DE GRUYTER Anal. Geom. Metr. Spaces 2021; 9:167–185

Research Article **Open Access**

Toni Ikonen*

Quasiconformal Jordan Domains

https://doi.org/10.1515/agms-2020-0127
Received November 13, 2020; accepted July 8, 2021

Abstract: We extend the classical Carathéodory extension theorem to quasiconformal Jordan domains (Y, d_Y). We say that a metric space (Y, d_Y) is a *quasiconformal Jordan domain* if the completion $\overline{Y}$ of (Y, d_Y) has finite Hausdorff 2-measure, the *boundary* $\partial Y = \overline{Y} \setminus Y$ is homeomorphic to $\mathbb{S}^1$, and there exists a homeomorphism $\phi\colon \mathbb{D} \to (Y, d_Y)$ that is quasiconformal in the geometric sense.
We show that ϕ has a continuous, monotone, and surjective extension $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$. This result is best possible in this generality. In addition, we find a necessary and sufficient condition for Φ to be a quasiconformal homeomorphism. We provide sufficient conditions for the restriction of Φ to $\mathbb{S}^1$ being a quasisymmetry and to ∂Y being bi-Lipschitz equivalent to a quasicircle in the plane.

Keywords: quasiconformal; metric surface; Carathéodory; Beurling–Ahlfors

MSC: Primary 30L10, Secondary 30C65, 28A75, 51F99.

1 Introduction

Let (X, d_X) be a metric space with locally finite Hausdorff 2-measure. If X is also homeomorphic to a 2-manifold, we say that (X, d_X) is a *metric surface*. A homeomorphism $\phi\colon (X, d_X) \to (Y, d_Y)$ between metric surfaces is *quasiconformal* if there exists $K \geq 1$ such that for all path families Γ,

$$K^{-1} \operatorname{mod} \Gamma \leq \operatorname{mod} \phi\Gamma \leq K \operatorname{mod} \Gamma, \tag{1.1}$$

where $\operatorname{mod} \Gamma$ is the *conformal modulus* of Γ, see Section 2.3.

We say that a metric surface (Y, d_Y) is a *metric Jordan domain* if the metric completion $\overline{Y}$ is homeomorphic to the closed unit disk $\overline{\mathbb{D}}$, the *boundary* $\partial Y = \overline{Y} \setminus Y$ is homeomorphic to the unit circle $\mathbb{S}^1$, and the Hausdorff 2-measure of $\overline{Y}$ is finite.

A metric Jordan domain is a *quasiconformal Jordan domain* if there exists a quasiconformal homeomorphism $\phi\colon \mathbb{D} \to (Y, d_Y)$. A metric Jordan domain is a quasiconformal one if and only if (Y, d_Y) is *reciprocal* as introduced in [18, Theorem 1.4]; see Definition 2.5. This uses the facts that $\mathcal{H}^2_{\overline{Y}}(\overline{Y}) < \infty$ and that ∂Y is a non-trivial continuum.

In general, it is not true that the completion $\overline{Y}$ of a quasiconformal Jordan domain is a quasiconformal image of the closed unit disk $\overline{\mathbb{D}}$. We illustrate this with an example after Theorem 1.1. Contrast this with the classical case when Y is a Jordan domain in the plane $\mathbb{R}^2$. Then any 1-quasiconformal homeomorphism $\phi\colon \mathbb{D} \to Y$, i.e., any Riemann map from the unit disk onto Y extends to a homeomorphism $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$ by a result known as the Carathéodory extension theorem [9, Chapter I, Theorem 3.1]. In fact, the extension still satisfies (1.1) with $K = 1$. Additionally, if $\phi\colon \mathbb{D} \to Y$ is K-quasiconformal for some $K \geq 1$, it still has a homeomorphic extension to the boundary.

***Corresponding Author: Toni Ikonen:** Department of Mathematics and Statistics,University of Jyvaskyla, Jyvaskyla, Finland, E-mail: toni.m.h.ikonen@jyu.fi

1.1 Carathéodory's theorem

We prove the following generalization of the classical Carathéodory extension theorem of quasiconformal maps.

Theorem 1.1. Let $\phi\colon \mathbb{D} \to Y$ be a quasiconformal map onto a quasiconformal Jordan domain. Then there exists an extension $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$ of ϕ that is surjective, monotone and $\Phi(\mathbb{S}^1) = \partial Y$.

Here we say that a map is *monotone* if it is continuous and the preimage of every point is a *continuum*, i.e., a compact and connected set.

The map Φ might fail to be a homeomorphism. As an example, consider the length space X homeomorphic to $\mathbb{R}^2$ obtained by collapsing the Euclidean square $[0,1]^2$ in $\mathbb{R}^2$ to a point. Let $\pi\colon \mathbb{R}^2 \to X$ denote the associated 1-Lipschitz quotient map. We define $Y = \pi((1,2)\times(0,1))$. Then $\partial Y = \pi(\partial\,[1,2]\times[0,1])$. The restriction of π to $(1,2)\times(0,1)$ is a 1-quasiconformal map, but its extension collapses the arc segment $\{1\}\times[0,1]$ to the singleton $\pi([0,1]^2)$. By considering a Riemann map $f\colon \mathbb{D} \to (0,1)^2$, the claim follows by setting $\phi = \pi\circ f$.

Next, we investigate when the extension in Theorem 1.1 is a quasiconformal homeomorphism. To this end, for every $y \in \overline{Y}$ and $\operatorname{diam} \overline{Y} \geq R > r > 0$, we let $\Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y})$ denote the family of paths joining $\overline{B}_{\overline{Y}}(y,r)$ to $\overline{Y} \setminus B_{\overline{Y}}(y,R)$.

Proposition 1.2. The extension Φ in Theorem 1.1 is quasiconformal if and only if for every $y \in \partial Y$ and $R > 0$ for which $\overline{Y} \setminus B_{\overline{Y}}(y,R) \neq \emptyset$,

$$\lim_{r\to 0^+} \operatorname{mod} \Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y}) = 0. \tag{1.2}$$

Moreover, if (1.2) holds at each $y \in \partial Y$ and ϕ is K-quasiconformal, then Φ is K-quasiconformal.

A well-known fact is that if there exists $C_U > 0$ such that for all $y \in \partial Y$ and $0 < r < \operatorname{diam} \partial Y$,

$$\mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y,r)) \leq C_U r^2, \tag{1.3}$$

then (1.2) holds; see Lemma 2.8. The condition (1.2) has a close link to the reciprocality condition introduced in [18]; see Definition 2.5. The aforementioned example of the collapsed disk $[0,1]^2$ fails (1.2) at exactly one point.

It can happen that the extension Φ in Theorem 1.1 is a homeomorphism, but not quasiconformal; see [14, Example 6.1]. There we have a metric space X for which there exists a 1-Lipschitz homeomorphism $\pi\colon \mathbb{R}^2 \to X$ which is 1-quasiconformal outside a Cantor set $K \subset [0,1]\times\{0\}$, but $\pi|_{(0,1)^2}$ does not extend to a 1-quasiconformal homeomorphism on $[0,1]^2$. The claim follows by setting $Y = \pi((0,1)^2)$ and setting $\phi = \pi\circ f$ for any Riemann map $f\colon \mathbb{D} \to (0,1)^2$.

1.2 Quasicircles

Consider a quasiconformal Jordan domain Y whose boundary points satisfy the area growth inequality (1.3). We know from Proposition 1.2 that the extension $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$ of any quasiconformal homeomorphism $\phi\colon \mathbb{D} \to Y$ is a quasiconformal homeomorphism. In particular, the *boundary map* $g_\phi = \Phi|_{\mathbb{S}^1}\colon \mathbb{S}^1 \to \partial Y$ is a homeomorphism.

We are especially interested when we can deduce that ∂Y is a *quasicircle*, i.e., a quasisymmetric image of $\mathbb{S}^1$. We refer the reader to Section 2 for definitions.

Theorem 1.3 (Beurling–Ahlfors extension)**.** Suppose that Y is a quasiconformal Jordan domain whose boundary points satisfy the area growth (1.3).

If $\phi\colon \mathbb{D} \to Y$ is a quasiconformal homeomorphism, then the boundary map g_ϕ is a quasisymmetry if and only if ∂Y has bounded turning. If ∂Y has bounded turning, then any quasisymmetry $g\colon \mathbb{S}^1 \to \partial Y$ is the boundary map of some quasiconformal map $\phi\colon \mathbb{D} \to Y$.

Theorem 1.3 has a parallel in the classical literature. Ahlfors and Beurling proved in [2] that every quasisymmetry $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ is the boundary homeomorphism of some quasiconformal map $\phi\colon \mathbb{D} \to \mathbb{D}$. In fact, we apply their result in proving that quasisymmetries $g\colon \mathbb{S}^1 \to \partial Y$ extend like claimed. It is also known that the boundary homeomorphisms of quasiconformal maps $\phi\colon \mathbb{D} \to \mathbb{D}$ are quasisymmetries. So we also recover this result with the assumptions of Theorem 1.3.

We now know from Theorem 1.3 that ∂Y is a quasicircle in some situations. We are interested whether or not ∂Y can be bi-Lipschitz embedded into the plane. We say that a quasicircle Z is *planar*, or a *planar quasicircle*, if there exists a bi-Lipschitz embedding $h\colon Z \to \mathbb{R}^2$.

One of the main results obtained in [13] states that a quasicircle is planar if and only if its *Assouad dimension* is strictly less than two, see Definition 2.10. There are quasicircles for every Assouad dimension between 1 and ∞ since $Z = (\mathbb{S}^1, \|\cdot\|_2^\alpha)$ for $0 < \alpha \leq 1$ has Assouad dimension α^{-1}.

Proposition 1.4. Let Y be a quasiconformal Jordan domain. If ∂Y is a quasicircle and the boundary points satisfy the area growth (1.3), then the Assouad dimension of ∂Y is at most two.

It is not clear if ∂Y in the above statement must be planar. However, if Y is *annularly linearly locally connected (ALLC)* and *Ahlfors* 2*-regular*, then ∂Y is a planar quasicircle [17, Theorems 8.1 and 8.2]; see [17] for the proofs and terminology. Quasiconformal Jordan domains satisfying these stronger assumptions appear in [24] and [3].

We localize these assumptions in the following statement and obtain the same conclusion.

Theorem 1.5. Let Y be a quasiconformal Jordan domain such that ∂Y is a quasicircle and its boundary points satisfy the area growth (1.3). Then the Assouad dimension of ∂Y is strictly less than two if the following two conditions are satisfied for some $r_0 > 0$, $C > 0$ and $\lambda > 1$:

(a) For every $y \in \partial Y$ and $0 < 2r < R < r_0$ and any pair $a, b \in \overline{B}_{\overline{Y}}(y, R) \setminus B_{\overline{Y}}(y, r)$, there exists a path $|\alpha| \subset \overline{B}_{\overline{Y}}(y, \lambda R) \setminus B_{\overline{Y}}(y, \lambda^{-1} r)$ containing a and b.
(b) For every $z \in Y$ with $0 < r < d(z, \partial Y) \leq r_0$, $\mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(z, r)) \geq C^{-1} r^2$.

In particular, ∂Y is planar.

If (a) holds we say that ∂Y is *relatively ALLC* and if (b) holds, we say that Y satisfies the *Ahlfors lower bound* near ∂Y. The main point of Theorem 1.5 is to only restrict the geometry of $\overline{Y}$ near the boundary ∂Y.

The relative ALLC guarantees that ∂Y is *porous* in $\overline{Y}$ below the given scale r_0 (Lemma 5.2). The porosity allows us to pack many balls in Y, well-disjoint from ∂Y, near all points of ∂Y at all scales below r_0. Now the Ahlfors lower bound, valid for such balls, combined with the upper bound (1.3) allow us to control quantitatively the total amount of such non-overlapping balls in a given interval of scales. This quantification allows us to prove planarity for ∂Y. This idea appears in [3, Lemma 3.12], where the authors prove that a compact set in an Ahlfors regular space is porous if and only if its Assouad dimension is strictly smaller than the homogeneous dimension of the space. A similar argument also works in the setting of Theorem 1.5.

1.3 Outline

In Section 2, we introduce the notations we use and some preliminary results. In Section 3, we prove Theorem 1.1. Theorem 1.3 and Proposition 1.4 are proved in Section 4. Theorem 1.5 is proved in Section 5. Section 6 contains some concluding remarks.

2 Preliminaries

2.1 Notation

Let (Y, d_Y) be a metric space. The open ball centered at a point $y \in Y$ of radius $r > 0$ with respect to the metric d is denoted by $B_Y(y, r)$. The closed ball is denoted by $\overline{B}_Y(y, r)$. We sometimes omit the subscript from d_Y, from B_Y, and from $\overline{B}_Y$, respectively.

We recall the definition of Hausdorff measure. Let (Y, d) be a metric space. For all $Q \geq 0$, the *Q-dimensional Hausdorff measure (Hausdorff Q-measure)* is defined by

$$\mathcal{H}^Q_Y(B) = \frac{\alpha(Q)}{2^Q} \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} B_i)^Q \colon B \subset \bigcup_{i=1}^{\infty} B_i, \operatorname{diam} B_i < \delta \right\}$$

for all sets $B \subset Y$, where $\alpha(Q) = \pi^{\frac{Q}{2}} \left(\Gamma\left(Q/2 + 1\right)\right)^{-1}$. The constant $\alpha(Q)$ is chosen in such a way that $\mathcal{H}^n_{\mathbb{R}^n}$ coincides with the Lebesgue measure $\mathcal{L}^n$ for all positive integers.

The *length* of a path $\gamma\colon [a, b] \to Y$ is defined as

$$\ell_d(\gamma) = \sup \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)),$$

the supremum taken over all finite partitions $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$. A path is *rectifiable* if it has finite length.

The *metric speed* of a path $\gamma\colon [a, b] \to Y$ at the point $t \in [a, b]$ is defined as

$$v_\gamma(t) = \lim_{h\to 0^+} \frac{d(\gamma(t+h), \gamma(t))}{h}$$

whenever this limit exists. If γ is rectifiable, its metric speed exists at $\mathcal{L}^1$-almost every $t \in [a, b]$ [7, Theorem 2.1].

A rectifiable path $\gamma\colon [a, b] \to Y$ is *absolutely continuous* if for all $a \leq s \leq t \leq b$,

$$d(\gamma(t), \gamma(s)) \leq \int_s^t v_\gamma(u)\, d\mathcal{L}^1(u)$$

with $v_\gamma \in L^1([a, b])$ and $\mathcal{L}^1$ the Lebesgue measure on the real line. Equivalently, γ is absolutely continuous if it maps sets of $\mathcal{L}^1$-measure zero to sets of $\mathcal{H}^1_Y$-measure zero in its image [7, Section 3].

Let $\gamma\colon [a, b] \to X$ be an absolutely continuous path. Then the *path integral* of a Borel function $\rho\colon X \to [0, \infty]$ *over* γ is

$$\int_\gamma \rho\, ds = \int_a^b (\rho \circ \gamma) v_\gamma\, d\mathcal{L}^1. \tag{2.1}$$

If γ is rectifiable, then the *path integral* of ρ over γ is defined to be the path integral of ρ over the arc length parametrization γ_s of γ; see for example Chapter 5 of [12].

2.2 Quasiconformal Jordan domains

We assume that Y is a quasiconformal Jordan domain. In particular, its completion $\overline{Y}$ is homeomorphic to $[0, 1]^2$ and has finite Hausdorff 2-measure.

Given a Borel set $A \subset Y$, the *length* of a path $\gamma\colon [a, b] \to Y$ *in* A is defined as $\int_Y \chi_A(y)\#(\gamma^{-1}(y))\, d\mathcal{H}^1_Y(y)$, where $\#(\gamma^{-1}(x))$ is the counting measure of $\gamma^{-1}(x)$. This formula makes sense for paths that are not necessarily rectifiable [8, Theorem 2.10.13]. When γ is rectifiable, for every Borel function $\rho\colon \overline{Y} \to [0, \infty]$,

$$\int_\gamma \rho\, ds = \int_{\overline{Y}} \rho(x)\#(\gamma^{-1}(x))\, d\mathcal{H}^1_{\overline{Y}}(x). \tag{2.2}$$

The equality (2.2) follows from [8, Theorem 2.10.13] via a standard approximation argument using simple functions.

We recall a special case of the coarea inequality [8, 2.10.25]. Let $\alpha \in \{0, 1\}$, $B \subset \overline{Y}$ Borel, and $f\colon \overline{Y} \to \mathbb{R}$ 1-Lipschitz. Then

$$\int_{\mathbb{R}}^{*} \mathcal{H}^{\alpha}_{\overline{Y}}(B \cap f^{-1}(t))\, d\mathcal{L}^1(t) \leq C_\alpha \mathcal{H}^{\alpha+1}_{\overline{Y}}(B), \tag{2.3}$$

where $C_0 = 1$ and $C_1 = 4/\pi$. Here $\int^*$ refers to the upper integral [8, 2.4.2]. If $\mathcal{H}^{\alpha+1}_{\overline{Y}}(B) < \infty$, the upper integral can be replaced with the usual one [8, 2.10.26].

Via a standard approximation argument using simple functions, we obtain the following.

Theorem 2.1. Let $f\colon \overline{Y} \to \mathbb{R}$ be 1-Lipschitz, $\alpha \in \{0, 1\}$ and C_α as in (2.3). Then, for every Borel function $g\colon \overline{Y} \to [0, \infty]$,

$$\int_{\mathbb{R}}^{*} \int_{f^{-1}(t)} g(y)\, d\mathcal{H}^{\alpha}_{\overline{Y}}(y)\, d\mathcal{L}^1(t) \leq C_\alpha \int_{\overline{Y}} g(y)\, d\mathcal{H}^{\alpha+1}_{\overline{Y}}(y).$$

When g is $\mathcal{H}^{\alpha+1}_{\overline{Y}}$-integrable, the upper integral can be replaced with the usual one.

In the following, we say that $C \subset \overline{Y}$ is a *continuum* if C is compact and connected. A compact set $F \subset \overline{Y}$ *separates* $x, y \in \overline{Y}$ if $x, y \in \overline{Y} \setminus F$ and the points are in different connected components of $\overline{Y} \setminus F$.

Lemma 2.2. Let $F \subset \overline{Y}$ be compact and $x, y \in \overline{Y}$ separated by F. Then there exists a continuum $C \subset F$ that separates x and y.

Proof. Let x, y and F be as in the claim. We consider a homeomorphism $h\colon \overline{Y} \to Z \subset \mathbb{S}^2$, where Z is the union of the equator and the southern hemisphere of $\mathbb{S}^2$. Then there exists a nested sequence of quadrilaterals $Z_n \supset Z_{n+1} \supset Z$ such that $h(x), h(y) \in Z_n$ is an interior point of Z_n for each $n \in \mathbb{N}$ and $Z = \bigcap_{n=1}^{\infty} Z_n$.

Since $F_n = \partial Z_n \cup h(F) \subset Z_n$ separates $h(x)$ and $h(y)$ in $\mathbb{S}^2$, there exists a continuum $C_n \subset F_n$ separating $h(x)$ and $h(y)$ in $\mathbb{S}^2$ [23, Chapter 2, Lemma 5.20]. In particular, for every path $\gamma\colon [0, 1] \to Z$ joining $h(x)$ to $h(y)$, there exists $z_n \in C_n \cap |\gamma|$ for every $n \in \mathbb{N}$. Up to passing to a subsequence and relabeling, the continua $(C_n)_{n=1}^{\infty}$ converge to a continuum $C' \subset \bigcap_{n=1}^{\infty} h(F) \cap Z_n = h(F)$ in the Hausdorff convergence [1, Theorems 4.4.15 and 4.4.17]. If γ and $(z_n)_{n=1}^{\infty}$ are as above, the accumulation points of $(z_n)_{n=1}^{\infty}$ are contained in $C' \cap |\gamma|$ [1, Proposition 4.4.14]. Consequently, $C' \cap |\gamma| \neq \emptyset$ for every such γ. Hence $C = h^{-1}(C') \subset F$ is a continuum separating x and y. □

2.3 Metric Sobolev spaces

In this section we give an overview of Sobolev theory in the metric surface setting, and refer to [12] for a comprehensive introduction.

Let $\Gamma \subset \mathcal{C}([0, 1]\,; Y)$ be a family of rectifiable paths in Y. A Borel function $\rho\colon Y \to [0, \infty]$ is *admissible* for Γ if the path integral $\int_\gamma \rho\, ds \geq 1$ for all rectifiable paths $\gamma \in \Gamma$. The *modulus* of Γ is

$$\operatorname{mod} \Gamma = \inf \int_Y \rho^2\, d\mathcal{H}^2_Y,$$

where the infimum is taken over all admissible functions ρ. Observe that if Γ_1 and Γ_2 are path families and every path $\gamma_1 \in \Gamma_1$ contains a subpath $\gamma_2 \in \Gamma_2$, then $\operatorname{mod} \Gamma_1 \leq \operatorname{mod} \Gamma_2$. In particular, this holds if $\Gamma_1 \subset \Gamma_2$. A property holds for *almost every* path if the family of paths for which the property fails has zero modulus.

Let $\psi\colon (Y, d_Y) \to (Z, d_Z)$ be a mapping between metric spaces Y and Z. A Borel function $\rho\colon Y \to [0, \infty]$ is an *upper gradient of* ψ if

$$d_Y(\psi(x), \psi(y)) \leq \int_\gamma \rho \, ds$$

for every rectifiable path $\gamma\colon [0, 1] \to Y$ connecting x to y. The function ρ is a *weak upper gradient of* ψ if the same holds for almost every rectifiable path.

A weak upper gradient $\rho \in L^2_{\mathrm{loc}}(Y)$ of ψ is *minimal* if it satisfies $\rho \leq \widetilde{\rho}$ almost everywhere for all weak upper gradients $\widetilde{\rho} \in L^2_{\mathrm{loc}}(Y)$ of ψ. If ψ has a weak upper gradient $\rho \in L^2_{\mathrm{loc}}(Y)$, then ψ has a minimal weak upper gradient, which we denote by ρ_ψ. We refer to Section 6 of [12] and Section 3 of [26] for details.

Fix a point $z \in Z$, and let $d_z = d_Z(\cdot, z)$. The space $L^2(Y, Z)$ is defined as the collection of measurable maps $\psi\colon Y \to Z$ such that $d_z \circ \psi$ is in $L^2(Y)$.

Moreover, $L^2_{\mathrm{loc}}(Y, Z)$ is defined as those measurable maps $\psi\colon Y \to Z$ for which, for all $y \in Y$, there is an open set $U \subset Y$ containing y such that $\psi|_U$ is in $L^2(U, Z)$.

The metric Sobolev space $N^{1,2}_{\mathrm{loc}}(Y, Z)$ consists of those maps $\psi\colon Y \to Z$ in $L^2_{\mathrm{loc}}(Y, Z)$ that have a minimal weak upper gradient $\rho_\psi \in L^2_{\mathrm{loc}}(Y)$.

For open $\emptyset \neq U \subset Y$, we say that $\psi \in N^{1,2}(U, Z)$ if $\psi|_U \in N^{1,2}_{\mathrm{loc}}(U, Z)$, $\rho_{\psi|_U} \in L^2(U)$ and $\psi|_U \in L^2(U, Z)$.

Given a homeomorphism $\psi\colon Y \to Z$, the pullback measure $\psi^*\mathcal{H}^2_Z$ is defined by $\psi^*\mathcal{H}^2_Z(B) = \mathcal{H}^2_Z(\psi(B))$ for each Borel set $B \subset Y$. The pullback measure has a decomposition $\psi^*\mathcal{H}^2_Z = J_\psi \mathcal{H}^2_Y + \mu^\perp$, where J_ψ is locally integrable with respect to $\mathcal{H}^2_Y$, and the measures $\mathcal{H}^2_Y$ and $\mu^\perp$ are singular [5, Sections 3.1-3.2 in Volume I]. We call the density J_ψ the *Jacobian* of ψ.

2.4 Quasiconformal mappings

We define quasiconformal maps and recall some basics.

Definition 2.3. Let (Y, d_Y) and (Z, d_Z) be metric spaces with locally finite Hausdorff 2-measures. We say that a homeomorphism $\psi\colon (Y, d_Y) \to (Z, d_Z)$ is *quasiconformal* if there exists $K \geq 1$ such that for all path families Γ in Y

$$K^{-1} \operatorname{mod} \Gamma \leq \operatorname{mod} \psi\Gamma \leq K \operatorname{mod} \Gamma, \tag{2.4}$$

where $\psi\Gamma = \{\psi \circ \gamma\colon \gamma \in \Gamma\}$. If (2.4) holds with a constant $K \geq 1$, we say that ψ is *K-quasiconformal*.

Definition 2.3 is sometimes called the *geometric* definition of quasiconformality. A special case of [26, Theorem 1.1] yields the following.

Theorem 2.4. Let Y and Z be metric spaces with locally finite Hausdorff 2-measure and $\psi\colon Y \to Z$ a homeomorphism. The following are equivalent for the same constant $K > 0$:

(i) $\operatorname{mod} \Gamma \leq K \operatorname{mod} \psi\Gamma$ for all path families Γ in Y.
(ii) $\psi \in N^{1,2}_{\mathrm{loc}}(Y, Z)$ and satisfies $\rho^2_\psi(y) \leq K J_\psi(y)$ for $\mathcal{H}^2_Y$-almost every $y \in Y$.

The *outer dilatation* of ψ is the smallest constant $K_O \geq 0$ for which the modulus inequality $\operatorname{mod} \Gamma \leq K_O \operatorname{mod} \psi\Gamma$ holds for all Γ in Y. The *inner dilatation* of ψ is the smallest constant $K_I \geq 0$ for which $\operatorname{mod} \psi\Gamma \leq K \operatorname{mod} \Gamma$ holds for all Γ in Y. The number $K(\psi) = \max\{K_I(\psi), K_O(\psi)\}$ is the *maximal dilatation* of ψ.

For a set $G \subset Y$ and disjoint sets $F_1, F_2 \subset G$, let $\Gamma(F_1, F_2; G)$ denote the family of paths that start from F_1, end in F_2 and whose images are contained in G. A *quadrilateral* is a set Q homeomorphic to $[0, 1]^2$ with boundary ∂Q consisting of four boundary arcs, overlapping only at the end points, labelled $\xi_1, \xi_2, \xi_3, \xi_4$ in cyclic order.

Definition 2.5. A metric surface Y is *reciprocal* if there exists a constant $\kappa \geq 1$ such that

$$\kappa^{-1} \leq \operatorname{mod} \Gamma(\xi_1, \xi_3; Q) \operatorname{mod} \Gamma(\xi_2, \xi_4; Q) \leq \kappa \tag{2.5}$$

for every quadrilateral $Q \subset Y$, and

$$\lim_{r\to 0^+} \operatorname{mod} \Gamma\left(\overline{B}_Y(y,r), Y \setminus B_Y(y,R); \overline{B}_Y(y,R)\right) = 0 \tag{2.6}$$

for all $y \in Y$ and $R > 0$ such that $Y \setminus B_Y(y,R) \neq \emptyset$.

We note that the product in (2.5) is always bounded from below by a universal constant $\kappa_0 > 0$ [19]. We also have the following.

Proposition 2.6 (Corollary 12.3 of [18])**.** Let Y be a metric surface, $U \subset Y$ a domain, and $\psi\colon U \to \Omega \subset \mathbb{R}^2$ a homeomorphism. If $K_O(\psi) < \infty$, then $K_I(\psi) \leq (2 \cdot \kappa_0)\, K_O(\psi) < \infty$.

Recall the definition of quasiconformal Jordan domain from the introduction.

Proposition 2.7. Let Y be a quasiconformal Jordan domain and $\Psi\colon \overline{Y} \to \overline{\mathbb{D}}$ a homeomorphism. Then $K_O(\Psi) \leq K < \infty$ if and only if there exists a constant $C > 0$ such that

$$\liminf_{r\to 0^+} \operatorname{mod} \Psi^{-1}\Gamma\left(\overline{B}_{\overline{\mathbb{D}}}(x,r), \overline{\mathbb{D}} \setminus B_{\overline{\mathbb{D}}}(x,2r); \overline{B}_{\overline{\mathbb{D}}}(x,2r)\right) \leq C \tag{2.7}$$

for every $x \in \overline{\mathbb{D}}$. The constants K and C depend on each other quantitatively. Moreover, $K_I(\Psi) \leq (2 \cdot \kappa_0) \cdot K$.

Proof. Since the Lebesgue 2-measure on $\overline{\mathbb{D}}$ is doubling, Theorem 1.2 of [26] states that an upper bound $K_O(\Psi) \leq K$ is quantitatively equivalent to the following statement: there exists $C' \geq 1$ such that for every $x \in \overline{\mathbb{D}}$,

$$\liminf_{r\to 0^+} \frac{r^2 \operatorname{mod} \Psi^{-1}\Gamma\left(\overline{B}_{\overline{\mathbb{D}}}(x,r), \overline{\mathbb{D}} \setminus B_{\overline{\mathbb{D}}}(x,2r); \overline{B}_{\overline{\mathbb{D}}}(x,2r)\right)}{\mathcal{L}^2(\overline{B}_{\overline{\mathbb{D}}}(x,r))} \leq C'.$$

Since $\mathcal{L}^2(B_{\overline{\mathbb{D}}}(x,r))$ is comparable to r^2, $K_O(\Psi) \leq K$ if and only if (2.7) holds for some C, with K and C depending on one another quantitatively.

It remains to prove that $K_I(\Psi) \leq C_0 K$. To this end, consider $\psi = \Psi|_Y$ and $\phi = \psi^{-1}$. Proposition 2.6 implies that $K_I(\psi) = K_O(\phi) \leq C_0 K$. Then Proposition 2.4 implies that $\phi \in N^{1,2}(\mathbb{D}; Y)$ since the Jacobian J_ϕ of ϕ is integrable. Observe that the extension $\Phi = \Psi^{-1}$ of ϕ is an element of $N^{1,2}(\overline{\mathbb{D}}; \overline{Y})$. This can be seen by extending Φ to a neighbourhood of $\overline{\mathbb{D}}$ via reflection over $\mathbb{S}^1$ and by applying [15, Theorem 1.12.3]. The minimal weak upper gradient of Φ has a representative that vanishes in $\mathbb{S}^1$ since $\mathcal{L}^2(\mathbb{S}^1) = 0$. Therefore $\rho_\Phi^2 \leq C_0 K J_\Phi$ holds $\mathcal{L}^2$-almost everywhere in $\overline{\mathbb{D}}$. This implies that Φ satisfies the second condition in Proposition 2.4 with the constant $C_0 K$. □

We recall a sufficient condition for (1.2) for later use.

Lemma 2.8. Suppose that there exists $C_U > 0$ such that for all $y \in \partial Y$ and $0 < r < \operatorname{diam} \partial Y$,

$$\mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y,r)) \leq C_U r^2. \tag{2.8}$$

Then for $\widetilde{C}_U = 8C_U/\log 2$ and for every $y \in \partial Y$ and $0 < 2r < R < 2^{-1} \operatorname{diam} \partial Y$,

$$\operatorname{mod} \Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y}) \leq \frac{\widetilde{C}_U}{\log \frac{R}{r}}. \tag{2.9}$$

In particular, (1.2) in Proposition 1.2 holds under the assumption (2.8).

Proof. The inequality (2.9) follows from (2.8) by considering the admissible function $\rho(x) = \frac{1}{\log \frac{R}{r}} \frac{1}{d(y,x)} \chi_{\{r \leq d(y,x) \leq R\}}$. We claim that ρ is admissible for the family $\Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y})$. To this end, fix a rectifiable $\gamma \in \Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y})$.

We denote $f(x) = d(y,x)$. Whenever $x \in f^{-1}(t)$, we have $\rho(x)\#(\gamma^{-1}(x)) \geq \frac{1}{\log \frac{R}{r}} \frac{1}{t} \chi_{\{r \leq t \leq R\}}$. Therefore

$$\int_{\mathbb{R}}^{\star} \int_{f^{-1}(t)} \rho(x)\#(\gamma^{-1}(x))\, d\mathcal{H}^0_{\overline{Y}}(x)\, d\mathcal{L}^1(t) \geq \int_r^R \frac{1}{\log \frac{R}{r}} \frac{1}{t}\, d\mathcal{L}^1(t) = 1.$$

Then Theorem 2.1 implies

$$\int_{\overline{Y}} \rho(x)\#(\gamma^{-1}(x))\, d\mathcal{H}^1_{\overline{Y}}(x) \geq \int_{\mathbb{R}}^{\star} \int_{f^{-1}(t)} \rho(x)\#(\gamma^{-1}(x))\, d\mathcal{H}^0_{\overline{Y}}(x)\, d\mathcal{L}^1(t).$$

The equality (2.2) yields $\int_\gamma \rho\, ds = \int_{\overline{Y}} \rho(x)\#(\gamma^{-1}(x))\, d\mathcal{H}^1_{\overline{Y}}(x)$. Hence ρ is admissible for $\Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y})$.

The L^2-norm of ρ is estimated from above by applying the area growth (2.8) on the annuli $A_l = \left\{2^l r \leq d(y,x) < 2^{l+1} r\right\}$ for $l = 0, 1, 2, \dots, k$ for $2^k r < R \leq 2^{k+1} r$, $k \in \mathbb{N}$. That is,

$$\begin{aligned}\int_{\overline{Y}} \rho^2(x)\, d\mathcal{H}^2_{\overline{Y}}(x) &\leq \sum_{l=0}^{k} \int_{A_l} \rho^2(x)\, d\mathcal{H}^2_{\overline{Y}}(x) \leq \frac{1}{\log^2(\frac{R}{r})} \sum_{l=0}^{k} \frac{\mathcal{H}^2_{\overline{Y}}(\overline{B}(y, 2^{l+1}r))}{2^{2l} r^2} \\ &\leq \frac{1}{\log^2(\frac{R}{r})} \sum_{l=0}^{k} \frac{C_U 2^{2l+2} r^2}{2^{2l} r^2} = 4C_U \frac{k+1}{\log^2 \frac{R}{r}} \leq \frac{8C_U/\log 2}{\log \frac{R}{r}}\end{aligned}$$

since $k + 1 \leq (2/\log 2) \log \frac{R}{r}$. The inequality (2.9) follows.

We claim now that (1.2) in Proposition 1.2 holds. Let $y \in \partial Y$ and $R' > R > 0$ such that $\overline{Y} \setminus B_{\overline{Y}}(y, R') \neq \emptyset$ and $2^{-1} \operatorname{diam} \partial Y > R$. Then for every $0 < 2r < R$, every path in $\Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R'); \overline{Y})$ has a subpath in $\Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y})$. Hence

$$\operatorname{mod} \Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R'); \overline{Y}) \leq \operatorname{mod} \Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y}).$$

The right-hand side converges to zero as $r \to 0^+$, given (2.9). This establishes (1.2).

□

2.5 Quasicircles

In this section we recall some basic properties of quasisymmetries and quasicircles. If $g\colon (Y, d_Y) \to (Z, d_Z)$, we denote

$$L_g(y,r) = \sup_{w \in \overline{B}_Y(y,r)} d_Z(g(y), g(w)) \quad \text{and} \quad \ell_g(y,r) = \inf_{w \in Y \setminus B_Y(y,r)} d_Z(g(y), g(w)).$$

Definition 2.9. Let $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism. A homeomorphism $g\colon (Y, d_Y) \to (Z, d_Z)$ between metric spaces is *η-quasisymmetric* if for every $y \in Y$ and $0 < r_1, r_2 < \operatorname{diam} Y$,

$$L_g(y, r_1) \leq \eta\left(\frac{r_1}{r_2}\right) \ell_g(y, r_2). \tag{2.10}$$

A homeomorphism g is *quasisymmetric* if it is η-quasisymmetric for some homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$.

A set $S \subset Y$ is *r-separated* if for every $x, y \in S$ with $x \neq y$, $d_Y(x,y) \geq r$, and an *r-net* if for every $y \in Y$, there exists $x \in S$ for which $d_Y(x,y) < r$. An r-separated set is *maximal* if it is also an r-net.

Definition 2.10. A metric space (Y, d_Y) has its *Assouad dimension* bounded from above by $Q > 0$ if for every $0 < \epsilon < 1$ and every $(y,r) \in Y \times (0, \operatorname{diam} Y)$, any ϵr-separated set $S \subset B_Y(y,r)$ satisfies

$$\#S \leq C\epsilon^{-Q}, \tag{2.11}$$

where C is a constant independent of ϵ, y, r and S. Here $\#S$ refers to the counting measure of S. The *Assouad dimension* of Y is the infimum of such Q.

A metric space (Y, d_Y) is said to be *doubling* if its Assouad dimension is finite.

Definition 2.11. Let $\lambda \geq 1$. A metric space (Y, d_Y) has λ-*bounded turning* if for every $y, z \in Y$ there exists a compact and connected set $E \subset Y$ containing y and z such that $\operatorname{diam} E \leq \lambda d_Y(y, z)$.

We recall that a metric space $\mathcal{C}$ homeomorphic to $\mathbb{S}^1$ is a quasisymmetric image of $\mathbb{S}^1$ if and only if $\mathcal{C}$ has bounded turning and is doubling [21]. We refer to any quasisymmetric image of $\mathbb{S}^1$ as a *quasicircle*.

3 Carathéodory's theorem

3.1 Proof of Theorem 1.1

We fix a quasiconformal homeomorphism $\phi\colon \mathbb{D} \to Y$ and claim that it has a monotone and surjective extension $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$. We are assuming that $\overline{Y}$ is homeomorphic to $[0, 1]^2$ and has finite Hausdorff 2-measure.

Fix $x_0 \in \mathbb{S}^1$. For each $0 < r < 2^{-1}$, denote $E_r = \mathbb{S}^1(x_0, r) \cap \overline{\mathbb{D}}$. Let V_r be the component of $\mathbb{D} \setminus E_r$ whose closure contains $\{x_0\}$.

Let $U_r = \overline{\phi(V_r)}$. Since V_r is connected, so is U_r. Moreover, as $V_{r'} \subset V_r$ whenever $r' < r$, we have $U_{r'} \subset U_r$. Therefore

$$\emptyset \neq \widetilde{C} = \bigcap_{0<r<2^{-1}} U_r \quad \text{is compact and connected [22, Theorem 28.2].} \tag{3.1}$$

Notice that $\widetilde{C} \subset \partial Y$.

Lemma 3.1 implies that $\widetilde{C}$ is a singleton. Let y_0 denote the unique element. We define $\Phi(x_0) := y_0$. We repeat the argument for every $x_0 \in \mathbb{S}^1$. By setting $\Phi(x) = \phi(x)$ for every $x \in \mathbb{D}$, we obtain a mapping

$$\Phi\colon \overline{\mathbb{D}} \to \overline{Y}.$$

We prove in Lemma 3.4 that Φ is continuous and surjective. Lemma 3.5 shows the monotonicity of Φ. Hence Theorem 1.1 follows after we verify these lemmas.

Lemma 3.1. Let d_r denote the diameter of U_r. Then $d_r \to \operatorname{diam} \widetilde{C} = 0$ for every $x_0 \in \mathbb{S}^1$.

Before proving Lemma 3.1, we prove a couple of technical lemmas. In the following, an *arc* refers to a set homeomorphic to $[0, 1]$.

Lemma 3.2. Let $C' \subset Y$ be an arc and $C \subset \partial Y$ a compact and connected set. Then

$$\operatorname{diam} C > 0 \quad \text{implies} \quad \operatorname{mod} \Gamma(C, C'; Y \cup C) > 0. \tag{3.2}$$

Proof of Lemma 3.2. Since C and C' are disjoint, there are Borel functions $\rho \in L^2(\overline{Y})$ admissible for $\Gamma(C, C'; Y \cup C)$. We fix such a function ρ and find a lower bound for the L^2-norm of ρ, depending only on C and C'. The claim (3.2) follows from this.

We argue as in the proof of [18, Proposition 3.5]. First, we join C and C' with an arc $\gamma\colon [0, 1] \to Y \cup C$ for which $r_0 = d(|\gamma|, \partial Y \setminus C) > 0$, and consider the Lipschitz function $f(z) = d(|\gamma|, z)$. Since C and C' are arcs, we can choose γ in such a way that $\gamma(0)$ separates C into two arcs J_1 and J_2, $\gamma(1)$ separates C' into two arcs J_3 and J_4, and $\gamma(t) \in Y \setminus (C \cup C')$ for every $0 < t < 1$.

Fix $0 < r_1 < r_0$ such that every J_i intersects $f^{-1}(r)$ for every $0 < r < r_1$. For every such $r > 0$, the level set $f^{-1}(r)$ separates $|\gamma|$ from the arc $\overline{\partial Y \setminus C}$. Then Theorem 2.2 provides us with a continuum $\Gamma_r \subset f^{-1}(r)$ that separates $|\gamma|$ from $\overline{\partial Y \setminus C}$. The continuum Γ_r must intersect every J_i, since otherwise we find a path joining $|\gamma|$ to $\overline{\partial Y \setminus C}$ that does not intersect Γ_r.

By applying Theorem 2.1 to the function $g(y) = \chi_{\overline{Y}}(y)$, we conclude that the level set $f^{-1}(r)$ has finite Hausdorff 1-measure for $\mathcal{L}^1$-almost every $0 < r < r_1$. In particular, the continuum Γ_r has finite Hausdorff

1-measure. Then every pair of points from Γ_r can be joined with a rectifiable path within Γ_r [20, Proposition 15.1]. Consequently, there exists a rectifiable arc $\theta\colon [0,1] \to \Gamma_r$ joining $J_1 \subset C$ to $J_3 \subset C'$. Since $0 < r < r_1 < r_0$, we have $\theta \in \Gamma(C, C'; Y \cup C)$. Hence

$$1 \leq \int_\theta \rho \, ds \leq \int_{f^{-1}(r)} \rho \, d\mathcal{H}^1_Y.$$

Then Theorem 2.1 and Hölder's inequality imply

$$r_1 \leq \frac{4}{\pi} \left(\mathcal{H}^2_Y(\overline{Y}) \right)^{1/2} \|\rho\|_{L^2(\overline{Y})}.$$

Rearranging this inequality establishes the claim. □

Lemma 3.3. Let $\theta\colon (0,1) \to G \subset \overline{Y}$ be a homeomorphism and suppose that for every $0 < s < t < 1$,

$$\ell(\theta|_{[s,t]}) \leq \int_s^t h(a) \, d\mathcal{L}^1(a)$$

for some $h \in L^1([0,1])$. Then there exists an absolutely continuous extension $\overline{\theta}\colon [0,1] \to \overline{G}$ of θ that is surjective.

Proof of Lemma 3.3. Let $0 < s < t < 1$. Then

$$d(\theta(s), \theta(t)) \leq \int_0^1 \left(\chi_{[0,s]}(a) + \chi_{[0,t]}(a) \right) h(a) \, d\mathcal{L}^1(a). \tag{3.3}$$

By the absolute continuity of the integral, given $\epsilon > 0$, there exists $\delta > 0$ for which

$$|s|, |t| < \delta \quad \text{implies} \quad \int_0^1 \left(\chi_{[0,s]} + \chi_{[0,t]} \right)(a) h(a) \, d\mathcal{L}^1(a) < \epsilon. \tag{3.4}$$

This fact and (3.3) imply that for any given $(s_j)_{j=1}^\infty \subset (0,1)$ converging to zero, the sequence $(\theta(s_j))_{j=1}^\infty$ is Cauchy. Since $\overline{Y}$ is complete, the sequence converges to some $y_0 \in \overline{G}$. We define $\overline{\theta}(0) := y_0$. The inequalities (3.3) and (3.4) imply that y_0 is independent of the sequence $(s_j)_{j=1}^\infty$, and setting $\overline{\theta}(s) = \theta(s)$ for $0 < s$ defines a continuous extension of θ to $[0,1)$.

By arguing similarly for $t = 1$, we find a continuous extension $\overline{\theta}\colon [0,1] \to \overline{G}$ of θ. The inequality (3.3) extends to every $0 \leq s < t \leq 1$, implying the absolute continuity of $\overline{\theta}$. Notice that for every $y \in \overline{G}$, there exists a sequence $(t_j)_{j=1}^\infty \subset (0,1)$ such that $\overline{\theta}(t_j) \to y$. By passing to a subsequence and relabeling, we may assume that $(t_j)_{j=1}^\infty$ has a limit in $[0,1]$. This implies that $\overline{\theta}$ is surjective. □

Proof of Lemma 3.1. Since $\widetilde{C}$ is the intersection of the U_r and U_r are nested, we have $d_r \to \operatorname{diam} \widetilde{C}$. Hence the difficulty lies in proving $\operatorname{diam} \widetilde{C} = 0$.

Fix an arc $C' \subset Y$ for which $\phi^{-1}(C') \subset \mathbb{D} \setminus (E_r \cup V_r)$ for every $0 < r < 2^{-1}$. We assume $\operatorname{diam} \widetilde{C} > 0$ and derive a contradiction. Since $\operatorname{diam} \widetilde{C} > 0$, there exist a subarc $C \subset \widetilde{C}$ such that $r_0 = d(C, C' \cup (\partial Y \setminus \widetilde{C})) > 0$. We claim that

$$\operatorname{mod} \Gamma(C, C'; Y \cup C) = 0. \tag{3.5}$$

If (3.5) holds for C, we obtain a contradiction with Lemma 3.2.

So it suffices to prove (3.5). We claim that there exists a sequence $r_n \to 0^+$ such that every $\theta \in \Gamma(C, C'; Y \cup C)$ has a subpath in $\Gamma(\phi(E_{r_n} \cap \mathbb{D}), C'; Y)$. If this can be proved, then the K-quasiconformality of ϕ yields

$$\operatorname{mod} \Gamma(C, C'; Y \cup C) \leq K \operatorname{mod} \Gamma(E_{r_n} \cap \mathbb{D}, \phi^{-1}(C'); \mathbb{D}).$$

Given Lemma 2.8, the right-hand side converges to zero as $n \to \infty$, and we conclude (3.5). The rest of the proof is spent on finding the sequence of radii $(r_n)_{n=1}^\infty$.

The K-quasiconformality of ϕ yields that the minimal weak upper gradient ρ_ϕ satisfies $\rho_\phi^2 \le KJ_\phi \in L^1(\mathbb{D})$. The integrability of J_ϕ follows from the fact that Y has finite Hausdorff 2-measure. This implies that ϕ has an $L^2(\mathbb{D})$-integrable upper gradient g [12, Lemma 6.2.2].

We consider $g_0 = g\chi_{\mathbb{D}} \in L^2(\overline{\mathbb{D}})$. Polar coordinates centered at x_0 yield

$$\infty > \|g_0\|_{L^1(\overline{\mathbb{D}})} \ge \int_0^{2^{-1}} \int_{E_r} g_0 \, d\mathcal{H}^1 \, d\mathcal{L}^1(r). \tag{3.6}$$

In particular, g_0 has a finite path integral over E_r for almost every $0 < r < 2^{-1}$. Let I denote those $0 < r < 2^{-1}$ for which this holds.

Let Γ_0 be the family of non-constant rectifiable paths in $\overline{\mathbb{D}}$ along which g_0 is not integrable. Consider an absolutely continuous non-constant path $\gamma\colon [a, b] \to \overline{Y}$ with image in Y and which is not an element of Γ_0. Since g is an upper gradient of ϕ, we have that

$$\ell(\phi \circ \gamma) \le \int_\gamma g \, ds = \int_\gamma g_0 \, ds; \text{ see [12, Proposition 6.3.2].} \tag{3.7}$$

Consider a surjective Lipschitz $\gamma_r\colon [0, 1] \to E_r$ for $r \in I$. Then, for every $0 < s < t < 1$, (3.7) implies

$$\ell((\phi \circ \gamma_r)|_{[s,t]}) \le \int_{\gamma_r|_{[s,t]}} g_0 \, ds \le \int_{E_r} g_0 \, d\mathcal{H}^1,$$

where the last term on the right is finite. Therefore $\theta_r = \phi \circ \gamma_r\colon (0, 1) \to \phi(E_r \cap \mathbb{D})$ satisfies the assumptions of Lemma 3.3. Hence there exists a continuous extension $\overline{\theta_r}\colon [0, 1] \to F_r$ onto $F_r := \overline{\phi(E_r \cap \mathbb{D})}$.

Since ϕ is a homeomorphism, $\overline{\theta_r}(s) \notin Y$ for both $s = 0, 1$. Hence F_r is homeomorphic to a circle or an arc. We note that $\phi(V_r)$ is one of the connected components of $Y \setminus F_r$. In particular, $U_r = \overline{\phi(V_r)}$ is homeomorphic to $[0, 1]^2$, $U_r \cap \partial Y$ is a point or an arc, and $C \subset \widetilde{C} \subset U_r \cap \partial Y$. As the ends of $U_r \cap \partial Y$ and F_r coincide, $d(C, \partial Y \setminus \widetilde{C}) > 0$ implies that $U_r \cap \partial Y$ is an arc and $C \cap F_r = \emptyset$. This means that every path $\theta \in \Gamma(C, C'; Y \cup C)$ has a subpath in $\Gamma(F_r \cap Y, C'; Y)$. Then (3.5) follows by taking a sequence $(r_n)_{n=1}^\infty \subset I$ converging to zero. □

Lemma 3.4. The mapping $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$ is continuous and surjective.

Proof. Let $x_0 \in \mathbb{S}^1$. If $\mathbb{D} \ni x_n \to x_0$, the accumulation points of $(\Phi(x_n))_{n=1}^\infty$ are contained in the intersection of U_r, where U_r are as in the definition of $\widetilde{C}$ in (3.1). Lemma 3.1 shows that the intersection is a singleton, which we defined to be $\Phi(x_0)$. This implies $\Phi(x_n) \to \Phi(x_0)$.

More generally, if $\overline{\mathbb{D}} \ni x_n \to x_0$, we find for every x_n an element $z_n \in \mathbb{D}$ such that $d_{\overline{Y}}(\Phi(z_n), \Phi(x_n)) \le 2^{-n}$ and $\|z_n - x_n\| \le 2^{-n}$. Then $\mathbb{D} \ni z_n \to x_0$ and $\Phi(z_n) \to \Phi(x_0)$. Since $d_Y(\Phi(z_n), \Phi(x_n)) \to 0$, we have $\Phi(x_n) \to \Phi(x_0)$. This implies that Φ is continuous.

Consider now $y_0 \in \overline{Y}$. Then there exists a sequence $Y \ni y_n \to y_0$. Up to passing to a subsequence and relabeling, $(\Phi^{-1}(y_n))_{n=1}^\infty$ converges to some $x_0 \in \overline{\mathbb{D}}$. The continuity of Φ implies $\Phi(x_0) = y_0$. □

Lemma 3.5. The mapping $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$ is monotone.

Proof. Let $y \in \partial Y$ and suppose that there are two distinct points x_1 and x_2 from $\mathbb{S}^1$ such that $\Phi(x_1) = y = \Phi(x_2)$. Let I_i be radial lines from x_i to 0 for $i = 1, 2$ and define $I = I_0 \cup I_1$. Here $\Phi(I)$ is a Jordan loop intersecting ∂Y exactly at y. Let U denote the component of $Y \setminus \Phi(I)$ whose closure intersects ∂Y only at y. Then $V = \Phi^{-1}(U)$ is one of the components of $\mathbb{D} \setminus I$. By construction, $\overline{V} \cap \mathbb{S}^1$ is connected and is mapped to the singleton y. Therefore x_1 and x_2 can be joined with a path in $\Phi^{-1}(y)$. The monotonicity of Φ follows. □

3.2 Proof of Proposition 1.2

Recall that we fix a quasiconformal homeomorphism $\phi\colon \mathbb{D} \to Y$ and study its extension Φ under the assumption (1.2). This is done in several parts. First, Lemma 3.6 proves that the extension is a homeomorphism.

Lemma 3.8 implies that Φ is quasiconformal. After this is verified, Lemma 3.9 proves that $K_O(\Phi) = K_O(\phi)$ and $K_I(\Phi) = K_I(\phi)$. In particular, the maximal dilatations of Φ and ϕ coincide.

To finish up the proof of Proposition 1.2, we also need to verify that if Φ is a quasiconformal homeomorphism, then the assumption (1.2) holds. But this follows from the corresponding Euclidean result. So Proposition 1.2 follows after we prove the lemmas mentioned above.

Lemma 3.6. The mapping Φ is a homeomorphism.

Proof. Let $y \in \partial Y$ and suppose that there exists a non-trivial continuum $E \subset \Phi^{-1}(y)$. Let $F \subset Y \setminus \overline{B}_{\overline{Y}}(y, R)$ be a non-trivial arc for some $R > 0$. For $0 < r < R$, let $\Gamma(y, r, R) = \Gamma(\overline{B}_{\overline{Y}}(y, r), \overline{Y} \setminus B_{\overline{Y}}(y, R); \overline{Y})$. Then

$$\operatorname{mod}\Gamma(y, r, R) \geq \operatorname{mod}(\Gamma(y, r, R) \cap \mathrm{AC}(Y)),$$

where $\mathrm{AC}(Y)$ refers to those absolutely continuous paths in $\overline{Y}$ whose images lie in Y. Since $\phi = \Phi|_{\mathbb{D}}$ is K-quasiconformal, we have that

$$\operatorname{mod}(\Gamma(y, r, R) \cap \mathrm{AC}(Y)) \geq K^{-1} \operatorname{mod}(\Gamma(E, \Phi^{-1}(F); \mathbb{D} \cup E \cup \Phi^{-1}(F))).$$

The right-hand side is a strictly positive lower bound; recall Lemma 3.2. Therefore $\operatorname{mod}\Gamma(y, r, R) \geq C_0 > 0$ for a constant independent of r. By passing to the limit $r \to 0^+$, we find a contradiction with (1.2). The injectivity of Φ follows. Since Φ is continuous, surjective, and monotone, we conclude that Φ is a homeomorphism. □

Next we claim that Φ is quasiconformal. To this end, we let $\Psi = \Phi^{-1}$. Due to Proposition 2.7, it is sufficient to find a constant C_0 such that for every $x \in \mathbb{S}^1$,

$$\liminf_{r \to 0^+} \operatorname{mod} \Psi^{-1}\Gamma\left(\overline{B}_{\overline{\mathbb{D}}}(x, r), \overline{\mathbb{D}} \setminus B_{\overline{\mathbb{D}}}(x, 2r); \overline{B}_{\overline{\mathbb{D}}}(x, 2r)\right) \leq C_0. \tag{3.8}$$

We denote $\Gamma(x, r, 2r) = \Gamma\left(\overline{B}_{\overline{\mathbb{D}}}(x, r), \overline{\mathbb{D}} \setminus B_{\overline{\mathbb{D}}}(x, 2r); \overline{B}_{\overline{\mathbb{D}}}(x, 2r)\right)$ for the rest of the section.

Fix $0 < r < 1/4$. Let $\xi_1 = \Psi^{-1}\left(\mathbb{S}^1(x, r) \cap \overline{\mathbb{D}}\right)$ and $\xi_3 = \Psi^{-1}\left(\mathbb{S}^1(x, 2r) \cap \overline{\mathbb{D}}\right)$. Let ξ_2 and ξ_4 denote the subarcs of ∂Y joining ξ_1 and ξ_3 in such a way that the arcs $\xi_1, \xi_2, \xi_3, \xi_4$ form the boundary decomposition of a quadrilateral Q in $\overline{Y}$. Then

$$\operatorname{mod} \Psi^{-1}\Gamma(x, r, 2r) = \operatorname{mod}\Gamma(\xi_1, \xi_3; Q) =: M.$$

Lemma 3.7. There exists a homeomorphism $f\colon Q \to [0, 1] \times [0, M]$ with the following properties: First, $f(\xi_1) = \{0\} \times [0, M]$ and $f(\xi_3) = \{1\} \times [0, M]$. Whenever $0 < a < b < M$ and $I = [a, b]$, let $Q^0 = f^{-1}([0, 1] \times I)$,

$$\xi_1^0 = f^{-1}(\{0\} \times I), \quad \xi_2^0 = f^{-1}([0, 1] \times \{a\}),$$
$$\xi_3^0 = f^{-1}(\{1\} \times I), \quad \xi_4^0 = f^{-1}([0, 1] \times \{b\}).$$

Then $b - a = \operatorname{mod}\Gamma(\xi_1^0, \xi_3^0; Q^0)$.

Proof. Proposition 9.1 [18] and [18, equation (57), Lemma 10.2] provide us with f having the stated properties. Notice that [18, Proposition 9.1] is applicable since (2.6) holds for every $x_0 \in \overline{Y}$ and the product in (2.5) is always bounded from below by a universal constant $\kappa_0 > 0$ [19]. These facts allow us to apply [18, equation (57), Lemma 10.2] as well. □

Lemma 3.8. The inequality (3.8) holds for a constant $C_0 = 2KC_1$, where C_1 depends only on $\mathbb{D}$ and K is the maximal dilatation of ϕ.

Proof. We let $b = 3M/4$ and $a = M/4$ in Lemma 3.7. Since the restriction of Ψ to Y is K-quasiconformal,

$$\operatorname{mod}\Gamma(\xi_1^0, \xi_3^0; Q^0) \leq K \operatorname{mod}\Gamma(\Psi(\xi_1^0), \Psi(\xi_3^0); \Psi(Q^0)).$$

Observe that $\Gamma(\Psi(\xi_1^0), \Psi(\xi_3^0); \Psi(Q^0)) \subset \Gamma(x, r, 2r)$. Therefore

$$\operatorname{mod}\Gamma(\Psi(\xi_1^0), \Psi(\xi_3^0); \Psi(Q^0)) \leq \operatorname{mod}\Gamma(x, r, 2r).$$

Here $\operatorname{mod}\Gamma(x, r, 2r) \le C_1$ for a constant depending only on $\overline{\mathbb{D}}$. Then Lemma 3.7 yields $M \le 2KC_1$. The claim follows by passing to the limit $r \to 0^+$. □

Lemma 3.9. The outer (resp. inner) dilatation of Φ coincides with the outer (resp. inner) dilatation of ϕ.

Proof. Lemmas 3.6 and 3.8 prove that Φ is a quasiconformal homeomorphism. In $\mathbb{D}$, the minimal weak upper gradients of Φ and ϕ coincide. This is also true for their Jacobians. Therefore they satisfy (ii) in Proposition 2.4 with the same constant. Hence $K_O(\Phi) = K_O(\phi)$.

Consider the Borel functions $g = \chi_{\mathbb{S}^1}$ and $\widetilde{g} = (g \circ \Phi^{-1})\rho_{\Phi^{-1}}$. The property (ii) in Proposition 2.4 implies

$$\|\widetilde{g}\|^2_{L^2(\overline{Y})} \le K_O(\Phi^{-1})\, \|g\|^2_{L^2(\overline{\mathbb{D}})} = 0.$$

Hence $\widetilde{g} = 0$ $\mathcal{H}^2_{\overline{Y}}$-almost everywhere in $\overline{Y}$. We conclude that $\rho_{\Phi^{-1}}(y) = 0$ for $\mathcal{H}^2_{\overline{Y}}$-almost every $y \in \partial Y$. Hence $\rho_{\Phi^{-1}} = \rho_{\phi^{-1}}\chi_Y$ $\mathcal{H}^2_{\overline{Y}}$-almost everywhere in $\overline{Y}$. We conclude that ϕ and Φ satisfy (ii) in Proposition 2.4 with the same constant. In other words, $K_I(\Phi) = K_I(\phi)$. □

4 Beurling–Ahlfors extension

For this section we fix a quasiconformal Jordan domain Y satisfying (1.3) and a quasiconformal homeomorphism $\phi\colon \mathbb{D} \to Y$. Let $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$ denote the quasiconformal homeomorphic extension of ϕ, obtained from Proposition 1.2. We refer to $g_\phi = \Phi|_{\mathbb{S}^1}$ as the boundary map of ϕ. The goal of this section is to prove Theorem 1.3. We reduce the proof to Proposition 4.1.

Observe that if g_ϕ is a quasisymmetry, then ∂Y has bounded turning as this property is preserved by quasisymmetries [21]. Moreover, if we fix an arbitrary quasisymmetry $g\colon \mathbb{S}^1 \to \partial Y$, then $h = g_\phi^{-1} \circ g\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a quasisymmetry. The Beurling–Ahlfors extension theorem [2] yields the existence of a quasiconformal map $H\colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ whose boundary map equals h. Then $G = \Phi \circ H\colon \overline{\mathbb{D}} \to \overline{Y}$ is the quasiconformal extension of g whose existence we wanted to establish. So Theorem 1.3 is a consequence of the following result.

Proposition 4.1. If ∂Y has bounded turning and satisfies the mass upper bound (1.3), then the boundary map $g_\phi = \Phi|_{\mathbb{S}^1}$ is a quasisymmetry.

We start the proof of Proposition 4.1 by first establishing Proposition 1.4. There we claim that ∂Y has Assouad dimension at most two.

Lemma 4.2. Suppose that ∂Y has bounded turning with constant $\lambda > 1$. Let $C = (4\lambda)^3 \frac{2}{\pi}$. Then for all $y \in \partial Y$ and all $0 < r < \operatorname{diam} \partial Y$, $\mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y, r)) \ge C^{-1} r^2$.

Proof. Consider $y \in \partial Y$ and the 1-Lipschitz function $f(z) = d(y, z)$. For every $0 < r < (4\lambda)^{-1} \operatorname{diam} \partial Y$, we have that $f^{-1}(r) \cap \partial Y \neq \emptyset$. Let $y_0 \in \partial Y \setminus \overline{B}_{\overline{Y}}(y, 2\lambda r)$. We obtain from Theorem 2.2 a continuum $C_r \subset f^{-1}(r)$ separating y and y_0.

Let $E \subset \partial Y$ be the (closure of the) component of $\partial Y \setminus C_r$ that contains y and let a and b denote the ends of E. Here $a, b \in E \cap C_r \subset f^{-1}(r)$ so $d(a, y) = r = d(b, y)$.

Let $E_a \subset \partial Y$ be the arc ending at a and y with $\operatorname{diam} E_a \le \lambda r$, and let E_b denote the corresponding arc for b and y. Then $E_a \cup E_b = E$. Indeed, otherwise $\max\{\operatorname{diam} E_a, \operatorname{diam} E_b\} \ge d(y, y_0) > 2\lambda r$.

Let $E' \subset \partial Y$ be the arc ending at a and b with smaller diameter. Then $\operatorname{diam} E' \le 2^{-1} \operatorname{diam} \partial Y$. Since $\operatorname{diam} E \le \operatorname{diam} E_a + \operatorname{diam} E_b \le 2\lambda r < 2^{-1} \operatorname{diam} \partial Y$, we have $E' = E$.

We conclude that $\operatorname{diam} C_r \ge d(a, b) \ge \lambda^{-1} \operatorname{diam} E \ge \lambda^{-1} r$. Since C_r is a continuum, $\mathcal{H}^1_{\overline{Y}}(f^{-1}(r)) \ge \mathcal{H}^1_{\overline{Y}}(C_r) \ge \operatorname{diam} C_r \ge \lambda^{-1} r$ for each $0 < r < (4\lambda)^{-1} \operatorname{diam} \partial Y$ [8, 2.10.12].

By integrating over the interval $(0, r)$ and by applying Theorem 2.1, we conclude that

$$\mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y, r)) \ge (4\lambda)^{-1} \frac{\pi}{2} r^2$$

whenever $0 < r < (4\lambda)^{-1} \operatorname{diam} \partial Y$. If $(4\lambda)^{-1} \operatorname{diam} \partial Y \leq r < \operatorname{diam} \partial Y$, we have

$$\mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y, r)) \geq \mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y, (4\lambda)^{-1} r)) \geq (4\lambda)^{-1} \frac{\pi}{2} \left((4\lambda)^{-1} r \right)^2 .$$

The claim follows for $C = (4\lambda)^3 \frac{2}{\pi}$. □

Proof of Proposition 1.4. Let $0 < r < \operatorname{diam} \partial Y$ and $0 < \epsilon < 1$. We consider a point $y \in \partial Y$ and an ϵr-separated set $S \subset B_{\overline{Y}}(y, r) \cap \partial Y$. We conclude from Lemma 4.2 and (1.3) that for some $C \geq 1$

$$C(2r)^2 \geq \mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y, 2r)) \geq \mathcal{H}^2_{\overline{Y}} \left(\bigcup_{x \in S} \overline{B}_{\overline{Y}}(x, \epsilon r) \right) \geq C^{-1} \left(\frac{\epsilon r}{2} \right)^2 \#S.$$

Therefore $\#S \leq C\epsilon^{-2}$ for some constant C independent of r, y and ϵ. Hence the Assouad dimension of ∂Y is at most 2. □

Proof of Proposition 4.1. Let $0 < r_0$ be such that for every $x \in \mathbb{S}^1$ and every $0 < r \leq r_0$, we have that $4L_g(x, 2r) < \operatorname{diam} \partial Y$.

Fix $x \in \mathbb{S}^1$ and $0 < r < r_0$. Let $z, a, b \in \mathbb{S}^1 \cap \mathbb{D}(x, r)$ be such that $0 < \|a - z\|_2 \leq \|b - z\|_2$. We proved in Proposition 1.4 that ∂Y is doubling, so by a result of Tukia–Väisälä [21, Theorems 2.15 and 2.23], the quasisymmetry of $g = g_\phi$ follows if there exists a constant $H > 0$ depending only on the constants in (1.3), Lemma 4.2, the bounded turning constant λ of ∂Y, and the maximal dilatation K of ϕ such that $d(g(a), g(z)) \leq Hd(g(b), g(z))$.

Let $\ell = d(g(a), g(z))$ and let $M > 0$ be such that $\ell > Md(g(b), g(z))$. If we find H_0 such that $M \leq H_0$ independently of a, b and z, we may set $H = H_0$. If $M \leq (2\lambda)^2$, any choice $H_0 \geq (2\lambda)^2$ suffices. So we may assume $M > (2\lambda)^2$.

Fix $z' \in \mathbb{S}^1$ with $d(g(z'), g(x)) > 2L_g(x, 2r)$. Then

$$2L_g(x, 2r) < d(g(z'), g(z)) + d(g(z), g(x)) \leq d(g(z'), g(z)) + L_g(x, 2r) \quad \text{and}$$
$$2^{-1}\ell \leq 2^{-1}(d(g(a), g(x)) + d(g(x), g(z))) \leq L_g(x, 2r).$$

Therefore $2^{-1}\ell < d(g(z'), g(z))$. We conclude that

$$g(a), g(z') \in \partial Y \setminus \overline{B}_{\overline{Y}} \left(g(z), \frac{\ell}{2} \right) \quad \text{and} \quad g(b) \in \partial Y \cap \overline{B}_{\overline{Y}} \left(g(z), \frac{\ell}{M} \right).$$

Let A' be the subarc of ∂Y joining $g(a)$ to $g(z')$ that does not contain $g(z)$. Then any arc joining A' to $g(z)$ within ∂Y must pass through either $g(a)$ or $g(z')$. Using this fact, the bounded turning of ∂Y yields

$$A' \subset \partial Y \setminus \overline{B}_{\overline{Y}} \left(g(z), \lambda^{-1} \frac{\ell}{2} \right). \tag{4.1}$$

Let B' be the subarc of ∂Y with smallest diameter which ends at $g(b)$ and $g(z)$. The bounded turning of ∂Y implies that

$$B' \subset \overline{B}_{\overline{Y}} \left(g(z), \lambda \frac{\ell}{M} \right). \tag{4.2}$$

The inclusions (4.1) and (4.2) imply that every path $\gamma \in \Gamma(A', B'; \overline{Y})$ has a subpath joining $\overline{Y} \setminus \overline{B}_{\overline{Y}}(g(z), \lambda^{-1} \frac{\ell}{2})$ to $\overline{B}_{\overline{Y}} \left(g(z), \lambda \frac{\ell}{M} \right)$ within $\overline{B}_{\overline{Y}}(g(z), \lambda^{-1} \frac{\ell}{2})$. Then Lemma 2.8 yields

$$\frac{\widetilde{C}_U}{\log \frac{M}{2\lambda^2}} \geq \operatorname{mod} \Gamma(A', B'; \overline{Y}). \tag{4.3}$$

Let $A = g^{-1}(A')$ and $B = g^{-1}(B')$. The relative distance $\Delta(A, B)$ satisfies

$$\Delta(A, B) := \frac{d(A, B)}{\min \{\operatorname{diam} A, \operatorname{diam} B\}} \leq 2. \tag{4.4}$$

First, $d(A,B) \leq \|a-z\|_2$ since $a \in A$ and $z \in B$. Second, $\operatorname{diam} A \geq \left\|a-z'\right\|_2 \geq r$, so $\|a-x\|_2 \leq r$ and $\|x-z\|_2 \leq r$ imply $2^{-1}\|a-z\|_2 \leq r$. Lastly, $\operatorname{diam} B \geq \|b-z\|_2 \geq \|a-z\|_2$. These imply (4.4).

The 2-Loewner property of $\overline{\mathbb{D}}$ [11, Example 8.24] states that there exists a constant $C_2 > 0$ for which

$$\operatorname{mod} \Gamma(A,B;\overline{\mathbb{D}}) \geq C_2 \tag{4.5}$$

depending only on the upper bound (4.4). The K-quasiconformality of the extension Φ implies that $\operatorname{mod}\Gamma(A',B';\overline{Y}) \geq K^{-1} \operatorname{mod}\Gamma(A,B;\overline{\mathbb{D}})$. Combining this inequality with (4.3) and (4.5) yields an upper bound on M in terms of C_2, K, $\widetilde{C}_U$, and λ. Setting H_0 to be the maximum of this bound and $(2\lambda)^2$ establishes the claim. □

5 Planar quasicircles

We prove Theorem 1.5 in this section. The main result of this section is the following.

Proposition 5.1. Under the assumptions of Theorem 1.5, the Assouad dimension of ∂Y is strictly less than 2.

Proof of Theorem 1.5 assuming Eq. (5.1).. Equation (5.1) states that ∂Y has Assouad dimension strictly less than 2. Having verified this, [13] yields the existence of a bi-Lipschitz embedding $h\colon \partial Y \to \mathbb{R}^2$, i.e., ∂Y is planar. □

So Theorem 1.5 follows from Proposition 5.1. We split the proof of Proposition 5.1 into a couple of sublemmas.

Lemma 5.2. Suppose that ∂Y has λ-bounded turning and satisfies (a) in Theorem 1.5. Then there exists a constant $C_p \geq 1$ depending only on λ such that for every $x \in \partial Y$ and $0 < r < \min\{r_0, \operatorname{diam}\partial Y\}$, there exists $y \in Y$ with $\overline{B}_{\overline{Y}}(y, C_p^{-1}2r) \subset B_{\overline{Y}}(x,r) \setminus \partial Y$.

Proof. This is only a small modification of the proof of Theorem 8.2 of [17] but we include the details here for the convenience of the reader. Let $s = 8\lambda^2(2\lambda+1)$ and $C_p = 8\lambda s$. Let $x \in \partial Y$ and $0 < r < \min\{r_0, \operatorname{diam}\partial Y\}$. We claim that there exists a point $v \in Y$ such that

$$B_{\overline{Y}}\left(v, C_p^{-1}2r\right) \subset B_{\overline{Y}}(x,r) \setminus \partial Y. \tag{5.1}$$

Suppose for now that $r < (4\lambda)^{-1}\min\{r_0, \operatorname{diam}\partial Y\}$. Then there exists a point $z \in \partial Y$ for which $d(x,z) \geq 2\lambda r$. We fix such a z.

Let $|\gamma|$ denote the (closure of the) subarc of $\partial Y \setminus \left\{w \in \partial Y\colon d(x,w) = \frac{r}{4\lambda}\right\}$ that contains x. Let a and b denote the end points of $|\gamma|$ labelled in such a way that $\{x,a,z,b\}$ is cyclically ordered on ∂Y. We have that $d(x,a) = d(x,b) = \frac{r}{4\lambda}$ and $|\gamma| \subset \overline{B}_{\overline{Y}}\left(x, \frac{r}{4\lambda}\right)$.

The relative ALLC condition of ∂Y implies that there exists a path α joining a to b in $\overline{B}_{\overline{Y}}\left(x,\frac{r}{4}\right) \setminus B_{\overline{Y}}\left(x,\frac{r}{8\lambda^2}\right)$. We assume without loss of generality that α is an arc.

Let $|\gamma_a|$ denote the (closure of the) component of $\partial Y \setminus \{x,z\}$ joining x and z that contains a and let $|\gamma_b|$ be the other component. Observe that $d(a,|\gamma_b|) \geq (8\lambda^2)^{-1}r$ since otherwise we would find an arc $|\gamma'|$ joining $|\gamma_b|$ to a within ∂Y for which $(8\lambda)^{-1}r \geq \operatorname{diam}|\gamma'|$. This would imply the contradiction $(8\lambda)^{-1}r \geq d(a,\{x,z\})$.

The lower bound on $d(a,|\gamma_b|)$ and connectedness of $|\alpha|$ imply the existence of $v \in |\alpha|$ such that $d(v,|\gamma_b|) = \frac{r}{s}$. Fix such a v. Suppose that there exists $w \in |\gamma_a|$ for which $d(v,w) < \frac{r}{s}$. Then $d(w,|\gamma_b|) < 2\frac{r}{s}$ and there exists a path β' joining w to $|\gamma_b|$ within ∂Y for which $\operatorname{diam}|\beta'| < 2\lambda\frac{r}{s}$. Since $|\beta'|$ contains either x or z, we have

$$d(v,\{x,z\}) \leq d(w,\{x,z\}) + d(v,w) < 2\lambda\frac{r}{s} + \frac{r}{s} = \frac{r}{8\lambda^2}.$$

This is a contradiction with the facts $d(x,z) > 2\lambda r$ and $v \in |\alpha| \subset \overline{B}_{\overline{Y}}\left(x,\frac{r}{4}\right) \setminus B_{\overline{Y}}\left(x,\frac{r}{8\lambda^2}\right)$. Since no such w exists, $d(v,|\gamma_a|) \geq \frac{r}{s}$. Consequently, $B_{\overline{Y}}\left(v,\frac{r}{s}\right) \subset B_{\overline{Y}}(x,r) \setminus \partial Y$.

If $(4\lambda)^{-1}\min\{r_0, \operatorname{diam}\partial Y\} \le r < \min\{r_0, \operatorname{diam}\partial Y\}$, then there exists a point $v \in Y$ such that

$$B_{\overline{Y}}\left(v, \frac{r}{4\lambda s}\right) \subset B_{\overline{Y}}\left(x, \frac{r}{4\lambda}\right) \setminus \partial Y \subset B_{\overline{Y}}(x, r) \setminus \partial Y.$$

In either case, (5.1) holds. □

Let $r_1 = \min\{r_0, \operatorname{diam}\partial Y\}$, where r_0 is the parameter from the assumptions of Theorem 1.5. There exists a constant $C \ge 1$ with the following properties:

(i) For every $y \in \partial Y$ and every $0 < r$, $\mathcal{H}^2_{\overline{Y}}(B_{\overline{Y}}(y, r)) \le Cr^2$. Moreover, for every $y \in \partial Y$ and every $0 < r < r_1$, $\mathcal{H}^2_{\overline{Y}}(B_{\overline{Y}}(y, r)) \ge C^{-1}r^2$.
(ii) For every $y \in \partial Y$ and $0 < r < r_1$, there exists $z \in Y$ such that $\overline{B}_{\overline{Y}}(z, C^{-1}2r) \subset B_{\overline{Y}}(y, r) \setminus \partial Y$.
(iii) For every $z \in Y$ with $d(z, \partial Y) \le r_1$ and every $0 < r < d(z, \partial Y)$, $C^{-1}r^2 \le \mathcal{H}^2_{\overline{Y}}(B_{\overline{Y}}(z, r))$.

Remark 5.3. We have assumed that for all radii $0 < r < \operatorname{diam}\partial Y$ and every $y \in \partial Y$, $\mathcal{H}^2_{\overline{Y}}(B_{\overline{Y}}(y, r)) \le C'r^2$ for some $C' > 0$. Then the upper bound in (i) holds for every $r > 0$ if we replace C' with $C'' = \max\left\{C', \mathcal{H}^2_{\overline{Y}}(\overline{Y})/(\operatorname{diam}\partial Y)^2\right\}$. The lower bound for such balls follows from Lemma 4.2.

The property (ii) follows from Lemma 5.2 for some constant. Recall that, under the assumptions of Theorem 1.5, the lower bound in (iii) holds for each $0 < r < d(z, \partial Y) \le r_0$ for some constant C'. Hence (iii) follows.

Our claim is qualitative, so we use a uniform constant C for these various conditions.

In the following, if B is a ball and $\xi > 0$, ξB refers to the ball with the same center and ξ times the radius.

Lemma 5.4. There exists a collection $\mathcal{B}$ of pairwise disjoint balls in $Y = \overline{Y} \setminus \partial Y$ such that for every $(y, r) \in \partial Y \times (0, r_1)$ there exists a ball $B \in \mathcal{B}$ with

$$r_1/2 > \max\left\{d(\partial Y, B), \operatorname{diam} B\right\} \quad \text{and} \quad \operatorname{diam} B \simeq r \simeq d(y, B) \simeq d(\partial Y, B), \tag{5.2}$$

where $A_1 \simeq A_2$ means that there exists a *constant of comparability* $D > 0$ for which $D^{-1}A_1 \le A_2 \le DA_1$. Here the constants of comparability depend only on the constant C.

Proof of Lemma 5.4. For each $x \in \partial Y$ and each integer m such that $0 < 2^m < r_1$, consider a point $v_{x,m} \in Y = \overline{Y} \setminus \partial Y$ with $\overline{B}_{\overline{Y}}(v_{x,m}, C^{-1}2^{m+1}) \subset B_{\overline{Y}}(x, 2^m) \setminus \partial Y$. Then the ball $B_{x,m} := \overline{B}_{\overline{Y}}(v_{x,m}, C^{-1}2^m)$ satisfies

$$\operatorname{diam} B_{x,m} \simeq 2^m \simeq d(x, B_{x,m}) \simeq d(\partial Y, B_{x,m}) \tag{5.3}$$

with the constants of comparison depending only on C.

For each $m \in \mathbb{Z}$ with $2^m < r_1$, let $\mathcal{B}_m$ denote the collection of the $B_{x,m}$ as $x \in \partial Y$ varies. The $5r$-covering theorem [8, 2.8.4] states that there exists subcollection $\mathcal{B}'_m \subset \mathcal{B}_m$ whose elements are pairwise disjoint and

$$\bigcup_{B \in \mathcal{B}_m} B \subset \bigcup_{B \in \mathcal{B}'_m} 5B. \tag{5.4}$$

Let $1 \le N \in \mathbb{N}$ and $m_1 \in \mathbb{Z}$ with $2^{m_1} < r_1 \le 2^{m_1+1}$. Consider the collection $\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}'_{m_1 - kN}$.

By choosing a sufficiently large N, the elements of $\mathcal{B}$ are pairwise disjoint and each element satisfies $\max\left\{d(\partial Y, B), \operatorname{diam} B\right\} < r_1/2$. The choice of N depends only on the constants in (5.3). The inclusion (5.4) implies that for each $0 < r < r_1$ and $x \in \partial Y$, there exists $B \in \mathcal{B}$ such that (5.3) holds with $B_{x,m}$ replaced by B and 2^m by r, with constants of comparison depending only on C. Then (5.2) follows. □

We fix arbitrary $0 < \epsilon < 1$, $0 < r < r_1$ and $y \in \partial Y$. Let $N = B_{\overline{Y}}(y, 2r) \cap \left\{x \in \overline{Y} : d(\partial Y, x) < 2^{-1}\epsilon r\right\}$. Let $\mathcal{B}_0 \subset \mathcal{B}$ denote the subcollection consisting of all $B \in \mathcal{B}$ with $B \subset B_{\overline{Y}}(y, 4r)$ with diameter at least $a\epsilon r$ for $a > 0$.

Lemma 5.5. There exist $1 > a > 0$, $b > 0$ and $\xi > 1$, with the choices depending only on the constants of comparability in (5.2), such that

$$T(x) := \sum_{B \in \mathcal{B}_0} \chi_{\xi B}(x) \ge -b\log(\epsilon) \quad \text{for all } x \in N, \tag{5.5}$$

where $\chi_{\xi B}$ is the characteristic function of the ball ξB.

Proof of Lemma 5.5. For each $(w, s) \in B(y, 3r) \cap \partial Y \times (0, r_1)$, let $B_s(w) \in \mathcal{B}$ be the ball obtained from Lemma 5.4. Lemma 5.4 implies the existence of $A > 1$ for which $A^{-1}s \le \operatorname{diam} B_s(w) < As$ and for the center $c_s(w)$ of $B_s(w)$, $A^{-1}s \le d(w, c_s(w)) < As$.

Let $0 < a < A^{-2}$. Then for every $Aa\epsilon r \le s < r/A$, we have $B_s(w) \in \mathcal{B}_0$. Moreover, if $\xi \ge 3a^{-1}$, the radius of $\xi B_s(w)$ is bounded from below by $3a^{-1} \operatorname{diam} B_s(w)/2$. Given $z \in B(w, 2^{-1}\epsilon r)$,

$$d(c_s(w), z) < s/(2Aa) + d(c_s(w), w) \le (a^{-1} + 2A^2)\frac{\operatorname{diam} B_s(w)}{2}.$$

Thus $\xi B_s(w) \supset B(w, 2^{-1}\epsilon r)$.

Let $k \in \mathbb{Z}$ be the largest integer for which $Aa\epsilon r < r/A^{2k+1}$. Let $\{\widetilde{s}_i\}_{i=1}^k$ be a strictly increasing sequence in the interval $\left(Aa\epsilon r, r/A^{2k+1}\right)$. Denote $s_i = A^{2(i-1)}\widetilde{s}_i$ for each $i = 1, 2, \ldots k$. Here $s_k < r/A$ and $As_{i-1} < A^{-1}s_i$ for each i. Hence the collection $\{B_{s_i}(w)\}_{i=1}^k$ contains k different balls. This implies

$$T(z) \ge k \quad \text{for every } z \in B(w, 2^{-1}\epsilon r). \tag{5.6}$$

We set now $a = A^{-4}$, $\xi = 3a^{-1}$, and $b = 1/(2\log(A))$. The maximality of k implies $k \ge -b\log(\epsilon)$.

If $z \in N$, there exists $w_z \in \partial Y$ such that $d(w_z, z) = d(\partial Y, z) < 2^{-1}\epsilon r$. In particular, $w_z \in B(y, 3r) \cap \partial Y$ and $z \in B(w_z, 2^{-1}\epsilon r)$. Then (5.6) implies (5.5) for the constants a, ξ, and b. □

Lemma 5.6. Let a, b, ξ, and $\mathcal{B}_0$ be as in Lemma 5.5. There exists a constant $d \ge 1$, depending only on the constants of comparability in (5.2) and C, such that $\mathcal{H}^2_{\overline{Y}}(5\xi B) \le d\mathcal{H}^2_{\overline{Y}}(B)$ for every $B \in \mathcal{B}_0$.

Proof of Lemma 5.6. Consider $B \in \mathcal{B}_0$. Then $5\xi B \subset B_{\overline{Y}}(y, \rho)$ for some $y \in \partial Y$ such that $5\xi \operatorname{diam} B \simeq 5\xi d(y, B) \simeq 5\xi d(\partial Y, B) \simeq \rho$, depending only on the constants of comparability in (5.2). The mass upper bound (i) yields $\mathcal{H}^2_{\overline{Y}}(5\xi B) \le C\rho^2 \simeq C25\xi^2(\operatorname{diam} B)^2$. Given (5.2), we have $\max\{\operatorname{diam} B, d(B, \partial Y)\} < r_1/2$. Hence the lower bound from (iii) implies $(\operatorname{diam} B)^2 \le C\mathcal{H}^2_{\overline{Y}}(B)$. The existence of d follows. □

Proof of Proposition 5.1. The claim is that there exist $0 < \delta < 2$ and $\widetilde{C} > 0$ such that for every $0 < \epsilon < 1$, every $0 < r < \operatorname{diam} \partial Y$, and every $y \in \partial Y$, any ϵr-separated set $E \subset B(y, r) \cap \partial Y$ satisfies $\#E \le \widetilde{C}\epsilon^{-\delta}$.

Suppose that the claim holds whenever $0 < r < r_1$. Consider $r_1 \le r < \operatorname{diam} \partial Y$. Let $E_0 \subset \partial Y$ be a maximal $r_1/2$-separated net. For every $f \in E_0$, the set $E_f = B(f, r_1/2) \cap E$ is $\epsilon r_1/2$-separated. Since $E = \bigcup_{f \in E_0} E_f$, we have $\#E \le \sum_{f \in E_0} \#E_f \le \#E_0 \widetilde{C}\epsilon^{-\delta}$. So the general case follows from the special one.

Now we prove the claim for each $0 < r < r_1$, $0 < \epsilon < 1$, and $y \in \partial Y$. We choose a, b and ξ as in Lemma 5.5 and let $\mathcal{B}_0 \subset \mathcal{B}$ be the collection defined before the statement Lemma 5.5. The collection $\mathcal{B}_0$ has the following properties:

A_1. $\sup_{B \in \mathcal{B}_0} \operatorname{diam} B < \infty$;
A_2. $0 < \mathcal{H}^2_{\overline{Y}}(5\xi B) \le d\mathcal{H}^2_{\overline{Y}}(B)$ for each $B \in \mathcal{B}_0$;
A_3. the balls in $\mathcal{B}_0$ are pairwise disjoint;
A_4. the measure of $S := \bigcup_{B \in \mathcal{B}_0} B \subset B_{\overline{Y}}(y, 4r)$ is finite.

Lemma 5.6 implies that the constant d in A_2 can be chosen to be independent of y, r, and ϵ.

Having verified properties A_1-A_4, [4, Theorem 9.6 (b)] explicitly states for $\mu = 1/12d^2$ the following:

$$\mathcal{H}^2_{\overline{Y}}\left(\{x \in \overline{B}_{\overline{Y}}(y, 2r) \colon T(x) \ge -b\log(\epsilon)\}\right) \le (1 + d)\mathcal{H}^2_{\overline{Y}}(S)e^{-\mu(-b\log(\epsilon))}.$$

Given that $S \subset B_{\overline{Y}}(y, 4r)$, the upper bound in (i) implies

$$\mathcal{H}^2_{\overline{Y}}\left(\{x \in \overline{B}_{\overline{Y}}(y, 2r) \colon T(x) \ge -b\log(\epsilon)\}\right) \le (1 + d)(C(4r)^2)\epsilon^{\mu b}. \tag{5.7}$$

Let $E \subset B_{\overline{Y}}(y, r) \cap \partial Y$ be an ϵr-separated set. We see from (5.5) and the lower bound in (i) that

$$\mathcal{H}^2_{\overline{Y}}\left(\{x \in \overline{B}_{\overline{Y}}(y, 2r) \colon T(x) \ge -b\log(\epsilon)\}\right) \tag{5.8}$$

$$\geq \mathcal{H}^2_{\overline{Y}}(N) \geq \mathcal{H}^2_{\overline{Y}}\left(\bigcup_{z\in E} B(z, 2^{-1}\epsilon r)\right) \geq (\#E)C^{-1}(2^{-1}\epsilon r)^2.$$

Now (5.7) and (5.8) yield $\#E \leq \widetilde{C}\epsilon^{-(2-\mu b)}$, where $\widetilde{C}$ is independent of y, r, and ϵ. We denote $\delta := 2 - \mu b < 2$. The claim follows. □

6 Concluding remarks

Consider any quasiconformal Jordan domain Y and ϕ as in Theorem 1.1. Since ϕ is a homeomorphism, the Jacobian J_ϕ of ϕ satisfies

$$\text{Area}(\phi) := \int_{\mathbb{D}} J_\phi \, d\mathcal{L}^2 \leq \mathcal{H}^2_{\overline{Y}}(Y) < \infty. \tag{6.1}$$

The number Area(ϕ) is called the *parametrized area* of ϕ.

Lytchak and Wenger consider in [16, Section 1.2] the class $\Lambda(\partial Y, \overline{Y})$ of those $u \in N^{1,2}(\mathbb{D}; \overline{Y})$ whose trace $u' : \mathbb{S}^1 \to \overline{Y}$ is a (weakly) monotone parametrization of ∂Y. Associated to such maps, one defines Area(u) by integrating a Jacobian of u [16, Section 1.2]. Also, $E(u) = \int_{\mathbb{D}} \rho_u^2 \, d\mathcal{L}^2$ is the corresponding energy.

Theorem 1.1 implies that every quasiconformal homeomorphism $\phi : \mathbb{D} \to Y$ defines an element of $\Lambda(\partial Y, \overline{Y})$. If ϕ is K-quasiconformal, then $E(\phi) \leq K\text{Area}(\phi) < \infty$ due to (6.1). Then [16, Theorem 7.6] yields the existence of $u_e \in \Lambda(\partial Y, \overline{Y})$ with minimal $E(u)$ among all $u \in \Lambda(\partial Y, \overline{Y})$, referred to as an energy minimizer. Similarly, [16, Theorem 1.1] yields the existence of $u_a \in \Lambda(\partial Y, \overline{Y})$ of minimal parametrized area.

For a general quasiconformal Jordan domain Y, it is not clear whether or not u_e (or u_a) is a quasiconformal homeomorphism. However, if we also assume that $\overline{Y}$ is geodesic, ∂Y is rectifiable, and $\overline{Y}$ satisfies a quadratic isoperimetric inequality and (1.2) at the boundary points of Y, the energy minimizer u_e is a quasiconformal homeomorphism [6, Theorem 1.3]. We refer the interested reader to [16] and [6] for further reading.

There are some ways to construct quasiconformal Jordan domains. For example, if X is a metric surface satisfying local versions of annular linear local connectivity and Ahlfors 2-regularity, Theorem 4.17 of [25] yields the existence of many quasiconformal Jordan domains $Y \subset X$ satisfying the assumptions of Theorem 1.5. Some examples of quasiconformal Jordan domains can also be obtained from [10, Theorem 2].

Acknowledgements: The author was supported by the Academy of Finland, project number 308659 and by the Vilho, Yrjö and Kalle Väisälä Foundation. The author wishes to thank the referee for their helpful suggestions.

References

[1] L. Ambrosio and P. Tilli. *Topics on analysis in metric spaces*, volume 25 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.
[2] A. Beurling and L. Ahlfors. The boundary correspondence under quasiconformal mappings. *Acta Math.*, 96:125–142, 1956.
[3] M. Bonk, J. Heinonen, and S. Rohde. Doubling conformal densities. *J. Reine Angew. Math.*, 541:117–141, 2001.
[4] M. Bonk, P. Koskela, and S. Rohde. Conformal metrics on the unit ball in Euclidean space. *Proc. London Math. Soc. (3)*, 77(3):635–664, 1998.
[5] V. I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007.
[6] P. Creutz and M. Romney. The branch set of minimal disks in metric spaces. *arXiv e-prints*, page arXiv:2008.07413, August 2020.
[7] J. Duda. Absolutely continuous functions with values in a metric space. *Real Anal. Exchange*, 32(2):569–581, 2007.
[8] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.
[9] J. B. Garnett and D. E. Marshall. *Harmonic measure*, volume 2 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2005.

[10] P. Haïssinsky. A sewing problem in metric spaces. *Ann. Acad. Sci. Fenn. Math.*, 34(2):319–345, 2009.
[11] J. Heinonen. *Lectures on analysis on metric spaces*. Universitext. Springer-Verlag, New York, 2001.
[12] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. T. Tyson. *Sobolev spaces on metric measure spaces: An approach based on upper gradients*, volume 27 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2015.
[13] D. Herron and D. Meyer. Quasicircles and bounded turning circles modulo bi-Lipschitz maps. *Rev. Mat. Iberoam.*, 28(3):603–630, 2012.
[14] T. Ikonen and M. Romney. Quasiconformal geometry and removable sets for conformal mappings. *J. Anal. Math.*, to appear.
[15] N. J. Korevaar and R. M. Schoen. Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.*, 1(3-4):561–659, 1993.
[16] A. Lytchak and S. Wenger. Area minimizing discs in metric spaces. *Arch. Ration. Mech. Anal.*, 223(3):1123–1182, 2017.
[17] S. Merenkov and K. Wildrick. Quasisymmetric Koebe uniformization. *Rev. Mat. Iberoam.*, 29(3):859–909, 2013.
[18] K. Rajala. Uniformization of two-dimensional metric surfaces. *Invent. Math.*, 207(3):1301–1375, 2017.
[19] K. Rajala and M. Romney. Reciprocal lower bound on modulus of curve families in metric spaces. *Ann. Acad. Sci. Fenn. Math.*, 44:681–692, 2019.
[20] S. Semmes. Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincaré inequalities. *Selecta Math. (N.S.)*, 2(2):155–295, 1996.
[21] P. Tukia and J. Väisälä. Quasisymmetric embeddings of metric spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 5(1):97–114, 1980.
[22] S. Willard. *General topology*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.
[23] R. L. Wilder. *Topology of manifolds*, volume 32 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, R.I., 1979. Reprint of 1963 edition.
[24] K. Wildrick. Quasisymmetric parametrizations of two-dimensional metric planes. *Proc. Lond. Math. Soc. (3)*, 97(3):783–812, 2008.
[25] K. Wildrick. Quasisymmetric structures on surfaces. *Trans. Amer. Math. Soc.*, 362(2):623–659, 2010.
[26] M. Williams. Geometric and analytic quasiconformality in metric measure spaces. *Proc. Amer. Math. Soc.*, 140(4):1251–1266, 2012.