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5-Point CAT(0) Spaces after Tetsu Toyoda

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Abstract: We give another proof of Toyoda's theorem that describes 5-point subspaces in CAT(0) length spaces.

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1 Introduction

The CAT(0) comparison is a certain inequality for 6 distances between 4 points in a metric space. The following descriptions, the so-called *(2+2)-comparison*, is the most standard, we refer to [3, 4] for other definitions and their equivalences.

Given a quadruple of points p, q, x, y in a metric space X , consider two *model triangles* (that is, a plane triangle with the same sides) $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\Delta}(pxy)$ and $[\tilde{q}\tilde{x}\tilde{y}] = \tilde{\Delta}(qxy)$ with common side $[\tilde{x}\tilde{y}]$.

If the inequality

$$|p - q|_X \leq |\tilde{p} - \tilde{z}| + |\tilde{z} - \tilde{q}|$$

holds for any point $\tilde{z} \in [\tilde{x}\tilde{y}]$, then we say that the quadruple p, q, x, y satisfies CAT(0) comparison; here $|p - q|_X$ denotes the distance from p to q in X .

If CAT(0) comparison holds for any quadruple (and any of its relabeling) in a metric space X , then we say that X is CAT(0).

It is not hard to check that if a quadruple of points satisfies CAT(0) comparison for all relabeling, then it admits a distance-preserving inclusion into a length CAT(0) space. The following theorem generalizes this statement to 5-point metric spaces.

1.1. Toyoda's theorem. *Let P be a 5-point metric space that satisfies CAT(0) comparison. Then P admits a distance-preserving inclusion into a length CAT(0) space X .*

Moreover, X can be chosen to be a subcomplex of a 4-simplex such that (1) each simplex in X has Euclidean metric and (2) the inclusion maps the 5 points on P to the vertexes of the simplex.

A slightly weaker version of this theorem was proved by Tetsu Toyoda [7]. Our proof is shorter; it uses the fact that convex spacelike hypersurfaces in $\mathbb{R}^{3,1}$ equipped with the induced length metrics are CAT(0) spaces

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[1]. We construct a distance-preserving inclusion ι of P into $\mathbb{R}^4$ or $\mathbb{R}^{3,1}$. In the case of $\mathbb{R}^4$ the convex hull K of $\iota(P)$ can be taken as X ; in the case of $\mathbb{R}^{3,1}$ we take as X a spacelike part of the boundary of K .

It is expected that any 5 point metric space P as in the theorem admits a distance-preserving inclusion in a product of trees.

An analog of Toyoda's theorem does not hold for 6-point sets. It can be seen by using the so-called (4+2)-comparison introduced in [2]; this comparison holds for any length CAT(0) space, but may not hold for a space with CAT(0) comparison (if it is not a length space).

The (4+2)-comparison is not a sufficient condition for 6-point spaces. More precisely, there are 6-point metric spaces that satisfy (4+2) and (2+2)-comparisons but do not admit a distance-preserving embedding into a length CAT(0) space. An example was constructed by the first author; it is described in [2] right after 7.2. See the final section for related questions.

2 5-point arrays in 3-space

Denote by $\mathcal{A}$ the space of all 5 point arrays in $\mathbb{R}^3$ that is nondegenerate in the following sense: (1) all 5 points do not lie on one plane and (2) no three points lie on one line. Note that $\mathcal{A}$ is connected.

A 5 point array $x_1, \dots, x_5 \in \mathbb{R}^3$ defines an affine map from a 4-simplex to $\mathbb{R}^3$. Fix an orientation of the 4-simplex and consider the induced orientations on its 5 facets. Each facet may be mapped in an orientation-preserving, degenerate, or orientation-reversing way. For each array consider the triple of integers (n_+, n_0, n_-) , where n_+ , n_0 , and n_- denote the number of orientation-preserving, degenerate, or orientation-reversing facets respectively.

Clearly $n_+ + n_0 + n_- = 5$ and since all 5 points cannot lie in one plane, we have that $n_+ \geq 1$, $n_- \geq 1$, and $n_0 \leq 1$. Therefore, the value $m = n_- - n_+$ can take an integer value between -3 and 3 ; in this case, we say that an array belongs to $\mathcal{A}_m$.

It defines a subdivision of $\mathcal{A}$ into 7 subsets $\mathcal{A}_{-3}, \dots, \mathcal{A}_3$ with combinatorial configuration as on the diagram; quadruples in one plane are marked in gray and the triple (n_+, n_0, n_-) is written below.

Every two quadrilaterals in the array have 3 common points that define a plane. If the remaining two points lie on opposite sides from the plane, then the corresponding facets have the same orientation; if they lie on one side, then the orientations are opposite. Therefore, the 7 subsets $\mathcal{A}_{-3}, \dots, \mathcal{A}_3$ can be described in the following way:

- $\mathcal{A}_{-3}$ — a tetrahedron with preserved orientation and one point inside.
- $\mathcal{A}_{-2}$ — a tetrahedron with preserved orientation and one point on a facet.
- $\mathcal{A}_{-1}$ — a double triangular pyramid formed by two tetrahedrons with preserved orientation.
- $\mathcal{A}_0$ — a pyramid over a convex quadrilateral
- $\mathcal{A}_1$ — a double triangular pyramid formed by two tetrahedrons with reversed orientation.
- $\mathcal{A}_2$ — a tetrahedron with reversed orientation and one point on a facet.
- $\mathcal{A}_3$ — a tetrahedron with reversed orientation and one point inside.

Note that the complement $\mathcal{A} \setminus \mathcal{A}_0$ has two connected components formed by $\mathcal{A}_- = \mathcal{A}_{-3} \cup \mathcal{A}_{-2} \cup \mathcal{A}_{-1}$ and $\mathcal{A}_+ = \mathcal{A}_3 \cup \mathcal{A}_2 \cup \mathcal{A}_1$. Observe that each array in $\mathcal{A}_-$ has at least 3 positively oriented facets and each array in $\mathcal{A}_+$ has at least 3 negatively oriented facets.

2.1. Observation. Let Q be a connected subset of $\mathcal{A}$ that does not intersect $\mathcal{A}_0$. Then either $Q \subset \mathcal{A}_+$ or $Q \subset \mathcal{A}_-$.

3 Associated form

In this section we recall some facts about the so-called *associated* form introduced in [6]; it is a quadratic form $W_{\mathbf{x}}$ on $\mathbb{R}^{n-1}$ associated to a given n -point array $\mathbf{x} = (x_1, \dots, x_n)$ in a metric space X .

Construction. Let $\triangle$ be the standard simplex $\triangle$ in $\mathbb{R}^{n-1}$; that is, the first $(n-1)$ of its vertices $v_1, \dots, v_n$ form the standard basis on $\mathbb{R}^{n-1}$, and $v_n = 0$.

Recall that $|a - b|_X$ denotes the distance between points a and b in the metric space X . Set

$$W_{\mathbf{x}}(v_i - v_j) = |x_i - x_j|_X^2$$

for all i and j . Note that this identity defines $W_{\mathbf{x}}$ uniquely.

The constructed quadratic form $W_{\mathbf{x}}$ will be called the *form associated to the point array $\mathbf{x}$* .

Note that an array $\mathbf{x} = (x_1, \dots, x_n)$ in a metric space X is isometric to an array in Euclidean space if and only if $W_{\mathbf{x}}(v) \geq 0$ for any $v \in \mathbb{R}^{n-1}$.

In particular, the condition $W_{\mathbf{x}} \geq 0$ for a triple $\mathbf{x} = (x_1, x_2, x_3)$ means that all three triangle inequalities for the distances between x_1, x_2 , and x_3 hold. For an n -point array, it implies that $W_{\mathbf{x}}(v) \geq 0$ for any vector v in a plane spanned by a triple x_i, x_j, x_k . In particular, we get the following:

3.1. Observation. Let $W_{\mathbf{x}}$ be a form on $\mathbb{R}^{n-1}$ associated with a point array $\mathbf{x} = (x_1, \dots, x_n)$. Suppose that L is a subspace of $\mathbb{R}^{n-1}$ such that $W_{\mathbf{x}}(v) < 0$ for any nonzero vector $v \in L$. Then the projections of any 3 vertices of $\triangle$ to the quotient space $\mathbb{R}^{n-1}/L$ are not collinear.

CAT(0) condition. Consider a point array $\mathbf{x}$ with 4 points. From 3.1, it follows that $W_{\mathbf{x}}$ is nonnegative on every plane parallel to a face of the tetrahedron $\triangle$. In particular, $W_{\mathbf{x}}$ can have at most one negative eigenvalue.

Assume $W_{\mathbf{x}}(w) < 0$ for some $w \in \mathbb{R}^3$. From 3.1, the line L_w spanned by w is transversal to each of 4 planes parallel to a face of $\triangle$.

Consider the projection of $\triangle$ along L_w to a transversal plane. The projection of the 4 vertices of $\triangle$ lie in general position; that is, no three of them lie on one line. Therefore, we can see one of two combinatorial pictures shown on the diagram. Since the set of lines L_w with $W_{\mathbf{x}}(w) < 0$ is connected, the combinatorics of the picture does not depend on the choice of w .

3.2. Claim. If CAT(0) comparison holds in X , then the diagram on the right cannot appear.

(The converse holds as well, but we will not need it.)

Proof. Suppose we see the picture on the right.

Let $[v_1, v_3]$ and $[v_2, v_4]$ be the line segments of $\triangle$ that correspond to the diagonals on the picture. Denote by m the point of $[v_1, v_3]$ that corresponds to the point of intersection.

In the plane spanned by $[v_2, v_4]$ and w , the vector w is timelike. Therefore we have the following reversed triangle inequality:

$$|v_2 - m| + |v_4 - m| < |v_2 - v_4|;$$

here we use shortcut $|a - b| = \sqrt{W(a - b)}$.

Note that the triangles $[v_1 v_2 v_3]$ and $[v_1 v_3 v_4]$ with metric induced by W are isometric to model triangles of $[x_1 x_2 x_3]$ and $[x_1 x_3 x_4]$. Whence (2+2)-point comparison does not hold. $\square$

The claim implies the following:

3.3. Observation. Suppose a metric on $\mathbf{x} = (x_1, \dots, x_n)$ satisfies CAT(0) comparison and $W_{\mathbf{x}}$ is its associated form on $\mathbb{R}^{n-1}$. Assume that L is a subspace of $\mathbb{R}^{n-1}$ such that $W_{\mathbf{x}}(v) < 0$ for any nonzero vector $v \in L$. Then if the projections of 4 vertices of $\triangle$ to the quotient space $\mathbb{R}^{n-1}/L$ lies in one plane, then its projection looks like the picture on the left; that is, one of the points lies in the triangle formed by the remaining three points.

3.4. Corollary. Suppose a metric on $\mathbf{x} = (x_1, \dots, x_5)$ satisfies CAT(0) comparison and $W_{\mathbf{x}}$ is its associated form on $\mathbb{R}^4$. Assume that L is a subspace of $\mathbb{R}^4$ such that $W_{\mathbf{x}}(v) < 0$ for any nonzero vector $v \in L$. Then $\dim L \leq 1$.

Moreover, if $\dim L = 1$, then the projections of the vertices of $\triangle$ to the quotient space $\mathbb{R}^3 = \mathbb{R}^4/L$ belong to $\mathcal{A} \setminus \mathcal{A}_0$ (defined in the previous section).

Proof. If $\dim L \geq 2$, then $\dim(\mathbb{R}^4/L) \leq 2$. By 3.1, these 5 projections lie in a general position; that is, no three of these projections lie on one line. Therefore, $\mathbb{R}^4/L = 2$ is the plane.

Any 5 points in a general position on the plane include 4 vertices of a convex quadrangle. The latter contradicts 3.3. $\square$

4 Convex spacelike surfaces

Let W be a quadratic form on $\mathbb{R}^4$. Suppose that W has exactly one negative eigenvalue. Choose future and past cones C^+ and C^- for W ; that is, C^+ and C^- are connected components of the set $\{v \in \mathbb{R}^4 \mid W(v) < 0\}$. A subset S in $\mathbb{R}^4$ will be called *spacelike* if $W(x - y) \geq 0$ for any $x, y \in S$.

Let K be a convex body in $\mathbb{R}^4$; denote by Σ the surface of K . A point p lies on the *upper side* of Σ (briefly $p \in \Sigma^+$) if there is a spacelike hyperplane in $\mathbb{R}^4$ that supports Σ at p from above; more precisely if the Minkowski sum $\{p\} + C^+$ does not intersect K .

Similarly, we define the *lower side* of Σ denoted by Σ^- . Note that Σ^+ and Σ^- might have common points. The subsets Σ^+ and Σ^- are spacelike; in particular, the length of any Lipschitz curve in these subsets can be defined and it leads to induced intrinsic pseudometrics on Σ^+ and Σ^- . Abusing notation, we will not distinguish a pseudometric space and the corresponding metric space.

4.1. Lemma. Let Σ be the surface of a convex set K in $\mathbb{R}^4$ and $C^\pm$ be the future and past cones for a quadratic form W . Then the upper and lower sides Σ^+ and Σ^- of Σ equipped with the induced intrinsic metric are CAT(0) length spaces.

Moreover, if a line segment $[pq]$ in $\mathbb{R}^4$ lies on Σ^+ , then $[pq]$ is a minimizing geodesic in Σ^+ ; that is,

$$|p - q|_{\Sigma^+}^2 = W(p - q).$$

This lemma is essentially stated by Anatolii Milka [1, Theorem 4]; we give a sketch of alternative proof based on smooth approximation.

Sketch. We can assume that W is nondegenerate; that is, after a linear change of coordinates it is the standard form on $\mathbb{R}^{3,1}$. If not, then there is a W -preserving projection of $\mathbb{R}^4$ to a W -nondegenerate subspace; apply this projection and note that this subspace is isometric a subspace of $\mathbb{R}^{3,1}$.

Assume S is a smooth strictly spacelike hypersurface in $\mathbb{R}^{3,1}$ with convex epigraph. By Gauss formula, S has nonpositive sectional curvature.

Suppose a strictly spacelike hyperplane Π cuts from S a disc D . Recall that Liberman's lemma [1, Theorem 3] implies that time coordinate is convex on any geodesic in S . We may assume that time is vanishing on Π ; therefore, by the lemma, D has a convex set in S . Therefore the Cartan–Hadamard theorem [4] implies that that D is CAT(0).

Now suppose D_n is a sequence of smooth discs of the described type that converges to a (possibly non-smooth) disc D . Note that the metric on D_n converges to the induced pseudometric on D . It follows that the metric space D' that corresponds to D is CAT(0).

The disc D might contain lightlike segments which have zero length. Note that every maximal lightlike segment in D starts at its interior point and goes to the boundary. Consider the map $\iota: D \rightarrow D$ that sends each maximal lightlike segment to its starting point. Note that the sublemma below implies that ι is length-nonincreasing. Since $|x - \iota(x)|_D = 0$, we get that the D' is isometric to the image of ι with the induced metric.

Consider the Minkowski sum

$$K^- = K + C^+;$$

it has a convex spacelike boundary ∂K^- . Choose a strictly spacelike hyperplane Π that lies above K . Denote by D the subset of ∂K^- below Π . Let us equip D with induced intrinsic pseudometric. By construction Σ^- is isometric to $\iota(D)$. It follows that Σ^- is CAT(0).

Now suppose a line segment $[pq]$ in $\mathbb{R}^4$ lies on Σ^- . Choose a supporting hyperplane Π at the midpoint of $[pq]$. Choose time coordinate that vanish on Π ; by Liberman's lemma, every shortest path in Σ^- between p and q has to lie on Π ; that is, the intersection $\Sigma^- \cap \Pi$ is a convex subset of Σ^- . Therefore $[pq]$ is convex in Σ^- which implies the second statement. $\square$

4.2. Sublemma. Let u and v be two lightlike vectors in $\mathbb{R}^{3,1}$. Suppose that the union of two half-lines $s \mapsto p + s \cdot u$ and $t \mapsto q + t \cdot v$ for $s, t \geq 0$ is a spacelike set. Then the function $(s, t) \mapsto |(p + s \cdot u) - (q + t \cdot v)|$ is nondecreasing in both arguments, where $|w| := \sqrt{\langle w, w \rangle}$ for a spacelike vector w .

Proof. Since u and v are lightlike, $\langle u, u \rangle = \langle v, v \rangle = 0$. Since the union of two half-lines is spacelike, $(p + s \cdot u) - (q + t \cdot v)$ is spacelike for any $s, t \geq 0$. It follows that

$$\begin{aligned} 0 &\leq |(p + s \cdot u) - (q + t \cdot v)|^2 = \\ &= |p - q|^2 - 2 \cdot s \cdot \langle u, q - p \rangle - 2 \cdot t \cdot \langle v, p - q \rangle - 2 \cdot s \cdot t \cdot \langle u, v \rangle \end{aligned}$$

for any $s, t \geq 0$. Therefore

$$\langle u, q - p \rangle \leq 0, \quad \langle v, p - q \rangle \leq 0 \quad \langle u, v \rangle \leq 0.$$

Whence the result. $\square$

Assume v is a nonzero vector in $\mathbb{R}^4$ and $p \in \Sigma$. We say that p lies on the *upper side* of Σ with respect to v (briefly $p \in \Sigma^+(v)$) if $p + t \cdot v \notin K$ for any $t > 0$. Correspondingly, p lies on the *lower side* of Σ with respect to v (briefly $p \in \Sigma^-(v)$) if $p + t \cdot v \notin K$ for any $t < 0$.

4.3. Observation. Let K be a compact convex set in $\mathbb{R}^4$ and $C^\pm$ be the future and past cones for a quadratic form W . Then the upper (lower) side of the boundary surface Σ of K can be described as the intersection of the upper (respectively lower) sides of Σ with respect to all vectors $v \in C^+$; that is,

$$\Sigma^\pm = \bigcap_{v \in C^+} \Sigma^\pm(v).$$

5 Proof assembling

Proof of Toyoda's theorem. Let $\{x_1, \dots, x_5\}$ be the points in P . Choose a 5-simplex $\triangle$ in $\mathbb{R}^4$; denote by W the form associated with the point array $(x_1, \dots, x_5)$.

If $W \geq 0$, then P admits a distance preserving embedding into Euclidean 4-space, so one can take the convex hull of its image as X .

Suppose $W(v) < 0$ for some $v \in \mathbb{R}^4$. Since P is CAT(0), 3.4 implies that W has exactly one negative eigenvalue. Moreover, if a line L is spanned by a vector v such that $W(v) < 0$, then the projection of the vertices of the simplex to $\mathbb{R}^3 = \mathbb{R}^4/L$ belongs to $\mathcal{A} \setminus \mathcal{A}_0$.

The space of such lines L is connected. By 2.1, we can assume that all the projections belong to $\mathcal{A}_-$. That is, we can choose timelike orientation such that for any $v \in C^+$ the lower part $\Sigma^-(v)$ of $\Sigma = \partial\triangle$ has at least 3 facets of $\triangle$.

In particular, $\Sigma^-(v)$ contains all edges of $\triangle$ for any $v \in C^+$. By 4.3, Σ^- contains all edges of $\triangle$. By 4.1, Σ^- with induced (pseudo)metric is a length CAT(0) space.

Since all edges of $\triangle$ lie in Σ^- , the inclusion $P \hookrightarrow \Sigma^-$ is distance preserving. Whence we can take $X = \Sigma^-$.

Finally, observe that in each case X is a subcomplex of $\triangle$ that includes all edges and has a model metric on each simplex. $\square$

6 Remarks

Let us recall the definition of *graph comparison* given by Vladimir Zolotov and the authors [5] and use it to formulate a few related questions.

Let Γ be a graph with vertices $v_1, \dots, v_n$. A metric space X is said to meet the Γ -comparison if for any set of points in X labeled by vertices of Γ there is a model configuration $\tilde{v}_1, \dots, \tilde{v}_n$ in the Hilbert space $\mathbb{H}$ such that if v_i is adjacent to v_j , then

$$|\tilde{v}_i - \tilde{v}_j|_{\mathbb{H}} \leq |v_i - v_j|_X$$

and if v_i is nonadjacent to v_j , then

$$|\tilde{v}_i - \tilde{v}_j|_{\mathbb{H}} \geq |v_i - v_j|_X.$$

The C_4 -comparison (for the 4-cycle C_4 on the diagram) defines CAT(0) comparison. Tetsu Toyoda have shown that C_4 -comparison implies graph comparisons for all cycles C_n [8]; remarkably, the metric space is *not* assumed to be intrinsic. The O_3 -comparison (for the octahedron graph O_3 on the diagram) defines another

comparison. Since O_3 contains C_4 as an induced subgraph, we get that O_3 -comparison is stronger than C_4 -comparison.

6.1. Open question. *Is it true that octahedron-comparison holds in any 6 points in a length $\text{CAT}(0)$ space?*

And, assuming the answer is affirmative, what about the converse: is it true that any 6-point metric space that satisfies octahedron-comparison admits a distance preserving embedding in a length $\text{CAT}(0)$ space?

The analogous questions for spaces with nonnegative curvature in the sense of Alexandrov (briefly $\text{CBB}(0)$) are open as well. The $\text{CBB}(0)$ comparison is equivalent to the 3-tree comparison (for the tripod-tree shown first on the following diagram). It turns out that any length $\text{CBB}(0)$ space satisfies the comparison for

the other trees on the diagram; it is formed by an infinite family of star-shaped trees and two trees with 6 vertices [2, 5]. (The 4-tree comparison (the second tree on the diagram) is equivalent to the so-called $(4+1)$ -point comparison in the terminology of [2].)

We expect that this comparison provides a necessary and sufficient condition for 5-point sets. Namely, we expect an affirmative answer to the following stronger question.

6.2. Question. *Suppose a 5-point metric space P satisfies the 4-tree comparison. Is it true that P admits a distance preserving embedding into a length $\text{CBB}(0)$ space?*

Finally, let us mention a related question about a 6-point condition.

6.3. Question. *Suppose a 6-point metric space P satisfies the 5-tree, $2(2)$ -tree, and $3(1)$ -tree comparisons. Is it true that P admits a distance preserving embedding into a length $\text{CBB}(0)$ space?*

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