2). By Theorem 4.15, we know that there exists a solution  $v \in \text{BV}(\Omega)$  for boundary data  $f$ . By Lemma 7.3,  $|I_f(v) - I_{f_k}(u_k)| \leq \|f - f_k\|_{L^1(\partial^* \Omega)} \rightarrow 0$  as  $k \rightarrow \infty$ . By the fact that  $u_k$  are solutions and Lemma 3.18, we get

$$\max\{P(E_1^k, X), P(E_2^k, X)\} \leq 2P(\Omega, X).$$

Thus by [38, Theorem 3.7], there are sets  $E_1, E_2 \subset \Omega$  such that  $\chi_{E_1^k} \rightarrow \chi_{E_1}$  and  $\chi_{E_2^k} \rightarrow \chi_{E_2}$  in  $L^1(X)$ , possibly having passed to a subsequence (not relabeled). Define  $u = \chi_{E_1} - \chi_{E_2}$ . Then, by the lower semicontinuity given in Lemma 4.10,

$$\begin{aligned} I_f(u) &= I_f(\chi_{E_1}) + I_f(-\chi_{E_2}) \leq \liminf_{k \rightarrow \infty} I_f(\chi_{E_1^k}) + \liminf_{k \rightarrow \infty} I_f(-\chi_{E_2^k}) \\ &\leq \liminf_{k \rightarrow \infty} (I_{f_k}(\chi_{E_1^k}) + \|f - f_k\|_{L^1(\partial^* \Omega)}) + \liminf_{k \rightarrow \infty} (I_{f_k}(-\chi_{E_2^k}) + \|f - f_k\|_{L^1(\partial^* \Omega)}) \\ &\leq \liminf_{k \rightarrow \infty} (I_{f_k}(u_k) + 2\|f - f_k\|_{L^1(\partial^* \Omega)}) = I_f(v). \end{aligned}$$

Now,  $v$  was a solution for  $I_f$  and hence so is  $u$ .  $\square$

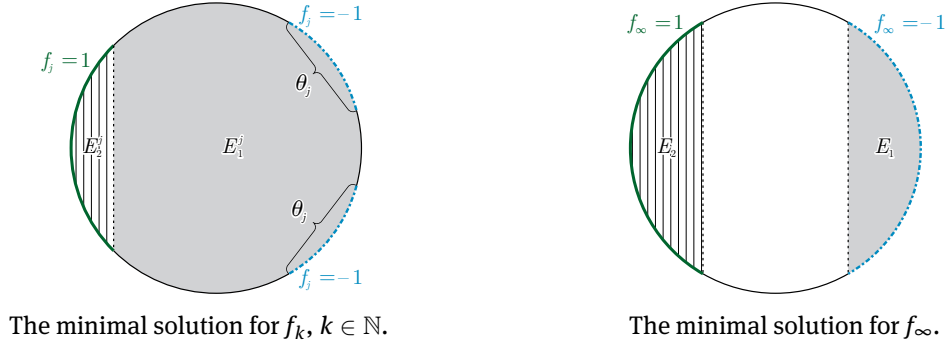


Figure 5: See Example 7.5

Observe that minimality of solutions need not be preserved when perturbing the boundary data. In Example 7.1, we saw that  $u_{2k} \rightarrow u$  as  $k \rightarrow \infty$ , where  $u$  was a solution for the limit boundary data. Nonetheless, while  $u_{2k}$  were the minimal solutions for the respective boundary value problems, that was not the case for  $u$ , since the minimal solution for the limit boundary data was given by  $u_\infty$ .

In the example, the boundary data were given as  $f_{2k} = \chi_{F_1^{2k}} - \chi_{F_2^{2k}}$  for decreasing sequences of sets  $\{F_1^{2k}\}_{k=1}^\infty$  and  $\{F_2^{2k}\}_{k=1}^\infty$ . One might also ask whether the minimality of a solution is preserved if the boundary data has the form  $f_k = \chi_{F_1^k} - \chi_{F_2^k}$  for increasing sequences of sets  $\{F_1^k\}_{k=1}^\infty$  and  $\{F_2^k\}_{k=1}^\infty$ . The next example shows that the minimality can be lost in this case as well.

**Example 7.5.** Let  $\Omega = B(0, 1) \subset \mathbb{R}^2 \cong \mathbb{C}$  (unweighted) and

$$f_k(e^{i\theta}) = \begin{cases} 1 & \text{when } \theta \in (\pi - \theta_k, \pi + \theta_k), \\ -1 & \text{when } \theta \in (\frac{\pi}{3} - \theta_k, \frac{\pi}{3}) \cup (-\frac{\pi}{3}, \theta_k - \frac{\pi}{3}), \\ 0 & \text{otherwise,} \end{cases}$$

where  $\theta_k = \frac{\pi k}{3(k+1)}$ . Then, the minimal solutions are given by  $u_k = \chi_{E_1^k} - \chi_{E_2^k}$ , where  $E_1^k = \{z \in \Omega : \Re z > -\cos \theta_k\}$  and  $E_2^k = \Omega \setminus E_1^k$ . The minimal solution for boundary data given by the limit function  $f_\infty$  is determined by the sets  $E_1 = \{z \in \Omega : \Re z > \frac{1}{2}\}$  and  $E_2 = \{z \in \Omega : \Re z < \frac{1}{2}\}$ . In particular,  $E_1 \subsetneq \bigcap_k E_1^k = \Omega \setminus E_2$ .

In light of the above example, we give one explicit construction of a solution (but not necessarily a minimal one) for limit boundary data. We first need the following more general lemma.

In what follows, for  $E \subset \Omega$  of finite perimeter in  $\Omega$ , we denote

$$I_f(E) := I_f(\chi_E) = \frac{1}{2} I_f(\chi_E - \chi_{\Omega \setminus E}).$$

**Lemma 7.6.** For each  $k \in \mathbb{N}$ , assume that  $f_k \in L^1(\partial^* \Omega)$  satisfies (3.2) and suppose that  $E_1^k, E_2^k \subset \Omega$  are disjoint sets such that  $\chi_{E_2^k} - \chi_{E_1^k}$  is a solution to the restricted Neumann problem with boundary data  $f_k$ . Denote  $E_k := E_1^k$ . Then for each  $n \in \mathbb{N}$  and for each choice of  $k_1, \dots, k_n \in \mathbb{N}$  with  $k_1 < \dots < k_n$ , we have

$$0 \leq I_{f_{k_n}}(E_{k_1} \cup \dots \cup E_{k_n}) - I_{f_{k_n}}(E_{k_n}) \leq 2 \sum_{j=1}^{n-1} \|f_{k_j} - f_{k_{j+1}}\|_{L^1(\partial^* \Omega)}.$$

*Proof.* The first inequality follows from Lemma 3.13. To prove the second, note first that for  $K \subset \Omega$  of finite perimeter in  $\Omega$  and for  $k \in \mathbb{N}$ , we have that  $I_{f_k}(E_k) \leq I_{f_k}(E_k \cap K)$ . Moreover, by Lemma 6.1 we know that

$$I_{f_k}(E_k \cup K) + I_{f_k}(E_k \cap K) \leq I_{f_k}(E_k) + I_{f_k}(K),$$

and so

$$I_{f_k}(E_k \cup K) \leq I_{f_k}(K). \tag{7.7}$$

Furthermore, if  $m \in \mathbb{N}$ , then

$$I_{f_k}(K) \leq I_{f_m}(K) + \|f_k - f_m\|_{L^1(\partial^*\Omega)}. \quad (7.8)$$

Now by an iterated  $((n-1)$ -times) application of (7.7) followed by (7.8), and finally by Lemma 7.3, we obtain

$$\begin{aligned} I_{f_{k_n}}(E_{k_1} \cup \dots \cup E_{k_n}) &\leq I_{f_{k_n}}(E_{k_1} \cup \dots \cup E_{k_{n-1}}) \\ &\leq I_{f_{k_{n-1}}}(E_{k_1} \cup \dots \cup E_{k_{n-1}}) + \|f_{k_{n-1}} - f_{k_n}\|_{L^1(\partial^*\Omega)} \\ &\leq \dots \\ &\leq I_{f_{k_1}}(E_{k_1}) + \sum_{j=1}^{n-1} \|f_{k_j} - f_{k_{j+1}}\|_{L^1(\partial^*\Omega)} \\ &\leq I_{f_{k_n}}(E_{k_n}) + \|f_{k_1} - f_{k_n}\|_{L^1(\partial^*\Omega)} + \sum_{j=1}^{n-1} \|f_{k_j} - f_{k_{j+1}}\|_{L^1(\partial^*\Omega)} \\ &\leq I_{f_{k_n}}(E_{k_n}) + 2 \sum_{j=1}^{n-1} \|f_{k_j} - f_{k_{j+1}}\|_{L^1(\partial^*\Omega)}. \end{aligned}$$

Thus we obtain the desired inequality.  $\square$

**Theorem 7.9.** *Suppose that  $\Omega$  satisfies the exterior measure density condition (3.1), and that  $\partial\Omega$  is codimension 1 Ahlfors regular as given in (4.3). For each  $k \in \mathbb{N}$ , suppose that  $f_k: \partial^*\Omega \rightarrow [-1, 1]$  satisfies (3.2), that  $\|f_k - f_{k+1}\|_{L^1(\partial^*\Omega)} \leq 2^{-k}$ , and that  $E_1^k, E_2^k \subset \Omega$  are disjoint sets such that  $\chi_{E_1^k} - \chi_{E_2^k}$  is a solution for boundary data  $f_k$ . Set  $E_k = E_1^k$ . Then the limit supremum*

$$E_+ := \bigcap_{n \in \mathbb{N}} \bigcup_{k=n}^{\infty} E_k$$

gives a solution  $\chi_{E_+} - \chi_{\Omega \setminus E_+}$  for the boundary data  $f := \lim_k f_k$ .

*Proof.* By Theorem 4.15 we know that there exists a solution  $v \in \text{BV}(\Omega)$  for boundary data  $f$ . From Lemma 7.3 we see that

$$|I_f(v) - 2I_{f_m}(E_m)| \leq \|f - f_m\|_{L^1(\partial^*\Omega)} \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (7.10)$$

For each  $n \in \mathbb{N}$ , set  $K_n = \bigcup_{k=n}^{\infty} E_k$ . By Lebesgue's dominated convergence theorem,  $\chi_{E_n \cup \dots \cup E_m} \rightarrow \chi_{K_n}$  in  $L^1(\Omega)$  as  $m \rightarrow \infty$ . Fix  $n \in \mathbb{N}$ . By Lemma 7.6 and Lemma 4.10,

$$\begin{aligned} \limsup_{m \rightarrow \infty} I_{f_m}(E_m) &\geq \limsup_{m \rightarrow \infty} I_{f_m}(E_n \cup \dots \cup E_m) - 2^{2-n} \\ &\geq \liminf_{m \rightarrow \infty} I_f(E_n \cup \dots \cup E_m) - \limsup_{m \rightarrow \infty} \|f - f_m\|_{L^1(\partial^*\Omega)} - 2^{2-n} \\ &\geq I_f(K_n) - 2^{2-n}. \end{aligned}$$

By letting  $n \rightarrow \infty$  and recalling (7.10), we obtain

$$\frac{I_f(v)}{2} = \lim_{m \rightarrow \infty} I_{f_m}(E_m) = \liminf_{n \rightarrow \infty} I_f(K_n) \geq I_f(E_+)$$

by Lemma 4.10, since  $\chi_{K_n} \rightarrow \chi_{E_+}$  in  $L^1(\Omega)$ . Thus  $\chi_{E_+}$  is also a solution for boundary data  $f$ .  $\square$

It can be seen from Example 7.5 that the set  $E_+$  constructed in Theorem 7.9 need not yield a *minimal* solution to the limit Neumann problem.

## 8 The unrestricted minimization problem

In this section we always assume that  $\Omega$  is a nonempty bounded domain with  $P(\Omega, X) < \infty$ , such that the trace operator  $T: \text{BV}(\Omega) \rightarrow L^1(\partial^*\Omega, P(\Omega, \cdot))$  is bounded.

So far we have looked at the most general situation where it is possible to have  $I_f(u) = \|Du\|(\Omega) + \int_{\partial^* \Omega} Tu f dP(\Omega, \cdot) < 0$  for some  $u \in BV(\Omega)$ . To overcome the fact that should  $I_f(u) < 0$  for some  $u$  then the minimal value of  $I_f$  is  $-\infty$ , we considered minimization only over  $u \in BV(\Omega)$  for which  $-1 \leq u \leq 1$ . In the special case where

$$\inf_{u \in BV(\Omega)} I_f(u) \geq 0,$$

the minimal energy must necessarily be 0; hence constant functions (and in particular, the zero function) would be a solution to the given Neumann boundary value problem with boundary data  $f$ . In this case we do not here need to restrict our attention to  $-1 \leq u \leq 1$  alone, but to *all* functions in the class  $BV(\Omega)$ . In this case it would be interesting to see under what conditions we would have *nonconstant* minimizers of  $I_f$  exist. If there is one, then there are infinitely many distinct (in the sense that they do not differ only by a constant) minimizers, as seen by multiplying by a scalar. In this study we take inspiration from [39]. We do not have a criterion that guarantees existence of a nonconstant minimizer. In the Euclidean setting, the PDE approach helps in forming such a guarantee, but we do not have such an approach in the metric setting. However, we do obtain a criterion under which there is no nonconstant minimizer, see Proposition 8.1 below. As a consequence of Proposition 8.3 we also obtain that if there are no minimizers for the unrestricted problem for the boundary data  $f$ , then there is a positive number  $\lambda(-f)$  such that the boundary data  $\lambda(-f)f$  does have a minimizer.

From now on, let  $g \in L^\infty(\partial^* \Omega, P(\Omega, \cdot))$  with  $\int_{\partial^* \Omega} g dP(\Omega, \cdot) = 0$ . We set  $\mathcal{M}_g$  to be the collection of all functions  $u \in BV(\Omega)$  such that  $\int_\Omega u d\mu = 0$  and  $\int_{\partial^* \Omega} Tu g dP(\Omega, \cdot) = 1$ , and

$$\lambda(g) := \inf_{u \in \mathcal{M}_g} \|Du\|(\Omega).$$

Note that if  $\lambda(g) < 1$ , then there is some  $u \in BV(\Omega)$  such that  $I_{-g}(u) < 0$ , and hence the unrestricted minimization problem for  $f = -g$  has no solution.

**Proposition 8.1.** *If  $\lambda(g) \geq 1$ , then there is a solution to the unrestricted minimization problem for the energy  $I_{-g}$  on  $\Omega$ . Furthermore, if  $\lambda(g) > 1$  then the only minimizers are constant functions.*

*Proof.* We will prove the claim of the proposition by showing that for each  $w \in BV(\Omega)$ , we have  $I_{-g}(w) \geq 0$ .

For  $w \in BV(\Omega)$ , we have two possibilities. The first possibility is that  $-\int_{\partial^* \Omega} Tw g dP(\Omega, \cdot) \geq 0$ ; in this case we have that  $I_{-g}(w) \geq 0$ . Thus it suffices to consider only the case that  $-\int_{\partial^* \Omega} Tw g dP(\Omega, \cdot) < 0$ . In this case we set

$$\alpha(w) = \int_{\partial^* \Omega} Tw g dP(\Omega, \cdot),$$

and note that  $\alpha(w) > 0$ . With  $c := \int_\Omega w d\mu$ , we have  $\alpha(w)^{-1}(w - c) \in \mathcal{M}_g$ , and so by the hypothesis of the proposition,

$$\alpha(w)^{-1} \|Dw\|(\Omega) \geq \lambda(g) \geq 1, \tag{8.2}$$

that is,

$$\|Dw\|(\Omega) \geq \alpha(w) = \int_{\partial^* \Omega} Tw g dP(\Omega, \cdot).$$

It then follows that

$$I_{-g}(w) = \|Dw\|(\Omega) + \int_{\partial^* \Omega} Tw [-g] dP(\Omega, \cdot) \geq 0.$$

Finally, suppose that  $\lambda(g) > 1$ . If  $-\int_{\partial^* \Omega} Tw g dP(\Omega, \cdot) < 0$ , then (8.2) implies that

$$\|Dw\|(\Omega) \geq \lambda(g) \int_{\partial^* \Omega} Tw g dP(\Omega, \cdot) > \int_{\partial^* \Omega} Tw g dP(\Omega, \cdot),$$

and hence  $I_{-g}(w) > 0$ . On the other hand, if  $-\int_{\partial^* \Omega} Tw g dP(\Omega, \cdot) \geq 0$ , then  $I_{-g}(w) \geq \|Dw\|(\Omega)$ . Since the least value of the functional  $I_{-g}$  is zero, its minimizer  $w \in BV(\Omega)$  satisfies  $\|Dw\|(\Omega) = 0$ , that is,  $w$  is constant.  $\square$

From the above proposition, it follows that if there is a nonconstant minimizer for  $I_{-g}$ , then necessarily  $\lambda(g) = 1$ . Observe that if  $\tau$  is a positive real number, then  $\lambda(\tau g) = \lambda(g)/\tau$ . Thus if  $\lambda(g) > 0$ , then  $I_{-\lambda(g)g}$  does have a minimizer from  $BV(\Omega)$ .

**Proposition 8.3.** *Suppose the trace operator  $T: BV(\Omega) \rightarrow L^1(\partial^* \Omega, P(\Omega, \cdot))$  is surjective and that there is a constant  $C > 0$  such that whenever  $u \in BV(\Omega)$  with  $\int_{\Omega} u \, d\mu = 0$ , we have*

$$\int_{\partial^* \Omega} |Tu| \, dP(\Omega, \cdot) \leq C \|Du\|(\Omega).$$

If  $\|g\|_{L^\infty(\partial^* \Omega, P(\Omega, \cdot))} > 0$ , then  $\lambda(g) > 0$ .

We refer the interested reader to [31, Theorem 5.5] together with [33, Theorem 1.2] for geometric conditions on  $\Omega$  that guarantee that the hypotheses of the above proposition hold. Note that if  $T$  is a bounded operator in the sense of [31, Theorem 5.5], then by the Poincaré inequality on  $\Omega$  we obtain the control of  $\int_{\partial^* \Omega} |Tu| \, dP(\Omega, \cdot)$  solely in terms of  $\|Du\|(\Omega)$  for  $u \in BV(\Omega)$  with  $\int_{\Omega} u \, d\mu = 0$ .

*Proof.* Since  $\|g\|_{L^\infty(\partial^* \Omega, P(\Omega, \cdot))} > 0$ , the class  $\mathcal{M}_g$  is non-empty. Indeed, we can choose a function  $w \in BV(\Omega)$  such that  $Tw = g$ , and then  $0 < \alpha(w) = \int_{\partial^* \Omega} Tw g \, dP(\Omega, \cdot) < \infty$ . With  $c := \int_{\Omega} w \, d\mu$ , we have  $\alpha(w)^{-1}(w - c) \in \mathcal{M}_g$ .

Now suppose that  $w \in \mathcal{M}_g$ . As  $\int_{\Omega} w \, d\mu = 0$ , we have by assumption

$$\int_{\partial^* \Omega} |Tw| \, dP(\Omega, \cdot) \leq C \|Dw\|(\Omega).$$

Hence,

$$\begin{aligned} 1 &= \int_{\partial^* \Omega} Tw g \, dP(\Omega, \cdot) \leq \|g\|_{L^\infty(\partial^* \Omega, P(\Omega, \cdot))} \int_{\partial^* \Omega} |Tw| \, dP(\Omega, \cdot) \\ &\leq C \|g\|_{L^\infty(\partial^* \Omega, P(\Omega, \cdot))} \|Dw\|(\Omega). \end{aligned}$$

Thus we must have

$$\lambda(g) \geq \frac{1}{C \|g\|_{L^\infty(\partial^* \Omega, P(\Omega, \cdot))}} > 0. \quad \square$$

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