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Ville Suomala

# Intermediate Value Property for the Assouad Dimension of Measures

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**Abstract:** Hare, Mendivil, and Zuberman have recently shown that if $X \subset \mathbb{R}$ is compact and of non-zero Assouad dimension $\dim_A X$, then for all $s > \dim_A X$, $X$ supports measures with Assouad dimension $s$. We generalize this result to arbitrary complete metric spaces.

**Keywords:** Assouad dimension; intermediate value property

**MSC:** Primary 28A80, 54E50.

## 1 Introduction and notation

The Assouad dimension has its origins in metric geometry and embedding problems, see e.g. [9, 13]. More recently, it has been investigated thoroughly in fractal geometry. For a few of the many recent advances, see [2, 4–6, 10, 15]. The *Assouad dimension* of a metric space $X = (X, d)$, denoted $\dim_A X$, is defined to be the infimum of those $\alpha \geq 0$ such that the following uniform bound holds for all $x \in X$, $0 < r < R < \infty$:

$$N(x, R, r) \leq C \left(\frac{R}{r}\right)^{\alpha} . \tag{1.1}$$

Here $N(x, R, r)$ denotes the minimal number of closed $r$-balls needed to cover the closed $R$-ball $B(x, R) = \{y \in X : d(x, y) \leq R\}$ and $0 < C < \infty$ is a constant independent of $x$, $r$, and $R$. If there is no $\alpha \in [0, \infty[$ fulfilling the requirement (1.1), we assign $\dim_A X$ the value $\infty$.

There is a close analogue of this concept for measures. Suppose that $\mu$ is a (Borel regular outer-) measure on $X$. For simplicity, we always assume that our measures are fully supported, that is $\operatorname{spt}\mu = X$. The *Assouad dimension* of $\mu$, denoted by $\dim_A \mu$ is the infimum of those $\alpha \geq 0$ such that for all $x \in X$, $0 < r < R < \infty$,

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C \left(\frac{R}{r}\right)^{\alpha} , \tag{1.2}$$

where again $C < \infty$ is a constant depending only on $\alpha$. If (1.2) is not fulfilled for any $\alpha < \infty$, we define $\dim_A \mu = \infty$. This concept has appeared in the literature also under the names upper regularity dimension and upper Assouad dimension and measures satisfying the condition (1.2) have been called $\alpha$-homogeneous.

It is well known and easy to see that $\dim_A X < \infty$ if and only if $X$ is geometrically doubling in the following sense: There is a constant $C < \infty$ such that

$$N(x, 2r, r) \leq C$$

for all $x \in X$, $0 < r < \infty$. Likewise, $\dim_A \mu < \infty$ if and only if

$$\frac{\mu\big(B(x, 2r)\big)}{\mu\big(B(x, r)\big)} \leq C ,$$

**Ville Suomala,** University of Oulu, Oulu, Finland, E-mail: ville.suomala@oulu.fi

for some constant $C < \infty$, and for all $x \in X$ and $0 < r < \infty$.

The investigation of the relation between the Assouad dimension of sets and that of measures supported on them was pioneered by Vol´berg and Konyagin [16, 17], who proved that for a compact metric space $X$ with finite Assouad dimension,

$$\dim_A X = \inf\{\dim_A \mu \,:\, \operatorname{spt}\mu \subset X\}\,. \tag{1.3}$$

Note that $\dim_A X$ is an obvious lower bound for $\dim_A \mu$, whenever $\mu$ is supported on $X$. Luukkainen and Saksman [14] later extended (1.3) for complete metric spaces. These results, however, left open the question whether all values $s > \dim_A X$ are attainable for $\dim_A \mu$. In other words, is it possible that the Assouad dimension of measures omits some values $s > \dim_A X$? Plausibly not, but an example of Hare, Mendivil, and Zuberman [7, Proposition 3.8] of an infinite compact set $X \subset \mathbb{R}$ such that

$$\{\dim_A \mu \,:\, \operatorname{spt}\mu \subset X\} = \{0, \infty\}$$

suggests that this could be a subtle issue. An analogous intermediate value problem for the Assouad dimension of subsets has been considered in [1, 18].

Quite recently, Hare, Mendivil, and Zuberman [7], have shown that if $X \subset \mathbb{R}$ is compact and $\dim_A X > 0$, then $\dim_A \mu$ indeed attains all values $s > \dim_A X$ when $\mu$ ranges over all measures supported on $X$. In this note, we prove

**Theorem 1.1.** *If $X$ is a complete metric space with $0 < \dim_A X < \infty$, then for all $s > \dim_A X$, there is a measure $\mu$ on $X$ such that $\dim_A \mu = s$.*

Theorem 1.1 extends the result of Hare, Mendivil, and Zuberman to arbitrary complete metric spaces and thus settles the problem raised in [7, Remark 4.3]. We stress that there is no imposed upper bound on $R$ in the definitions (1.1) and (1.2) and our result thus makes perfect sense also for unbounded $X$.

Our starting point to prove Theorem 1.1 is similar to that in [7] as we both rely on the construction of generalized cubes in [11], see Theorem 2.1 below. In [7], the authors make use of the geometry of the real-line and impose additional regularity assumptions on the generalized cubes. Our method works in the generality of [11] without any additional technical conditions allowing us to obtain the result in its full generality. Another difference is that while for $s > \dim_A X$, [7] provides an ad hoc construction yielding a measure with $\dim_A \mu = s$, we consider a natural family of measures $\mu_p$ and relying on the continuity of the map $p \mapsto \dim_A \mu_p$ verify that $\dim_A \mu_p$ attains all values in $]\dim_A X, \infty[$.

A result analogous to Theorem 1.1 for the lower dimension follows by essentially the same proof. We discuss this issue briefly in the last section of the paper.

## 2 Toolbox: Generalized cubes

We begin by restating [11, Theorem 2.1], which is our main tool. It provides us an analogue of the $M$-adic cubes in Euclidean spaces. These "generalized nested cubes" will be used throughout these notes.

**Theorem 2.1.** *Let $X \neq \varnothing$ be a metric space such that $N(x, 2r, r) < \infty$ for all $x \in X$, $r > 0$. For each $0 < \delta < \frac{1}{7}$, there exists at most countable collections $\mathcal{Q}_k$, $k \in \mathbb{Z}$, of Borel sets having the following properties:*

*(i)* $X = \bigcup_{Q \in \mathcal{Q}_k} Q$ *for every* $k \in \mathbb{Z}$,

*(ii)* *If* $Q \in \mathcal{Q}_k$, $Q' \in \mathcal{Q}_{k'}$, $k' \geq k$ *and* $Q \cap Q' \neq \varnothing$, *then* $Q' \subset Q$.

*(iii)* *For every* $Q \in \mathcal{Q}_k$, *there is a distinguished point* $x_Q \in X$ *so that*

$$B\left(x_Q, \frac{1}{3}\delta^k\right) \subset Q \subset B\left(x_Q, 2\delta^k\right)\,.$$

*(iv)* *There exists a point* $x_0 \in \bigcap_{k\in\mathbb{Z}}\{x_Q \,:\, Q \in \mathcal{Q}_k\}$.

*(v)* $\{x_Q \,:\, Q \in \mathcal{Q}_k\} \subset \{x_Q \,:\, Q \in \mathcal{Q}_{k+1}\}$ *for all* $k \in \mathbb{Z}$.

We make some remarks and introduce some further notation. We first note that the condition $0 < \delta < \frac{1}{7}$ is purely technical and the result holds also for any $\frac{1}{7} \leq \delta < 1$ with a cost of reducing the parameters $\frac{1}{3}$ and 2 in *(iii)*. We also note that, even if $\delta$ is fixed, the families $\mathcal{Q}_k$ are in no way unique. However, for the sake of simplicity, when we refer to Theorem 2.1 with a given value of the parameter $\delta$, we always assume the families $\mathcal{Q}_k$ have been fixed. We will call the elements of $\bigcup_{k\in\mathbb{Z}} \mathcal{Q}_k$ *cubes* and refer to the point $x_Q$ as the *center* of the cube $Q$. The distinguished point $x_0$ may be considered the "origin" of the space $X$. We also denote by $Q_0$ the unique element of $\mathcal{Q}_0$ that contains $x_0$. This is the "unit cube" of $X$.

If $Q \in \mathcal{Q}_k$, $Q' \in \mathcal{Q}_{k+m}$, $m \in \mathbb{N}$, and $Q' \subset Q$, we denote $Q' \prec_m Q$. We also abbreviate $\prec_1$ to $\prec$. If $Q' \prec Q$, we say that $Q$ is the *parent* of $Q'$, and that $Q'$ is a *child* of $Q$. More generally, we will refer to cubes $Q' \prec_m Q$, $m \in \mathbb{N}$, as *offspring* of $Q$. It is convenient to think about the child-cube $Q' \prec Q$ with $x_{Q'} = x_Q$ as a distinguished *central subcube* of $Q$. If $Q' \prec Q$ is not a central subcube, we say that $Q$ is a *boundary cube*. Let

$$M_Q = |\{Q' : Q' \prec Q\}|$$

stand for the number of children of $Q$. If $\dim_A X < \infty$, then $M_Q$ is bounded and we denote

$$M := \max_{n\in\mathbb{Z},\, Q\in\mathcal{Q}_n} M_Q .$$

Indeed, it is easy to check that

$$M \leq K\delta^{-\dim_A X}, \tag{2.1}$$

where $K = K(\delta)$ is such that $\log_\delta K(\delta) \longrightarrow 0$ as $\delta \downarrow 0$.

To demonstrate Theorem 2.1 and our later considerations, it is helpful to keep in mind the following simple example.

*Example* 2.2. Let $\delta = \frac{1}{3}$, $X = \mathbb{R}$ and let $\mathcal{Q}_k$ consist of the triadic half-open intervals

$$\mathcal{Q}_k = \left\{ \left[(j-\tfrac{1}{2})3^{-k}, (j+\tfrac{1}{2})3^{-k}\right[ \;:\; j \in \mathbb{Z} \right\} .$$

Now $x_Q$ is literally the center point of each interval $Q$ and we may, for instance, fix $x_0 = 0$ so that $Q_0 = [-\frac{1}{2}, \frac{1}{2}[$. Moreover, $M_Q = 3$ for all $Q \in \mathcal{Q}_k$, $k \in \mathbb{Z}$ and thus $M = 3 = \delta^{-\dim_A X}$.

Before proceeding further, we introduce our last bit of notation. For each $t > 0$, we denote by $n_t$ the smallest integer such that $\delta^{n_t} \leq t$, that is,

$$n_t = \ulcorner \log t / \log \delta \urcorner .$$

This notation allows us to switch to a logarithmic scale in the ratio $R/r$. Namely, if $N = n_r - n_R$, then

$$\frac{1}{C}\delta^{-N} \leq \frac{R}{r} \leq C\delta^{-N} . \tag{2.2}$$

Here, and in what follows, we denote by $C$ a positive and finite constant that only depends on $\delta$ and $M$ and whose precise value is of no importance.

Our next lemma provides us doubling measures that respect the construction of the generalized cubes. This is a quantitative version of the Vol´berg-Konyagin Theorem (1.3) in our situation.

**Lemma 2.3.** *Suppose that $X \neq \varnothing$ is a complete metric space. Let $0 < \delta < \frac{1}{7}$ and $0 < p < \frac{1}{M}$. Then there is a measure $\mu_p$ on $X$ such that $\dim_A \mu_p < \infty$ and so that for all $Q' \prec Q$ it holds that*

$$\mu_p(Q') = \begin{cases} p\mu_p(Q), & \text{if } Q' \text{ is a boundary cube,} \\ \big(1-(M_Q-1)p\big)\,\mu_p(Q), & \text{if } Q' \text{ is the central subcube of } Q . \end{cases} \tag{2.3}$$

*Moreover, if $t > \dim_A X$, then $\dim_A \mu_p < t$ for some choice of $\delta$ and $p$.*

*Proof.* The claim is implicitely derived in the proof of [11, Theorem 3.1] (see Remark 5.1 (2) in the same paper) but since in [11] it was stated in a less quantitative form, let us briefly sketch the main idea.

The construction of the measures $\mu_p$ satisfying (2.3), given the families $\mathcal{Q}_k$, is detailed in [11] and we thus take their existence for granted. It is easy to check that for the measures $\mu_p$, the Assouad dimension may be determined by looking at the ratios of the measures of cubes along offspring chains. In particular, if $\alpha > 0$ is such that

$$\frac{\mu(Q)}{\mu(Q')} \leq \delta^{-m\alpha}\,, \tag{2.4}$$

whenever $m \in \mathbb{N}$, $Q, Q' \in \bigcup_k \mathcal{Q}_k$, and $Q' \prec_m Q$, then we may conclude that $\dim_A \mu_p \leq \alpha$. This is a consequence of the fact that if $x \in X$, $t > 0$, and $x \in Q \in \mathcal{Q}_{n_t}$, then

$$\frac{1}{C}\mu_p(Q) \leq \mu_p\left(B(x,t)\right) \leq C\mu_p(Q)\,. \tag{2.5}$$

To verify (2.5) is an easy exercise using (2.3) and Theorem 2.1, or see the proof of [11, Theorem 3.1]. Thus, the task is to show that for suitably chosen $\delta$ and $p$, the upper bound (2.4) holds for $\alpha$ close to $\dim_A X$.

To that end, let us fix $0 < \delta < \frac{1}{7}$ and the families $\mathcal{Q}_k$ provided by Theorem 2.1. Consider $0 < p < \frac{1}{M}$ and let $\mu_p$ satisfy (2.3). Let $Q, Q' \in \bigcup_k \mathcal{Q}_k$ and suppose that $Q' \prec_m Q$. Using (2.1), we have

$$\begin{aligned}\frac{\mu_p(Q)}{\mu_p(Q')} &\leq p^{-m} = (pM)^{-m}M^m\\ &\leq (pM)^{-m}K^m\delta^{-m\dim_A X}\\ &= \delta^{-m(\dim_A X - \log_\delta K + \log_\delta(pM))}\end{aligned} \tag{2.6}$$

Thus, (2.4) holds for $\alpha = \dim_A X - \log_\delta K + \log_\delta(pM)$ and whence

$$\dim_A \mu_p \leq \dim_A X - \log_\delta K + \log_\delta(pM)\,.$$

The upper bound can be made arbitrarily close to $\dim_A X$ by first choosing $\delta$ small enough and then $p$ close enough to $\frac{1}{M}$. □

*Remark* 2.4. (1) The condition (2.3) implies that (having the nested cubes construction fixed), the measures $\mu_p$ are uniquely defined up to a multiplicative constant. If we further require that $\mu_p(Q_0) = 1$, the measure $\mu_p$ is thus completely determined by this condition.

(2) It is instructive to think about the measures $\mu_p$ in the setting of the Example 2.2, where they are uniquely defined for all $0 < p < \frac{1}{2}$ via (2.3) and the condition $\mu_p[-\frac{1}{2}, \frac{1}{2}[= 1$. In this case, the measure of each triadic interval is split among its triadic child-intervals in such a way that the central child inherits $(1-2p)$ times the mass of its parent and the rest is split equally between the two boundary children. It is an easy exercise to verify that $\dim_A \mu_p = -\log_3 p$, for all $0 < p \leq \frac{1}{3}$. Thus, $\dim_A \mu_p$ varies continuously in $p$ and attains all values $\geq 1 = \dim_A \mathbb{R}$. Our main result, Theorem 1.1, and its proof, reflect this phenomenon in the more abstract setting of Lemma 2.3.

To conclude this section, we provide the following variant of [7, Lemma 3.5].

**Lemma 2.5.** *Suppose that* $\dim_A X > 0$. *Then, for a suitably small* $\beta > 0$ *(depending only on* $\dim_A X$, $\delta$, *and* $M$*), there are arbitrarily large* $N \in \mathbb{N}$ *and cubes*

$$Q^N \prec Q^{N-1} \prec \ldots \prec Q^1 \prec Q^0 \tag{2.7}$$

*such that at least* $\beta N$ *of the cubes* $Q^1, \ldots, Q^N$ *are boundary cubes.*

*Proof.* Suppose that $\beta > 0$ does not satisfy the requirement of the lemma. We prove the lemma by deriving a lower bound for $\beta$. To that end, we fix $N_0$ so large that any cube sequence as in (2.7) with $N \geq N_0$ contains at most $\beta N$ boundary cubes. Then, for all $k \in \mathbb{Z}$, all $Q \in \mathcal{Q}_k$, and all $N \geq N_0$, it holds that

$$\left|\{Q' : Q' \prec_N Q\}\right| \leq \binom{N}{\lfloor \beta N \rfloor} M^{\beta N}\,. \tag{2.8}$$

To see this, label the offspring of $Q$ as follows. Denote by $Q_1$ the central child-cube of $Q$ and by $Q_2, \dots, Q_{M_Q}$ the boundary children. For each $Q_i$, denote by $Q_{i1}$ the central child-cube of $Q_i$ and by $Q_{i2}, \dots, Q_{iM_{Q_i}}$ the boundary children. Continuing this labelling for $N$ generations of offspring of $Q$, each $Q' \prec_N Q$ gets labelled with a sequence $\mathtt{i} \in \{1, \dots, M\}^N$. But, for each $Q_{\mathtt{i}} \prec_N Q$, $\mathtt{i} = i_1 \dots i_N$, only at most $\beta N$ of the symbols $i_k$ differ from 1, and for each such $i_k$, there are at most $M$ admissible values. Whence the bound (2.8).

Let $x \in X$, $0 < r < R < \infty$ and consider a cube $Q \in \mathcal{Q}_{n_R}$ with $Q \cap B(x, R) \neq \emptyset$. There are at most $C$ such cubes. Switching to the logarithmic scale $N = n_r - n_R$ and recalling (2.8), we observe that the ball $B(x, R)$ may be covered by

$$C \binom{N}{\lfloor \beta N \rfloor} M^{\beta N} \tag{2.9}$$

cubes $Q' \in \mathcal{Q}_{n_r}$. Using trivial bounds for factorials, e.g.

$$n^n e^{1-n} \leq n! \leq (n+1)^{n+1} e^{-n} ,$$

and taking logarithms in base $\delta$, we note that

$$\binom{N}{\lfloor \beta N \rfloor} \leq C \delta^{-\kappa(\beta) N} ,$$

where $\kappa(\beta) \to 0$ as $\beta \to 0$. Thus, (2.9) is bounded from above by

$$C \delta^{-N(\kappa(\beta) - \beta \log_\delta M)} .$$

Since each $Q' \in \mathcal{Q}_{n_r}$ may be covered by $C$ balls of radius $r$, and because $\frac{R}{r} \geq C \delta^{-N}$, recall (2.2), it follows that

$$N(x, R, r) \leq C \left( \frac{R}{r} \right)^{\kappa(\beta) - \beta \log_\delta M} .$$

This upper bound is valid irrespective of $x$, $r$ and $R$ and thus

$$\dim_A X \leq \kappa(\beta) - \beta \log_\delta M ,$$

yielding the required lower bound for $\beta$. □

# 3 Proof of Theorem 1.1

In this section, we prove the following two lemmas. Theorem 1.1 follows readily from these lemmas and the last claim of Lemma 2.3 along with the intermediate value theorem for continuous functions. We fix $0 < \delta < \frac{1}{7}$ and consider the families $\mathcal{Q}_k$ provided by Theorem 2.1 along with the measures $\mu_p$ provided by Lemma 2.3.

**Lemma 3.1.** *If* $\dim_A X > 0$, *then* $\lim_{p \downarrow 0} \dim_A \mu_p = \infty$.

**Lemma 3.2.** *The map* $p \mapsto \dim_A \mu_p$ *is continuous on the interval* $]0, \frac{1}{M}[$.

*Proof of Lemma 3.1.* We estimate $\dim_A \mu_p$ from below. Let $\beta > 0$ be the constant from Lemma 2.5. Then, there are arbitrarily long offspring chains

$$Q^N \prec Q^{N-1} \prec \dots \prec Q^1 \prec Q^0$$

with at least $\beta N$ of the cubes $Q^1, \dots, Q^N$ boundary cubes. Whence, using (2.3),

$$\mu_p(Q^N) \leq p^{\beta N} \mu_p(Q^0) .$$

Let $x = x_{Q^N}$ and pick $n \in \mathbb{Z}$ such that $Q^0 \in \mathcal{Q}_n$. Then, for $r = \frac{1}{C}\delta^{n+N}$, $R = C\delta^n$, we have $B(x,r) \subset Q^N$ and $Q^0 \subset B(x,R)$. Thus

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \geq \frac{\mu(Q^0)}{\mu(Q^N)} \geq p^{-\beta N} = \delta^{-N(\beta \log_\delta p)} \geq C\left(\frac{R}{r}\right)^{\beta \log_\delta p} .$$

Since $N$, and thus the ratio $R/r$, may be arbitrarily large, this shows that

$$\dim_A \mu_p \geq \beta \log_\delta p \longrightarrow \infty ,$$

as $p \downarrow 0$. □

*Proof of Lemma 3.2.* Recall from (2.5) that for all $x \in X$, $t > 0$, and $x \in Q \in \mathcal{Q}_{n_t}$, we have

$$\frac{1}{C}\mu_p(Q) \leq \mu_p\left(B(x,t)\right) \leq C\mu_p(Q) . \tag{3.1}$$

More precisely, the constant $C$ may be taken to be uniform in $p$, if $p$ is bounded away from 0, which we can of course assume.

Consider $x \in X$ and $0 < r < R < \infty$. We switch to the logarithmic scale $N = n_r - n_R$. Let $Q_R \in \mathcal{Q}_{n_R}$, $Q_r \in \mathcal{Q}_{n_r}$ be the unique cubes containing $x$. Then $Q_r \prec_N Q_R$ and by virtue of (3.1),

$$\frac{1}{C}\frac{\mu_p(Q_R)}{\mu_p(Q_r)} \leq \frac{\mu_p(B(x,R))}{\mu_p(B(x,r))} \leq C\frac{\mu_p(Q_R)}{\mu_p(Q_r)} . \tag{3.2}$$

Consider the parental line of cubes from $Q_R$ to $Q_r$ by denoting $Q_r = Q^N \prec Q^{N-1} \prec \ldots \prec Q^1 \prec Q^0 = Q_R$. Let $\mathfrak{B} = \{i \in \{1,\ldots,N\} : Q^i \text{ is a boundary cube}\}$ and $\mathfrak{C} = \{1,\ldots,N\} \setminus \mathfrak{B}$. Using (2.3), it follows that

$$\frac{\mu_p(Q_r)}{\mu_p(Q_R)} = p^{|\mathfrak{B}|}\prod_{n\in\mathfrak{C}}\left(1 + (1 - M_{Q^{n-1}})p\right) .$$

Thus, given $0 < p < p' < \frac{1}{M}$, we have

$$\left(\frac{1-(M-1)p'}{1-(M-1)p}\right)^N \leq \frac{\mu_p(Q_R)/\mu_p(Q_r)}{\mu_{p'}(Q_R)/\mu_{p'}(Q_r)} \leq \left(\frac{p'}{p}\right)^N .$$

Recalling (2.2) and (3.2) and expressing these upper and lower bounds as

$$\left(\frac{p'}{p}\right)^N = \delta^{N(\log_\delta p' - \log_\delta p)} ,$$

$$\left(\frac{1-(M-1)p'}{1-(M-1)p}\right)^N = \delta^{N\left(\log_\delta\left(1-(M-1)p'\right) - \log_\delta\left(1-(M-1)p\right)\right)} ,$$

we note that if one of the measures $\mu_p$, $\mu_p'$ satisfies the condition (1.2) with the exponent $\alpha$, then the other satisfies it with the exponent $\alpha + \varepsilon$, where

$$\varepsilon = \max\left\{\log_\delta p - \log_\delta p', \log_\delta\left(1-(M-1)p'\right) - \log_\delta\left(1-(M-1)p\right)\right\} . \tag{3.3}$$

Thus, we observe a quantitative modulus of continuity for $p \mapsto \dim_A \mu_p$. □

*Remark* 3.3. The key estimate derived in the proof of Lemma 3.2 is the following: Given $0 < p < \frac{1}{M}$, and $\varepsilon > 0$, we have

$$\delta^{\varepsilon N} \leq \frac{\mu_p(Q')/\mu_p(Q)}{\mu_{p'}(Q')/\mu_{p'}(Q)} \leq \delta^{-\varepsilon N} \qquad \text{for all} \qquad Q \prec_N Q' , \tag{3.4}$$

provided $|p' - p|$ is small enough, see (3.3). This estimate actually implies the continuity of $p \mapsto \dim \mu_p$ for a variety of Assouad type dimensions dim in addition to $\dim_A$ and the lower dimension $\dim_L$ (considered in Section 4 below). For some variants of the Assouad dimension such as the upper and lower Assouad spectrums introduced in [8], and the related quasi-Assouad dimensions, the continuity is obvious from (3.4). Moreover, if $X$ is compact, then it is relatively easy to derive the continuity of $p \mapsto \dim_q \mu_p$, $p \mapsto \dim_M \mu_p$, $p \mapsto \dim_F \mu_p$ using (3.4). Here $\dim_q \mu$, where $1 \neq q > 0$, denotes the $L^q$-dimension, see e.g. [12] for the definition. Moreover, $\dim_M \mu$ and $\dim_F \mu$ denote the Minkowski and Frostman dimensions of $\mu$, respectively, as defined in [3].

# 4 Attainable values for the lower dimension

In this final section, we discuss the following "dual" result for Theorem 1.1.

**Theorem 4.1.** *If $X$ is a complete metric space with $\dim_A X < \infty$, then for all $0 < s < \dim_L X$, there is a measure $\mu$ with support $X$ such that $\dim_L \mu = s$.*

Here $\dim_L X$ is the lower dimension of $X$ defined as the supremum of the exponents $\alpha$, for which there is a constant $0 < C < \infty$ such that

$$N(x, R, r) \geq C \left(\frac{R}{r}\right)^{\alpha},$$

for all $x \in X$ and $0 < r < R < \operatorname{diam}(X)$. Analogously, $\dim_L \mu$ is the supremum of those $\alpha$, for which

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \geq C \left(\frac{R}{r}\right)^{\alpha}$$

holds irrespective of $x \in X$ and $0 < r < R < \operatorname{diam}(X)$.

For a compact set of reals, this result is also due to Hare, Mendivil, and Zuberman, see [7, Theorem 4.1]. In its full generality, Theorem 4.1 may be proved using ideas very similar to our proof of Theorem 1.1. Thus, we only sketch the idea and leave the details for the interested reader.

To begin with, we observe that $p \mapsto \dim_L \mu_p$ is continuous. This follows by the very same proof as Lemma 3.2, see (3.4). Likewise, switching to a chain of central cubes in the proof of Lemma 3.1 gives $\lim_{p\downarrow 0} \dim_L \mu_p = 0$. So if we knew that for $s < \dim_L X$, we had $\dim_L \mu_p > s$ for a suitably chosen $\delta$ and $p$, the claim would follow just as in the case of Theorem 1.1. This holds in some nice cases such as the Example 2.2, but unfortunately it does not hold in general if there is variation in the numbers $M_Q$.

However, the argument can still be saved by considering mass distributions more general than those defined by (2.3). Given $J \in \mathbb{N}$, let $0 < p < \frac{1}{M}$ and let $\eta = (\eta_1, \ldots, \eta_J)$ be a probability vector, where all the weights $\eta_i$ are non-zero. For each cube $Q \in \mathcal{Q}_k$ ($\delta^k < C \operatorname{diam}(X)$), we consider $J$ distinguished central subcubes $Q(1), \ldots, Q(J) \prec Q$ whose distance to the complement of $Q$ is comparable to $\delta^k$. Note that this is possible, provided $\delta$ is small enough depending on $J$ and $\dim_L X$, see e.g. [10, Proof of Theorem 3.2]. Again, each boundary cube $Q' \prec Q$ inherits $p$ times the mass of its parent and the rest is distributed among the central child-cubes $Q(1), \ldots, Q(J)$ in the proportion of the weights $\eta_1, \ldots, \eta_J$. Denote the resulting measure by $\mu_{p,\eta}$. Note that the measures $\mu_p$ considered earlier correspond to the case $J = 1$ and the trivial probability vector $\eta = (1)$.

The proof of Lemma 3.2 still works in this setting and implies that

$$(p, \eta) \mapsto \dim_L \mu_{p,\eta} \text{ is continuous.} \tag{4.1}$$

Moreover, if e.g. $p \to 0$ and $\eta \to (1, 0, \ldots, 0)$, then it is very easy to see, by following a cube chain $Q^N \prec Q^{N-1} \ldots \prec Q^1 \prec Q^0$, where $Q^{i+1} = Q^i(1)$ for each $i = 0, \ldots, N-1$, that

$$\dim_L \mu_{p,\eta} \longrightarrow 0\,. \tag{4.2}$$

Finally, a result of Käenmäki and Lehrbäck [10] implies that given $s < \dim_L X$, it is possible to choose $\delta$, $J$, $p$, and $\eta$ so that

$$\dim_L \mu_{p,\eta} > s\,. \tag{4.3}$$

In fact, for (4.3), it is enough to consider the uniform probability vector $\eta = (\frac{1}{J}, \frac{1}{J}, \ldots, \frac{1}{J})$ for a suitably chosen $\delta > 0$, $J \in \mathbb{N}$, and $p$, see [10, Theorem 3.4]. Theorem 4.1 now follows by combining (4.1)–(4.3) and the intermediate value theorem for continuous functions.

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