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Variable Anisotropic Hardy Spaces with Variable Exponents

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Abstract: Let $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ be a variable exponent function satisfying the globally log-Hölder continuous and let Θ be a continuous multi-level ellipsoid cover of $\mathbb{R}^n$ introduced by Dekel et al. [12]. In this article, we introduce highly geometric Hardy spaces $H^{p(\cdot)}(\Theta)$ via the radial grand maximal function and then obtain its atomic decomposition, which generalizes that of Hardy spaces $H^p(\Theta)$ on $\mathbb{R}^n$ with pointwise variable anisotropy of Dekel et al. [16] and variable anisotropic Hardy spaces of Liu et al. [24]. As an application, we establish the boundedness of variable anisotropic singular integral operators from $H^{p(\cdot)}(\Theta)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ in general and from $H^{p(\cdot)}(\Theta)$ to itself under the moment condition, which generalizes the previous work of Bownik et al. [6] on $H^p(\Theta)$.

Keywords: Hardy space; continuous ellipsoid cover; maximal function; atomic decomposition; singular integral operator

MSC: 42B35, 42B30, 42B25, 42B20

1 Introduction

The main purpose of this article is to introduce and to investigate the variable anisotropic Hardy spaces $H^{p(\cdot)}(\Theta)$ with variable exponents. Due to the celebrated work [19] of Fefferman and Stein on classical isotropic Hardy spaces, there has been an increasing interest in extending classical Hardy spaces. In 2002, Bownik [3] investigated a special form of Hardy spaces $H^p(\mathbb{R}^n)$, i.e., anisotropic Hardy spaces $H^p_A(\mathbb{R}^n)$ defined over $\mathbb{R}^n$, where the Euclidian balls are replaced by images of the unit ball by powers of a fixed expansion matrix A. In 2011, Dekel et al. [16] introduced a more general Hardy space $H^p(\Theta)$ defined over $\mathbb{R}^n$, where the Euclidian balls are replaced by continuous ellipsoid cover Θ including the anisotropic Hardy space of Bownik [3]. Different from Bownik's spaces, the anisotropy in $H^p(\Theta)$ can change rapidly from point to point in $\mathbb{R}^n$ and from level to level in depth.

As we all know, the variable function spaces have found their applications in fluid dynamics [1, 2], image processing [9, 22], partial differential equations, variational calculus [8, 18] and harmonic analysis [10]. In 2012, Nakai and Sawano [27] introduced the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$, via the radial grand maximal function, and then obtained some real-variable characterizations of the space, such as the characterizations in terms of the atomic and the molecular decompositions. Then, in 2018, Liu et al. [24] introduced the variable anisotropic Hardy space $H^{p(\cdot)}_A(\mathbb{R}^n)$ defined over $\mathbb{R}^n$, where A is a general expansive matrix on $\mathbb{R}^n$, it extends the theory of Hardy spaces on $\mathbb{R}^n$ of Bownik [3].

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Inspired by Dekel et al. [16] and Liu et al. [24], the first goal is to further introduce the variable anisotropic Hardy space $H^{p(\cdot)}(\Theta)$ with variable exponent defined via the radial grand maximal function and then obtain its atomic decomposition.

The theory of singular integral operators plays an important role in harmonic analysis and partial differential equations; see, for example, [20, 21, 26, 30]. In the classical isotropic setting of $\mathbb{R}^n$ we consider Calderón-Zygmund operators T with regularity s of the form

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy, \qquad x \notin \operatorname{supp} f,\ f \in C_c^\infty(\mathbb{R}^n),$$

whose kernel $K(x,y)$ satisfies the bound

$$|\partial_y^\alpha K(x,y)| \le C|x-y|^{-n-|\alpha|} \qquad \text{for all } x \neq y \text{ and multi-indices } |\alpha| \le s. \tag{1.1}$$

It is well-known that operators T are bounded on isotropic Hardy spaces $H^p(\mathbb{R}^n)$ provided that $s > n(1/p-1)$ and T preserves vanishing moments $T^*(x^\alpha) = 0$ for $|\alpha| < s$, see [25, Proposition 7.4.4], [26, Theorem III.4]. Bownik [3] introduced anisotropic Calderón-Zygmund operators associated with expansive dilations and has shown their boundedness on anisotropic Hardy spaces, where the anisotropy is fixed and global on $\mathbb{R}^n$. An extension of these results to anisotropic Hardy spaces with variable exponents was done in [29].

Recently, Bownik et al. [6] further introduced the following class of singular integral operators adapted to variable anisotropy depending on a point $x \in \mathbb{R}^n$ and a scale $t \in \mathbb{R}$. Precisely, suppose that Θ is a continuous ellipsoid cover of Dekel et al. [12] consisting of ellipsoids $\theta_{x,t}$ with center $x \in \mathbb{R}^n$ and scale $t \in \mathbb{R}$ of the form $\theta_{x,t} = M_{x,t}(\mathbb{B}^n) + x$, where $M_{x,t}$ is a nonsingular matrix and $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$, see Definition 2.1. This ellipsoid cover Θ defines a space of homogeneous type [12] with quasi-distance ρ_Θ defined as infimum of ellipsoid volumes

$$\rho_\Theta(x,y) := \inf_{\theta\in\Theta} \{|\theta| : x, y \in \theta\}.$$

An anisotropic analogue of the bound (1.1). It takes the form

$$\left|\partial_y^\alpha [K(\cdot, M_{y,m}\cdot)](x, M_{y,m}^{-1}y)\right| \le C/\rho_\Theta(x,y) \qquad \text{for all } x \neq y \text{ and multi-indices } |\alpha| \le s, \tag{1.2}$$

where $m = -\log_2 \rho_\Theta(x,y)$. Then, they obtain the boundedness of the extended variable anisotropic singular integral operator T from $H^p(\Theta)$ to $L^p(\mathbb{R}^n)$ in general and from $H^p(\Theta)$ to itself under the moment condition.

Inspired by this, and also as an application of the atomic decomposition of $H^{p(\cdot)}(\Theta)$, the second goal is to extend these boundedness results to the variable exponents setting.

To be precise, this article is organized as follows.

In Section 2, we first recall notation, definitions and properties of continuous ellipsoid cover Θ, and quasi-distance ρ_Θ that are used throughout the paper. We also introduce the variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and some related properties. In Section 3, we define the variable Hardy space $H^{p(\cdot)}(\Theta)$ by means of the radial grand maximal function and show some relate properties.

In Section 4 via borrowing some ideas from those used in the proofs of [16, Theorems 4.4 and 4.19] and [24, Theorem 4.8], we obtain the atomic characterization of $H^{p(\cdot)}(\Theta)$. For this purpose, we first introduce the variable anisotropic atomic Hardy space $H^{p(\cdot)}_{q,l}(\Theta)$ (see Definition 4.2 below) and then prove

$$H^{p(\cdot)}(\Theta) = H^{p(\cdot)}_{q,l}(\Theta)$$

with equivalent quasi-norms (see Theorem 4.3 below). Indeed, we first present a key lemma (i.e. Lemma 4.4) about pointwise estimation of $M^\circ a$ which denotes the radial grand maximal function of $(p(\cdot), q, l)$-atom a. It is worth pointing out that the proof of Lemma 4.4 is motivated by the proofs of [16, Theorem 4.3] and [23, pp. 1686-1687], fortunately, the estimate plays an important role to simplify the proof of Theorem 4.3 below. Then, using this key lemma, the anisotropic Fefferman-Stein vector-valued inequality of the Hardy-Littlewood maximal operator M_Θ on $L^{p(\cdot)}(\mathbb{R}^n)$ (see Lemma 3.1 below) and Lemma 4.5 below, we prove that $H^{p(\cdot)}_{q,l}(\Theta)$ is continuously embedded into $H^{p(\cdot)}(\Theta)$. Conversely, we first present the density of the subset $H^{p(\cdot)}(\Theta)\cap L^q(\mathbb{R}^n)$ in $H^{p(\cdot)}(\Theta)$ for any $q\in(p_+,\infty)\cap[1,\infty)$ (see Proposition 4.7 below). By this density and the Calderón-Zygmund decomposition associated with radial grand maximal functions on anisotropic $\mathbb{R}^n$ from [16, Section 4.2] as well as an argument similar to that used in the proofs of [3, p. 38, Theorem 6.4], [16, Theorem 4.19] and [24, Theorem 4.8], we then prove that $H^{p(\cdot)}(\Theta)$ is continuously embedded into $H^{p(\cdot)}_{\infty,l}(\Theta)$ and hence also into $H^{p(\cdot)}_{q,l}(\Theta)$ due to the fact that each $(p(\cdot),\infty,l)$-atom is also a $(p(\cdot),q,l)$-atom for any $q\in(1,\infty)$.

Section 5 is devoted to showing the boundedness of variable anisotropic singular integral operators T from $H^{p(\cdot)}(\Theta)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ in general (see Theorem 5.7 below) or from $H^{p(\cdot)}(\Theta)$ to itself under the moment condition (see Theorem 5.8 below). Worthwhile to point out that the crucial pointwise estimate of $M^\circ Ta(x)$ simplifies the corresponding pointwise estimate of $M^\circ T\tilde{a}(x)$ in [6] even when $(p(\cdot),\infty)$-atom a reduced to (p,∞)-atom $\widetilde{a}$. Precisely, we draw more on the estimate of $M^\circ a(x)$ in the proof of Lemma 4.4.

Finally, we make some conventions on notation. Let $\mathbb{N}:=\{1,2,\dots\}$ and $\mathbb{N}_0:=\{0\}\cup\mathbb{N}$. For any $\alpha:=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}_0^n$, $|\alpha|:=\alpha_1+\cdots+\alpha_n$, and $\partial^\alpha:=(\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(\frac{\partial}{\partial x_n})^{\alpha_n}$. Throughout the whole paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $D\lesssim F$ means that $D\le CF$. If $D\lesssim F$ and $F\lesssim D$, we then write $D\sim F$. For any sets $E\subset\mathbb{R}^n$, we use $E^\complement$ to denote the set $\mathbb{R}^n\setminus E$. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of Schwartz functions, $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions, and $C^N(\mathbb{R}^n)$ the space of continuously differentiable functions of order N.

2 Preliminaries

In this section we recall some notion, notations and basic properties on continuous ellipsoid covers (see, for example, [12, 16]) and variable Lebesgue spaces (see, for example, [10, 11]), respectively. An *ellipsoid* θ in $\mathbb{R}^n$ is an image of the Euclidean unit ball $\mathbb{B}^n:=\{x\in\mathbb{R}^n:|x|<1\}$ under an affine transform, i.e.,

$$\theta:=M_\theta(\mathbb{B}^n)+c_\theta,$$

where M_θ is a nonsingular matrix and c_θ is the center.

Let us begin with the definition of continuous ellipsoid covers, which is from [12, Definition 2.4].

Definition 2.1. We say that

$$\Theta:=\{\theta_{x,t}:x\in\mathbb{R}^n,t\in\mathbb{R}\}$$

is a *continuous ellipsoid cover* of $\mathbb{R}^n$, or shortly a cover, if there exist constants $\mathbf{p}(\Theta):=\{a_1,\dots,a_6\}$ such that:

(i) For every $x\in\mathbb{R}^n$ and $t\in\mathbb{R}$, there exists an ellipsoid $\theta_{x,t}:=M_{x,t}(\mathbb{B}^n)+x$, where $M_{x,t}$ is a nonsingular matrix and x is the center, satisfying

$$a_1 2^{-t}\le|\theta_{x,t}|\le a_2 2^{-t}. \tag{2.1}$$

(ii) Intersecting ellipsoids from Θ satisfy a "shape condition", i.e., for any $x,y\in\mathbb{R}^n$, $t\in\mathbb{R}$ and $s\ge0$, if $\theta_{x,t}\cap\theta_{y,t+s}\ne\emptyset$, then

$$a_3 2^{-a_4 s}\le\frac{1}{\|(M_{y,t+s})^{-1}M_{x,t}\|}\le\|(M_{x,t})^{-1}M_{y,t+s}\|\le a_5 2^{-a_6 s}. \tag{2.2}$$

Here, $\|\cdot\|$ is the matrix norm of M given by $\|M\|:=\max_{|x|=1}|Mx|$.

There are many examples and results for ellipsoid cover in [7, 12–14]. Let us show one example from [12] to explain exactly why we need to range x as well in ellipsoid cover. Via suitable ellipsoid cover needing range x, Dahmen, Dekel and Petrushev showed a higher adaptive anisotropic Besov spaces (B-spaces) smoothness than their regular Besov space smoothness. To better illustrate this, in [12, Section 7.1], they showed that, for a suitable ellipsoid cover Θ, the B-space smoothness of the characteristic function of the unit ball $B(0,1)\subset\mathbb{R}^2$ in $\dot{B}^\alpha_{\tau\tau}(\Theta)$ is essentially $4/p$, while in the corresponding (classical isotropic) Besov spaces it is $2/p$. More strikingly, in the adaptive B-space scales $\dot{B}^{\alpha}_{\tau\tau}(\Theta)$, the smoothness of the characteristic function of any square $Q \subset \mathbb{R}^2$ is arbitrarily high, i.e., can be any $\alpha > 0$, while in the corresponding isotropic Besov spaces it is essentially $2/p$ (see [12, Section 7.2]). However, it is important to note that the cover Θ needed to describe that level of smoothness depends on α.

Next we collect results about ellipsoid covers from [12, 16] which will be used through the whole paper.

Proposition 2.2. *Let Θ be a continuous ellipsoid cover.*

(i) [16, Lemma 2.3] Then there exists $J := J(\mathbf{p}(\Theta)) \geq 1$ such that, for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

$$\theta_{x,t} \subset \frac{1}{2}\theta_{x,t-J}.$$

(ii) [12, Lemma 2.8] For any $x, y \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ with $t \leq s$, if $\theta_{x,t} \cap \theta_{y,s} \neq \emptyset$, there exists a constant $\gamma > 0$ such that

$$\theta_{y,s} \subset \theta_{x,t-\gamma}.$$

Definition 2.3. A *quasi-distance* on a set X is a mapping $\rho : X \times X \to [0, \infty)$ that satisfies the following conditions for all $x, y, z \in X$:

(i) $\rho(x, y) = 0 \Leftrightarrow x = y$;
(ii) $\rho(x, y) = \rho(y, x)$;
(iii) For some $\kappa \geq 1$,

$$\rho(x, y) \leq \kappa(\rho(x, z) + \rho(y, z)).$$

Dekel, Han, and Petrushev have shown that an ellipsoid cover Θ induces a quasi-distance ρ_Θ on $\mathbb{R}^n$, see [12, Proposition 2.7]. Moreover, $\mathbb{R}^n$ equipped with the quasi-distance ρ_Θ and the Lebesgue measure is a space of homogeneous type, [12, Proposition 2.10], which implies in the following Proposition 2.4.

Proposition 2.4. *Let Θ be a continuous ellipsoid cover.*

(i) [12, Proposition 2.7] The function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined by

$$\rho_\Theta(x, y) := \inf_{\theta \in \Theta} \{|\theta| : x, y \in \theta\}$$

is a quasi-distance on $\mathbb{R}^n$.

(ii) [12, Proposition 2.10] Let

$$B_{\rho_\Theta}(x, r) := \{y \in \mathbb{R}^n : \rho_\Theta(x, y) < r\}. \tag{2.3}$$

Then

$$|B_{\rho_\Theta}(x, r)| \sim r \qquad \text{for all } x \in \mathbb{R}^n,\ r > 0,$$

where the constants of equivalence depend only on $\mathbf{p}(\Theta)$.

Proposition 2.4(ii) shows that those balls induced by ρ_Θ satisfy 1-Ahlfors-regularity.

For any measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty]$, let

$$p_- := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} p(x), \qquad p_+ := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \qquad \underline{p} := \min\{p_-, 1\}. \tag{2.4}$$

Define $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot)$ satisfying $0 < p_- \leq p_+ < \infty$.

For any measurable function $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the *variable Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ denotes the set of measurable functions f on $\mathbb{R}^n$ such that, for some $\lambda > 0$,

$$\varrho_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0, \infty) : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

Let $C^{\log}(\mathbb{R}^n)$ be the set of all functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the *globally log-Hölder continuous condition*, namely, there exist $C_{\log}(p)$, $C_\infty \in (0, \infty)$ and $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \le \frac{C_{\log}(p)}{\log(e + \frac{1}{|x-y|})}$$

and

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)}.$$

Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, define the conjugate exponent $p'(\cdot)$ by the equation

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

Lemma 2.5. *[15, Theorem 3.2.7] Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space.*

The following Lemma 2.6 shows Hölder's inequality adapted to the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 2.6. *[10, Theorem 2.26] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies $1 \le p_-$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then $fg \in L(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \le C(p(\cdot))\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.7. *(i) [11, Lemma 2.3] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then, for any $s \in (0, \infty)$ and $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\||f|^s\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|^s_{L^{sp(\cdot)}(\mathbb{R}^n)}.$$

(ii) [11, Lemma 2.7] In addition, for any $\lambda \in \mathbb{C}$ and $f,\ g \in L^{p(\cdot)}(\mathbb{R}^n)$, it holds true that

$$\|\lambda f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = |\lambda|\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \text{ and } \|f + g\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} + \|g\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $\underline{p}$ is as in (2.4).

Proposition 2.8. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\{f_j\}_{j\in\mathbb{N}}$ are non-negative $L^{p(\cdot)}(\mathbb{R}^n)$ integrable functions. Then*

$$\left\|\sum_{j=1}^{\infty} f_j\right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \le \sum_{j=1}^{\infty} \|f_j\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $\underline{p}$ is as in (2.4).

Proof. For any $f_j \in L^{p(\cdot)}(\mathbb{R}^n)$, by Fatou's lemma on $L^{p(\cdot)}(\mathbb{R}^n)$ (see [10, Theorem 2.61]) and Lemma 2.7(ii), we have

$$\left\|\sum_{j=1}^{\infty} f_j\right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} = \left\|\lim_{n\to\infty}\sum_{j=1}^{n} f_j\right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \le \liminf_{n\to\infty}\left\|\sum_{j=1}^{n} f_j\right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\le \lim_{n\to\infty}\sum_{j=1}^{n} \|f_j\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} = \sum_{j=1}^{\infty} \|f_j\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

We finish the proof of Proposition 2.8. □

3 Variable Anisotropic Hardy Spaces with Variable Exponents

In this section we recall the definitions and properties of Hardy spaces with variable anisotropy which were originally introduced by Dekel, Petrushev and Weissblat [16].

Let Θ be a continuous ellipsoid cover. For any locally integrable function f on $\mathbb{R}^n$, the *Hardy-Littlewood maximal operators* $M_{B_{\rho_\Theta}}$ and M_Θ are defined, respectively, to be

$$M_{B_{\rho_\Theta}} f(x) := \sup_{r>0} \sup_{B_{\rho_\Theta}(y,r)\ni x} \frac{1}{|B_{\rho_\Theta}(y,r)|} \int_{B_{\rho_\Theta}(y,r)} |f(z)|\, dz \tag{3.1}$$

and

$$M_\Theta f(x) := \sup_{t\in\mathbb{R}} \sup_{\theta_{y,t}\ni x} \frac{1}{|\theta_{y,t}|} \int_{\theta_{y,t}} |f(z)|\, dz, \tag{3.2}$$

where $B_{\rho_\Theta}(y,r)$ is as in (2.3). By [16, Lemma 3.2], these two maximal functions are pointwise equivalent

$$M_{B_{\rho_\Theta}} f(x) \sim M_\Theta f(x) \qquad \text{for all } f \in L_{\text{loc}}(\mathbb{R}^n),\ x \in \mathbb{R}^n. \tag{3.3}$$

By Proposition 2.4, we know that $(\mathbb{R}^n, \rho, dx)$ is an RD-space. From this, [31, Theorem 2.7] and (3.3), we deduce the following Fefferman-Stein vector-valued inequality of the maximal operator M_Θ on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 3.1. *Let $r \in (1,\infty]$. Assume that $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfies $1 < p_- \le p_+ < \infty$. Then there exists a positive constant $C := C(r, n, p(\cdot))$ such that, for any sequence $\{f_k\}_{k\in\mathbb{N}}$ of measurable functions,*

$$\left\| \left\{ \sum_{k\in\mathbb{N}} [M_\Theta(f_k)]^r \right\}^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \left\| \left(\sum_{k\in\mathbb{N}} |f_k|^r \right)^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

with the usual modification made when $r = \infty$, where M_Θ denotes the Hardy-Littlewood maximal operator as in (3.2).

Definition 3.2. Let $N, \widetilde{N} \in \mathbb{N}_0$ with $N \le \widetilde{N}$ and

$$\mathcal{S}_{N,\widetilde{N}}(\mathbb{R}^n) := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) :\ \|\varphi\|_{N,\widetilde{N}} := \max_{|\alpha|\le N} \sup_{y\in\mathbb{R}^n} (1+|y|)^{\widetilde{N}} |\partial^\alpha \varphi(y)| \le 1 \right\}.$$

For each $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\theta_{x,t} = M_{x,t}(\mathbb{B}^n) + x \in \Theta$, denote

$$\varphi_{x,t}(y) := \varphi_{M_{x,t}}(y) := \left|\det(M_{x,t}^{-1})\right| \varphi(M_{x,t}^{-1} y).$$

Definition 3.3. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The *radial maximal function* of f is defined by

$$M_\varphi^\circ f(x) := \sup_{t\in\mathbb{R}} |f * \varphi_{x,t}(x)| \quad \text{for all } x \in \mathbb{R}^n.$$

For any $N, \widetilde{N} \in \mathbb{N}_0$ with $N \le \widetilde{N}$, the *radial grand maximal function* of f is defined by

$$M_{N,\widetilde{N}}^\circ f(x) := \sup_{\varphi\in\mathcal{S}_{N,\widetilde{N}}(\mathbb{R}^n)} M_\varphi^\circ f(x) \quad \text{for all } x \in \mathbb{R}^n.$$

By a proof similar to that of [4, Proposition 2.11(i)], we obtain the following lemma.

Lemma 3.4. *Let $N, \widetilde{N} \in \mathbb{Z}_+$ with $N \le \widetilde{N}$. If $q \in [1,\infty)$ and $f \in L^q(\mathbb{R}^n)$, then,*

$$|f(x)| \le M_{N,\widetilde{N}}^\circ f(x) \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{3.4}$$

where $M_{N,\widetilde{N}}^\circ f(x)$ is as in Definition 3.3.

Lemma 3.5. *[16, Theorem 3.8] For any cover Θ and $f \in L^q(\mathbb{R}^n)$ with $q \in [1, \infty)$, there exist constants $c_1, c_2 > 0$ depending on the parameters of the cover such that*

$$M^\circ f(x) \le c_1 \sup_{\varphi \in \mathcal{S}_{N,\widetilde{N}}(\mathbb{R}^n),\, \operatorname{supp}(\varphi) \subseteq \mathbb{B}^n} M^\circ_\varphi f(x), \quad x \in \mathbb{R}^n,$$

$$M^\circ f(x) \le c_2 M_\Theta f(x), \quad x \in \mathbb{R}^n.$$

Let Θ be a continuous ellipsoid cover of $\mathbb{R}^n$ with parameters $\mathbf{p}(\Theta) = \{a_1, \dots, a_6\}$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and let $\underline{p}$ be as in (2.4). We define $N_{p(\cdot)} := N_{p(\cdot)}(\Theta)$ as the minimal integer satisfying

$$N_{p(\cdot)}(\Theta) > \frac{\max\{1, a_4\} n + 1}{a_6 \underline{p}}, \tag{3.5}$$

and then $\widetilde{N}_{p(\cdot)} := \widetilde{N}_{p(\cdot)}(\Theta)$ as the minimal integer satisfying

$$\widetilde{N}_{p(\cdot)}(\Theta) > \frac{a_4 N_{p(\cdot)}(\Theta) + 1}{a_6}.$$

Definition 3.6. Let Θ be a continuous ellipsoid cover, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $M^\circ := M^\circ_{N_{p(\cdot)}, \widetilde{N}_{p(\cdot)}}$. The *variable anisotropic Hardy space with variable exponent* is defined as

$$H^{p(\cdot)}(\Theta) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : M^\circ f \in L^{p(\cdot)}(\mathbb{R}^n) \right\}$$

with the quasi-norm $\|f\|_{H^{p(\cdot)}(\Theta)} := \|M^\circ f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Lemma 3.7. *Let Θ be a continuous ellipsoid cover and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$.*

(i) The inclusion $H^{p(\cdot)}(\Theta) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ is continuous;
(ii) $H^{p(\cdot)}(\Theta)$ is complete.

Proof. (i) For any $f \in H^{p(\cdot)}(\Theta)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, by [17, Formula (5.8)], Lemma 2.6, Definition 3.6 and Lemma 2.7(i), we have

$$|\langle f, \varphi\rangle|^{\underline{p}} \lesssim \int_{\theta_{0,0}} (M^\circ f(x))^{\underline{p}}\, dx \lesssim \|(M^\circ f)^{\underline{p}}\|_{L^{p(\cdot)/\underline{p}}(\mathbb{R}^n)} \|\chi_{\theta_{0,0}}\|_{L^{(p(\cdot)/\underline{p})'}(\mathbb{R}^n)} \lesssim \|f\|^{\underline{p}}_{H^{p(\cdot)}(\Theta)},$$

which implies that the inclusion $H^{p(\cdot)}(\Theta) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ is continuous.

(ii) By (i) and repeating the proof of [11, Proposition 4.1], we can obtain Lemma 3.7(ii) holds true. □

4 Atomic Characterization of $H^{p(\cdot)}(\Theta)$

In this section, we establish the atomic characterization of variable anisotropic Hardy space $H^{p(\cdot)}(\Theta)$ with variable exponent. Motivated by Liu et al. [24, Definition 4.1] and Dekel et al. [16, Definition 4.1], we first introduce the definition of $(p(\cdot), q, l)$-atom.

Definition 4.1. For a continuous ellipsoid cover Θ, we say that $(p(\cdot), q, l)$ is *admissible* if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $1 < q \le \infty$ and $l \in [N_{p(\cdot)}, \infty) \cap \mathbb{N}_0$. An *anisotropic $(p(\cdot), q, l)$-atom* is a function $a : \mathbb{R}^n \to \mathbb{R}$ such that

(i) $\operatorname{supp} a \subset \theta_{x,t}$ for some $\theta_{x,t} \in \Theta$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$;
(ii) $\|a\|_{L^q(\mathbb{R}^n)} \le \frac{|\theta_{x,t}|^{1/q}}{\|\chi_{\theta_{x,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$;
(iii) $\int_{\mathbb{R}^n} a(y) y^\alpha\, dy = 0$ for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \le l$.

Definition 4.2. Let Θ be a continuous ellipsoid cover, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $(p(\cdot),\ q,\ l)$ an admissible triple as in Definition 4.1. The *variable anisotropic atomic Hardy space* $H^{p(\cdot)}_{q,\,l}(\Theta)$ associated with Θ is defined to be the set of all tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ of the form $f = \sum_{i=1}^{\infty} \lambda_i a_i$, where the series converges in $\mathcal{S}'(\mathbb{R}^n)$, $\{\lambda_i\}_i \subset \mathbb{C}$ and $\{a_i\}_i$ are $(p(\cdot),\ q,\ l)$-atoms respectively, supported, on $\{\theta_{x_i,\,t_i}\}_i \subset \Theta$. Moreover, the quasi-norm of $f \in H^{p(\cdot)}_{q,\,l}(\Theta)$ is defined by

$$\|f\|_{H^{p(\cdot)}_{q,\,l}(\Theta)} := \inf \left\| \left\{ \sum_i \left[\frac{|\lambda_i|\chi_{\theta_{x_i,\,t_i}}}{\|\chi_{\theta_{x_i,\,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where the infimum is taken over all admissible decompositions of f as above.

Now we state the main result of this section as follows.

Theorem 4.3. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $q \in (\max\{p_+, 1\}, \infty]$ with p_+ as in (2.4), and let Θ be a continuous ellipsoid cover and $(p(\cdot),\ q,\ l)$ an admissible triple as in Definition 4.1, then $H^{p(\cdot)}_{q,\,l}(\Theta) = H^{p(\cdot)}(\Theta)$ with equivalent quasi-norms.*

To establish the atomic characterization of $H^{p(\cdot)}(\Theta)$, we need several technical lemmas as follows.

Lemma 4.4. *Let Θ be a continuous ellipsoid cover, $(p(\cdot),\ q,\ l)$ an admissible triple as in Definition 4.1, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $\beta := \frac{\max\{1, a_4\} n+1}{\underline{p}}$, where $\underline{p}$ is as in (2.4). Then there exists a positive constant C, such that, for any $x \in (\theta_{z,\,t-J})^{\complement}$,*

$$M^{\circ} a(x) \le C \frac{[M_{\Theta}(\chi_{\theta_{z,\,t}})(x)]^{\beta}}{\|\chi_{\theta_{z,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

Proof. We show this lemma by borrowing some ideas from the proofs of [16, Theorem 4.3] and [23, pp. 1686-1687]. Let $\theta_{z,\,t}$ be the ellipsoid associated with an atom a, where $z \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We estimate the integral of the function $M^{\circ} a(x)$ on $x \in (\theta_{z,\,t-J})^{\complement}$, where J is from Proposition 2.2(i). By Definition 3.3 and Lemma 3.5, we estimate $\left|\int_{\mathbb{R}^n} a(y)\varphi_{x,\,s}(y)\,dy\right|$, where $\varphi \in \mathcal{S}_{N_{p(\cdot)},\,\widetilde{N}_{p(\cdot)}}(\mathbb{R}^n)$ with support in $\mathbb{B}^n$, $s \in \mathbb{R}$ and $x \in \theta_{z,\,t-kJ} \backslash \theta_{z,\,t-kJ+J}$, $k \ge 2$. It is easy to see that if $\theta_{z,\,t} \cap \theta_{x,\,s} = \emptyset$ then $\int_{\mathbb{R}^n} a(y)\varphi_{x,\,s}(y)dy = 0$. Thus, we may assume

$$\theta_{z,\,t} \cap \theta_{x,\,s} \ne \emptyset. \tag{4.1}$$

Suppose P is a polynomial of degree $N_{p(\cdot)} - 1$, by repeating the proof of [16, pp. 1077-1079], we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^n} a(y)\varphi_{x,\,s}(y)\,dy \right| &= |\det M^{-1}_{x,\,s}| \left| \int_{\mathbb{R}^n} a(y)\varphi(M^{-1}_{x,\,s}(x-y))\,dy \right| \\ &\lesssim 2^s \|a\|_{L^q(\mathbb{R}^n)} \left(\int_{\theta_{z,\,t}} \left|\varphi(M^{-1}_{x,\,s}(x-y)) - P(M^{-1}_{x,\,s}(x-y))\right|^{q'} dy \right)^{1/q'} \\ &\lesssim \frac{2^{(s-t)/q}}{\|\chi_{\theta_{z,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left|M^{-1}_{x,\,s}(x-\theta_{z,\,t})\right|^{1/q'} \sup_{y \in F(\theta_{z,\,t})} |\varphi(y) - P(y)|, \end{aligned} \tag{4.2}$$

where $1/q + 1/q' = 1$ and

$$F(\theta_{z,\,t}) := M^{-1}_{x,\,s}(x - (M_{z,\,t}(\mathbb{B}^n) + z)) = M^{-1}_{x,\,s}(x-z) - M^{-1}_{x,\,s} M_{z,\,t}(\mathbb{B}^n).$$

Moreover, by [16, Formula (4.6)], we also know that

$$F(\theta_{z,\,t}) \subset \left(1/2 M^{-1}_{x,\,s} M_{z,\,t-kJ+J}(\mathbb{B}^n)\right)^{\complement}. \tag{4.3}$$

Case 1. $t \le s$. We choose $P = 0$ and estimate the term $|M_{x,s}^{-1}M_{z,t}(\mathbb{B}^n)|^{1/q'}$. From (2.2) and (4.1), we deduce that

$$M_{x,s}^{-1}M_{z,t}(\mathbb{B}^n) \subset a_3^{-1}2^{a_4(s-t)}(\mathbb{B}^n),$$

which implies that

$$|M_{x,s}^{-1}M_{z,t}(\mathbb{B}^n)|^{1/q'} \lesssim 2^{a_4(s-t)n/q'}. \tag{4.4}$$

Repeating the estimate of [16, pp. 1078-1079], we conclude that

$$\sup_{y\in M_{x,s}^{-1}(x-\theta_{z,t})} \left|\varphi(y) - P(y)\right| = \sup_{y\in F(\theta_{z,t})} |\varphi(y)| \le \sup_{y\in F(\theta_{z,t})} (1+|y|)^{-N_{p(\cdot)}} \lesssim 2^{-a_6 N_{p(\cdot)}(s-t+kJ)}.$$

From this, (4.2), (3.5) and (4.4), we conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} a(y)\varphi_{x,s}(y)\,dy\right| &\lesssim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} 2^{\frac{(s-t)}{q}} 2^{a_4\frac{(s-t)n}{q'}} 2^{-a_6 N_{p(\cdot)}(s-t+kJ)} \\ &\lesssim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} 2^{(s-t)\left[\frac{1}{q}+\frac{a_4 n}{q'}-\frac{\max\{1,a_4\}n+1}{\underline{p}}\right]} 2^{-kJ\frac{\max\{1,a_4\}n+1}{\underline{p}}}.\end{aligned} \tag{4.5}$$

Since $q > 1$, $q' > 1$ and $\underline{p} \le 1$, we have

$$\frac{1}{q} + \frac{a_4 n}{q'} - \frac{\max\{1, a_4\}n + 1}{\underline{p}} < 0.$$

Therefore, the (4.5) over $s \ge t$ has the largest value when $s = t$ and hence

$$\left|\int_{\mathbb{R}^n} a(y)\varphi_{x,s}(y)\,dy\right| \lesssim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} 2^{-kJ\frac{\max\{1,a_4\}n+1}{\underline{p}}}. \tag{4.6}$$

Case 2. $t \ge s$. We estimate the term $\left|M_{x,s}^{-1}M_{z,t}(\mathbb{B}^n)\right|^{1/q'}$. By (4.1) and (2.2), we obtain

$$\left|M_{x,s}^{-1}M_{z,t}(\mathbb{B}^n)\right|^{1/q'} \lesssim 2^{-a_6(t-s)n/q'}. \tag{4.7}$$

Repeating the estimate of [16, pp. 1079-1080], we conclude that

$$\sup_{y\in F(\theta_{z,t})} \left|\varphi(y) - P(y)\right| \lesssim 2^{-a_6(t-s)N_{p(\cdot)}} \sup_{y\in F(\theta_{z,t})} (1+|y|)^{-N_{p(\cdot)}}. \tag{4.8}$$

We have two cases: $t - kJ + J \le s$ and $t - kJ + J \ge s$. We start with the first case. By (2.2), we have

$$|M_{x,s}^{-1}M_{z,t-kJ+J}| \lesssim 2^{-a_6(t-kJ-s)}.$$

Combining this and (4.3), it follows that

$$(1+|y|)^{-N_{p(\cdot)}} \lesssim 2^{a_6 N_{p(\cdot)}(t-kJ-s)}.$$

Therefore, by this and (4.8), we have

$$\sup_{y\in F(\theta_{z,t})} \left|\varphi(y) - P(y)\right| \lesssim 2^{-a_6(t-s)N_{p(\cdot)}} 2^{a_6 N_{p(\cdot)}(t-kJ-s)}.$$

Inserting this and (4.7) into (4.2) and using (3.5), we conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} a(y)\varphi_{x,s}(y)\,dy\right| &\lesssim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} 2^{\frac{(s-t)}{q}} 2^{-a_6(t-s)n\frac{1}{q'}} 2^{-a_6(t-s)N_{p(\cdot)}} 2^{a_6 N_{p(\cdot)}(t-s-kJ)} \\ &\lesssim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} 2^{(t-s)(-\frac{1}{q}-\frac{a_6 n}{q'})} 2^{-N_{p(\cdot)}a_6 kJ}\end{aligned} \tag{4.9}$$

$$\lesssim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} 2^{(t-s)(-\frac{1}{q}-\frac{a_6 n}{q'})} 2^{\frac{\max\{1,a_4\}n+1}{\underline{p}}(-kJ)}.$$

Since $t-s \geq 0$ and $-\frac{1}{q}-\frac{a_6 n}{q'} < 0$. Therefore, the (4.9) over $s \leq t$ has the largest value when $s = t$ and hence

$$\left|\int_{\mathbb{R}^n} a(y)\varphi_{x,s}(y)\,dy\right| \lesssim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} 2^{-kJ\frac{\max\{1,a_4\}n+1}{\underline{p}}}. \tag{4.10}$$

For the second case, when $t-kJ+J \geq s$, inserting (4.7) and (4.8) into (4.2), and using (3.5), we conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} a(y)\varphi_{x,s}(y)\,dy\right| &\lesssim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} 2^{\frac{1}{q}(s-t)} 2^{-a_6(t-s)n\frac{1}{q'}} 2^{-a_6(t-s)N_{p(\cdot)}}\\ &\lesssim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} 2^{(t-s)(-\frac{1}{q}-\frac{a_6 n}{q'}-a_6 N_{p(\cdot)})}\\ &\lesssim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} 2^{(kJ-J)(-\frac{1}{q}-\frac{a_6 n}{q'}-a_6 N_{p(\cdot)})}\\ &\lesssim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} 2^{kJ(-\frac{1}{q}-\frac{a_6 n}{q'}-\frac{\max\{1,a_4\}n+1}{\underline{p}})}.\end{aligned}$$

Since

$$2^{kJ(-\frac{1}{q}-\frac{a_6 n}{q'}-\frac{\max\{1,a_4\}n+1}{\underline{p}})} \leq 2^{-kJ\frac{\max\{1,a_4\}n+1}{\underline{p}}},$$

we choose the upper estimate of the first case, i.e., $t-kJ+J \leq s$.

Let

$$\beta := \frac{\max\{1,a_4\}n+1}{\underline{p}}.$$

From (4.6), (4.10), (3.2) and $x \in \theta_{z,t-kJ-\gamma}$, we deduce that

$$\begin{aligned}M^{\circ}a(x) &\lesssim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} 2^{-kJ\beta} \sim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left(\frac{1}{|\theta_{x,t-kJ-\gamma}|}\int_{\theta_{x,t-kJ-\gamma}} \chi_{\theta_{z,t}}(y)\,dy\right)^{\beta}\\ &\lesssim \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} [M_{\Theta}(\chi_{\theta_{z,t}})(x)]^{\beta},\end{aligned}$$

where $\underline{p}$ is as in (2.4). Therefore, we finish the proof of Lemma 4.4. □

The following Lemma 4.5 is essentially from [28, Theorem 1.1], which plays an important role in this section and is also of independent interest. Since its proof remains the same with that of [28, Theorem 1.1], the details being omitted.

Lemma 4.5. *Let $r(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $q \in [1,\infty] \cap (r_+,\infty]$ with r_+ as in (2.4). Assume that $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$, $\{\theta_{x_i,t_i}\}_{i\in\mathbb{N}} \subset \Theta$ and $\{a_i\}_{i\in\mathbb{N}} \subset L^q(\mathbb{R}^n)$ satisfy, for any $i \in \mathbb{N}$, $\operatorname{supp} a_i \subset \theta_{x_i,t_i}$,*

$$\|a_i\|_{L^q(\mathbb{R}^n)} \leq \frac{|\theta_{x_i,t_i}|^{1/q}}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{r(\cdot)}(\mathbb{R}^n)}}$$

and

$$\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\chi_{\theta_{x_i,t_i}}}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{r(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{r}}\right\}^{1/\underline{r}}\right\|_{L^{r(\cdot)}(\mathbb{R}^n)} < \infty.$$

Then

$$\left\|\left[\sum_{i\in\mathbb{N}}|\lambda_i a_i|^{\underline{r}}\right]^{\frac{1}{\underline{r}}}\right\|_{L^{r(\cdot)}(\mathbb{R}^n)} \leq C\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\chi_{\theta_{x_i,t_i}}}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{r(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{r}}\right\}^{1/\underline{r}}\right\|_{L^{r(\cdot)}(\mathbb{R}^n)},$$

where C is a positive constant independent of λ_i, a_i and θ_{x_i,t_i}.

We say that an ellipsoid cover Θ is *pointwise continuous* if for every $t \in \mathbb{R}$, the matrix valued function $x \mapsto M_{x,t}$ is continuous. That is,

$$\|M_{x',t} - M_{x,t}\| \to 0 \text{ as } x' \to x. \tag{4.11}$$

The condition (4.11) is implicitly used in [16] to guarantee that the superlevel set Ω corresponding to the grand maximal function, which is given by (4.13), is open. In this paper, since it is always possible to construct an equivalent ellipsoid cover

$$\Xi := \{\xi_{x,\,t} : x \in \mathbb{R}^n, t \in \mathbb{R}\}$$

such that Ξ is pointwise continuous and Ξ is equivalent to Θ (see [6, Theorem 2.2]). We say that two ellipsoid covers Θ and Ξ are *equivalent* if there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we have

$$\frac{1}{C}\xi_{x,t} \subset \theta_{x,t} \subset C\xi_{x,t}. \tag{4.12}$$

Therefore, in this paper, we always assume that the ellipsoid cover is pointwise continue.

Now, let us recall the Calderón-Zygmund decomposition established in [16]. Throughout this section for a given continuous ellipsoid cover Θ, we consider $f \in L^q(\mathbb{R}^n)$, $q \geq 1$, for every $\lambda > 0$, let

$$\Omega := \{x : M^\circ f(x) > \lambda\}. \tag{4.13}$$

We shall assume that Θ is pointwise continuous, that (4.11) holds. By [6, Lemma 3.7], the set Ω is open. Since M° is bounded from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$ and bounded on $L^r(\mathbb{R}^n)$, $r > 1$(see [16, Theorem 3.8]), then, we have

$$\left|\{x : M^\circ f(x) > \lambda\}\right| < \infty,$$

where M° is the grand maximal function as in Definition 3.6.

By [16, Section 4.2 (Whitney covering lemma)], there exist sequences $\{x_i\}_{i\in\mathbb{N}} \subset \Omega$ and $\{t_i\}_{i\in\mathbb{N}}$, such that

$$\Omega = \bigcup_{i\in\mathbb{N}} \theta_{x_i,\,t_i}, \tag{4.14}$$

$$\theta_{x_i,\,t_i+\gamma} \cap \theta_{x_j,\,t_j+\gamma} = \emptyset \qquad \forall\, i \neq j, \tag{4.15}$$

$$\theta_{x_i,\,t_i-J-2\gamma} \cap \Omega^{\complement} = \emptyset \qquad \forall\, i \in \mathbb{N}, \tag{4.16}$$

$$\theta_{x_i,\,t_i-J-2\gamma-1} \cap \Omega^{\complement} \neq \emptyset \qquad \forall\, i \in \mathbb{N}, \tag{4.17}$$

where J and γ are as in Proposition 2.2. Moreover, there exists a constant $L > 0$ such that

$$\sharp\{j \in \mathbb{N}_0 : \theta_{x_j,\,t_j-J-\gamma} \cap \theta_{x_i,\,t_i-J-\gamma} \neq \emptyset\} \leq L \qquad \forall\, i \in \mathbb{N}, \tag{4.18}$$

where $\sharp E$ denotes the cardinality of a set E.

Fix $\phi \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\phi \subset 2\mathbb{B}^n$, $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on $\mathbb{B}^n$. For every $i \in \mathbb{N}_0$, define $\widetilde{\phi_i} := \phi(M^{-1}_{x_i,\,t_i}(x-x_i))$. Obviously, $\widetilde{\phi_i} \equiv 1$ on $\theta_{x_i,\,t_i}$. By Proposition 2.2(i), we have $\operatorname{supp}\widetilde{\phi_i} \subset x_i + 2M_{x_i,\,t_i}(\mathbb{B}^n) \subset \theta_{x_i,\,t_i-J}$. For every $i \in \mathbb{N}_0$, define

$$\phi_i(x) := \begin{cases} \frac{\widetilde{\phi}_i(x)}{\sum_j \widetilde{\phi}_j(x)}, & \text{if } x \in \Omega, \\ 0, & \text{if } x \notin \Omega. \end{cases} \tag{4.19}$$

Observe that ϕ_i is well defined since by (4.14) and (4.18), $1 \leq \sum_i \widetilde{\phi_i}(x) \leq L$ for every $x \in \Omega$. Also $\phi_i \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp}\phi_i \subset \theta_{x_i,\,t_i-J}$. By (4.14) and (4.19), we have $\sum_i \phi_i(x) = \chi_\Omega(x)$, which implies that the family $\{\phi_i\}_{i\in\mathbb{N}}$ forms a smooth partition of unitary subordinate to the cover of Ω by the ellipsoids $\{\theta_{x_i,\,t_i-J}\}_{i\in\mathbb{N}}$.

Let $\mathbb{P}_l(\mathbb{R}^n)$ denote the space of polynomials of n variables with degree $\leq l$, where $N_{p(\cdot)} \leq l$, see (3.5). For each $i \in \mathbb{N}_0$ we introduce an Hilbert space structure on the space $\mathbb{P}_l(\mathbb{R}^n)$ by setting

$$\langle P, Q\rangle_i := \frac{1}{\int \phi_i} \int_{\mathbb{R}^n} P(x)Q(x)\phi_i(x)\,dx \quad \text{for any } P,\ Q \in \mathbb{P}_l(\mathbb{R}^n). \tag{4.20}$$

The distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ induces a linear functional on $\mathbb{P}_l(\mathbb{R}^n)$ given by

$$\mathbb{P}_l(\mathbb{R}^n) \ni Q \mapsto \langle f, Q\rangle_i.$$

By Riesz's lemma it is represented by a unique polynomial $P_i \in \mathbb{P}_l(\mathbb{R}^n)$ such that

$$\langle f, Q\rangle_i = \langle P_i, Q\rangle_i \quad \text{for any } Q \in \mathbb{P}_l(\mathbb{R}^n).$$

Definition 4.6. For every $i \in \mathbb{N}_0$, define the locally "bad part" $b_i := (f - P_i)\phi_i$ and the "good part" $g := f - \sum_i b_i$. The representation $f = g + \sum_i b_i$, where g and b_i are as above, is a Calderón-Zygmund decomposition of degree l and height λ associated with M°.

Proposition 4.7. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (p_+, \infty) \cap [1, \infty)$ with p_+ as in (2.4). Then $H^{p(\cdot)}(\Theta) \cap L^q(\mathbb{R}^n)$ is dense in $H^{p(\cdot)}(\Theta)$.*

To prove Proposition 4.7, we will use the following three results which are from [16, Lemma 4.13, Lemma 4.9 and Lemma 4.10]. Lemma 4.8 gives control for the good part g. Lemmas 4.9 and 4.10 show the estimates for the bad parts b_i.

Lemma 4.8. *[16, Lemma 4.13] Suppose $\sum_i b_i$ converges in $\mathcal{S}'(\mathbb{R}^n)$. Then there exists a positive constant C, independent of $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\lambda > 0$, such that*

$$M^\circ g(x) \le C\lambda \sum_i v^{-k_i(x)} + M^\circ f(x)\chi_{\Omega^\complement}(x),$$

where $v := 2^{a_6 JN}$ and

$$k_i(x) := \begin{cases} k, & x \in \theta_{x_i,\, t_i - J(k+2)} \backslash \theta_{x_i,\, t_i - J(k+1)} \ \text{ for some } \ k \in \mathbb{N}_0, \\ 0, & x \in \theta_{x_i,\, t_i - J}. \end{cases}$$

Lemma 4.9. *[16, Lemma 4.9] There exists a positive constant C such that*

$$M^\circ b_i(x) \le CM^\circ f(x) \qquad \textit{for any } x \in \theta_{x_i,\, t_i - J}.$$

Lemma 4.10. *[16, Lemma 4.10] There exists a positive constant C such that, for all $i \in \mathbb{N}$, $k \ge 0$ and $x \in \theta_{x_i,\, t_i - J(k+2)} \backslash \theta_{x_i,\, t_i - J(k+1)}$,*

$$M^\circ b_i(x) \le C\lambda v^{-k},$$

where $v := 2^{a_6 JN}$.

Proof of Proposition 4.7. From Lemma 4.8, we know that there exists a positive constant C, independent of $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\lambda > 0$, such that

$$M^\circ g^\lambda(x) \le C\lambda \sum_i v^{-k_i(x)} + M^\circ f(x)\chi_{\Omega^\complement}(x).$$

For any $x \in \theta_{x_i,\, t_i - J(k+2)} \backslash \theta_{x_i,\, t_i - J(k+1)}$ with $k \in \mathbb{N}_0$, by (3.3), we have

$$2^{-kJ} \le C \frac{1}{|\theta_{x_i,\, t_i - J(k+2)}|} \int\limits_{\theta_{x_i,\, t_i - J(k+2)}} \chi_{\theta_{x_i,\, t_i}}(y)\, dy \le CM_\Theta(\chi_{\theta_{x_i,\, t_i}})(x),$$

where M_Θ is as in (3.2). Then, we have

$$M^\circ g^\lambda(x) \lesssim \lambda \sum_i \left[M_\Theta(\chi_{\theta_{x_i,\, t_i}})(x)\right]^{a_6 N} + M^\circ f(x)\chi_{\Omega^\complement}(x). \tag{4.21}$$

Then, by a proof similar to that of [31, Lemma 4.8] with [31, Proposition 4.7 and Theorem 2.7] replaced by (4.21), Lemmas 4.9, 4.10 and 3.1, we finish the proof of Proposition 4.7. □

Following [16, Section 4.3], for each $k \in \mathbb{Z}$, we consider the Calderón-Zygmund decomposition of f of degree $l \geq N_{p(\cdot)}$ at height 2^k associated with M°,

$$f = g^k + \sum_i b_i^k, \tag{4.22}$$

where

$$\Omega^k := \{x : M^\circ f > 2^k\}, \qquad b_i^k := (f - P_i^k)\phi_i^k \quad \text{and} \qquad \theta_i^k := \theta_{x_i^k, t_i^k}.$$

Here, sequences $\{x_i^k\}_{i\in\mathbb{N}} \subset \Omega^k$ and $\{t_i^k\}_{i\in\mathbb{N}} \subset \mathbb{R}$ satisfy (4.14)-(4.18) for Ω^k, functions $\{\phi_i^k\}_{i\in\mathbb{N}}$ are defined as in (4.19), and polynomials $\{P_i^k\}_{i\in\mathbb{N}}$ are projections of f onto $\mathbb{P}_l(\mathbb{R}^n)$ with respect to the inner product given by (4.19).

Let $l \in \mathbb{N}$ with $l \geq N_{p(\cdot)}$. For each $i \in \mathbb{N}$ and $P,\ Q \in \mathbb{P}_l(\mathbb{R}^n)$, define

$$\langle P, Q\rangle_{i,k} := \frac{1}{\int_{\mathbb{R}^n} \phi_i^k(x)\,dx} \int_{\mathbb{R}^n} P(x)Q(x)\phi_i^k(x)\,dx, \tag{4.23}$$

which induces a finite dimensional Hilbert space $(\mathbb{P}_l(\mathbb{R}^n), \langle\cdot,\cdot\rangle_{i,k})$. The distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ induces a linear functional on $\mathbb{P}_l(\mathbb{R}^n)$ by

$$Q \mapsto \langle f, Q\rangle_{i,k} \ \text{ for any } \ Q \in \mathbb{P}_l(\mathbb{R}^n),$$

which by Riesz's lemma is represented by a unique polynomial $P_i^k \in \mathbb{P}_l(\mathbb{R}^n)$ such that

$$\langle f, Q\rangle_{i,k} = \langle P_i^k, Q\rangle_{i,k} \ \text{ for any } \ Q \in \mathbb{P}_l(\mathbb{R}^n). \tag{4.24}$$

Obviously P_i^k is the orthogonal projection of f onto $\mathbb{P}_l(\mathbb{R}^n)$ with respect to the inner product induced by (4.23). That is, P_{ij}^{k+1} is the unique polynomial in $\mathbb{P}_l(\mathbb{R}^n)$ such that, for all $Q \in \mathbb{P}_l(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (f(y) - P_j^{k+1}(y))\phi_i^k(y)Q(y)\phi_j^{k+1}(y)\,dy = \int_{\mathbb{R}^n} P_{ij}^{k+1}(y)Q(y)\phi_j^{k+1}(y)\,dy. \tag{4.25}$$

In particular, if $\theta_{x_i^k, t_i^k - J} \cap \theta_{x_j^{k+1}, t_j^{k+1} - J} = \emptyset$, then $P_{ij}^{k+1} = 0$.

For each $k \in \mathbb{Z}$, define the index set

$$I_k := \{(i,j) \in \mathbb{N}_0 : \theta_{x_i^k, t_i^k - J} \cap \theta_{x_j^{k+1}, t_j^{k+1} - J} \neq \emptyset\}.$$

The following Lemmas 4.11, 4.13 and 4.14 show some properties of the smooth partition of unity ϕ_i^k. Lemma 4.12 gives some results for these ellipsoids from the Whitney covering lemma. These lemmas play an important role in the proof of $H^{p(\cdot)}(\Theta) \subset H_{q,l}^{p(\cdot)}(\Theta)$.

Lemma 4.11. *[16, Lemma 4.8] There exists a positive constant C such that*

$$\sup_{y\in\mathbb{R}^n} |P_i^k(y)\phi_i^k(y)| \leq C2^k.$$

Lemma 4.12. *[16, Lemma 4.16] The following holds for any $k \in \mathbb{Z}$.*

(i) For any $(i,j) \in I_k$, we have $\theta_{x_j^{k+1}, t_j^{k+1} - J} \subset \theta_{x_i^k, t_i^k - J - 3\gamma - 1}$,

(ii) There exists $L' > 0$, which does not depend on k, such that

$$\sharp\{i \in \mathbb{N}_0 : (i,j) \in I_k\} \leq L' \qquad \text{for any } j \in \mathbb{N}_0.$$

Lemma 4.13. *[16, Lemma 4.17] There exists a constant $C > 0$, such that for every $i,\ j \in \mathbb{N}_0$ and $k \in \mathbb{Z}$,*

$$\sup_{x\in\mathbb{R}^n} \left|P_{ij}^{k+1}(x)\phi_j^{k+1}(x)\right| \leq C2^{k+1}.$$

Moreover, $P_{ij}^{k+1} = 0$ if $(i,j) \notin I_k$.

Lemma 4.14. *[16, Lemma 4.18] Let $k \in \mathbb{Z}$. Then $\sum_{i\in\mathbb{N}}(\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\phi_j^{k+1}) = 0$, where the series converges pointwise and in $\mathcal{S}'(\mathbb{R}^n)$.*

Proof of Theorem 4.3. First, we show that

$$H_{q,l}^{p(\cdot)}(\Theta) \subset H^{p(\cdot)}(\Theta). \tag{4.26}$$

Let $f \in H_{q,l}^{p(\cdot)}(\Theta)$. By Definition 4.2, we know that there exist $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(p(\cdot),\ q,\ l)$-atoms $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, on $\{\theta_{x_i,\,t_i}\}_{i\in\mathbb{N}} \subset \Theta$ such that

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \qquad \text{in } \mathcal{S}'(\mathbb{R}^n) \tag{4.27}$$

and

$$\|f\|_{H_{q,l}^{p(\cdot)}(\Theta)} \sim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{\theta_{x_i,\,t_i}}}{\|\chi_{\theta_{x_i,\,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{4.28}$$

Fix an $x \in \mathbb{R}^n$ for the time being. By (4.27) and Lemma 4.4, we have

$$\begin{aligned} M^\circ f(x) &\le \sum_{i\in\mathbb{N}} |\lambda_i| M^\circ(a_i)(x)\chi_{\theta_{z,\,t-J}}(x) + \sum_{i\in\mathbb{N}} |\lambda_i| M^\circ(a_i)(x)\chi_{(\theta_{z,\,t-J})^\complement}(x) \\ &\lesssim \left\{ \sum_{i\in\mathbb{N}} [|\lambda_i| M^\circ(a_i)(x)\chi_{\theta_{z,\,t-J}}(x)]^{\underline{p}} \right\}^{1/\underline{p}} + \sum_{i\in\mathbb{N}} |\lambda_i| \frac{1}{\|\chi_{\theta_{z,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} [M_\Theta(\chi_{\theta_{z,\,t}})(x)]^\beta \\ &=: \mathrm{I}_1 + \mathrm{I}_2, \end{aligned}$$

where

$$\beta = \frac{\max\{1, a_4\}n + 1}{\underline{p}} > \frac{1}{\underline{p}} \tag{4.29}$$

and M_Θ denotes the Hardy-Littlewood maximal operator as in (3.2).

For the term I_1, from the $L^q(\mathbb{R}^n)$ boundedness of M° (since $M^\circ f(x) \lesssim M_\Theta f(x)$ (see [16, Lemma 3.2 and Theorem 3.8(ii)])) and $M_\Theta f(x)$ is bounded on $L^q(\mathbb{R}^n)$ and Definition 4.1(ii), we conclude that

$$\left\| M^\circ(a_i)\chi_{\theta_{z,\,t-J}} \right\|_{L^q(\mathbb{R}^n)} \lesssim \|a_i\|_{L^q(\mathbb{R}^n)} \lesssim \frac{|\theta_{x_i,\,t_i}|^{1/q}}{\|\chi_{\theta_{x_i,\,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}. \tag{4.30}$$

By Lemma 2.7(i), Lemma 3.1 and the fact that, for any $\{\theta_{x_i,\,t_i}\}_{i\in\mathbb{N}} \subset \Theta$ and $r \in (0,\ \underline{p})$,

$$\chi_{\theta_{x_i,\,t_i-J}} \le \frac{a_2}{a_1} 2^{J/r} [M_\Theta(\chi_{\theta_{x_i,\,t_i}})]^{1/r}, \tag{4.31}$$

we obtain

$$\begin{aligned} \left\| \left\{ \sum_{i\in\mathbb{N}} \chi_{\theta_{x_i,\,t_i-J}}^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le \left\| \left\{ \sum_{i\in\mathbb{N}} \left(\frac{a_2}{a_1} 2^{J/r} [M_\Theta(\chi_{\theta_{x_i,\,t_i}})] \right)^{\underline{p}/r} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \left\| \left\{ \sum_{i\in\mathbb{N}} [M_\Theta(\chi_{\theta_{x_i,\,t_i}})]^{\underline{p}/r} \right\}^{r/\underline{p}} \right\|_{L^{p(\cdot)/r}(\mathbb{R}^n)}^{1/r} \\ &\lesssim \left\| \left\{ \sum_{i\in\mathbb{N}} [\chi_{\theta_{x_i,\,t_i}}]^{\underline{p}/r} \right\}^{r/\underline{p}} \right\|_{L^{p(\cdot)/r}(\mathbb{R}^n)}^{1/r} \\ &\lesssim \left\| \left\{ \sum_{i\in\mathbb{N}} \chi_{\theta_{x_i,\,t_i}}^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{4.32}$$

Combining this and $f \in H^{p(\cdot)}_{q,l}(\Theta)$, we have

$$\begin{aligned} &\left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{\theta_{x_i,t_i-J}}}{\|\chi_{\theta_{x_i,t_i-J}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\quad \lesssim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{\theta_{x_i,t_i}}}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty. \end{aligned} \tag{4.33}$$

Thus, by Lemma 4.5 with (4.30) and (4.33), and (4.28), we obtain

$$\begin{aligned} \|\mathrm{I}_1\|_{L^{p(\cdot)}(\mathbb{R}^n)} &= \left\| \left\{ \sum_{i\in\mathbb{N}} \left[|\lambda_i| M^\circ(a_i)\chi_{\theta_{z,t-J}}\right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{\theta_{x_i,t_i}}}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{H^{p(\cdot)}_{q,l}(\Theta)}. \end{aligned} \tag{4.34}$$

To deal with I_2, by Lemma 2.7(i), Lemma 3.1 and (4.29), we have

$$\begin{aligned} \|\mathrm{I}_2\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\sim \left\| \sum_{i\in\mathbb{N}} |\lambda_i| \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} [M_\Theta(\chi_{\theta_{z,t}})]^\beta \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\sim \left\| \left\{ \sum_{i\in\mathbb{N}} |\lambda_i| \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} [M_\Theta(\chi_{\theta_{z,t}})]^\beta \right\}^{1/\beta} \right\|^\beta_{L^{\beta p(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \left\| \left\{ \sum_{i\in\mathbb{N}} |\lambda_i| \frac{1}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \chi^\beta_{\theta_{z,t}} \right\}^{1/\beta} \right\|^\beta_{L^{\beta p(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{\theta_{z,t}}}{\|\chi_{\theta_{z,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{H^{p(\cdot)}_{q,l}(\Theta)}, \end{aligned} \tag{4.35}$$

which together with Definition 3.6 and (4.34) implies that

$$\|f\|_{H^{p(\cdot)}(\Theta)} \sim \|M^\circ f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}_{q,l}(\Theta)}.$$

Thus, (4.26) holds true.

We now prove that $H^{p(\cdot)}(\Theta) \subset H^{p(\cdot)}_{q,l}(\Theta)$. To this end, it suffices to show that

$$H^{p(\cdot)}(\Theta) \subset H^{p(\cdot)}_{\infty,l}(\Theta), \tag{4.36}$$

due to the fact that each $(p(\cdot), \infty, l)$-atom is also a $(p(\cdot), q, l)$-atom and hence

$$H^{p(\cdot)}_{\infty,l}(\Theta) \subset H^{p(\cdot)}_{q,l}(\Theta).$$

Next we prove (4.36) by two steps.

Step 1. In this step, we show that, for any

$$f \in H^{p(\cdot)}(\Theta) \cap L^q(\mathbb{R}^n),\ q \in (\max\{p_+, 1\}, \infty],$$

$$\|f\|_{H^{p(\cdot)}_{\infty,l}(\Theta)} \lesssim \|f\|_{H^{p(\cdot)}(\Theta)} \tag{4.37}$$

holds true.

To prove (4.37), we borrow some ideas from those used in the proofs of [24, Theorem 4.8] and [16, Theorem 4.19]. Let $f \in H^{p(\cdot)}(\Theta) \cap L^q(\mathbb{R}^n)$. For each $k \in \mathbb{Z}$, we consider the Calderón-Zygmund decomposition of f of degree $l \geq N_{p(\cdot)}$ at height 2^k associated with M°, $f = g^k + \sum_{i\in\mathbb{N}} b_i^k$. From this, the definition of b_i^k and $\sum_{i\in\mathbb{N}} \phi_i^k = \chi_{\Omega_k}$, it follows that

$$g^k = f - \sum_i b_i^k = f - \sum_i [f - P_i^k]\phi_i^k = f\chi_{\Omega_k^\complement} - \sum_i P_i^k \phi_i^k.$$

By this, (3.4), Lemma 4.11 and (4.18) we have

$$\|g^k\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^k \text{ and } \|g^k\|_{L^\infty(\mathbb{R}^n)} \to 0 \text{ as } k \to -\infty. \tag{4.38}$$

Notice that $f \in L^q(\mathbb{R}^n) = H^q(\Theta)$ with $q \in (\max\{p_+, 1\}, \infty)$ (see [16, p. 1075]). Then, repeating the proof of [16, Lemma 4.11] with some slight modifications, we find that, for any $k \in \mathbb{Z}$, $\left\{\sum_{i=1}^m b_i^k\right\}_{m\in\mathbb{N}}$ converges in $L^q(\mathbb{R}^n)$ and hence converges in $\mathcal{S}'(\mathbb{R}^n)$. By this, Lemma 4.9, Lemma 4.10, (4.14), (4.16), (4.18) and (3.2), we conclude that, for any $k \in \mathbb{Z}$, $x \in \mathbb{R}^n$ and $r \in (0, \underline{p})$ close to $\underline{p}$,

$$\begin{aligned}
M^\circ\left(\sum_{i\in\mathbb{N}} b_i^k\right)(x) &\leq \sum_{i\in\mathbb{N}} M^\circ(b_i^k)(x)\chi_{\theta_{x_i^k,\, t_i^k-J}}(x) + \sum_{i\in\mathbb{N}} M^\circ(b_i^k)(x)\chi_{(\theta_{x_i^k,\, t_i^k-J})^\complement}(x) \\
&\lesssim M^\circ(f)(x)\chi_{\Omega_k}(x) + 2^k \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} v^{-j}\chi_{\theta_{x_i^k,\, t_i^k-J(j+1)}\setminus\theta_{x_i^k,\, t_i^k-Jj}}(x) \\
&\lesssim M^\circ(f)(x)\chi_{\Omega_k}(x) + 2^k \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} 2^{-a_6 N_{p(\cdot)}Jj}\left[\chi^r_{\theta_{x_i^k,\, t_i^k-J(j+1)}}(x)\right]^{1/r} \\
&\lesssim M^\circ(f)(x)\chi_{\Omega_k}(x) + 2^k \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} 2^{-(a_6 N_{p(\cdot)}Jj - J(j+1)/r)}\left\{\frac{1}{|\theta_{x_i^k,\, t_i^k-J(j+1)-\gamma}|}\int_{\theta_{x_i^k,\, t_i^k-J(j+1)-\gamma}} \chi_{\theta_{x_i^k,\, t_i^k}}(y)\,dy\right\}^{1/r} \\
&\lesssim M^\circ(f)(x)\chi_{\Omega_k}(x) + 2^k \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} 2^{-Jj(a_6 N_{p(\cdot)}-1/r)}\left[M_\Theta\left(\chi_{\theta_{x_i^k,\, t_i^k}}\right)(x)\right]^{1/r} \\
&\lesssim M^\circ(f)(x)\chi_{\Omega_k}(x) + 2^k \sum_{i\in\mathbb{N}}\left[M_\Theta\left(\chi_{\theta_{x_i^k,\, t_i^k}}\right)(x)\right]^{1/r},
\end{aligned}$$

where the last inequality holds due to $a_6 N_{p(\cdot)} - 1/r > 0$. From this, the fact that $r \in (0, \underline{p})$, Lemma 2.7(i), Lemma 3.1, (4.14), (4.18) and the definition of Ω_k, it follows that, for any $k \in \mathbb{Z}$,

$$\begin{aligned}
&\left\|M^\circ\left(\sum_{i\in\mathbb{N}} b_i^k\right)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad\lesssim \|M^\circ(f)\chi_{\Omega_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\|2^k\left\{\sum_{i\in\mathbb{N}}\left[M_\Theta\left(\chi_{\theta_{x_i^k,\, t_i^k}}\right)\right]^{1/r}\right\}^r\right\|^{1/r}_{L^{p(\cdot)/r}(\mathbb{R}^n)} \\
&\quad\lesssim \|M^\circ(f)\chi_{\Omega_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\|2^k\sum_{i\in\mathbb{N}}\chi_{\theta_{x_i^k,\, t_i^k}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad\lesssim \|M^\circ(f)\chi_{\Omega_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\|2^k\chi_{\Omega_k}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|M^\circ(f)\chi_{\Omega_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

This together with (4.22) and Definition 3.6 further implies that,

$$\begin{aligned}
\left\|f - g^k\right\|_{H^{p(\cdot)}(\Theta)} &= \left\|\sum_{i\in\mathbb{N}} b_i^k\right\|_{H^{p(\cdot)}(\Theta)} = \left\|M^\circ\left(\sum_{i\in\mathbb{N}} b_i^k\right)\right\|_{L^{p(\cdot)}(\Theta)} \\
&\lesssim \|M_\Theta(f)\chi_{\Omega_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \to 0 \text{ as } k \to +\infty.
\end{aligned}$$

From this, the fact that $\|g^k\|_{L^\infty(\mathbb{R}^n)} \to 0$ as $k \to -\infty$ (see (4.38)), and Lemma 3.7, we deduce that

$$f = \sum_{k\in\mathbb{Z}} \left(g^{k+1} - g^k\right) \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

On the other hand, by an argument same as that used in [16, Theorem 4.19], we obtain

$$f = \sum_{k\in\mathbb{Z}} \left(g^{k+1} - g^k\right) := \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} h_i^k \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

where

$$h_i^k := \left(f - P_i^k\right) \phi_i^k - \sum_{j\in\mathbb{N}} \left(f - P_j^{k+1}\right) \phi_j^{k+1} \phi_i^k + \sum_{j\in\mathbb{N}} P_{ij}^{k+1} \phi_j^{k+1} \tag{4.39}$$

satisfies

$$\int_{\mathbb{R}^n} h_i^k(x) Q(x)\, dx = 0 \qquad \text{for any } Q \in \mathbb{P}_l(\mathbb{R}^n), \tag{4.40}$$

$$\operatorname{supp}\ h_i^k \subset \theta_{x_i^k,\, t_i^k - J - 3\gamma - 1} \tag{4.41}$$

and

$$\|h_i^k\|_{L^\infty(\mathbb{R}^n)} \leq C2^k, \tag{4.42}$$

where C is a positive constant independent of k and i.

Now, for any $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, let

$$\lambda_i^k := C2^k \left\| \chi_{\theta_{x_i^k,\, t_i^k - J - 3\gamma - 1}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \qquad \text{and} \qquad a_i^k := (\lambda_i^k)^{-1} h_i^k, \tag{4.43}$$

where C is as in (4.42). Then, by (4.40), (4.41) and (4.42), we easily know that, for any $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, a_i^k is a $(p(\cdot),\ \infty,\ l)$-atom. Moreover, we have

$$f = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

In addition, from (4.43), (4.14), (4.18), the definition of Ω_k and Definition 3.6, we further deduce that

$$\begin{aligned}
&\left\| \left\{ \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i^k| \chi_{\theta_{x_i^k,\, t_i^k - J - 3\gamma - 1}}}{\|\chi_{\theta_{x_i^k,\, t_i^k - J - 3\gamma - 1}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\sim \left\| \left\{ \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \left[2^k \chi_{\theta_{x_i^k,\, t_i^k - J - 3\gamma - 1}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\| \left[\sum_{k\in\mathbb{Z}} \left(2^k \chi_{\Omega_k}\right)^{\underline{p}} \right]^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \left\| \left[\sum_{k\in\mathbb{Z}} \left(2^k \chi_{\Omega_k \setminus \Omega_{k+1}}\right)^{\underline{p}} \right]^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\| M^\circ(f) \left[\sum_{k\in\mathbb{Z}} \left(\chi_{\Omega_k \setminus \Omega_{k+1}}\right)^{\underline{p}} \right]^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \left\| M^\circ(f) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{H^{p(\cdot)}(\Theta)}.
\end{aligned} \tag{4.44}$$

This implies that (4.37) holds true.

Step 2. In this step, we prove that (4.37) also holds true for any $f \in H^{p(\cdot)}(\Theta)$.

To this end, let $f \in H^{p(\cdot)}(\Theta)$. Then, by Proposition 4.7, we know that there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset H^{p(\cdot)}(\Theta) \cap L^q(\mathbb{R}^n)$ with $q \in (\max\{p_+,\ 1\}, \infty)$ such that $f = \sum_{j\in\mathbb{N}} f_j$ in $H^{p(\cdot)}(\Theta)$ and for any $j \in \mathbb{N}$,

$$\|f_j\|_{H^{p(\cdot)}(\Theta)} \leq 2^{2-j} \|f\|_{H^{p(\cdot)}(\Theta)}.$$

Notice that, for any $j \in \mathbb{N}$, by the conclusion obtained in **Step 1**, we find that f_j has an atomic decomposition, namely,

$$f_j = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda_i^{j,k} a_i^{j,k} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

where $\{\lambda_i^{j,k}\}_{k\in\mathbb{N},\, i\in\mathbb{Z}}$ and $\{a_i^{j,k}\}_{k\in\mathbb{N}, i\in\mathbb{Z}}$ are constructed as in (4.43). Thus, $\{a_i^{j,k}\}_{k\in\mathbb{N},\, i\in\mathbb{Z}}$ are $(p(\cdot), \infty, l)$-atoms. By this, (4.44) and Proposition 2.8, we have

$$f = \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda_i^{j,k} a_i^{j,k} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

and

$$\begin{aligned}\|f\|^{\underline{p}}_{H^{p(\cdot)}_{\infty,l}(\Theta)} &\sim \left\| \left\{ \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i^{j,k}| \chi_{\theta_{x_i^{j,k},\, t_i^{j,k}-J-3\gamma-1}}}{\|\chi_{\theta_{x_i^{j,k},\, t_i^{j,k}-J-3\gamma-1}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \sum_{j\in\mathbb{N}} \|M^\circ(f_j)\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \sum_{j\in\mathbb{N}} \|f_j\|^{\underline{p}}_{H^{p(\cdot)}(\Theta)} \lesssim \|f\|^{\underline{p}}_{H^{p(\cdot)}(\Theta)},\end{aligned}$$

which implies that (4.37) holds true for any $f \in H^{p(\cdot)}(\Theta)$ and hence completes the proof of Theorem 4.3. □

5 Variable Anisotropic Singular Integral Operators

In this section, we introduce the notion of variable anisotropic singular integral operators associated with a continuous ellipsoid cover Θ and show that such operators are bounded from $H^{p(\cdot)}(\Theta)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ in general and from $H^{p(\cdot)}(\Theta)$ to itself under the moment condition.

Definition 5.1. [6, Definition 5.1] A locally square integrable function K on $\Omega := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ is called a *variable anisotropic singular integral kernel* with respect to a continuous ellipsoid cover Θ if there exist two positive constants $c > 1$ and C such that

$$\int_{B_{\rho_\Theta}(y,\, cr)^\complement} |K(x, y) - K(x, y')|\, dx \leq C \tag{5.1}$$

and

$$\int_{B_{\rho_\Theta}(x,\, cr)^\complement} |K(x, y) - K(x', y)|\, dy \leq C, \tag{5.2}$$

where $r > 0$, $y' \in B_{\rho_\Theta}(y, r)$, $y \in \mathbb{R}^n$ and $B_{\rho_\Theta}(y, r)$ is as in (2.3).

We say that T is a *variable anisotropic singular integral operator* (VASIO) of order 0 if $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator if there exists a kernel K satisfying (5.1) and (5.2) such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, dy \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n),\ x \notin \operatorname{supp}(f).$$

Example 5.2. (i) Let K be a locally integrable function on

$$\Omega := \{(x,\ y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$$

and there exist positive constants δ and C such that for all $x \neq y \in \mathbb{R}^n$ we have

$$|K(x, y)| \leq \frac{C}{\rho_\Theta(x, y)},$$

$$|K(x, y) - K(x, y')| \le C \frac{[\rho_\Theta(y, y')]^\delta}{[\rho_\Theta(x, y)]^{1+\delta}} \qquad \text{if } \rho_\Theta(y, y') \le \frac{1}{2\kappa}\rho_\Theta(x, y),$$

$$|K(x, y) - K(x', y)| \le C \frac{[\rho_\Theta(x, x')]^\delta}{[\rho_\Theta(x, y)]^{1+\delta}} \qquad \text{if } \rho_\Theta(x, x') \le \frac{1}{2\kappa}\rho_\Theta(x, y),$$

where $\kappa \ge 1$ is the triangle inequality constant of ρ_Θ. This kernel K satisfies (5.1) and (5.2) by referring to [6, Proposition 5.7]. And hence a bounded linear operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ associated with the above kernel K is a singular integral operator as in Definition 5.1.

(ii) Dekel, Han and Petrushev [13] introduced a Littlewood-Paley result of operators $\{D_m\}_{m\in\mathbb{Z}}$ defined via some ellipsoid cover Θ satisfying

$$|D_m(x, y)| \le c \frac{2^{-m\delta}}{[2^{-m} + \rho_\Theta(x, y)]^{1+\delta}} \qquad \text{for all } x, y \in \mathbb{R}^n \quad \text{(see [13, p. 654])},$$

where c and δ are positive constants. It is not hard to show that a vector-valued kernel defined via $\{D_m\}_{m\in\mathbb{Z}}$ satisfies (5.1) and (5.2) under vector-valued variants (see [5, Definition A.1]). Moreover, for any $f \in L^p(\mathbb{R}^n)$, it holds true that

$$\|f\|_{L^p(\mathbb{R}^n)} \sim \left\| \left(\sum_{m\in\mathbb{Z}} |D_m(f)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \qquad \text{(see [13, Proposition 4.1])}.$$

By [6, Theorem 5.2 and Remark 5.3], we have the following theorem.

Theorem 5.3. *Let T be a* VASIO *of order 0 and $1 < q < \infty$. Then T extends to a bounded linear operator $L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$.*

Since we are interested in the boundedness of singular integral operators on the Hardy spaces $H^{p(\cdot)}(\Theta)$, $0 < p(\cdot) \le 1$, we need to impose smoothness hypothesis on the kernel K, which is much stricter than that given by Definition 5.1. To this end we shall extend the definition of Calderón-Zygmund operators in anisotropic setting which was given in [6, Definition 5.4].

Definition 5.4. Let $s \in \mathbb{N}_0$ and let T be a VASIO as in Definition 5.1 with kernel $K(x, y)$ in the class $\mathcal{C}^s(\mathbb{R}^n)$ as a function of y. Then we say that T is a *VASIO of order s* if there exists a constant $C > 0$ such that for any $(x, y) \in \Omega$ and for any multi-index $|\alpha| \le s$ we have

$$\left|\partial_y^\alpha [K(\cdot, M_{y,m}\cdot)](x, M_{y,m}^{-1} y)\right| \le C/\rho_\Theta(x, y) \qquad \text{where } m := -\log_2 \rho_\Theta(x, y). \tag{5.3}$$

More precisely, the left hand side of (5.3) means $|\partial_y^\alpha \widetilde{K}(x, M_{y,m}^{-1} y)|$, where $\widetilde{K}(x, y) := K(x, M_{y,m}\, y)$. The smallest constant C satisfying (5.3) is called a Calderón-Zygmund norm of T, which is denoted by $\|T\|_{(s)}$.

Our ultimate goal is to show that anisotropic Calderón-Zygmund operators T are bounded on $H^{p(\cdot)}(\Theta)$. Generally, we can not expect this unless we also assume that T preserves vanishing moments. Hence, we adopt the following definition motivated by [6, Definition 5.9].

Definition 5.5. Let $s \in \mathbb{N}$ and $1 < q < \infty$. We say that a VASIO T of order s satisfies

$$T^*(x^\alpha) = 0 \qquad \text{for all } |\alpha| \le l,$$

where $l < a_6 s/a_4$, if for any $f \in L^q(\mathbb{R}^n)$ with compact support with vanishing moments $\int_{\mathbb{R}^n} f(x)x^\alpha dx = 0$ for all $|\alpha| < s$, we have

$$\int_{\mathbb{R}^n} Tf(x)x^\alpha dx = 0 \qquad \text{for all } |\alpha| \le l.$$

The actual value of q is not relevant in Definition 5.5 as we merely need that $T : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded (see [6, Theorem 5.2 and Remark 5.3]). However, the requirement that $l < a_6 s/a_4$ is essential to guarantee that the integrals $\int_{\mathbb{R}^n} Tf(x)x^\alpha\,dx$ are well defined for all $|\alpha| \le l$. This is a consequence of the following lemma.

Lemma 5.6. *[6, Lemma 5.10] Let $l, s \in \mathbb{N}$, $1 < q < \infty$. Let T be a* VASIO *of order s. Suppose that $f \in L^q(\mathbb{R}^n)$ satisfies $\operatorname{supp} f \subset \theta_{z,t}$ for some $z \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $\int_{\mathbb{R}^n} f(x)x^\alpha dx = 0$ for all $|\alpha| < s$. Then, for some $C > 0$ depending only on $\|T\|_{(s)}$ and $\mathbf{p}(\Theta)$,*

$$|Tf(x)| \le C\|f\|_{L^q(\mathbb{R}^n)}|\theta_{z,\,t}|^{-1/q}2^{-k\gamma(1+a_6 s)} \qquad \text{for } x \in \theta_{z,\,t-(k+1)\gamma} \setminus \theta_{z,\,t-k\gamma},\ k \in \mathbb{N}. \tag{5.4}$$

In particular, if $l < a_6 s/a_4$, then

$$\int_{\mathbb{R}^n} |Tf(x)|(1+|x|^l)\,dx < \infty. \tag{5.5}$$

We are now ready to state the main results of the paper, Theorems 5.7 and 5.8. There are generalizations of [6, Theorems 5.12 and 5.11] from $H^{p(\cdot)}(\Theta)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ and from $H^{p(\cdot)}(\Theta)$ to itself.

Theorem 5.7. *Let Θ be a continuous ellipsoid cover with parameters $\mathbf{p}(\Theta) = \{a_1, \dots, a_6\}$ and let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Suppose T is a* VASIO *of order s with*

$$s > \frac{1/\underline{p} - 1}{a_6}, \tag{5.6}$$

where $\underline{p}$ is as in (2.4). *Then, T extends to a bounded linear operator from $H^{p(\cdot)}(\Theta)$ to $L^{p(\cdot)}(\mathbb{R}^n)$.*

Theorem 5.8. *Let Θ be a continuous ellipsoid cover with parameters $\mathbf{p}(\Theta) = \{a_1, \dots, a_6\}$ and let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Suppose that T is a* VASIO *of order s such that*

$$s > \frac{a_4}{a_6}N_{p(\cdot)}, \qquad \text{where } N_{p(\cdot)} = \left\lfloor \frac{\max(1, a_4)n + 1}{a_6\underline{p}} \right\rfloor + 1, \tag{5.7}$$

$$T^\star(x^\alpha) = 0 \qquad \text{for all } \alpha \in \mathbb{N}_0^n,\ |\alpha| \le N_{p(\cdot)}. \tag{5.8}$$

Then, T extends to a bounded linear operator from $H^{p(\cdot)}(\Theta)$ to itself.

To prove Theorems 5.7 and 5.8, we need the following definition and technical lemmas.

Definition 5.9. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $1 < q \le \infty$, $\varepsilon \in (0, \infty)$ and $l \in [N_{p(\cdot)}, \infty) \cap \mathbb{N}_0$. For a continuous ellipsoid cover Θ, we say that a measurable function m is called an *anisotropic $(p(\cdot),\ q,\ l,\ \epsilon)$-molecule* if

(i) $\|m\|_{L^q(\theta_{z,\,t})} \le \frac{|\theta_{z,\,t}|^{1/q}}{\|\chi_{\theta_{z,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$ for some $\theta_{z,\,t} \in \Theta$ with $z \in \mathbb{R}^n$ and $t \in \mathbb{R}$;

(ii) $|m(x)| \le \frac{2^{-kJ\varepsilon}}{\|\chi_{\theta_{z,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$ for $x \in \theta_{z,\,t-(k+1)\gamma} \setminus \theta_{z,\,t-k\gamma}$ with $k \in \mathbb{N}$;

(iii) $\int_{\mathbb{R}^n} m(y)y^\alpha\,dy = 0$ for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \le l$.

Lemma 5.10. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $1 < q < \infty$, and $l \ge N_{p(\cdot)}$. Then, for any $f \in L^q(\mathbb{R}^n) \cap H^{p(\cdot)}(\Theta)$, there exist a sequence of $(p(\cdot), \infty, l)$-atoms $\{a_i^k\}_{k\in\mathbb{Z}, i\in\mathbb{N}}$, a sequence $\{\lambda_i^k\}_{k\in\mathbb{Z}, i\in\mathbb{N}} \subset \mathbb{C}$, and a positive constant C independent of f such that*

$$\left\| \left\{ \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i^k|\chi_{\theta_{x_i^k,\,t_i^k}}}{\|\chi_{\theta_{x_i^k,\,t_i^k}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{H^{p(\cdot)}(\Theta)} \tag{5.9}$$

and

$$f = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \quad \text{converges in } L^q(\mathbb{R}^n) \text{ and almost everywhere,} \tag{5.10}$$

where $\underline{p}$ is as in (2.4).

Proof. By checking the proof of Theorem 4.3, we obtain (5.9) and (5.10) holds almost everywhere. By referring to the proof of [6, Theorem 4.10] with the existing estimates in the proof of Theorem 4.10, we conclude that (5.10) also holds in $L^q(\mathbb{R}^n)$.

□

Lemma 5.11. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (1, \infty]$, $l \in [N_{p(\cdot)}, \infty) \cap \mathbb{N}_0$ and $s \in \mathbb{N}$ such that $s > \frac{a_4}{a_6} l$. Suppose that T is an anisotropic Calderón-Zygmund operator of order $s \in \mathbb{N}$ satisfying*

$$T^\star(x^\alpha) = 0$$

for all $|\alpha| \le l$. Then, for any $(p(\cdot),\ q,\ s-1)$-atom a supported on some $\theta_{z,\,t}$ with $z \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $T(a)$ is a harmless constant multiple of a $(p(\cdot),\ q,\ l,\ \varepsilon)$-molecule associated with $\theta_{z,\,t-\gamma}$, where $\epsilon := a_6 s + 1$.

Proof. Without lose of generality, we may assume that a is a $(p(\cdot),\ q,\ s-1)$-atom supported on some $\theta_{z,\,t}$ and $s > \frac{a_4}{a_6} N_{p(\cdot)}$. Then, for the anisotropic Calderón-Zygmund operator T of order s satisfying $T^\star(x^\alpha) = 0$ for all $|\alpha| \le l$, by the vanishing moments of a, we know that $T(a)$ has the vanishing moments up to order l. To prove that $T(a)$ is a harmless constant multiple of a $(p(\cdot),\ q,\ l,\ \varepsilon)$-molecule, we still need to show that

$$\|T(a)\|_{L^q(\theta_{z,\,t-\gamma})} \lesssim \frac{|\theta_{z,\,t-\gamma}|^{1/q}}{\|\chi_{\theta_{z,\,t-\gamma}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \quad \text{for some } \theta_{z,\,t-\gamma} \in \Theta \text{ with } z \in \mathbb{R}^n \text{ and } t \in \mathbb{R} \tag{5.11}$$

and

$$|T(a)(x)| \lesssim \frac{2^{-kJ\varepsilon}}{\|\chi_{\theta_{z,\,t-\gamma}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \quad \text{for } x \in \theta_{z,\,t-(k+1)\gamma} \setminus \theta_{z,\,t-k\gamma} \text{ with } k \in \mathbb{N}. \tag{5.12}$$

Indeed, on the one hand, for any $x \in \theta_{z,\,t}$, applying the fact that T is bounded on $L^q(\mathbb{R}^n)$ for all $q \in (1,\ \infty)$ (see Theorem 5.3), supp $a \subset \theta_{z,\,t}$ and the size condition of a, we obtain that

$$\|T(a)\|_{L^q(\theta_{z,\,t-\gamma})} \lesssim \|a\|_{L^q(\mathbb{R}^n)} \lesssim \frac{|\theta_{z,\,t}|^{1/q}}{\|\chi_{\theta_{z,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim \frac{|\theta_{z,\,t-\gamma}|^{1/q}}{\|\chi_{\theta_{z,\,t-\gamma}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}},$$

where the last inequality holds due to (4.32) and hence (5.11) holds true.

On the other hand, for any $x \in \theta_{z,\,t-(k+1)\gamma} \setminus \theta_{z,\,t-k\gamma}$ with $k \in \mathbb{N}$. By Lemma 5.6, the size condition of a and (4.32), we obtain

$$|Ta(x)| \le C 2^{-k\gamma(a_6 s+1)} 2^{t/q} \|a\|_{L^q(\mathbb{R}^n)} \lesssim \frac{2^{-k\gamma(a_6 s+1)}}{\|\chi_{\theta_{z,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim \frac{2^{-k\gamma\epsilon}}{\|\chi_{\theta_{z,\,t-\gamma}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}},$$

where $1/q + 1/q' = 1$ and $\epsilon := a_6 s + 1$. This finishes the proof of Lemma 5.11. □

Proof of Theorem 5.7. First, we show that $\|T(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}(\Theta)}$ holds true for any $f \in H^{p(\cdot)}(\Theta) \cap L^q(\mathbb{R}^n)$ with $q \in (1,\ \infty) \cap (p_+,\ \infty)$. For $f \in H^{p(\cdot)}(\Theta) \cap L^q(\mathbb{R}^n)$, by Lemma 5.10, we know that there exist $\{\lambda_i^k\}_{k\in\mathbb{Z},i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(p(\cdot),\ \infty,\ s-1)$-atoms, $\{a_i^k\}_{k\in\mathbb{Z},i\in\mathbb{N}}$, supported, respectively, on $\{\theta_{x_i^k,\,t_i^k}\}_{k\in\mathbb{Z},i\in\mathbb{N}} \subset \Theta$ such that

$$f = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \quad \text{converges almost everywhere}$$

and

$$\left\| \left\{ \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i^k| \chi_{\theta_{x_i^k,\,t_i^k}}}{\|\chi_{\theta_{x_i^k,\,t_i^k}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}(\Theta)}. \tag{5.13}$$

Then, by Lemma 2.7, we obtain

$$\|T(f)\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} = \left\|T\left(\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k\right)\right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\|\left\{\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\left[|\lambda_i^k|\left|T(a_i^k)\chi_{\theta_{x_i^k,\,t_i^k-\gamma}}\right|\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\|\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}|\lambda_i^k|\left|T(a_i^k)\chi_{(\theta_{x_i^k,\,t_i^k-\gamma})^\complement}\right|\right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} =: \mathrm{J}_1 + \mathrm{J}_2. \tag{5.14}$$

For the term J_1, repeating the estimate of I_1 of the proof of Theorem 4.3 with the boundedness of T on $L^q(\mathbb{R}^n)$ with $q \in (\max\{p_+,\ 1\},\ \infty)$ (see Theorem 5.3) and (5.13), we conclude that

$$\mathrm{J}_1 \lesssim \left\|\left\{\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i^k|\chi_{\theta_{x_i^k,\,t_i^k}}}{\|\chi_{\theta_{x_i^k,\,t_i^k}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|^{\underline{p}}_{H^{p(\cdot)}(\Theta)}.$$

To deal with J_2, $(p(\cdot),\ \infty,\ s-1)$-atom a_i^k is shortly wrote as a supported on $\theta_{z,\,t} := \theta_{x_i^k,\,t_i^k}$. By Lemma 5.6, Definition 4.1 and (4.31), we conclude that, for any $x \in \mathbb{R}^n$ and $r \in (0, \underline{p})$ close to $\underline{p}$,

$$\begin{aligned}|Ta(x)\chi_{(\theta_{z,\,t-\gamma})^\complement}(x)| &\lesssim \sum_{j=1}^{\infty}|Ta(x)|\chi_{\theta_{z,\,t-(j+1)\gamma}\setminus\theta_{z,\,t-j\gamma}}(x)\\ &\lesssim \frac{1}{\|\chi_{\theta_{z,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\sum_{j=1}^{\infty}2^{-j\gamma(1+a_6 s)}2^{\gamma(j+1)/r}[M_\Theta(\chi_{\theta_{z,\,t}})(x)]^{1/r}\\ &\lesssim \frac{1}{\|\chi_{\theta_{z,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\sum_{j=1}^{\infty}2^{-j\gamma(1+a_6 s-1/r)}[M_\Theta(\chi_{\theta_{z,\,t}})(x)]^{1/r}\\ &\lesssim \frac{[M_\Theta(\chi_{\theta_{z,\,t}})(x)]^{1/r}}{\|\chi_{\theta_{z,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}},\end{aligned} \tag{5.15}$$

where the last inequality holds due to $1 + a_6 s - 1/r > 0$.

Then inserting (5.15) into J_2 and repeating the estimate of (4.35), we obtain

$$\mathrm{J}_2 \lesssim \left\|\left\{\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i^k|\chi_{\theta_{x_i^k,\,t_i^k}}}{\|\chi_{\theta_{x_i^k,\,t_i^k}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|^{\underline{p}}_{H^{p(\cdot)}(\Theta)},$$

where the last inequality holds due to (5.13).

Combining (5.14) and the estimates of J_1 and J_2, we further conclude that

$$\|T(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}(\Theta)}.$$

Next, we prove that $\|T(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}(\Theta)}$ also holds true for any $f \in H^{p(\cdot)}(\Theta)$. Let $f \in H^{p(\cdot)}(\Theta)$. By Proposition 4.7, we know that there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset H^{p(\cdot)}(\Theta) \cap L^q(\mathbb{R}^n)$ with $q \in (1,\ \infty) \cap (p_+,\ \infty)$ such that $f_j \to f$ in $H^{p(\cdot)}(\Theta)$ as $j \to +\infty$. Therefore $\{f_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $H^{p(\cdot)}(\Theta)$. From this, we see that, for any $j,\ k \in \mathbb{N}$,

$$\|T(f_j) - T(f_k)\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|T(f_j - f_k)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f_j - f_k\|_{H^{p(\cdot)}(\Theta)}.$$

Thus $\{T(f_j)\}_{j\in\mathbb{N}}$ is also a Cauchy sequence in $L^{p(\cdot)}(\mathbb{R}^n)$. Applying Lemma 2.5, we conclude that there exists a $g \in L^{p(\cdot)}(\mathbb{R}^n)$ such that $T(f_j) \to g$ in $L^{p(\cdot)}(\mathbb{R}^n)$ as $j \to +\infty$. Let $T(f) := g$. We claim that $T(f)$ is well defined.

Indeed, for any other sequence $\{h_j\}_{j\in\mathbb{N}} \subset H^{p(\cdot)}(\Theta)\cap L^q(\mathbb{R}^n)$ satisfying $h_j \to f$ in $H^{p(\cdot)}(\Theta)$ as $j\to+\infty$, by Lemma 2.7(ii), we have

$$\begin{aligned}\|T(h_j)-T(f)\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} &\le \|T(h_j)-T(f_j)\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} + \|T(f_j)-g\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\lesssim \|h_j-f_j\|^{\underline{p}}_{H^{p(\cdot)}(\Theta)} + \|T(f_j)-g\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\lesssim \|h_j-f\|^{\underline{p}}_{H^{p(\cdot)}(\Theta)} + \|f-f_j\|^{\underline{p}}_{H^{p(\cdot)}(\Theta)} + \|T(f_j)-g\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \to 0\end{aligned}$$

as $j\to+\infty$, which is wished.

From this, we see that, for any $f\in H^{p(\cdot)}(\Theta)$,

$$\|T(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \lim_{j\to+\infty}\|T(f_j)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \lim_{j\to+\infty}\|f_j\|_{H^{p(\cdot)}(\Theta)} \sim \|f\|_{H^{p(\cdot)}(\Theta)}$$

and hence completes the proof of Theorem 5.7. □

Proof of Theorem 5.8 . First, we show that $\|T(f)\|_{H^{p(\cdot)}(\Theta)} \lesssim \|f\|_{H^{p(\cdot)}(\Theta)}$ holds true for any $f\in H^{p(\cdot)}(\Theta)\cap L^q(\mathbb{R}^n)$ with $q\in(1,\infty)\cap(p_+,\infty)$. For any $f\in H^{p(\cdot)}(\Theta)\cap L^q(\mathbb{R}^n)$, by Lemma 5.10, we know that there exist $\{\lambda_i^k\}_{k\in\mathbb{Z},i\in\mathbb{N}}\subset\mathbb{C}$ and a sequence of $(p(\cdot),\infty,s-1)$-atoms, $\{a_i^k\}_{k\in\mathbb{Z},i\in\mathbb{N}}$, supported, respectively, on $\{\theta_{x_i^k,t_i^k}\}_{k\in\mathbb{Z},i\in\mathbb{N}}\subset\Theta$ such that

$$f=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k \quad \text{converges in } L^q(\mathbb{R}^n)$$

and

$$\left\|\left\{\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i^k|\chi_{\theta_{x_i^k,t_i^k}}}{\|\chi_{\theta_{x_i^k,t_i^k}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}(\Theta)}. \tag{5.16}$$

Since T is bounded on $L^q(\mathbb{R}^n)$ (see Theorem 5.3), it follows that $Tf=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^k Ta_i^k$ converges in $L^q(\mathbb{R}^n)$ and hence

$$Tf=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^k Ta_i^k \ \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

From this, it is easy to see that, for all $x\in\mathbb{R}^n$,

$$\begin{aligned}\|M^\circ(T(f))\|^{\underline{p}}_{L^{p(\cdot)}} &\le \left\|\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}|\lambda_i^k|M^\circ(T(a_i^k))\right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \left\|\left\{\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\left[|\lambda_i^k|M^\circ(T(a_i^k))\chi_{\theta_{x_i^k,t_i^k-\gamma}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\quad+\left\|\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}|\lambda_i^k|M^\circ(T(a_i^k))\chi_{(\theta_{x_i^k,t_i^k-\gamma})^\complement}\right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} =: \mathrm{K}_1+\mathrm{K}_2\end{aligned} \tag{5.17}$$

For K_1, from the $L^q(\mathbb{R}^n)$ boundedness of M° (since $M^\circ f(x)\lesssim M_\Theta f(x)$ (see [16, Lemma 3.2 and Theorem 3.8(ii)])) and the $L^q(\mathbb{R}^n)$ boundedness of T (see Theorem 5.3) for all $q\in(1,\infty)$. Repeating the estimate of I_1 of the proof of Theorem 4.3, we know that

$$\mathrm{K}_1\lesssim\left\|\left\{\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i^k|\chi_{\theta_{x_i^k,t_i^k}}}{\|\chi_{\theta_{x_i^k,t_i^k}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}(\Theta)},$$

where the last inequality holds due to (5.16).

For K_2, let $\varphi\in\mathcal{S}_{N_{p(\cdot)},\widetilde{N}_{p(\cdot)}}(\mathbb{R}^n)$ with support in $\mathbb{B}^n$, and $s\in\mathbb{R}$. Suppose P is a polynomial of degree $N_{p(\cdot)}-1$. For any $(p(\cdot),\infty,s-1)$-atom a_i^k, shortly wrote as a supported on some $\theta_{z,t}:=\theta_{x_i^k,t_i^k}$, by Lemma 5.11, we see that $T(a)$ is a harmless constant multiple of a $(p(\cdot),\ q,\ l,\ \varepsilon)$-molecule associated with $\theta_{z,\,t-\gamma}$, where $\epsilon := a_6 s + 1$. From this, Definition 5.9, Hölder's inequality and an argument similar to that used in the proof of Lemma 4.4, we know that, for any $x \in (\theta_{z,\,t-\gamma})^{\complement}$ and $r \in \mathbb{R}$,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} T(a)(y)\varphi_{x,\,r}(y)\,dy\right| &\lesssim 2^r \left|\int_{\mathbb{R}^n} T(a)(y)\left[\varphi(M_{x,\,r}^{-1}(x-y)) - P(M_{x,\,r}^{-1}(x-y))\right] dy\right| \\ &\lesssim 2^r \|T(a)\|_{L^q(\theta_{z,\,t-\gamma})}\left(\int_{\theta_{z,\,t-\gamma}} \left|\varphi(M_{x,\,r}^{-1}(x-y)) - P(M_{x,\,r}^{-1}(x-y))\right|^{q'} dy\right)^{1/q'} \\ &\quad + 2^r \sum_{k\in\mathbb{N}} \int_{\theta_{z,\,t-(k+1)\gamma}} \frac{2^{-k\gamma\epsilon}}{\|\chi_{\theta_{z,\,t-\gamma}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left|\varphi(M_{x,\,r}^{-1}(x-y)) - P(M_{x,\,r}^{-1}(x-y))\right| dy \\ &\lesssim \frac{2^{(r-t)/q}}{\|\chi_{\theta_{z,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left|M_{x,\,r}^{-1}(x-\theta_{z,\,t-\gamma})\right|^{1/q'} \sup_{y\in F(\theta_{z,\,t-\gamma})} |\varphi(y) - P(y)| \\ &\quad + \sum_{k\in\mathbb{N}} \frac{2^{-k\gamma\epsilon}}{\|\chi_{\theta_{z,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left|M_{x,\,r}^{-1}(x-\theta_{z,\,t-(k+1)\gamma})\right| \sup_{y\in F(\theta_{z,\,t-(k+1)\gamma})} |\varphi(y) - P(y)|,\end{aligned}$$

where $1/q + 1/q' = 1$,

$$F(\theta_{z,\,t-\gamma}) := M_{x,\,r}^{-1}(x - (M_{z,\,t-\gamma}(\mathbb{B}^n) + z)) = M_{x,\,r}^{-1}(x-z) - M_{x,\,r}^{-1}M_{z,\,t-\gamma}(\mathbb{B}^n)$$

and

$$F(\theta_{z,\,t-(k+1)\gamma}) := M_{x,\,r}^{-1}(x - (M_{z,\,t-(k+1)\gamma}(\mathbb{B}^n) + z)) = M_{x,\,r}^{-1}(x-z) - M_{x,\,r}^{-1}M_{z,\,t-(k+1)\gamma}(\mathbb{B}^n).$$

Then, by repeating the proof of Lemma 4.4 with regular modifications, we obtain

$$M^{\circ}(T(a))(x) \le C\frac{[M_{\Theta}(\chi_{\theta_{z,\,t}})(x)]^{\beta}}{\|\chi_{\theta_{z,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}, \tag{5.18}$$

where β is as in Lemma 4.4. By this and an argument same as that used in the estimate of (4.35), we obtain

$$\mathrm{K}_2 \lesssim \left\|\left\{\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i^k|\chi_{\theta_{x_i^k,\,t_i^k}}}{\|\chi_{\theta_{x_i^k,\,t_i^k}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}_{q,\,l}(\Theta)},$$

where the last inequality holds due to (5.16).

Combining (5.17) and the estimates of K_1 and K_2, we further conclude that

$$\|T(f)\|_{H^{p(\cdot)}(\Theta)} \lesssim \|f\|_{H^{p(\cdot)}(\Theta)}.$$

From Lemmas 3.7 and 2.7 and a similar proof of Theorem 5.7, we finally deduce that $\|T(f)\|_{H^{p(\cdot)}(\Theta)} \lesssim \|f\|_{H^{p(\cdot)}(\Theta)}$ also holds true for any $f \in H^{p(\cdot)}(\Theta)$. □

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