Anal. Geom. Metr. Spaces 2018; 6:48–63 DE GRUYTER

Research Article Open Access

Mark Medwid and Xiangdong Xie*

Rigidity of Local Quasisymmetric Maps on Fibered Spaces

https://doi.org/10.1515/agms-2018-0003
Received August 29, 2017; revised March 7, 2018; accepted April 23, 2018

Abstract: We give conditions under which a fiber-preserving quasisymmetric map between open subsets of fibered metric spaces is locally biLipschitz. We also show that a quasiconformal map between open subsets of 2-step Carnot groups with reducible first layer is locally biLipschitz.

Keywords: rigidity; fibered space; Carnot group; quasiconformal map; quasisymmetric map

MSC: 30L10, 22E25, 53C17

1 Introduction

In this paper we study the rigidity property of local quasisymmetric maps between open subsets of fibered spaces that send fibers to fibers.

The Euclidean spaces admit a rich class of quasiconformal maps. However, if the quasiconformal maps (or quasisymmetric maps) preserve additional structures, then they are often forced to be biLipschitz. Le Donne-Xie [7] showed that if a quasisymmetric map permutes the fibers of fibered metric spaces, then it is biLipschitz. The result of Le Donne-Xie is valid only for globally defined quasisymmetric maps and the proof uses the assumption that the fibers are unbounded. It is natural to ask whether fiber-preserving quasisymmetric maps between open subsets of fibered metric spaces are locally biLipschitz. This note is an attempt in this direction. We provide two results (Theorem 1.1 and Theorem 1.2) that are applicable in different situations.

One often has to deal with quasimetrics instead of metrics while studying the ideal boundary of negatively curved spaces. For this reason we state our result on fibered spaces for quasimetric spaces. See Section 2 for the definition of a quasimetric space.

To state our first result, we use the following notion of fibered spaces. We stress that our definition of "fibered space" is different from the one in [7], see the remark after the definition for more information.

Definition 1.1. Let $L \geq 1$, $\alpha \in (0, 1]$, $C : X \to [1, \infty)$ a function, and $\varphi : [0, \infty) \to [1, \infty)$ a continuous function. We say that a quasimetric space (X, d) is an (L, α, C, φ)-*fibered space* if X admits a partition $\mathcal{F}$ into closed sets (members of $\mathcal{F}$ are called *fibers*) with the following properties:

(1) *Fibers are snow-flake equivalent to geodesic spaces*:
For each fiber F, there is a geodesic metric space $(\tilde{F}, d)$ such that F is L-biLipschitz to $(\tilde{F}, d^{\alpha})$;

(2) *Locally uniformly perfect leaf space*:
For any $x \in X$ and all sufficiently small $r > 0$, there is a fiber F such that $\frac{r}{C(x)} \leq d(x, F) \leq r$; moreover, for any bounded subset $B \subset X$, $C_B := \sup\{C(x) | x \in B\} < \infty$.

Mark Medwid: Mathematics and Computer Science Department, Rhode Island College, Providence, RI, U.S.A., E-mail: mmedwid@ric.edu
***Corresponding Author: Xiangdong Xie:** Dept. of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH, U.S.A., E-mail: xiex@bgsu.edu

(3) *Slow divergence of fibers*:
If F is a fiber of X and $x, y \in F$, then for any fiber F':

$$\frac{1}{\varphi(d(x,y))} \cdot d(y, F') \le d(x, F') \le \varphi(d(x,y)) \cdot d(y, F'). \tag{1.2}$$

Remark 1.3. Comparison of our notion of "fibered space" with the one in [7]:
Both require the fibers to be snow-flake equivalent to geodesic spaces. The definition in [7] requires the fibers to be unbounded and also requires every fiber to be the limit of parallel fibers. We do not have these assumptions. On the other hand, we have the "Slow divergence of fibers" condition that is not assumed in [7]. In particular, our Theorem 1.1 applies (while the main result in [7] does not) to the power maps $x \mapsto |x|^s x$ away from the origin and infinity (the fibers in the domain are horizontal lines and the fibers in the range are images of horizontal lines under the power map). However, our Theorem 1.1 does not apply to general Carnot groups since generally the "Slow divergence of fibers" condition is not satisfied in the Carnot setting.

Let $K \ge 1$ and $C > 0$. A bijection $f : \tilde{X} \to X$ is called a (K, C) *quasisimilarity* if

$$\frac{C}{K} \cdot d(\tilde{x}_1, \tilde{x}_2) \le d(f(\tilde{x}_1), f(\tilde{x}_2)) \le KC \cdot d(\tilde{x}_1, \tilde{x}_2)$$

for all $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$. It is clear that f is a quasisimilarity if and only if f is biLipschitz. The notion of quasisimilarity is sometimes more useful since it involves two constants and often there is control on K but not on C (in such cases one can not get control on the biLipschitz constant). Throughout this paper we shall use the terms "quasisimilarity" and "biLipschitz" interchangeably.

We also need the following:

Definition 1.4. Let $f : X \to Y$ be a map between quasimetric spaces and $x \in X$ a non-isolated point. Define:

$$L_f(x) = \limsup_{y \to x} \frac{d(f(x), f(y))}{d(x,y)}, \qquad l_f(x) = \liminf_{y \to x} \frac{d(f(x), f(y))}{d(x,y)}.$$

The quantities $L_f(x)$ and $l_f(x)$ are the *upper* and *lower pointwise Lipschitz constants* of f at x, respectively. A point $x \in X$ is called a *K-quasisimilarity point* (for some $K \ge 1$) with respect to f if $0 < l_f(x) \le L_f(x) \le K l_f(x) < \infty$.

Quasisimilarity points are related to differentiability, see Section 3.1.

Our first result is the following:

Theorem 1.1. *Let X, Y be fibered spaces. Let $U \subset X$, $V \subset Y$ be open. Suppose $f : U \to V$ is a quasisymmetry that satisfies the following:*
(1) For each fiber F of X, f maps each component of $F \cap U$ onto an open subset of a fiber of Y. Similarly for f^{-1}.
(2) For each $x \in U$, there is an open subset $U_x \subset U$ and some $K_x \ge 1$ such that for every fiber F of X, $F \cap U_x$ contains a dense set of K_x-quasisimilarity points with respect to f.

Then f is a local quasisimilarity.

Theorem 1.1 applies when all the fibers are parallel. In this case the statement is quantitative. See Section 3.4. A particular example is provided by quasiconformal maps between open subsets of Heisenberg groups that send vertical line segments to vertical line segments.

For the proof of Theorem 1.1, we will first show that f is a local quasisimilarity when restricted to a particular fiber. This is achieved by comparing the stretch of f along fibers as well as between fibers to obtain an upper bound for the upper pointwise Lipschitz constant L_f. Then, we shall compare L_f across fibers, which will allow us to conclude that f is a local quasisimilarity.

Our second result relates to 2-step Carnot groups. In [7] it is shown that a globally defined quasiconformal map of a Carnot group with reducible first layer is biLipschitz. The following result is a version of that result for quasiconformal maps defined on open subsets of 2-step Carnot groups.

Theorem 1.2. *Let G be a 2-step Carnot group identified with its Lie algebra $\mathfrak{g} = V_1 \oplus V_2$. Suppose there is a nontrivial proper subspace W_1 of V_1 such that $A(W_1) = W_1$ for all graded automorphisms A of $\mathfrak{g}$. Then every quasiconformal map $f : U \to V$ between open subsets of G is locally biLipschitz.*

As indicated in Remark 1.3, in general the left cosets of connected subgroups of a Carnot group do not satisfy the "Slow divergence of fibers" condition, as the calculation in Section 4.1 shows. So Carnot groups with left cosets of a connected subgroup is not a fibered space in the sense of Definition 1.1. As a result, Theorem 1.2 requires a separate argument. The proof of Theorem 1.2 follows the same two basic steps as the proof of Theorem 1.1: first, we show the restriction to a particular fiber is a local quasisimilarity, which we then use to show that f is a local quasisimilarity. The key difference between the two proofs is the way in which both arguments address the first step. In particular, in the 2-step Carnot group case, we show that the divergence of left cosets is at worst quadratic and that the worst case is realized (Section 4.1). We then consider left cosets with the worst case divergence. The divergence of the image left cosets is no worse than the divergence of the original left cosets, and this leads to an upper bound on the Lipschitz constant.

The theme of the paper is rigidity of quasiconformal/quasisymmetric maps. A major motivation for the study of rigidity of quasiconformal maps is the rigidity question of quasiisometries between negatively curved spaces: every negatively curved space has an ideal boundary, and the rigidity of quasiisometries between negatively curved spaces corresponds to the rigidity of quasiconformal maps between their ideal boundaries.

The first main result on the rigidity of quasiconformal maps on Carnot groups is due to Pansu [8]. He showed that every global quasiconformal map of the quarternionic Heisenberg group is a similarity. Capogna-Cowling [3] proved that 1-quasiconformal maps on Carnot groups are smooth. Recently Cowling-Ottazzi [4] further showed that 1-quasiconformal maps on Carnot groups are boundary maps of isometries of the negatively curved homogeneous manifolds associated to the Carnot groups. Other rigidity results include [5], [9], [10]. Conjecturally, every global quasiconformal map of a Carnot group G is biLipschitz, provided G is not a Euclidean group or a Heisenberg group. The result in [7] verifies this conjecture for Carnot groups with reducible first layer.

Related to the current paper, a natural question is whether Theorem 1.2 holds for all Carnot groups with reducible first layer. There is evidence to suggest that quasiconformal maps on Carnot groups satisfy an even stronger rigidity property (stronger than quasiconformal maps being biLipschitz). For example, there is a description of all global quasiconformal maps on the higher model Filiform groups in [10]. A more ambitious question is to identify all quasiconformal maps (local and global) on all Carnot groups other than Euclidean groups and Heisenberg groups (it is known that quasiconformal maps on Euclidean spaces and Heisenberg groups are very rich [1], [6]).

This paper is organized as follows. In section 2 we fix the notation and terminology. In section 3 we prove Theorem 1.1 and give some applications. The proof of Theorem 1.2 is in section 4.

2 Notation and terminology

In this section we fix the notation and terminology. The reader is referred to Section 2 of [7] for definitions and theorems not recalled here.

A function $d : X \times X \to [0, \infty)$ is a *quasimetric* on a set X if

(1) d is symmetric, that is, $d(x_1, x_2) = d(x_2, x_1)$ for all $x_1, x_2 \in X$;
(2) $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$;
(3) d satisfies the generalized triangle inequality, that is, there exists some constant $M \geq 1$ such that for all $x_1, x_2, x_3 \in X$ the inequality $d(x_1, x_3) \leq M \cdot (d(x_1, x_2) + d(x_2, x_3))$ holds.

We call M a quasimetric constant of d.

An example of a quasimetric is a symmetric function $d : X \times X \to [0, \infty)$ that is biLipschitz equivalent with a metric. Homogeneous distances on Carnot groups are biLipschitz equivalent with Carnot metrics and so are quasimetrics.

For $x \in X$ and $r > 0$, denote $B(x, r) = \{x' \in X | d(x, x') < r\}$. For any $B = B(x, r)$ and any $\lambda > 0$, denote $\lambda B = B(x, \lambda r)$. A subset $U \subset X$ of a quasimetric space X is *open* if for any $x \in U$, there is some $r > 0$ such that $B(x, r) \subset U$. A subset $F \subset X$ is *closed* if its complement is open. It is easy to check that a subset F is closed if and only if for any $x \in X$ and any sequence $\{x_j\} \subset F$, the condition $d(x_j, x) \to 0$ implies $x \in F$.

Let G be a 2-step Carnot group with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$. We shall identify G with $\mathfrak{g}$ via the exponential map. Under this identification the group law on $\mathfrak{g}$ is given by:

$$X * Y = X + Y + \frac{1}{2}[X, Y], \quad \text{for} \quad X, Y \in \mathfrak{g}. \tag{2.1}$$

In addition to the Carnot metric, we can also equip $\mathfrak{g}$ with a homogeneous distance. Fix any norms on V_1, V_2. Then, for $X = X_1 + X_2$, with $X_1 \in V_1, X_2 \in V_2$, we define the norm of X to be:

$$||X|| = |X_1| + |X_2|^{\frac{1}{2}}.$$

For $X, Y \in \mathfrak{g}$, we set $d(X, Y) = ||(-X) * Y||$. The homogeneous distance d on $\mathfrak{g}$ is left-invariant and is biLipschitz equivalent with the Carnot-Carathéodory metric.

3 Fiber-preserving quasisymmetric maps

The goal of this section is to first prove Theorem 1.1 and then derive some consequences.

3.1 Quasisimilarity points

Here we explain that the assumption in Theorem 1.1 of a dense set of quasisimilarity points is reasonable.

Quasisimilarity points are related to differentiability. For example, if $f : U \to V$ is a map between open subsets of a Carnot group and f is Pansu differentiable at $p \in U$ with the Pansu differential $df(p)$ a graded isomorphism, then p is a quasisimilarity point with respect to f. By Pansu's differentiability theorem, if f is a quasiconformal map, then f is Pansu differentiable a.e. Hence a.e. point is a quasisimilarity point. If f is η-quasisymmetric, then a.e. point is an $\eta(1)$-quasisimilarity point.

The bad news is that there are biLipschitz maps between metric spaces that are nowhere differentiable. Here is how to construct one. Equip $\mathbb{R}^2$ with the product metric d_α $(0 < \alpha < 1)$ where the metric on the first factor is the usual Euclidean metric $|\cdot|$ and the metric on the second factor is $|\cdot|^\alpha$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an α Hölder continuous function. Let $f : (\mathbb{R}^2, d_\alpha) \to (\mathbb{R}^2, d_\alpha)$ be given by $f(x, y) = (x + \phi(y), y)$. Then f is biLipschitz. However, for suitable choices of ϕ (suitable nowhere differentiable Hölder continuous functions) the map f is nowhere differentiable. The map f is nowhere differentiable not only in the classical sense, but also in the Pansu sense. In other words, if $\delta_t : \mathbb{R}^2 \to \mathbb{R}^2$ $(t > 0)$ is the dilation given by $\delta_t(x, y) = (tx, t^{\frac{1}{\alpha}} y)$ and $L_{(a,b)} : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the translation by (a, b), then for any $(x_0, y_0) \in \mathbb{R}^2$, the maps $\delta_{1/t} \circ L_{-f(x_0,y_0)} \circ f \circ L_{(x_0,y_0)} \circ \delta_t$ have no limit as $t \to 0$.

The good news is that the above mentioned lack of differentiability can be corrected sometimes. Take the above example. We may view $(\mathbb{R}^2, d_\alpha)$ as a fibered space whose fibers are horizontal lines. Then the restrictions of f to the fibers F are just translations along the real line and so are certainly differentiable. This implies that every $p \in F$ is a quasisimilarity point with respect to $f|_F$. The following lemma says that if $f : X \to Y$ is a fiber preserving quasisymmetric map between fibered spaces and $x \in X$ lies on a fiber F, then x is a quasisimilarity point with respect to f if and only if x is a quasisimilarity point with respect to $f|_F$. The proof is easy. The point is that oftentimes the restricted quasisymmetric map $f|_F : F \to f(F)$ is differentiable a.e. (in some sense) and so a.e. point of F is a quasisimilarity point with respect to $f|_F$, despite the possibility that the map f is nowhere differentiable.

Lemma 3.1. *Let X, Y be fibered spaces and $U \subset X$, $V \subset Y$ be open sets. Let $f : U \to V$ be η-quasisymmetric for some η. Assume for each fiber F in X, each component of $F \cap U$ is mapped onto an open subset of a fiber of Y. Let F be a fiber in X and $x \in F \cap U$.*

(1) If x is a K-quasisimilarity point with respect to f, then x is also a K-quasisimilarity point with respect to $f|_{F\cap U}$;

(2) If x is a K-quasisimilarity point with respect to $f|_{F\cap U}$, then x is a $K\eta(1)^2$-quasisimilarity point with respect to f.

Proof. (1) This is clear since by definition we have $l_f(x) \le l_{f|_{F\cap U}}(x) \le L_{f|_{F\cap U}}(x) \le L_f(x)$.
(2) Now suppose x is a K-quasisimilarity point with respect to $f|_{F\cap U}$. So $0 < L_{f|_{F\cap U}}(x) \le Kl_{f|_{F\cap U}}(x) < \infty$. Now let $x' \in U$ be close to x. Pick any $x'' \in F \cap U$ with $d(x, x'') = d(x, x')$. Since f is an η-quasisymmetry, we have:

$$\frac{1}{\eta(1)} d(f(x), f(x')) \le d(f(x), f(x'')) \le \eta(1) d(f(x), f(x')).$$

It follows that $l_{f|_{F\cap U}}(x) \le \eta(1) l_f(x)$ and $L_f(x) \le \eta(1) L_{f|_{F\cap U}}(x)$. Hence x is a $K\eta(1)^2$-quasisimilarity point with respect to f.

□

3.2 Criterion for a map to be Lipschitz

Here we recall that the existence of an upper bound for the upper pointwise Lipschitz constants implies the map is Lipschitz.

Lemma 3.2. *Let X, Y be quasimetric spaces. Suppose there exist constants $L \ge 1$, $\alpha \in (0, 1]$ and metric spaces $(\tilde{X}, d)$, $(\tilde{Y}, d)$ such that X is L-biLipschitz to $(\tilde{X}, d^\alpha)$ and Y is L-biLipschitz to $(\tilde{Y}, d^\alpha)$. Further suppose $(\tilde{X}, d)$ is a geodesic space. If $f : X \to Y$ is a map so that*

$$L_f(x) \le b$$

for all $x \in X$ and some fixed $b \ge 0$, then f is $L^4 b$-Lipschitz.

Proof. We first assume that X is a geodesic metric space and Y is a metric space (corresponding to $L = 1$, $\alpha = 1$). Fix $\epsilon > 0$. Let $p, q \in X$ with $p \ne q$. Consider the geodesic $\gamma : [0, d(p, q)] \to X$ joining p and q. For any $x \in \operatorname{im}\gamma$, there exists some $r_x > 0$ such that

$$\frac{d(f(z), f(x))}{d(z, x)} \le b + \epsilon$$

for all $z \in B(x, r_x)$. The collection of open balls $\{B(x, \frac{1}{2} r_x)\}_{x\in \operatorname{im}\gamma}$ forms an open cover of $\operatorname{im}\gamma$ and hence we get a finite subcover $\{B(x_j, \frac{1}{2} r_{x_j})\}_{j=1}^n$. We may assume none of the balls is contained in the union of the other balls and that $p, x_1, \cdots, x_n, q$ are in linear order on γ. Notice that $B(x_{j-1}, \frac{1}{2} r_{x_{j-1}}) \cap B(x_j, \frac{1}{2} r_{x_j}) \ne \emptyset$. So we have

$$d(x_{j-1}, x_j) < \frac{1}{2} r_{x_{j-1}} + \frac{1}{2} r_{x_j} \le \max\{r_{x_{j-1}}, r_{x_j}\}.$$

By the choice of r_x, $d(f(x_{j-1}), f(x_j)) \le (b + \epsilon) d(x_{j-1}, x_j)$. Now the triangle inequality implies $d(f(p), f(q)) \le (b + \epsilon) \cdot d(p, q)$. Since this is true for any $\epsilon > 0$, we have $d(f(p), f(q)) \le b \cdot d(p, q)$.

Now the general case. Let $f_1 : (\tilde{X}, d^\alpha) \to X$ and $f_2 : (\tilde{Y}, d^\alpha) \to Y$ be L-biLipschitz maps, where $(\tilde{X}, d)$ is a geodesic metric space and $(\tilde{Y}, d)$ is a metric space. Denote $\tilde{f} = f_2^{-1} \circ f \circ f_1$. Let $\tilde{x}_i \in \tilde{X}$ $(i = 1, 2)$ and set $x_i = f_1(\tilde{x}_i)$. The biLipschitz property of f_1, f_2 implies:

$$\frac{1}{L^2} \left(\frac{d(\tilde{f}(\tilde{x}_1), \tilde{f}(\tilde{x}_2))}{d(\tilde{x}_1, \tilde{x}_2)} \right)^\alpha \le \frac{d(f(x_1), f(x_2))}{d(x_1, x_2)} \le L^2 \left(\frac{d(\tilde{f}(\tilde{x}_1), \tilde{f}(\tilde{x}_2))}{d(\tilde{x}_1, \tilde{x}_2)} \right)^\alpha . \tag{3.1}$$

The assumption $L_f(x) \le b$ now implies $L_{\tilde{f}}(\tilde{x}) \le (L^2 b)^{\frac{1}{\alpha}}$ for all $\tilde{x} \in \tilde{X}$. By the special case proved above we conclude that $\tilde{f}$ is $(L^2 b)^{\frac{1}{\alpha}}$-Lipschitz. Now (3.1) implies that f is $L^4 b$-Lipschitz.

□

3.3 Proof of Theorem 1.1

In this subsection we prove Theorem 1.1.

We first fix the notation. Let X, Y be (L, α, C, φ)-fibered spaces for some $L \geq 1$, $\alpha \in (0, 1]$, some function $C : X \to [1, \infty)$ and some continuous function $\varphi : [0, \infty) \to [1, \infty)$. Let $M \geq 1$ be such that both X and Y are M-quasimetric spaces. Let $U \subset X$, $V \subset Y$ be open. Suppose $f : U \to V$ is an η-quasisymmetry such that the following hold:

(1) For each fiber F of X, f maps each component of $F \cap U$ onto an open subset of a fiber of Y. Similarly for f^{-1}.

(2) There is some $K \geq 1$ such that for every fiber F of X, $F \cap U$ has a dense set of K-quasisimilarity points (by assumption (2) in Theorem 1.1, the constant for the quasisimilarity points is locally bounded; since the conclusion of Theorem 1.1 is local, we may assume the same constant K works for all $x \in U$).

We need to show that f is a local quasisimilarity. To do so, we will first prove that f is a local quasisimilarity when restricted to a particular fiber. This is achieved by comparing the stretch of f along fibers as well as between fibers to obtain an upper bound for the upper pointwise Lipschitz constant L_f, at which point we will be able to apply Lemma 3.2. Then, we shall compare L_f across fibers, which will allow us to conclude that f is a local quasisimilarity.

The following lemma says that if $x \in U \backslash F$ and $p \in F \cap U$ almost realizes the distance from F to x, then $f(p) \in f(F \cap U)$ also almost realizes the distance from $f(F \cap U)$ to $f(x)$.

Lemma 3.3. *Let F be a fiber of X and $x \in U \backslash F$. Let $p \in F \cap U$ be such that $d(x, p) \leq 2\, d(x, F)$. Then*

$$d(f(x), f(p)) \leq \eta(2)\, d(f(x), f(F \cap U)).$$

Proof. Let $\bar{y} \in f(F \cap U)$. Then $\bar{y} = f(y)$ for some $y \in F \cap U$. By assumption, $d(x, p) \leq 2d(x, y)$. Since f is an η-quasisymmetry, we have $d(f(x), f(p)) \leq \eta(2)d(f(x), f(y)) = \eta(2)d(f(x), \bar{y})$. Since this is true for any $\bar{y} \in f(F \cap U)$, the lemma follows. □

In the next lemma we use the quasisymmetry condition to compare the stretch of f along the fibers and the stretch of f between fibers.

Lemma 3.4. *Let F_1 and F_2 be two distinct fibers of X. Let $x, y \in F_1$ be such that $d(x, F_2) \leq d(x, y) \leq Cd(x, F_2)$ for some $C \geq 1$. Assume $d(x, X \backslash U) > \max\{C, 2\} \cdot d(x, F_2)$. Then*

$$\frac{1}{\eta(2)\eta(C)} \cdot \frac{d(f(x), f(y))}{d(x, y)} \leq \frac{d(f(x), f(F_2 \cap U))}{d(x, F_2)} \leq C\eta(2) \cdot \frac{d(f(x), f(y))}{d(x, y)}.$$

Proof. Let $p \in F_2$ be such that $d(x, p) \leq 2d(x, F_2)$. The assumption implies $p \in U$. Now $d(x, p)/2 \leq d(x, y) \leq Cd(x, p)$. By the quasisymmetry condition we have

$$\frac{d(f(x), f(F_2 \cap U))}{\eta(2)} \leq \frac{d(f(x), f(p))}{\eta(2)} \leq d(f(x), f(y)) \leq \eta(C)d(f(x), f(p)).$$

On the other hand, by Lemma 3.3, $d(f(x), f(p)) \leq \eta(2)d(f(x), f(F_2 \cap U))$. Now

$$\frac{1}{C\eta(2)} \cdot \frac{d(f(x), f(F_2 \cap U))}{d(x, F_2)} \leq \frac{d(f(x), f(y))}{d(x, y)} \leq \eta(2)\eta(C) \cdot \frac{d(f(x), f(F_2 \cap U))}{d(x, F_2)}.$$

□

Given a fiber F of X, the intersection $F \cap U$ may be disconnected and different components of $F \cap U$ might be mapped into distinct fibers of Y. For this reason we shall restrict our attention to balls $B \subset U$ that are deep inside U where the behavior of f is better controlled.

From now until the end of this subsection, we fix $p \in U$. Since the fibers are uniformly snow-flake equivalent to geodesic metric spaces (condition (1) of a fibered space), for $r_0 > 0$ sufficiently small compared with

$d(p, X\backslash U)$, the ball $B = B(p, r_0) \subset U$ has the following property: for any fiber F of X, $F \cap B$ lies in a component of $F \cap U$. By assumption on the map f, $f(F \cap B)$ lies in a single fiber of Y. We denote by $F^\star$ the fiber of Y that contains $f(F \cap B)$. Similarly we pick $\hat{r}_0 > 0$ so that the ball $\hat{B} = B(f(p), \hat{r}_0) \subset V$ has the following property: for any fiber $\hat{F}$ of Y, $f^{-1}(\hat{F} \cap \hat{B})$ lies in a single fiber of X.

Set

$$C_1 = \sup\{\varphi(d(x_1, x_2)) : x_1, x_2 \in B\}, \quad C_2 = \sup\{\varphi(d(y_1, y_2)) : y_1, y_2 \in f(B)\},$$

$$\hat{C}_1 = \sup\{\varphi(d(y_1, y_2)) : y_1, y_2 \in \hat{B}\}, \quad \hat{C}_2 = \sup\{\varphi(d(x_1, x_2)) : x_1, x_2 \in f^{-1}(\hat{B})\}.$$

Since B is bounded and f is quasisymmetric, $f(B)$ is also bounded. The fact that φ is continuous implies $C_1, C_2 < \infty$. Similarly, $\hat{C}_1, \hat{C}_2 < \infty$.

Recall that as f is η-quasisymmetric, f^{-1} is η_1-quasisymmetric with $\eta_1(t) = (\eta^{-1}(t^{-1}))^{-1}$. Also recall the notation $\lambda B(x, r) = B(x, \lambda r)$ for any $\lambda > 0$ and any ball $B(x, r)$. It follows from the generalized triangle inequality that for any $y \in B(x, r)$ we have $B(y, r) \subset 2MB(x, r)$.

Set

$$\lambda_1 = \frac{1}{2} \sup\left\{\lambda \in (0, (2M)^{-1}] : f(\lambda B) \subset \frac{1}{2M}\hat{B}\right\},$$

$$\hat{\lambda}_1 = \frac{1}{2} \sup\left\{\lambda \in (0, (2M)^{-1}] : f^{-1}(\lambda \hat{B}) \subset \frac{1}{2M}B\right\}.$$

In the next lemma, we bound the Lipschitz constant of f along a fiber. This will be used in Lemma 3.6 to show that f is locally Lipschitz along a fiber.

Lemma 3.5. *Let F be a fiber in X and $x_0 \in F \cap \lambda_1 B$ be a K-quasisimilarity point. Then $L_{f|_F}(y) \le C_3 \cdot L_f(x_0)$ for all $y \in F \cap \lambda_1 B$, where $C_3 = 4\eta(2)^2\eta(C_B)C_1C_2$ and C_B is defined in Definition 1.1.*

A similar statement holds for f^{-1} with $\hat{\lambda}_1$ instead of λ_1.

Proof. Let $y \in F \cap \lambda_1 B$. By condition (2) of a fibered space, for $z \in F$ sufficiently close to y, there exists a fiber F_z such that $\frac{d(y,z)}{C_B} \le d(y, F_z) \le d(y, z)$. Let $y_z \in F_z$ be a point such that $d(y, y_z) \le 2d(y, F_z)$. Similarly, let $x_z \in F_z$ be a point such that $d(x_0, x_z) \le 2d(x_0, F_z)$. Let $x(z) \in F$ be such that $d(x_0, x(z)) = d(x_0, x_z)$. Now as $z \to y$, $x(z) \to x_0$. Since $B(x_0, \lambda_1 r_0) \subset B$, we have $x(z) \in B$ for z sufficiently close to y. We may choose z close enough to y so that $x(z)$ satisfies

$$\frac{d(f(x(z)), f(x_0))}{d(x(z), x_0)} \le 2L_f(x_0). \tag{3.2}$$

By Lemma 3.4,

$$\frac{1}{\eta(2)\eta(C_B)} \cdot \frac{d(f(y), f(z))}{d(y, z)} \le \frac{d(f(y), f(F_z \cap U))}{d(y, F_z)} \le \eta(2)C_B \cdot \frac{d(f(y), f(z))}{d(y, z)}, \tag{3.3}$$

$$\frac{1}{\eta(2)^2} \cdot \frac{d(f(x_0), f(x(z)))}{d(x_0, x(z))} \le \frac{d(f(x_0), f(F_z \cap U))}{d(x_0, F_z)} \le 2\eta(2) \cdot \frac{d(f(x_0), f(x(z)))}{d(x_0, x(z))}. \tag{3.4}$$

Now by (1.2), we have

$$\frac{1}{C_1} \cdot d(x_0, F_z) \le d(y, F_z) \le C_1 \cdot d(x_0, F_z), \tag{3.5}$$

and (1.2) applied to $f(x_0)$ and $f(y)$ gives

$$\frac{1}{C_2} \cdot d(f(x_0), F_z^\star) \le d(f(y), F_z^\star) \le C_2 \cdot d(f(x_0), F_z^\star). \tag{3.6}$$

Claim: $d(f(y), f(F_z \cap U)) = d(f(y), F_z^\star)$ for z sufficiently close to y.

Assuming the claim we first finish the proof. For z sufficiently close to y,

$$\begin{aligned}
\frac{d(f(y), f(z))}{d(y,z)} &\le \eta(2)\eta(C_B)\cdot\frac{d(f(y), f(F_z\cap U))}{d(y,F_z)} && \text{by (3.3)}\\
&= \eta(2)\eta(C_B)\cdot\frac{d(f(y), F_z^\star)}{d(y,F_z)} && \text{by Claim}\\
&\le \eta(2)\eta(C_B)C_1C_2\cdot\frac{d(f(x_0), F_z^\star)}{d(x_0,F_z)} && \text{by (3.5), (3.6)}\\
&\le \eta(2)\eta(C_B)C_1C_2\cdot\frac{d(f(x_0), f(F_z\cap U))}{d(x_0,F_z)} &&\\
&\le 2\eta(2)^2\eta(C_B)C_1C_2\cdot\frac{d(f(x_0), f(x(z)))}{d(x_0,x(z))} && \text{by (3.4)}\\
&\le 4\eta(2)^2\eta(C_B)C_1C_2L_f(x_0). && \text{by (3.2)}
\end{aligned}$$

This implies that $L_{f|_F}(y) \le 4\eta(2)^2\eta(C_B)C_1C_2L_f(x_0) = C_3L_f(x_0)$.

We now prove the claim. As $z \to y$, we have $y_z \to y$. This implies $y_z \in B$ for z sufficiently close to y. Since $y_z \in F_z$, we have $f(y_z) \in f(F_z \cap B) \subset F_z^\star$. Let $\hat{w}_z \in F_z^\star$ be such that $d(f(y), \hat{w}_z) \le 2d(f(y), F_z^\star)$. It suffices to show $\hat{w}_z \in f(F_z \cap U)$. Notice $\hat{w}_z \to f(y)$ as $z \to y$. The choice of λ_1 implies $f(y) \in f(\lambda_1 B) \subset \frac{1}{2M}\hat{B}$. Hence for z sufficiently close to y we have $\hat{w}_z \in B(f(y), (2M)^{-1}\hat{r}_0) \subset \hat{B}$. Similarly $f(y_z) \in \hat{B}$ for z sufficiently close to y. Now we have $f(y_z), \hat{w}_z \in F_z^\star \cap \hat{B}$. The choice of $\hat{r}_0$ now implies $y_z = f^{-1}(f(y_z)), f^{-1}(\hat{w}_z)$ lie in the same fiber of X. Since $y_z \in F_z$, we have $f^{-1}(\hat{w}_z) \in F_z \cap U$. It follows that $\hat{w}_z \in f(F_z \cap U)$.

□

The following lemma follows easily from Lemma 3.5 and Lemma 3.2. We just need to shrink the ball a little so that the estimate in Lemma 3.5 can be applied. Set $\lambda_2 = 2^{\alpha-2}L^{-2}M^{-2}\lambda_1$ and $\hat{\lambda}_2 = 2^{\alpha-2}L^{-2}M^{-2}\hat{\lambda}_1$.

Lemma 3.6. *Let F be a fiber in X and $x_0 \in F \cap \lambda_2 B$ be a K-quasisimilarity point. Then $f|_{F\cap\lambda_2 B}$ is $L^4C_3L_f(x_0)$-Lipschitz, where C_3 is the constant in Lemma 3.5.*

Proof. Let $g_F : (\tilde{F}, d^\alpha) \to F$ be a L-biLipschitz map, where $(\tilde{F}, d)$ is a geodesic metric space. Let $x, y \in F \cap \lambda_2 B$ and set $\tilde{x} = g_F^{-1}(x)$, $\tilde{y} = g_F^{-1}(y)$. Let $\tilde{\gamma}$ be a geodesic in $\tilde{F}$ between $\tilde{x}$ and $\tilde{y}$. A direct calculation shows that $g_F(\tilde{\gamma}) \subset F \cap \lambda_1 B$. Since we already obtained the upper bound estimate for $L_{f|_F}(y)$ in Lemma 3.5 for any $y \in F \cap \lambda_1 B$, the claim now follows from Lemma 3.2.

□

In the next lemma we show that f is locally biLipschitz along a fiber by applying Lemma 3.6 to both f and f^{-1}. Again we need to shrink the ball further. Let

$$\lambda_3 = \frac{1}{2}\sup\{\lambda \in (0, \lambda_2] | f(\lambda B) \subset \hat{\lambda}_2\hat{B}\}.$$

Lemma 3.7. *Let F be a fiber in X and $x_0 \in F \cap \lambda_3 B$ be a K-quasisimilarity point. Then $f|_{F\cap\lambda_3 B}$ is a $(K_2, L_f(x_0))$-quasisimilarity, where $K_2 = \max\{L^4C_3, KL^4\hat{C}_3\}$ and $\hat{C}_3 = 4\eta_1(2)^2\eta_1(C_{\hat{B}})\hat{C}_1\hat{C}_2$.*

Proof. Notice that $f(x_0) \in \hat{\lambda}_2\hat{B}$. Since $x_0 \in F \cap \lambda_3 B \subset F \cap B$, by the definition of $F^\star$ we have $f(x_0) \in F^\star$. So we have $f(x_0) \in F^\star \cap \hat{\lambda}_2\hat{B}$. The relations $L_f(x_0) = \frac{1}{l_{f^{-1}}(f(x_0))}$ and $l_f(x_0) = \frac{1}{L_{f^{-1}}(f(x_0))}$ imply that $f(x_0)$ is a K-quasisimilarity point with respect to f^{-1}. Lemma 3.6 applied to f^{-1} implies that $f^{-1}|_{F^\star\cap(\hat{\lambda}_2\hat{B})}$ is Lipschitz with constant

$$L^4\hat{C}_3L_{f^{-1}}(f(x_0)) = \frac{L^4\hat{C}_3}{l_f(x_0)} \le \frac{KL^4\hat{C}_3}{L_f(x_0)}.$$

The choice of λ_3 ensures $f(\lambda_3 B) \subset \hat{\lambda}_2\hat{B}$. Now the lemma follows from Lemma 3.6.

□

At this point, f is a quasisimilarity along each fiber. To show f itself is a quasisimilarity (locally), we need to relate the quasisimilarity constants along different fibers. This is done again by using the quasisymmetry condition. Set $\lambda_4 = \frac{1}{M(2M+1)}\lambda_3$. Recall that X, Y are M-quasimetric spaces.

Lemma 3.8. *For $i = 1, 2$, let F_i be a fiber of X and $x_i \in F_i \cap \lambda_4 B$ be a K-quasisimilarity point. Then $\frac{1}{\eta(1)^2K_2^2}L_f(x_1) \leq L_f(x_2) \leq \eta(1)^2K_2^2L_f(x_1)$.*

Proof. Pick $y_i \in F_i$ such that $d(x_1, y_1) = d(x_1, x_2) = d(x_2, y_2)$. The choice of λ_4 guarantees that $y_1, y_2 \in \lambda_3 B$. By Lemma 3.7 we have

$$\frac{L_f(x_1)}{K_2}d(x_1, y_1) \leq d(f(x_1), f(y_1)) \leq L_f(x_1)K_2d(x_1, y_1) \tag{3.7}$$

and

$$\frac{L_f(x_2)}{K_2}d(x_2, y_2) \leq d(f(x_2), f(y_2)) \leq L_f(x_2)K_2d(x_2, y_2). \tag{3.8}$$

The quasisymmetry condition applied to $d(x_1, y_1) = d(x_1, x_2)$ and $d(x_1, x_2) = d(x_2, y_2)$ implies

$$\frac{1}{\eta(1)}d(f(x_1), f(y_1)) \leq d(f(x_1), f(x_2)) \leq \eta(1)d(f(x_1), f(y_1)) \tag{3.9}$$

and

$$\frac{1}{\eta(1)}d(f(x_1), f(x_2)) \leq d(f(x_2), f(y_2)) \leq \eta(1)d(f(x_1), f(x_2)). \tag{3.10}$$

Using (3.7)–(3.10) we obtain

$$\begin{aligned}&\frac{L_f(x_1)}{K_2}d(x_1, y_1) \leq d(f(x_1), f(y_1))\\ &\leq \eta(1)d(f(x_1), f(x_2)) \leq \eta(1)^2d(f(x_2), f(y_2)) \leq \eta(1)^2L_f(x_2)K_2d(x_2, y_2).\end{aligned}$$

It follows that $L_f(x_1) \leq \eta(1)^2K_2^2L_f(x_2)$. Similarly we obtain $L_f(x_2) \leq \eta(1)^2K_2^2L_f(x_1)$.

□

Finally, the following lemma completes the proof of Theorem 1.1.

Lemma 3.9. *Let $x_0 \in \lambda_4 B$ be a K-quasisimilarity point. Then $f|_{\lambda_4 B}$ is a $(\eta(1)^3K_2^3, L_f(x_0))$–quasisimilarity.*

Proof. Let $x_1, x_2 \in \lambda_4 B$. Let F_i be the fiber containing x_i. Let $y_1 \in F_1$ be such that $d(x_1, y_1) = d(x_1, x_2)$. Then $y_1 \in \lambda_3 B$. The quasisymmetry condition implies

$$\frac{1}{\eta(1)}d(f(x_1), f(y_1)) \leq d(f(x_1), f(x_2)) \leq \eta(1)d(f(x_1), f(y_1)). \tag{3.11}$$

On the other hand, Lemmas 3.7 and 3.8 imply that for any fiber F with $F \cap \lambda_4 B \neq \emptyset$, $f|_{F\cap\lambda_4 B}$ is a $(\eta(1)^2K_2^3, L_f(x_0))$ quasisimilarity. So

$$\frac{L_f(x_0)d(x_1, y_1)}{\eta(1)^2K_2^3} \leq d(f(x_1), f(y_1)) \leq \eta(1)^2K_2^3L_f(x_0)d(x_1, y_1). \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\frac{L_f(x_0)d(x_1, x_2)}{\eta(1)^3K_2^3} \leq d(f(x_1), f(x_2)) \leq \eta(1)^3K_2^3L_f(x_0)d(x_1, x_2).$$

□

3.4 Parallel fibers

Here we apply Theorem 1.1 to the case when all the fibers are parallel.

Let X be a quasimetric space. Two closed sets $F_1, F_2 \subset X$ are *parallel* if there exists some $c > 0$ so that $d(x_1, F_2) = d(x_2, F_1) = c$ for all $x_1 \in F_1, x_2 \in F_2$. If all the fibers in a fibered space X are parallel, then the function $\varphi : [0, \infty) \to [1, \infty)$ (in the definition of a fibered space) is identically 1.

Corollary 3.10. *Let X, Y be quasimetric spaces that partition into fibers. Suppose:*
(1) the fibers of X and Y are (L, α)-snow-flake equivalent to geodesic metric spaces for some $L \geq 1$, $\alpha \in (0, 1]$;
(2) all fibers in X are parallel; the same for Y;
(3) there is a constant $C \geq 1$ such that for any $x \in X$ and all sufficiently small $r > 0$, there exists a fiber F satisfying $r/C \leq d(x, F) \leq r$. The same also holds for Y.
Let $U \subset X$, $V \subset Y$ be open subsets and $f : U \to V$ be an η-quasisymmetric map. Suppose
(4) for each fiber F of X, every component of $F \cap U$ is mapped onto an open subset of a fiber of Y; similarly for f^{-1};
(5) there exists some $0 < K < \infty$ such that for every fiber F of X, $F \cap U$ has a dense set of K-quasisimilarity points.

Then each point $p \in U$ has a neighbourhood U_p such that $f : U_p \to f(U_p)$ is a (K', c_p)–quasisimilarity for some $K' \geq 1$ and $c_p > 0$, where K' depends only on η, C, L, and K.

Proof. We use the notation in the proof of Theorem 1.1. Let $p \in U$. Since all the fibers are parallel, we have $C_1 = C_2 = \hat{C}_1 = \hat{C}_2 = 1$. Condition (3) implies $C_x = C$ for all $x \in X$ and so $C_B = C$. Similarly $C_{\hat{B}} = C$. Now the corollary follows from Lemma 3.9.

□

It is clear that condition (3) is satisfied when X is path connected. Condition (3) is also satisfied when the space of fibers is "uniformly perfect". For example, if $X = F \times Z$ with fibers $F \times \{z\}$ $(z \in Z)$ and Z is uniformly perfect, then condition (3) holds for X. A particular example is $X = \mathbb{R}^n \times C$, where C is the Cantor set.

Now consider the Heisenberg groups $\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R}$. The fibers are the vertical lines $\{x\} \times \mathbb{R}$ $(x \in \mathbb{R}^{2n})$.

Corollary 3.11. *Let $U, V \subset \mathbb{H}^n$ be open subsets and $f : U \to V$ a quasisymmetric map. If f maps vertical line segments to vertical line segments, then f is locally biLipschitz.*

Proof. A vertical line with the restriction of d is isometric to $(\mathbb{R}, |\cdot|^{\frac{1}{2}})$, where $|\cdot|$ is the Euclidean metric on $\mathbb{R}$. Hence condition (1) is satisfied. All fibers are parallel and so condition (2) is satisfied. It is easy to see that condition (3) holds for $C = 1$. Condition (4) holds by assumption. Pansu's differentiability theorem and Fubini's theorem imply that condition (5) holds for a.e. vertical line. The proof of Theorem 1.1 yields that f is locally biLipschitz when restricted to the union of these vertical lines. Now the conclusion follows from the continuity of f.

□

4 Quasiconformal maps on 2-step Carnot groups

In this section we prove Theorem 1.2. It will be a consequence of the following result:

Theorem 4.1. *Let G be a 2-step Carnot group identified with its Lie algebra $\mathfrak{g} = V_1 \oplus V_2$. Let W_1 be a nontrivial proper subspace of V_1 and define $W_2 = [W_1, W_1]$. The subspace $W := W_1 \oplus W_2$ corresponds to a Lie subgroup of G. Let $U, V \subset G$ be open subsets and $f : U \to V$ a quasiconformal map. Suppose that for each left coset F of W, each component of $F \cap U$ is mapped into a left coset of W. Then f is locally biLipschitz.*

In both Theorem 4.1 and Theorem 1.2 quasiconformality is defined with respect to the Carnot metric. Since the homogeneous distances are biLipschitz equivalent with the Carnot metrics and we are only interested in quasiconformal maps and biLipschitz maps, we can use homogeneous distances for the calculations. Notice that Theorem 1.2 follows from Theorem 4.1 due to the following result:

Proposition 4.2 ([9] Proposition 3.4). *Let N and N' be two Carnot groups, $W_1 \subset V_1$, $W_1' \subset V_1'$ be subspaces. Denote by $\mathfrak{n}_{W_1} \subset \mathfrak{n}$ and $\mathfrak{n}'_{W_1'} \subset \mathfrak{n}'$ respectively the Lie subalgebras generated by W_1 and W_1'. Let $H \subset N$ and $H' \subset N'$ respectively be the connected Lie subgroups of N and N' corresponding to $\mathfrak{n}_{W_1}$ and $\mathfrak{n}'_{W_1'}$. Let $U \subset N$, $U' \subset N'$ be open subsets and $F : U \longrightarrow U'$ be a quasiconformal homeomorphism. If $dF(x)(W_1) \subset W_1'$ for a.e. $x \in N$, then for every left coset L of H, F maps each component of $L \cap U$ into a left coset of H'.*

Remark 4.1. Proposition 3.4 in [9] is stated for global quasiconformal maps $F : N \to N'$. Its proof is valid for quasiconformal maps defined on open subsets.

4.1 Divergence of left cosets

In this subsection we study how two left cosets of W diverge from each other. This will be used in the next subsection to control the Lipschitz constant. The assumption and notation are as in Theorem 4.1.

We first estimate the distance from a point to a left coset of W. Fix an inner product on $\mathfrak{g}$. Define $W_1^\perp$ and $W_2^\perp$ to be the orthogonal complements of W_1, W_2 in V_1, V_2, respectively. Denote by π_2 the orthogonal projection of V_2 onto W_2 and $\pi_2^\perp$ the orthogonal projection of V_2 onto $W_2^\perp$. Notice that a left coset of W has the form $(x + y) * W$, where $x \in W_1^\perp$ and $y \in W_2^\perp$.

Note that there exists $K > 0$ so that for all $X, Y \in \mathfrak{g}$, $|[X, Y]| \le K|X||Y|$.

Lemma 4.3. *Let $w = w_1 + w_2 \in W$ with $w_1 \in W_1$ and $w_2 \in W_2$. Let $x \in W_1^\perp$ and $y \in W_2^\perp$. If $d(0, (x+y)*W) \le 1$, then $\frac{1}{C_1}(|x| + A) \le d(w_1 + w_2, (x + y) * W) \le (|x| + A)$, where $A = \sqrt{|y + \pi_2^\perp[x, w_1]|}$ and C_1 depends only on K.*

Proof. Recall $d(w, (x + y) * W) = \inf_{g\in(x+y)*W} d(w, g)$. Let $g \in (x + y) * W$. Then $g = (x + y) * (\tilde w_1 + \tilde w_2)$ for some $\tilde w_1 \in W_1$ and $\tilde w_2 \in W_2$. We have, by Equation (2.1), that $(-w_1 - w_2) * (x + y) * (\tilde w_1 + \tilde w_2)$ is equal to:

$$(\tilde w_1 - w_1 + x) + \left(\tilde w_2 - w_2 - \frac{1}{2}[w_1, \tilde w_1] + \frac{1}{2}\pi_2[x, \tilde w_1 + w_1]\right) + \left(y + \frac{1}{2}\pi_2^\perp[x, \tilde w_1 + w_1]\right). \tag{4.2}$$

Now, making the choice $\tilde w_1 = w_1$ and $\tilde w_2 = w_2 - \pi_2[x, w_1]$ simplifies (4.2) to

$$x + \left(y + \pi_2^\perp[x, w_1]\right), \tag{4.3}$$

which yields the right hand inequality.

Jensen's inequality implies that the norm of (4.2) is at least $2^{-\frac{3}{4}}B$, where

$$B = |\tilde w_1 - w_1 + x| + \sqrt{\left|\tilde w_2 - w_2 - \frac{1}{2}[w_1, \tilde w_1] + \frac{1}{2}\pi_2[x, \tilde w_1 + w_1]\right|} + \sqrt{\left|y + \frac{1}{2}\pi_2^\perp[x, \tilde w_1 + w_1]\right|}. \tag{4.4}$$

We must bound B from below. There are two cases.

Case 1. $|w_1 - \tilde w_1| \ge A$.
Since $\tilde w_1 - w_1 \in W_1$ and $x \in W_1^\perp$ are perpendicular to each other,

$$B \ge |\tilde w_1 - w_1 + x| \ge \frac{\sqrt{2}}{2}(|\tilde w_1 - w_1| + |x|) \ge \frac{\sqrt{2}}{2}(|x| + A).$$

Case 2. $|w_1 - \tilde w_1| \le A$.

This case can be broken into two subcases:

Case 2.1. $|x| \le \frac{1}{K}A$.
First we note that $|w_1 - \tilde{w}_1 + x| \ge |x|$ since $x \in W_1^\perp$ and $w_1 - \tilde{w}_1 \in W_1$. Furthermore,

$$\left|\frac{1}{2}\pi_2^\perp[x, w_1 - \tilde{w}_1]\right| \le \frac{1}{2}\left|[x, w_1 - \tilde{w}_1]\right| \le \frac{K}{2}|x| \cdot |w_1 - \tilde{w}_1| \le \frac{1}{2}A^2.$$

Since $|y + \pi_2^\perp[x, w_1]| = A^2$, we have

$$\left|y + \frac{1}{2}\pi_2^\perp[x, w_1 + \tilde{w}_1]\right| = \left|y + \pi_2^\perp[x, w_1] + \frac{1}{2}\pi_2^\perp[x, \tilde{w}_1 - w_1]\right| \ge \frac{1}{2}A^2.$$

Hence $B \ge |x| + \sqrt{|y + \frac{1}{2}\pi_2^\perp[x, w_1 + \tilde{w}_1]|} \ge \frac{1}{\sqrt{2}}(|x| + A)$.
Case 2.2. $|x| \ge \frac{1}{K}A$.
We have that $B \ge |x + w_1 - \tilde{w}_1| \ge |x| \ge \frac{1}{K+1}(|x| + A)$.

□

There is a dichotomy for the divergence of left cosets (of W) depending on whether W is normal in G. When W is normal in G, all left cosets of W are parallel to each other (see Lemma 4.4). Otherwise, the left cosets have quadratic divergence (see Lemma 4.6).

Lemma 4.4. *Suppose W is normal in G. Then all left cosets of W are parallel to each other.*

Proof. Here we will use the multiplicative notation. Assume W is normal in G. Let g_1W, g_2W be two left cosets. Let $g_1w \in g_1W$ be arbitrary. Then

$$d(g_1w, g_2W) = d(e, w^{-1}g_1^{-1}g_2W) = d(e, g_1^{-1}g_2 \cdot (g_1^{-1}g_2)^{-1}w^{-1}g_1^{-1}g_2W) = d(e, g_1^{-1}g_2W)$$

is independent of $w \in W$. Similarly $d(g_2w, g_1W) = d(e, g_2^{-1}g_1W)$ for any $w \in W$. Since $d(e, g) = d(g^{-1}, e)$ for any $g \in G$, we have

$$d(e, g_2^{-1}g_1W) = d(e, (g_2^{-1}g_1W)^{-1}) = d(e, Wg_1^{-1}g_2) = d(e, g_1^{-1}g_2W).$$

□

To express the divergence of left cosets when W is not normal in G, we define a quantity K_2, which vanishes precisely when W is normal in G (see Lemma 4.5). Define

$$K_2 = \max\{|\pi_2^\perp[x, w_1]| : x \in W_1^\perp, w_1 \in W_1, |x| = 1, |w_1| = 1\}.$$

The linearity of the bracket implies $|\pi_2^\perp[x, w_1]| \le K_2|x| \cdot |w_1|$ for any $x \in W_1^\perp, w_1 \in W_1$.

Lemma 4.5. *$K_2 = 0$ if and only if W is normal in G.*

Proof. Notice that W is normal in G if and only if W is an ideal of $\mathfrak{g}$. First assume W is an ideal of $\mathfrak{g}$. Then for any $x \in W_1^\perp, w_1 \in W_1$, we have $[x, w_1] \in V_2 \cap W = W_2$, hence $\pi_2^\perp[x, w_1] = 0$. So $K_2 = 0$.

Conversely assume $K_2 = 0$. Then $\pi_2^\perp[x, w_1] = 0$ and so $[x, w_1] \in W_2$ for any $x \in W_1^\perp, w_1 \in W_1$. Hence

$$[\mathfrak{g}, W] = [V_1, W_1] = [W_1 + W_1^\perp, W_1] = [W_1, W_1] + [W_1^\perp, W_1] \subset W_2 \subset W.$$

□

Since the Carnot metric d_c is a true metric (that is, it satisfies the triangle inequality) and the homogeneous distance d is biLipschitz equivalent with d_c, there is some $M \ge 1$ such that $d(x, z) \le M(d(x, y) + d(y, z))$ for any $x, y, z \in G$.

Statement (1) below says that the divergence of left cosets is at worst quadratic; statement (2) says that the worst case is achieved for suitably chosen points along the left coset and suitably chosen left cosets converging to the fixed left coset.

Lemma 4.6. *Assume W is not normal in G (so $K_2 > 0$). Let C_1 be the constant in Lemma 4.3.*
(1) Let $g * W$ and $h * W$ be distinct left cosets, and $q, q' \in g * W$. If $d(q, h * W) << 1$, then

$$d(q', h * W) \leq 2\sqrt{C_1 K_2} \cdot \sqrt{d(q, q')} \cdot \sqrt{d(q, h * W)};$$

(2) Let $x_0 \in W_1^{\perp}$ and $w_0 \in W_1$ with $|x_0| = |w_0| = 1$ be such that $|\pi_2^{\perp}[x_0, w_0]| = K_2$. Set $h_t = tx_0$, $t \in [0, 1]$. Then for any $r > 0$, any $\tilde{w}_1 \in W_1$ with $|\tilde{w}_1| \leq \frac{r}{2}$ and any $w_2 \in W_2$ so that $q' := ((rw_0 + \tilde{w}_1) + w_2)$ satisfies $d(0, q') \leq 2Mr$, the following holds:

$$d(q', h_t * W) \geq \frac{\sqrt{K_2}}{2\sqrt{M C_1}} \cdot \sqrt{d(0, q')} \cdot \sqrt{d(0, h_t * W)}.$$

Proof. (1) We may assume $q = 0$ since the distance is left-invariant. Then $g * W = W$. Let $q' = w_1 + w_2 \in W$, with $w_1 \in W_1$, $w_2 \in W_2$. Also $h * W = (x + y) * W$ with $x \in W_1^{\perp}$, $y \in W_2^{\perp}$. We have two cases.

Case 1. $\pi_2^{\perp}[x, w_1] = 0$.
In this case, by Lemma 4.3, $d(q', (x + y) * W) \leq |x| + \sqrt{|y|}$ and $d(0, (x + y) * W) \geq (|x| + \sqrt{|y|})/C_1$. In particular,

$$d(q', (x + y) * W) \leq C_1 \cdot d(0, (x + y) * W) \leq 2\sqrt{C_1 K_2 d(0, q')} \cdot \sqrt{d(0, (x + y) * W)}$$

when $d(0, (x + y) * W) < \frac{4K_2 d(0,q')}{C_1}$.

Case 2. $\pi_2^{\perp}[x, w_1] \neq 0$.
Notice $d(0, q') \geq |w_1|$. We have:

$$\begin{aligned} d(q', (x + y) * W) &\leq |x| + \sqrt{|y + \pi_2^{\perp}[x, w_1]|} \\ &\leq |x| + \sqrt{|y|} + \sqrt{|\pi_2^{\perp}[x, w_1]|} \\ &\leq |x| + \sqrt{|y|} + \sqrt{K_2} \cdot \sqrt{|x|} \cdot \sqrt{|w_1|} \\ &\leq |x| + \sqrt{|y|} + \sqrt{K_2 d(0, q')} \cdot \sqrt{|x|} \\ &\leq \left(\sqrt{|x| + \sqrt{|y|}} + \sqrt{K_2 d(0, q')}\right) \cdot \sqrt{|x| + \sqrt{|y|}} \\ &\leq 2\sqrt{C_1 K_2 d(0, q')} \cdot \sqrt{d(0, (x + y) * W)} \end{aligned}$$

when $|x| + \sqrt{|y|} \leq K_2 d(0, q')$.

(2) As $|\tilde{w}_1| \leq \frac{r}{2}$, we have $|\pi_2^{\perp}[tx_0, \tilde{w}_1]| \leq K_2 |tx_0| \cdot |\tilde{w}_1| \leq \frac{K_2 rt}{2}$. Recall $|\pi_2^{\perp}[x_0, w_0]| = K_2$. We have

$$|\pi_2^{\perp}[tx_0, rw_0 + \tilde{w}_1]| = |\pi_2^{\perp}[tx_0, rw_0] + \pi_2^{\perp}[tx_0, \tilde{w}_1]| = |rt\pi_2^{\perp}[x_0, w_0] + \pi_2^{\perp}[tx_0, \tilde{w}_1]| \geq \frac{K_2 rt}{2}.$$

Notice $d(0, h_t * W) = |tx_0| = t$. Now by Lemma 4.3

$$d(q', h_t * W) \geq \frac{1}{C_1}\left(|tx_0| + \sqrt{|\pi_2^{\perp}[tx_0, rw_0 + \tilde{w}_1]|}\right) \geq \frac{1}{C_1}\left(\sqrt{\frac{K_2 rt}{2}}\right) \geq \frac{\sqrt{K_2 d(0, q')}}{2\sqrt{M} C_1} \cdot \sqrt{t}.$$

In the last step above we used the assumption $d(0, q') \leq 2Mr$.

□

4.2 Proof of Theorem 4.1

Here we finish the proof of Theorem 4.1.

Let $U, V \subset G$ be open subsets and $f : U \to V$ a quasiconformal map. Suppose that for each left coset F of W, each component of $F \cap U$ is mapped into a left coset of W. We need to show that f is locally biLipschitz.

We shall do so by finding upper bound for the Lipschitz constant. The theorem then follows by considering f^{-1}.

Let K_2 be the constant defined before Lemma 4.5. By Lemma 4.5, $K_2 = 0$ if and only if W is normal in G. In this case all left cosets of W are parallel and Theorem 4.1 follows from Corollary 3.10 (see the proof of Corollary 3.11 for more details).

From now on we assume $K_2 > 0$. Let $p \in U$ be fixed. Since quasiconformal maps on Carnot groups are locally quasisymmetric, there is some ball $B = B(p, r_0) \subset U$ such that $f|_B$ is η-quasisymmetric for some η. By Pansu's differentiability theorem and Fubini's theorem, for almost every left coset F of W, f is Pansu differentiable at a.e. $x \in F \cap B$. In particular, for such a left coset F, a.e. $x \in F \cap B$ is an $\eta(1)$-quasisimilarity point.

Recall that there is a constant $M \geq 1$ such that $d(x, z) \leq M(d(x, y) + d(y, z))$ for any $x, y, z \in G$. Set $\lambda = \frac{1}{20M^3}$. Now we are ready to bound the Lipschitz constant.

Lemma 4.7. *Let F be a left coset of W such that f is Pansu differentiable at a.e. $x \in F \cap B$. Then $L_f(q) \leq C_2$ for a.e. $q \in F \cap \lambda B$, where*

$$C_2 = 1024\eta(1)^2 M^2 C_1^3 \cdot \frac{diam(f(B))}{r_0}$$

and C_1 is the constant in Lemma 4.3.

Proof. Let $q \in F \cap \lambda B$ be a point where f is Pansu differentiable, Then $F = q * W$. Let $q' \in F$ be a point of the form $q' = q * \left((\frac{1}{4M} r_0 w_0 + \bar{w}_1) + \bar{w}_2\right)$ with $\bar{w}_1 \in W_1, \bar{w}_2 \in W_2, |\bar{w}_1| << r_0, |\bar{w}_2| << r_0^2$, where w_0 is as in Lemma 4.6 (2). The requirements $|\bar{w}_1| << r_0, |\bar{w}_2| << r_0^2$ and the generalized triangle inequality for d imply $q' \in B$. As $\bar{w}_1$ varies in a small neighbourhood of 0 in W_1 and $\bar{w}_2$ varies in a small neighbourhood of 0 in W_2, the set of $q' = q * \left((\frac{1}{4M} r_0 w_0 + \bar{w}_1) + \bar{w}_2\right)$ form an open subset of $F \cap B$. Since f is Pansu differentiable at a.e. $x \in F \cap B$, we can pick such a q' so that f is Pansu differentiable at q'. So q' is an $\eta(1)$-quasisimilarity point with respect to f.

Denote $F_t = q * (tx_0) * W$. Notice $d(q, F_t) = t$. Since $(-q) * q' = (\frac{1}{4M} r_0 w_0 + \bar{w}_1) + \bar{w}_2$, and $|\bar{w}_1| << r_0$, $|\bar{w}_2| << r_0^2$, it is clear that the assumption in Lemma 4.6 (2) for q' is satisfied by $(-q) * q'$. It follows that

$$d(q', F_t) = d(q', q * (tx_0) * W) \geq \frac{\sqrt{K_2 d(q, q')}}{2\sqrt{M} C_1} \cdot \sqrt{d(q, F_t)} = \frac{\sqrt{K_2 d(q, q')}}{2\sqrt{M} C_1} \cdot \sqrt{t}. \tag{4.5}$$

Let $x_t \in F_t$ be such that $d(q, x_t) = d(q, F_t) = t$. Denote $s_t = d(f(q), f(F_t))$. Then we have, for t sufficiently small,

$$s_t \leq d(f(q), f(x_t)) \leq 2L_f(q) \cdot d(q, x_t) = 2L_f(q)t. \tag{4.6}$$

Let $x'_t \in F_t$ be such that $d(f(q'), f(x'_t)) = d(f(q'), f(F_t))$. For t sufficiently small, we have:

$$\begin{aligned}
d(f(q'), f(x'_t)) &\geq \frac{l_f(q')}{2} \cdot d(q', x'_t) \\
&\geq \frac{l_f(q')}{2} \cdot d(q', F_t) \\
&\geq \frac{l_f(q')}{2} \cdot \frac{\sqrt{K_2 d(q, q')}}{2\sqrt{M} C_1} \cdot \sqrt{t} \quad \text{by (4.5)} \\
&\geq \frac{l_f(q')}{2} \cdot \frac{\sqrt{K_2 d(q, q')}}{2\sqrt{M} C_1} \cdot \frac{1}{\sqrt{2L_f(q)}} \sqrt{s_t} \quad \text{by (4.6)} \\
&= \frac{\sqrt{K_2 d(q, q')}}{4\sqrt{2M} C_1} \cdot \frac{l_f(q')}{\sqrt{L_f(q)}} \cdot \sqrt{s_t}.
\end{aligned}$$

In summary,

$$d(f(q'), f(x'_t)) \geq \frac{\sqrt{K_2 d(q, q')}}{4\sqrt{2M} C_1} \cdot \frac{l_f(q')}{\sqrt{L_f(q)}} \cdot \sqrt{s_t}. \tag{4.7}$$

Now the “worst case" bound of divergence also applies to the fibers $f(F)$ and $f(F_t)$, so for sufficiently small t we have

$$d(f(q'), f(x_t')) = d(f(q'), f(F_t)) \le 2\sqrt{C_1 K_2 d(f(q), f(q'))} \cdot \sqrt{s_t}. \tag{4.8}$$

Combining inequalities (4.7) and (4.8) yields $l_f(q') \le C \cdot \sqrt{L_f(q)}$, where

$$C = 8\sqrt{2M} C_1^{\frac{3}{2}} \cdot \sqrt{\frac{d(f(q), f(q'))}{d(q, q')}}. \tag{4.9}$$

Recall that q' is an $\eta(1)$-quasisimilarity point with respect to f. We get $L_f(q') \le C\eta(1) \cdot \sqrt{L_f(q)}$.

Since $(-q') \star q = (-\frac{1}{4M} r_0 w_0 - \bar{w}_1) - \bar{w}_2$, the assumption in Lemma 4.6 (2) for q' is also satisfied by $(-q') \star q$. As both q and q' are $\eta(1)$-quasisimilarity points, the above argument can be repeated with the roles of q and q' switched. This gives $L_f(q) \le C\eta(1) \cdot \sqrt{L_f(q')}$. Combining this with $L_f(q') \le C\eta(1) \cdot \sqrt{L_f(q)}$, we get

$$L_f(q) \le C^2 \eta(1)^2 = 128\eta(1)^2 M C_1^3 \cdot \frac{d(f(q), f(q'))}{d(q, q')} \le 1024\eta(1)^2 M^2 C_1^3 \cdot \frac{\operatorname{diam}(f(B))}{r_0}.$$

In the last inequality we used the facts that $q, q' \in B$ and $d(q, q') \ge \frac{r_0}{8M}$.

□

Lemma 4.7 only provides an upper bound for $L_f(q)$ for a.e $q \in \lambda B$. We are not in a position to apply Lemma 3.2 and need a different argument to show f is locally Lipschitz.

A horizontal line segment is a curve of the form $\gamma_g(t) = g \star (tX)$, $t \in I$, where $0 \ne X \in V_1$, I is a closed interval and $g \in \mathfrak{g}$.

Lemma 4.8. *f is Lipschitz in a neighbourhood of p.*

Proof. Quasiconformal maps on Carnot groups are absolutely continuous along almost all curves (Theorem 1.1 of [2]). So for a fixed $0 \ne X \in V_1$, f is absolutely continuous along a.e. γ_g. On the other hand, Fubini's theorem and Pansu's differentiability theorem imply that for a.e. γ_g, the map f is Pansu differentiable at almost every point of γ_g. So for a.e. γ_g, f is absolutely continuous along γ_g and f is Pansu differentiable at almost every point of γ_g. Lemma 4.7 implies that the length of tangent vectors of the curve $f \circ \gamma_g$ is at most C_2. The absolute continuity allows us to conclude that $f \circ \gamma_g$ is C_2-Lipschitz. Since this is true for a.e. γ_g and f is continuous, we conclude that f is C_2-Lipschitz when restricted to every horizontal line segment.

For any $x, y \in G$, there is a path γ from x to y consisting of a finite number of horizontal line segments such that the length of γ is at most $C_3 \cdot d(x, y)$, where C_3 is a constant depending only on the Carnot group G. Combing this with the first paragraph we see that f is $C_2 C_3$-Lipschitz in a neighbourhood of p.

□

Completing the proof of Theorem 4.1. Let $p \in U$. By Lemma 4.8, f is Lipschitz in a neighbourhood of p. The same lemma applied to f^{-1} shows that f^{-1} is Lipschitz in a neighbourhood of $f(p)$. Hence f is biLipschitz in a neighbourhood of p.

□

Acknowledgements: The authors would like to thank the referee for comments and suggestions that greatly improved the exposition. The second author also acknowledges support from Simons Foundation grant #315130.

References

[1] Z. Balogh, *Hausdorff dimension distribution of quasiconformal mappings on the Heisenberg group,* J. Anal. Math. **83** (2001), 289—312.

[2] Z. Balogh, P. Koskela and S. Rogovin, *Absolute continuity of quasiconformal mappings on curves,* Geom. Funct. Anal. **17** (2007), no. 3, 645–664.

[3] L. Capogna, M. Cowling, *Conformality and Q-harmonicity in Carnot groups,* Duke Math. J. **135** (2006), no. 3, 455—479.

[4] M. Cowling, A. Ottazzi, *Global contact and quasiconformal mappings of Carnot groups,* Conform. Geom. Dyn. **19** (2015), 221–239.

[5] M. Cowling, M. Reimann, *Quasiconformal mappings on Carnot groups: three examples,* Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), 2003, pp. 111–118.

[6] A. Koranyi, H. M. Reimann, *Quasiconformal mappings on the Heisenberg group,* Invent. Math. **80** (1985), no. 2, 309–338.

[7] E. Le Donne, X. Xie, *Rigidity of fiber-preserving quasisymmetric maps,* Revista Matematica Iberoamericana **32** (2016), no. 4, 1407–1422.

[8] P. Pansu, *Metriques de Carnot-Caratheodory et quasiisometries des espaces symetriques de rang un,* Ann. of Math. (2) **129** (1989), no. 1, 1–60.

[9] X. Xie, *Quasisymmetric maps on reducible Carnot groups*, Pacific Journal of Mathematics **265** (2013), no. 1, 113–122.

[10] X. Xie, *Quasiconformal maps on model Filiform groups*, Michigan Mathematical Journal **64** (2015), no. 1, 169–202.