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Avicenna on Mathematical Infinity

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Abstract: Avicenna believed in mathematical finitism. He argued that magnitudes and sets of ordered numbers and numbered things cannot be actually infinite. In this paper, I discuss his arguments against the actuality of mathematical infinity. A careful analysis of the subtleties of his main argument, i. e., *The Mapping Argument*, shows that, by employing the notion of *correspondence* as a tool for comparing the sizes of mathematical infinities, he arrived at a very deep and insightful understanding of the notion of mathematical infinity, one that is much more modern than we might expect. I argue, moreover, that Avicenna's mathematical finitism is interwoven with his literalist ontology of mathematics, according to which mathematical objects are properties of existing physical objects.

1 Introduction

In the Aristotelian tradition the problem of infinity has two distinct aspects. Its negative aspect includes various arguments for the impossibility of the *actual* existence of infinity. Its positive aspect, on the other hand, justifies the merely *potential* existence of infinity and explains how something can have the potentiality of being infinite (*apeiron*), although this potentiality can never be actualized. Avicenna had some innovative ideas with respect to both of these aspects. Compared to most other Aristotelian philosophers, he had a more flexible approach to the impossibility of actual (*bil-fi'l*) infinity (*lā nihāya*).¹ Specifically, he preserves the possibility of the actual existence of a very specific type of infinity. He believes that an infinite *non-ordered* (*ghayr murattab*) set of immaterial objects, e. g., angels or souls, can (and, indeed, does) actually exist.² Nonetheless, this view does not take him very far from Aristotle's own position on the ordinary types

1 Nawar 2015, 2355 n15.

2 See Marmura 1960, Rashed 2005, 298 f., and McGinnis 2010.

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of physical or mathematical infinity we usually think about in scientific inquiry. Avicenna, like Aristotle, believes that they exist *only* in a potential (*bil-quwwa*) sense. The significance of Avicenna's views on physical or mathematical infinity lies, therefore, in the subtle and insightful ideas he adds to both the negative and the positive aspects of the problem to support the core idea of Aristotelian infinity, rather than in a rejection of Aristotle's view.

This paper aims to explain Avicenna's views on the negative aspect of the problem of mathematical infinity and to clarify their significance and novelty from the perspective of the history and philosophy of mathematics. I should, therefore, first specify what exactly I mean by 'mathematical infinity'. In the next section, I discuss this issue and elaborate the relation between mathematical and physical infinity in Avicenna's philosophy. Knowing about this relation helps us to realize better how Avicenna's arguments for the impossibility of the actual existence of mathematical infinity are interwoven with his arguments against the actuality of the physical infinite. Moreover, it sheds a new light on why Avicenna discusses mathematical infinity in the *Physics* parts of his works. In Section 3, I briefly review two of Avicenna's arguments against the actuality of infinity. The first, *The Collimation Argument* (*burhān al-musāmita*), appeals to the notion of *motion*, while the other, *The Ladder Argument* (*burhān al-sullam*), does not engage such physical notions. Our study of Avicenna's views on the negative aspect of the problem of mathematical infinity is completed in Section 4, by investigating the details of his main argument against the actuality of mathematical infinity; this is *The Mapping Argument* (*burhān al-taṭābuq* or *al-taṭbīq*). I will show that only this argument can be applied to the case of numerical (discrete) infinity. Elucidation of the philosophical and mathematical presuppositions of this argument reveals that the affinity between Avicenna's understanding of the notion of infinity and our modern understanding of this notion is stronger than we might have expected, or so I will argue there. I close, in Section 5, with some concluding reflections.

2 The Notion of Mathematical Infinity

There are two important points to make concerning the notion of mathematical infinity, before going through the details of Avicenna's views about this notion. First, I explain exactly what I mean by 'mathematical infinity'. Second, I discuss the connection of this notion with the notion of physical infinity from the perspective of Avicenna's philosophy.

Aristotle defines infinity as something that “if, taking it quantity by quantity, we can always take something outside.”³ Avicenna accepts this definition and mentions it in many different places of his oeuvre. For example, in *The Physics of the Healing* he says that infinity is “that which whatever you take from it – and any of the things equal to that thing you took from it – you [always] find something outside of it.”⁴ Nonetheless, Avicenna’s treatment of the more specific notion of mathematical infinity is not in complete accordance with that of Aristotle. In the Aristotelian tradition, the problem of mathematical infinity has been studied by analysing three different yet interrelated phenomena: (a) the infinity of numbers⁵ (i. e., the subject matters of arithmetic), (b) the infinity of magnitudes (i. e., the subject matters of geometry), and (c) the infinite divisibility of magnitudes.⁶ What I mean by ‘mathematical infinity’ is restricted to the two former types of infinity. I do not go, therefore, into the details of Avicenna’s views on the latter type of mathematical infinity. This departs somewhat from the general approach of contemporary Aristotle scholars, most of whom have paid more attention to the infinite divisibility of magnitudes.⁷ Specifically, they have tried to clarify Aristotle’s view about (a) and (b) by scrutinizing his views about (c). I think, nonetheless, that following the opposite strategy is more plausible, at least in the context of Avicenna’s philosophy. Let me justify why.

Aristotle believes that a thing – whatever it is – may be “infinite either by addition or by division.”⁸ As a result, the first two kinds of mathematical infinity

³ *Physics* II.6, 207a7–8. All translations of Aristotle’s texts in this paper are from *The Revised Oxford Translation of The Complete Works of Aristotle*. See Aristotle 1984.

⁴ Avicenna 2009, III.7 [2]. See also Avicenna 2009, III.7 [3] and III.9 [1]. As another example, in the letter to the vizier Abū Sa’d, Avicenna defines infinity as “a quantity or something possessing a quantity that if you take something from it, you still find something other than what you took and you never reach something beyond which there is nothing of it [i. e., of that infinity]” (Avicenna 2000, 28).

⁵ Throughout the paper, by ‘numbers’ I just mean ordinary natural numbers.

⁶ This classification is inspired by Aristotle’s *Physics* III, 206a9–12. However, in that passage he speaks of the infinity of *time* rather than *magnitude*. He confirms there that the infinite divisibility of magnitudes and the infinity of numbers and time are, *in a sense*, undeniable. Given the so-called ‘isomorphism thesis’ according to which one-dimensional magnitudes, one-dimensional motions, and time have the same mathematical structure, (b) and (c) have some strong connections, respectively, to the infinity of time and the infinite divisibility of temporal intervals. Fred Miller 1982, Sec. 5, has argued that Aristotle endorses this thesis. See Newstead 2001 for a discussion of this thesis from the perspective of modern mathematical theories of continuum. McGinnis 1999 shows that Avicenna endorses this thesis and his theory of time rests on it.

⁷ See, among others, Hintikka 1966, Bostock 1972 and 2012, Lear 1979, Bowin 2007, and Coope 2012.

⁸ *Physics* III, 206a14–8.

(i. e., the infinity of numbers and magnitudes) are cases of being *infinite by addition* while the latter (i. e., the infinite divisibility of magnitudes) is a case of being *infinite by division*. There are two things which motivate Aristotle scholars to base their discussions of mathematical infinity on the infinite by division rather than the infinite by addition. First, some of Aristotle's texts can, in principle, be interpreted in a way that represents him as believing that the problem of the infinite by addition is reducible to the problem of the infinite by division.⁹ Consequently, the latter notion has a priority over the former in discussion. Second, Aristotle's discussions of the infinity of numbers and magnitudes lack any *mathematically* remarkable feature and cannot, therefore, attract the attention of modern historians and philosophers of mathematics. By contrast, none of these points is true of Avicenna's account of mathematical infinity. On the one hand, the reducibility of the notion of infinity by addition to that of infinity by division does not play any influential role in Avicenna's discussion of the infinity of numbers and magnitudes. To be precise, he is sympathetic to the idea of discussing infinite divisibility prior to the infinite largeness of numbers and magnitudes,¹⁰ *but* the plausibility of his discussions of the latter does not really depend on that of the former. There is no obvious inferential connection between them. On the other hand, and perhaps more importantly, his discussions on the infinity of numbers and magnitudes include the introduction and application of some mathematical notions and the presentation of some arguments which are worthy of investigation by historians and philosophers of mathematics, or so I believe. Therefore what motivates Aristotle scholars is weakened in the context of Avicenna's philosophy of mathematics. Additionally, it seems that, at the end of the day, without a correct conception of the infinity of numbers it is impossible to illustrate the infinite divisibility of magnitudes. The infinite divisibility of a line is nothing other than that the number of divisions we can make in that line is infinite; ergo, the infinity of numbers should be discussed prior to the infinite divisibility of magnitude. These considerations are enough to show that we can base our study of Avicenna's views about mathematical infinity on (a) and (b), rather than (c). I think that Avicenna's views about the infinite divisibility of magnitudes should

⁹ See, for example, Lear 1979, 195, who interprets *Physics* III, 206b3–4, in this way. Bowin 2007, Sec. III, not only confirms this approach, but also construes *Physics* III, 207b10–13, as claiming the strong epistemological thesis “that our ability to think of ever larger natural numbers also depends upon the infinite divisibility of magnitudes” (Bowin 2007, 244).

¹⁰ In *The Physics of the Healing*, Avicenna says that “before we speak about finite bodies and their states with respect to largeness, we should speak about the finite and infinite with respect to smallness and divisibility” (Avicenna 2009, III.2 [1]). But, as we will see, his discussion of the former has no argumentative connection to that of the latter.

be studied in connection with the notion of the *mathematical continuum*, and I postpone this to an independent further work.

Another point worth mentioning about the notion of mathematical infinity is its relation to physical infinity. Aristotelian philosophers in general believe that the mathematical realm is connected to the physical realm, although the nature of this connection can be (and indeed is) construed in many different ways. Avicenna's views on the ontology of mathematics and the nature of mathematical objects show how strong this connection is for him. Familiarity with these views helps us to attain a more comprehensive understanding of Avicenna's position on mathematical infinity. Therefore, I briefly sketch his theory on the ontological status of mathematical objects. I have discussed the details and subtleties of this theory elsewhere.¹¹

According to Avicenna, mathematical objects, i.e. numbers (*a'dād*) and geometrical shapes (or magnitudes (*maqādir*) in general), are neither Platonic forms, nor independent material objects, nor even purely mental existents completely separated from matter. They are, in the first instance, accidents of actually existing material objects.¹² They are, therefore, mixed with particular materials (or, in other words, with particular kinds of matter) in the extramental realm. They are predicated upon the physical objects. By the aid of our estimation (*wahm*), we can separate mathematical objects, in our minds, from all those determinate kinds of matter to which they are attached outside the mind.¹³ Nonetheless, mathematical objects cannot be separated from materiality itself. Even in the mind they are mixed with materiality. They "absolutely do not dispense with matter, even though they can do with some kind of matter."¹⁴

The ontological status of mathematical objects, as the objects studied by mathematics, may become clearer in contrast with the ontological status of the objects studied and investigated by natural sciences and metaphysics. Whether in the external world or in the mind, the objects studied by natural sciences are inseparable from not only materiality itself but also from the particular matters with which they are mixed. We cannot separate humanness, for example, from

¹¹ See Zarepour 2016.

¹² See Avicenna 2005, III.3–4.

¹³ According to Avicenna's theory of knowledge, estimation is a bodily cognitive faculty which plays a protagonist role in the epistemology of mathematics. For a magnificent discussion on the other human or animal functions of this faculty, see Black 1993. See also Hall 2006 for a more recent study on the role of the estimative faculty in Avicenna's psychology.

¹⁴ Avicenna 2005, VII.2 [21]. I have corrected an oversight in Marmura's translation by putting the second 'with' in the above quote to replace his 'without'. To be precise, Avicenna says: "*al-hindisiyāt min al-ta'limiyāt lā tastaghni hudūduhā 'an al-mawād muṭlaqan, wa 'in istaghanat 'an naw' mā min al-mawād.*"

either materiality in general or even the particular matter it is mixed with, i. e., flesh and blood. Therefore, even in our estimation, we cannot detach humanness from flesh and blood. On the other hand, the objects studied by metaphysics, though they *may* be mixed with some particular matters in the external world, are separable from not only those particular matters but also from materiality itself in the mind.¹⁵ With respect to separability from matter, mathematical objects lie between these two groups of objects. More precisely, with respect to separability from determinate kinds of matter, mathematical objects are similar to the objects studied by metaphysics and dissimilar to the objects studied by natural sciences. In our minds, we can separate mathematical objects from those particular matters with which they are mixed in the extramental world. But with respect to separability from materiality itself, mathematical objects are similar to the objects studied by natural sciences and dissimilar to the objects studied by metaphysics, because even in our mind we cannot separate them from materiality itself. Mathematical objects, inasmuch as they are the subject matters of mathematical studies, are inseparable from the material form (*al-šūra al-māddiya*).¹⁶

15 Some objects studied by metaphysics, e. g., God and mind, are necessarily separated from matter. Therefore they cannot be mixed with matter. By contrast, others, e. g. numbers, can in principle be mixed with matter. However, if we consider them as mixed with matter then our study is no longer metaphysical. See Marmura's diagram of the classification of the objects studied by the different sciences at the end of his 1980 paper.

16 See Avicenna 2005, III.4 [2]. By 'material forms', or more precisely 'forms that belong to matter', Avicenna seems to mean the form of corporeality which is common to all corporeal things. Mathematical objects cannot be conceived separated from the corporeal form. Shihadeh 2014 discusses Avicenna's views on the corporeal form and its reception in the twelfth century. See especially page 367 for an explanation of the inseparability of mathematical objects from the corporeal form. The corporeal form can be conceptually separated from prime matter. Therefore, one might suggest that mathematical objects, though inseparable from the corporeal form, can be completely separated from matter in the estimation. If so, when Avicenna says that mathematical objects "do not dispense with matter, even though they can do with some kind of matter", by *dispensability with matter* (*istighnā 'an mādda*) he does mean nothing more than *separability from the corporeal form*. In other words, mathematical objects in the estimative faculty are detached from matter, though still attached to the corporeal form. However, it seems that mathematical objects should have a stronger connection to materiality. It is possible (and mathematicians often need to) consider different but qualitatively indistinguishable instances of each kind of geometrical shapes, e. g., two distinct circles of the same radius or two distinct squares of the same size. But the corporeal form cannot be the distinguishing feature of these distinct instances. This is because, as Shihadeh 2014, 385, clarifies, "Avicenna does not speak of multiple 'corporeities'." All corporeal things, Avicenna believes, share the same corporeal form; corporeality itself is not quantifiable. Therefore, indispensability with matter seem to be something more than mere attachment to the corporeal form. It is rather attachment to an unqualitative indeterminate kind of matter which we can call it, in Aristotelian term, intelligible matter (or

Even in our estimation, we should consider them as accidents of material objects and, therefore, attached to matter, albeit not to a specific kind of matter.¹⁷

According to the above picture, Avicenna should be described as a literalist with respect to the ontology of mathematics. He believes that mathematical objects are accidents and properties of physical objects that literally exist in the external world. The existence of such properties does not depend upon the specific kinds of matter of the objects of which they are properties. To study these properties, we can therefore ignore those specific kinds of matter. Mathematical objects are then abstracted by the estimative faculty from all specific kinds of matter. Nonetheless, it does not mean that they are studied *as if* they are not material properties. Mathematics for Avicenna is a specific way of studying a very specific group of physical properties. One might oppose this rendition of Avicenna's view by putting forward that mathematical objects are mental objects that are constructed by the abstraction mechanism. The quantity investigated by mathe-

estimative matter, to have a more Avicennian flavor). This intelligible matter (to which mathematical objects are attached in the estimative faculty) plays the role of the distinguishing element of distinct instances of the same kind of geometrical shapes. Admittedly, this account should be discussed and developed in more detail. Nonetheless, doing so is beyond the scope of this paper.

17 As I have shown in detail in Zarepour 2016, these views concerning the ontological status of the objects studied and investigated by the different sciences are deduced from Avicenna's discussions on the classification of the sciences, which appear, with slight differences, in several parts of his oeuvre; e. g., Ch. 2 of Bk. I of *Isagoge* (1952), Chs. 1–3 of Bk. I of *The Metaphysics of the Healing* (2005), and Chs. 1f. of *The Metaphysics of 'Alā'ī Encyclopedia* (2004). The idea of classification of the sciences according to the ontological status of the objects that they study goes back to Aristotle (*Metaphysics* VI.1, 1026a13–19) and has been discussed by Al-Fārābī in his *The Aims of Aristotle's Metaphysics*, of which Avicenna explicitly says, in his autobiography (cf. Gutas 2014, 17f.) that he has read it. To be more precise, for Avicenna the theoretical sciences are divided primarily according to whether or not the objects they study are related to *motion* (i. e., whether or not they are *movable*). But an object is movable if and only if it is associated with matter in the external world. That is how *movability/immovability* can be replaced with *inseparability/separability from matter* as a criterion for classifying the objects of the sciences. There is another thing which encourages me to be focused mainly on the latter distinction in my explanation of Avicenna's position about the nature of the objects studied by the different sciences. (In)separability from matter can be divided into two finer-grained kinds of (in)separability – i. e., (in)separability from specific kinds of matter and (in)separability from materiality itself – which play important roles in Avicenna's discussion of classification of the sciences but there is no parallel division with respect to (im)movability. The original Arabic text of *The Aims of Aristotle's Metaphysics* can be found in Al-Fārābī 1890, 34–8. Gutas 2014, 272–5, and McGinnis/Reisman 2007, 78–81 provide partial English translations of this work. Its complete English translation can be found in Bertolacci 2006, 66–72. Cleary 1994 has discussed Aristotle's classification of the sciences. See also Marmura 1980 and Gutas 2003 for two modern commentaries on Avicenna's classification of the sciences.

matics, the objector might discuss, is not the extramental quantity.¹⁸ However, I do not find this position convincing. Mathematical abstraction for Avicenna is not a machinery for creating the objects that otherwise do not exist. It is a cognitive mechanism which provides us with a suitable conceptual framework for thinking about some specific physical properties in a specific way. But the objects of mathematics are those physical properties themselves, rather than their conceptual/mental counterparts constructed by the abstraction mechanism. The existence of mathematical objects does not depend on the human mind.¹⁹

Avicenna thus believes that mathematical objects have some sort of dependency on or inseparability from materiality. However, for the case of geometrical objects (and magnitudes in general) this dependency is in some sense stronger than for numbers. Numbers, unlike magnitudes, are in principle separable from materiality itself. But numbers separated from materiality cannot be subject to increase and decrease. Consequently, they cannot be the subject of mathematical studies, and should therefore be studied in metaphysics. In other words, numbers, inasmuch as they are numbers, are not inseparable from materiality; but inasmuch as they are the subject of mathematical studies (i. e., inasmuch as they are the subject matters of mathematics) they should be considered as mixed with materiality. Numbers emancipated from any dependency on materiality are, therefore, the subject of metaphysical studies rather than mathematical studies. Otherwise put, Avicenna accepts that *numerosity* can in principle find a way into the domain of immaterial objects which are subjects of metaphysical studies; but he denies that this sort of numerosity can be the subject of mathematical studies.²⁰

Contrary to numbers, however, magnitudes and geometrical shapes, inasmuch as they are themselves (and, *a fortiori*, inasmuch as they are the subject of

18 There is a parallel debate on Aristotle's ontology of mathematics. Mueller 1970 and 1990 provides a literalist interpretation of Aristotle's view about mathematical objects. Lear 1982, on the other hand, emphasizes the role of mathematical abstraction in constructing mathematical objects.

19 Admittedly, there are still some issues to be clarified. For instance, there are *peculiar* or *perfect/exact* geometrical shapes that can be (and are indeed) studied by mathematics but do not exist in the extramental world; i. e., there are no physical objects of those shapes. One might say, for example, that there is no perfect circle or no closed-shape-with-5326-straight-sides in the physical world, but these objects are (or, at least, can be) studied by mathematics. Therefore, the objector might conclude, that literalism is false. I think, however, that there are plausible answers for these objections in Avicenna's philosophy of mathematics. Nonetheless, an adequate treatment of them here would take us far afield. See Zarepour 2016, Sec. 5.

20 Numbers separated from matter are not, according to Avicenna, receptive to decrease and increase. They are not capable of being subject to addition, subtraction, and other mathematical operations and, consequently, they cannot be the subject of mathematical studies. See Avicenna 2005, I.3 [17 f.].

mathematical studies), are inseparable from materiality. They cannot be detached from the material form. In other words, although there can be *numerous* immaterial objects, there cannot be any immaterial magnitude or, for example, immaterial triangle. This is the case, at least, if we interpret immateriality as meaning being separated from materiality itself, even from estimative matter, as well as from the material form; not, therefore, simply as being separated from the particular matters upon which they are predicated in the extramental realm. In brief, there is no place for magnitudes in the realm of immaterial objects. This contrast between the ontological status of numbers and magnitudes has some interesting consequences for Avicenna's views about the problem of infinity to which I return at the end of this paper.

But insofar as our concern is merely with the relationship between mathematical and physical infinity, the aforementioned contrast between numbers and magnitudes has no importance. The only important thing is that mathematical objects, either numbers or magnitudes, inasmuch as they are the subject matters of mathematics, are (or, at least, should be considered as being) inseparable from materiality.²¹ Therefore, it is necessarily true that there actually exists a mathematically infinite magnitude if and only if there exists a physically infinite object upon which that magnitude is predicated. Similarly, it is necessarily the case that there actually exists an infinite set of numbers if and only if there exists an infinite set of numbered physical objects upon which those numbers are predicated. In this sense, the problem of mathematical infinity for Avicenna is a special case of the problem of physical infinity.

3 Two Arguments against the Actuality of Infinity

Avicenna investigated the problem of infinity extensively, in all of his main encyclopaedic works as well as in several other places.²² He proposed several argu-

²¹ For two alternative interpretations of Avicenna's views about the nature of mathematical objects see Ardeshir 2008 and Tahiri 2016, who believe that mathematical objects are 'mental' (Ardeshir's term) or 'intentional' (Tahiri's term) objects completely separated from materiality. However, I have argued that these interpretations are not sufficiently accurate. See Zarepour 2016.

²² These are some of the places in which Avicenna *directly* discusses the problem of infinity: (1) *The Physics of the Healing* (2009, III.7–11), (2) *The Physics of the Salvation* (1985, IV.2, 244–52), (3) *The Physics of Pointers and Reminders* (1957, Namaṭ I, Ch. 11, 160–67), (4) *The Metaphysics of 'Alā'ī Encyclopedia* (2004, Ch. 16, 58–61), (5) *The Physics of Fountains of Wisdom* (1980, Ch. 3, 19f.), and (6) *The Letter to the Vizier Abū Sa'd* (2000, 27–36). As we will mention shortly, there are other places in which Avicenna *indirectly* considers this problem.

ments, some of which are more faithful to the structure of Aristotelian arguments against an actual infinity. Almost all of Aristotle's arguments are based on (a) the application of physical notions such as *motion* and *traversability*, and (b) the presupposition of certain Aristotelian doctrines in physics and cosmology. As a result, they hold little interest for someone who is looking at the problem of infinity from a purely mathematical perspective.²³ Moreover, most of those Aristotelian physical or cosmological presuppositions have lost plausibility for modern readers. This is why contemporary philosophers of mathematics usually avoid discussing the details of Aristotle's argument against mathematical infinity.²⁴ By contrast, in addition to some (by and large Aristotelian) physical arguments, Avicenna proposed some mathematical arguments in which he does not appeal to physical notions and Aristotelian presuppositions. In this section I briefly review a physical argument in which Avicenna appeals to the notion of *circular motion*, and a mathematical argument. In Section 4 I focus on another mathematical argument, this being Avicenna's main argument against the actuality of mathematical infinity, which includes the introduction and application of some mathematically significant notions and methods.

3.1 The Collimation Argument

There are some places in which Avicenna provides an indirect discussion of the problem of infinity. For example, in his discussions of the void in *The Physics of the Healing*²⁵ and *The Physics of the Salvation*,²⁶ he appeals to the impossibility of circular motion in an *infinite* void as one of his premises in arguing for the impossibility of the void. To justify this premise, he proposes an auxiliary argument

23 It is true that, according to Avicenna, ontology of mathematics somehow depends on the ontology of physics, but this does not entail either that methodology of mathematical studies is the same as that of physical studies or that every physical notion has something to do with mathematics. We can look at mathematical properties of physical objects by employing a methodology which does not appeal to some physical notions such as mass, weight, and motion. In this sense, physical arguments can be separated from mathematical arguments, even if mathematical ontology cannot be entirely detached from physical ontology.

24 For example, David Bostock's reluctance to discuss these arguments was expressed in this way: "I shall not rehearse his [i. e., Aristotle's] arguments [for the claim that there is a definite limit even to the *possible* sizes of things], which – unsurprisingly – carry no conviction for one who has been brought up to believe in the Newtonian infinity of space. I merely note that this is his view" (2012, 479 f.).

25 Avicenna 2009, II.8 [8].

26 Avicenna 1985, IV.2, 233–44.

that is known as *The Collimation Argument* (*burhān al-musāmita*) or *The Parallelism Argument* (*burhān al-muwāzāt*). This argument likely originates in Aristotle's *De Caelo*.²⁷ Aristotle's original argument was proposed to show that the infinite "cannot revolve in a circle; nor could the world, if it were infinite." Avicenna extensively revised this argument to show, primarily, that circular motion in an infinite void is impossible. Coupling this result with the claim that the void, if it exists, cannot be finite, Avicenna concludes that the void does not exist.²⁸ But in other places such as *The Physics of Fountains of Wisdom*,²⁹ he also proposed this argument as an independent argument against the actual infinitude of intervals (*ab'ād*). The argument goes as follows:

Consider the line L which is infinite in one direction; it starts from the center O of a finite circle C , intersects the circumference of the circle, and extends infinitely. Consider, moreover, another line L' which is parallel to but distinct from L , and extends infinitely in both directions. Now, suppose that the circle C together with L start to rotate around O , while L' remains motionless and fixed. As a result of this circular motion, these two lines intersect. Therefore, there is a moment of time in which these lines are parallel and there is a moment of time in which they intersect with each other. From this fact, Avicenna concludes that there should be a moment of time T and, accordingly, a point P on L' in which these lines intersect each other for the first time (after the beginning of the circular motion). But there is obviously no such point. For every point P which we consider as the first intersection point of these lines, there are infinitely many points on L' prior to P which would have been passed and intersected by L (Fig. 1). Since Avicenna believes that circular motion undeniably can happen, he concludes that what should be rejected is the existence of infinite lines and intervals.³⁰

²⁷ *De Caelo* I.5, 272a8–20.

²⁸ For an explanation of this argument, as it appeared in *The Physics of the Healing*, see McGinnis 2007.

²⁹ Avicenna 1980, Ch. 3, 20.

³⁰ A variation of this argument was employed by Abū Sahl Al-Quhī (940–1000) to show that a principal characteristic of Aristotelian infinity, i. e., the claim that "the infinite magnitude will not be traversed in a finite time" (*Physics* VI.7, 238a20–31), is wrong. I will shortly sketch how this argument can show that the infinite is traversable in a finite time. Rashed 1999 has discussed the details of Al-Quhī's argument.

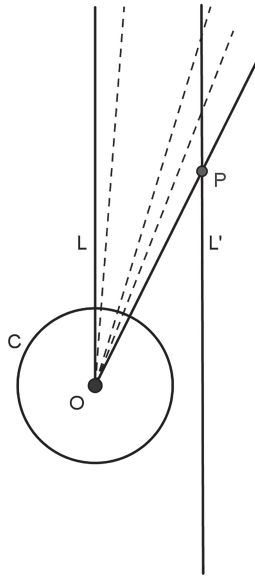


Fig. 1

Although this argument is intuitively powerful, it is not convincing. The argument is not only invalid from the point of view of modern mathematics, but is also incompatible with some of Avicenna's own views. From the fact that there is a moment of time in which L and L' are parallel and there is another moment in which they intersect, we cannot conclude that there is a moment in which these lines intersect with each other for the first time, nor that there is a point on L in which these lines intersect for the first time. Here Avicenna seems to suppose that the set of all temporal moments in which L and L' are intersected has a least element (with respect to the natural order and succession we consider for temporal moments). Correspondingly, there is a first point of intersection on these lines.³¹ However, this supposition is false. It is not the case that every subset of temporal moments or every subset of the set of points on a straight line has a least or first element.³²

³¹ By 'the first point of intersection' I mean the point on which those lines intersected each other for the first time.

³² It is accepted in modern mathematics that the set of points on a straight line that is infinite in both directions and the set of temporal moments with their own natural orders and successions of their elements are isomorphic to the set of real numbers with its natural order. The above supposition can therefore be paraphrased in the language of modern mathematics as the claim that every subset of real numbers has a least element with respect to their own natural order. Equivalently, the natural order on real numbers makes this set *well-ordered* in its technical sense. However, this claim can be mathematically proven to be false.

Interestingly, while Avicenna endorses the above supposition in the context of *The Collimation Argument*, he seems to reject it in his discussion of time and linear motions. If the above supposition were true, we could argue similarly against the possibility of even *linear* motions on *finite* spatial intervals. Consider a finite interval AB . Suppose that an object X is first motionless on A , then it moves and goes without stopping from A to B , and finally stops on B . Therefore, there is a time in which X is at rest and there is a time in which X is in motion. Now, if every subset of the points on a line (a spatial or temporal interval) had a least member, then there would be a first intermediary point of AB on which X does not stop (i. e., it merely passes that point). Correspondingly, there would be a moment in which X is on an intermediary point for the first time. But, given the continuity of time, space and motion – which Avicenna holds – there is neither such point nor such moment.³³ Therefore, if the supposition under discussion were true, we should conclude, in a way parallel to what we had in *The Collimation Argument*, that it is impossible for X to move from A to B . It means that not only is circular motion in an infinite space impossible, but linear motion on a finite magnitude is impossible as well. However, Avicenna accepts, not surprisingly of course, that continuous linear motion is possible. It indicates that he should reject the supposition that every subset of temporal moments or every subset of points on a line has a least or first element. He does so, indeed. McGinnis has elegantly shown that, according to Avicenna's theory of time, "if one takes some instant t as a limit, then for any other instant t' , no matter how close one wants to take t' to t , then there is another instant t'' that is not identical with t , but is closer to t than t' . Since this same analysis will be true of t'' , t''' and so on, one can get indefinitely close to t without actually being at t ."³⁴ It means that the set of all temporal moments between t and t' has no least number. So the aforementioned supposition is rejected.³⁵ But rejecting this supposition renders *The Collimation Argument* invalid. It seems therefore that this argument is controversial even with respect to Avicenna's own philosophical framework. At least, Avicenna owes us an explanation of why this argument is based on a supposition that he rejects in another context.

³³ See Avicenna's discussions of the continuity of magnitudes and motions in, for example, *The Physics of the Healing* (2009, III.2 and IV.8).

³⁴ McGinnis 2004, 60.

³⁵ Considering the isomorphism between the structures of time, space and motion, the rejection of the aforementioned supposition with respect to one of them entails its rejection with respect to the others. For discussions of Avicenna's theories of time and motion, see, respectively, McGinnis 1999 and Ahmed 2016. Ahmed has explicitly pointed out that Avicenna rejects that supposition with respect to motion – i. e., he holds that there is no first part of motion (Ahmed 2016, 236 n50).

Avicenna could however slightly revise this argument such that it validly entails the impossibility of the actual infinity, at least in an Aristotelian framework which would be acceptable for him. Avicenna accepts, following Aristotle, that the infinite cannot be traversed in a finite time.³⁶ Therefore, he could argue that if the circle C is finite, then it takes a finite time for it to rotate once around O . Accordingly, it takes a finite time for L to rotate once around O . But in each round of rotating around O , L traverses the whole line L' . This means that the infinite length of L' can be traversed in a finite time (equal to half the time of L 's rotating around O). This argument shows that the conjunction of (a) the principle that the infinite cannot be traversed in a finite time, (b) the possibility of having infinite intervals or lines, and (c) the possibility of circular motion, entails a contradiction. Avicenna could, therefore, argue that since (a) and (c) are obviously true (according to him), (b) must be rejected; there is no actually infinite interval. This argument seems perfectly sound, at least in an Aristotelian framework. However, Avicenna did not propose such a revised version of *The Collimation Argument*, and, as I mentioned, his own original version is problematic.

A more careful inspection of the original version of *The Collimation Argument* (especially an inspection of how this argument is related to the argument I proposed against the impossibility of linear motion) reveals some interesting aspects of Avicenna's understanding of the notion of continuity; but discussing this issue would take us too far from our main concerns.³⁷ I turn, therefore, to another argument against the actuality of infinite magnitudes.

3.2 The Ladder Argument

The idea of *motion* plays an important role in *The Collimation Argument*; that argument should therefore be categorized as a physical argument in the aforementioned sense. Now we briefly review a mathematical argument against the actuality of infinite intervals that does not appeal to such physical notions. This argument, known as *The Ladder Argument* (*burhān al-sullam*), is first proposed in *The Physics of the Healing*³⁸ as a potential rehabilitation of one of Aristotle's

³⁶ Avicenna 2009, III.4 [1] and III.8 [5f.].

³⁷ After Avicenna, many influential figures in Islamic philosophy discussed this argument, and from many different perspectives. For example, Abu l-Barakāt Al-Baghdādī (1080–1165), Naṣīr Al-Dīn Al-Ṭūsī (1201–1274), and Al-Ḥillī (1250–1325) criticized the argument. On the other hand, Fakhr Al-Dīn Al-Rāzī (1149–1209) and Mulla Ṣadrā (1572–1640) defended the argument.

³⁸ Avicenna 2009, III.8 [5–7].

physical arguments in *De Caelo*.³⁹ *The Ladder Argument* is also Avicenna's only argument against the actuality of infinity in *Pointers and Reminders*.⁴⁰ The argument, as appeared in *The Healing*, goes as follows:

TEXT # 1: Let us posit a certain interval between two opposite points on two lines extending infinitely. Now, let us connect the [points] by a line that is a chord of the intersecting angle. So, because the extension of the two lines, which is infinite, is proportional to the increase of the interval [that is, the length of the chord], the increases to that interval are infinite. [Those increases] can also exist together equally, because the increases that are below will actually be joined to those that are above. For instance, the [amount that] the second increases the first will belong to the third, together with any other increase. So the infinite increases must actually exist in one of the intervals, and that is because the increases actually exist, and every actual increase will exist and so will belong to a certain one [of the intervals]. In that case, it necessarily follows that some interval will exist in which there is an actual infinity of equal increases. So that interval would increase the first finite [interval] by an infinite [amount], in which case there would be an infinite interval [...]. This infinite can exist only between two lines, in which case it is finite and infinite, which is absurd.⁴¹

To have a more diagrammatic understanding of this argument, suppose that L and L' are two distinct lines that start at the same point A and extend infinitely to make an acute angle with infinite sides. Now, consider two arbitrary points D_1 and E_1 on L and L' respectively. As a result, D_1E_1 is an interval which lies between L and L' . Furthermore, consider all intervals D_2E_2, D_3E_3, D_4E_4 , etc., parallel to D_1E_1 , such that D_i and E_i (for every natural number $1 \leq i$) lie respectively on L and L' and the difference between the lengths of every two consecutive intervals is constant. If we suppose that this difference is d , then for every natural number $1 \leq i$, $D_{i+1}E_{i+1} - D_iE_i = d$. So we have a hierarchy of intervals in which every interval is formed by adding an interval of the length d to the previous interval.⁴² For short, every interval is formed by an *increase* to the previous interval. Therefore, every interval is formed by a number of increases to D_1E_1 . Moreover, if an increase belongs to an interval, then all the previous increases belong to the same interval too. As a result, every interval somehow includes all the previous intervals. For instance

³⁹ *De Caelo* I.5, 271b26–272a7.

⁴⁰ Avicenna 1957, Namaṭ I, Ch. 11, 160–167.

⁴¹ Avicenna 2009, III.8 [7].

⁴² To be more precise, each interval is formed by adding an interval of the length d to a copy of the previous interval. However, Avicenna does not consider any difference between an interval and its copies. For the sake of simplicity, I follow the same practice in my discussion on *The Ladder Argument*. So instead of saying that, for example, the interval I_2 includes a copy of the interval I_1 , I simply say that I_2 includes I_1 .

both the first and the second increases belong to D_3E_3 and, in a sense, it includes both D_1E_1 and D_2E_2 , i. e.,

$$D_3E_3 = D_2E_2 + d = D_1E_1 + d + d.$$

The number of the increases is infinite, and all of them actually exist, since otherwise there is an upper bound for the lengths of D_iE_i and, as a result, L and L' would be finite. From these premises, Avicenna concludes that there should be an interval BC (in such a way that B and C lie respectively on L and L') which includes all of the infinitely many increases. BC is, therefore, larger than any D_iE_i we consider. It should be itself infinite. However, BC is restricted to L and L' , and terminates at them, which indicates that it is finite (Fig. 2). Consequently, it is both finite and infinite. Contradiction. Ergo, there are no infinite lines such as L and L' .

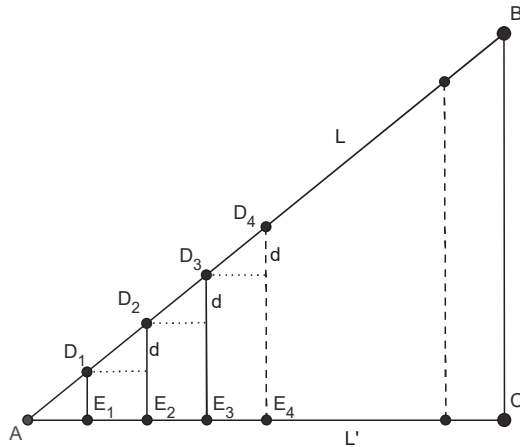


Fig. 2

This argument again seems to be controversial. It is based on three premises:

- (1) Every increase belongs to an interval. In other words, for every increase there is a D_iE_i to which the increase belongs.
- (2) If an increase belongs to an interval, then all of the previous increases also belong to the same interval. In other words, if the n^{th} increase belong to D_iE_i , then all the first, the second, [...], and the $(n-1)^{\text{th}}$ increases also belong to the same interval.
- (3) All of the infinitely many increases actually exist.

From these premises, which seem to be uncontroversial, Avicenna then concludes that:

- (4) The infinite increases all together must actually exist in one of the intervals. In other words, some interval will exist in which there is an actual infinity of equal increases.

It is, however, far from clear how (4) can be validly entailed from premises (1)–(3). These premises indicate that there exists an actually infinite hierarchy of finite intervals with increasing lengths. Each interval is longer than the previous by the amount d , and each interval is formed by a finite number of increases to $D_i E_i$. Nonetheless, these facts do not seem to imply that there is an actually infinite interval which includes all the increases.

As I mentioned earlier, *The Ladder Argument* is discussed in *The Physics of the Healing* as a rehabilitation of one of Aristotle's physical arguments which Avicenna finds weak. However, it seems that *The Ladder Argument* is still vulnerable to one of the very objections that he himself puts forward against Aristotle's argument. Roughly speaking, Aristotle's argument goes as follows:⁴³ Consider a finite circle in an infinite space and suppose that it takes a finite time for it to rotate once around its center. Suppose, moreover, that two distinct radii of this circle are extended infinitely. Now, Aristotle argues that there should be an infinite interval – e. g., an infinite arc parallel to the circumference of the circle – between these two radii. But if so, this infinite interval would be traversed in a finite time by one of the two infinitely extended radii during the rotation of the circle. This however leads to a contradiction, since the infinite is not traversable in a finite time. Given the assumption that circular motion is possible, Aristotle concludes that space cannot be actually infinite. Consequently, there is no magnitude. In criticizing Aristotle's argument, Avicenna says:

TEXT # 2: It is not the case that an infinite interval [[e. g., an infinite arc]] must occur [[between two infinitely extended radii of a circle]]; rather, the increase [[of the lengths of the intervals between those radii]] will proceed infinitely, where every increase will involve one finite [amount] being added to another, in which case the interval will be finite. This is just like what you learned concerning number – namely, that [number] is susceptible to infinite addition, and yet, any number that occurs is finite without some number actually being infinite, since any given number in an infinite sequence exceeds some earlier number [in that sequence] only by some finite [number].⁴⁴

⁴³ See Aristotle's *De Caelo* I.5, 271b26–272a7 and Avicenna 2009, III.8 [5].

⁴⁴ Avicenna 2009, III.8 [6]. Words inside the single and double-square-brackets are added by McGinnis and me respectively.

I think, however, that the same point can be made concerning *The Ladder Argument*. By borrowing Avicenna's own wording, we can say: The increase of the length of the intervals D_iE_i will proceed infinitely, where every increase will involve one finite interval of the length d being added to another finite interval, in which case the outcome will be finite. Therefore, although we have actually an infinite hierarchy of finite intervals with increasing lengths, it is not the case that there is an actually infinite interval terminated at L and L' which includes all of the intervals D_iE_i . This is exactly what happens in the case of numbers. The supposition that all numbers actually exist does not imply that there actually exist a number which is infinite. Even if the chain of consecutive numbers extends infinitely and all numbers actually exist, this does not entail that infinity itself is a number and occupies a place in the chain of numbers. Similarly, the actual existence of infinitely many intervals terminated at L and L' does not necessarily entail the actual existence of an infinite interval terminated at these two lines.

It is not clear how Avicenna intended to save *The Ladder Argument* from the above objection. This is why the soundness of this argument was the subject of a longstanding discussion in the post-Avicennan Islamic philosophy.⁴⁵ But regardless of this historical debate, the argument does not seem to be convincing for modern readers. From the perspective of contemporary mathematics, the above objection is quite compelling and reveals a fatal flaw in *The Ladder Argument*. Avicenna seem to believe that the existence of an infinite plane entails the existence of an infinite triangle ABC whose sides are infinite. He argues, then, that this is impossible. On the one hand, all three sides of this triangle should be infinite; so, BC is infinite. On the other hand, BC is restricted to AB and AC , since it terminates at B and C ; so, BC is finite. Consequently, BC is both finite and infinite. Contradiction. Ergo, there is no infinite plane and no infinite interval at all. However, this is unsound. The existence of an infinite plane does not imply the existence of an infinite triangle, or any other infinite shape with a closed boundary on a plane. Therefore, even this mathematical argument – which, unlike *The Collimation Argument*, does not appeal to physical notions such as motion – does not work, at least from our modern point of view. I now turn to Avicenna's main argument for the non-actuality of mathematical infinity.

⁴⁵ For example, Abu l-Barakāt Al-Baghdādi criticized the argument, and Naṣir Al-Dīn Al-Ṭūsī and Mullā Ṣadrā defended it. See McGinnis 2018 for a detailed discussion of the different aspects of the historical debate concerning *The Ladder Argument*.

4 Avicenna's Main Argument: The Mapping Argument

Concerns may be raised about the aforementioned arguments. One of them has to do with the genuine relation of these arguments to the problem of mathematical infinity. Considering the *contexts* of these arguments (which appear mostly in the *Physics* parts of Avicenna's works), it might seem that the target of these discussions is *exclusively* a rejection of the actual existence of infinitely large bodies (or a rejection of the infinity of the world). Therefore, one might conclude that these discussions have no decisive outcome for the problem of *mathematical* infinity. One might claim, therefore, that although Avicenna rejects the actuality of physical infinity, we have no evidence to suppose that he does not accept the actual mathematical infinite. We cannot say based merely on these arguments that Avicenna rejects the actual existence of an infinite set of numbers or an infinitely long line (as an object of geometry).

Admittedly, there are some phrases that *might* motivate an interpretation of Avicenna as believing that mathematical infinity should not be discussed in *Physics*. For example, he says: “*fa’inna al-naẓar fī al-’umūr ḡhayr al-ṭabī’īya, wa annahā hal takūn ḡhayr mutanāhiya fī al-’adad aw fī al-quwwa, aw ḡhayr dhālik, falays al-kalām fīhā lā’iqa bihādihā al-mawḍi’.*”⁴⁶ Jon McGinnis has translated this phrase as, “For now, this [i. e., *The Physics of the Healing*] is not the place to investigate things outside of natural philosophy – that is, to discuss whether there is an infinite with respect to number, power, or the like.” McGinnis adds, in a footnote, that “the proper place for such a discussion would seem to be the science of metaphysics, and while Avicenna has no appreciable discussion of the infinite in number in book 3 of his *Ilāhīyāt*, which is his most extended account of the philosophy of mathematics, he does have scattered, extended discussions of the infinite in book 6 (particularly chapters 2 and 4) where he discusses causes.”⁴⁷ Therefore, it seems that McGinnis interprets Avicenna as believing that *Physics* is not the proper place for discussing mathematical infinity.

Despite my undeniable debt to McGinnis's works on Avicenna, my interpretation differs from his. My discussion in the second section shows that mathematical infinity is not something entirely distinct from physical infinity. According to Avicenna, mathematical objects cannot exist independently from physical objects. Therefore, if the numbers and magnitudes of physical objects cannot be actually infinite (non-actuality of physical infinity), then numbers

⁴⁶ Avicenna 2009, III.7 [1].

⁴⁷ Avicenna 2009, III.7, 320 n1.

and magnitude, inasmuch as they are mathematical objects, cannot be actually infinite either (non-actuality of mathematical infinity). This is because, according to Avicenna, there is no number or magnitude fully separated from physical objects which can still be considered as a mathematical object (i. e., a subject of mathematical study). Therefore, the claim that the above arguments cannot be employed to attack the actuality of mathematical infinity seems to be not tenable.

My disagreement with McGinnis's view arises, among other things,⁴⁸ from the different ways we translate the Arabic phrase cited. According to his translation, Avicenna believes that the infinity of numbers, which are subject matters of mathematics and therefore stand outside natural philosophy, should be discussed somewhere other than *Physics*. I think, however, that the phrase should be translated as something like this: "This [i. e., *The Physics of the Healing*] is not the place to investigate non-natural (*ghayr al-ṭabī'īya*) things, and to discuss whether they are infinite with respect to number, power, or the like." Therefore, according to my translation, Avicenna simply claims that the infinity of non-natural things (which I understand to mean *things completely separated from matter, not things outside of natural philosophy*)⁴⁹ with respect to numbers (not *the infinity of numbers inasmuch as they are the subject matters of mathematics*) should be discussed elsewhere. As we will see, Avicenna believes that there are some infinite sets of fully immaterial objects. What he wants to clarify here, therefore, is simply that his argument against the actuality of infinity does not apply to the objects that are completely separated from matter and, in this sense, non-natural (*ghayr al-ṭabī'īya*). It seems to me, therefore, that 'non-natural' does not refer to numbers and magnitudes which are the subject matters of mathematics and, therefore, attached to matter. As a result, the phrase quoted is not evidence

48 One of these things may be his view about the nature of mathematical objects. McGinnis believes that Avicenna sees "mathematical objects as mental constructs abstracted from concrete physical objects" (McGinnis 2006, 68), and that Avicenna "invokes an account of conceptual analysis and mathematical objects that has certain affinities with the thoughts of some contemporary modal metaphysicians and mathematical constructivists or intuitionists (or perhaps better, "anti-Platonist mathematicians")" (McGinnis 2006, 64). I agree that Avicenna is anti-Platonist in his ontology, but he does not believe that mathematical objects are mental objects/constructs, or so it seems to me. Therefore, his ontology of mathematics cannot be interpreted as a constructivist ontology. For Avicenna's arguments against mathematical Platonism see Zarepour 2019.

49 Mathematical objects are, by definition, the subject matter of mathematics. Therefore, in this sense, they lie outside of natural philosophy. However, they are not completely separable from materiality. As a result, they are not entirely non-natural. Thus the claim that the (in)finity of non-natural things should be discussed somewhere other than *Physics* has no immediate consequence for the problem of mathematical infinity and whether it can be discussed in *Physics*.

against the idea that the afore-discussed arguments can be applied against the actuality of mathematical infinity.

There is yet another concern about *The Collimation Argument* and *The Ladder Argument*. It can be convincingly argued that the actual existence of an infinite number of physical objects entails the actual existence of an infinite interval; therefore, one of the indirect conclusions of these two arguments (albeit, if they were sound) could be to reject the actuality of numerical infinity. But it should be admitted that these arguments are *not* intended to be *directly* applied to the case of numerical (i. e., discrete) infinity. These arguments are primarily against the actuality of infinite magnitudes and intervals, rather than the infinity of numbers or the infinity of a set of numbered things. Now, one might wonder if Avicenna has proposed any argument which can be *directly* applied to the case of numerical infinity. Fortunately, the answer is positive.

Avicenna's main argument against actual infinity is *The Mapping Argument* or *The Correspondence Argument*⁵⁰ (*burhān al-taṭābuq* or *al-taṭbiq*), which is proposed and discussed in many different places in his oeuvre.⁵¹ This argument is a substantially revised version of an argument originally proposed by Al-Kindī (c. 801–873).⁵² A significant advantage of *The Mapping Argument* over the two aforementioned arguments, and over Al-Kindī's original argument, is that *The Mapping Argument* can be applied simultaneously to both numbers and magnitudes. Almost whenever Avicenna mentions this argument, he explicitly states that he intends to show (by this argument) that both *numbers* and *magnitudes* (in addition to some other things) cannot be infinite. For example, in his discussion of this argument in *The Physics of the Healing*, Avicenna says:

⁵⁰ In this paper, I use 'mapping' and 'correspondence' synonymously.

⁵¹ See, for example, (1) *The Physics of the Healing* (2009, III.8 [1]), (2) *The Physics of the Salvation* (1985, IV.2, 244–245), (3) *The Metaphysics of 'Alā'ī Encyclopedia* (2004, Ch. 16, 58–60), and (4) *The Physics of Fountains of Wisdom* (1980, Ch. 3, 19 f.).

⁵² Al-Kindī's argument has appeared, with different wordings, in his *On First Philosophy* and three other essays which are exclusively dedicated to the discussion of infinity: (1) *On the Quiddity of What Cannot Be Infinite, and What Is Said to Be Infinite*, (2) *On the Oneness of God and the Finiteness of the Body of the World*, and (3) *Al-Kindī's Epistle to Aḥmad ibn Muḥammad Al-Khurāsānī, Explaining the Finiteness of the Body of the World*. For an English translation of these works, see Al-Kindī 2012. Rescher/Khatchadourian 1965 have discussed Al-Kindī's views about mathematical infinity by translating and analysing the third essay. Shamsi 1975 provides a translation of the first essay and discusses Al-Kindī's views on the finitude of the world and the time. For a more detailed discussion of the various aspects of Al-Kindī's position on infinity, see Adamson 2007, Ch. 4.

TEXT # 3: The first thing we say is that it is impossible that there exist as wholly actualized some unlimited (*ghayra dhī nihāya*) magnitude, number, or [set of] numbered things having an order in either nature or position (*waḍ'*).⁵³

Similarly, in the beginning of his discussion of *The Mapping Argument* in *The Physics of the Salvation*, Avicenna says:

TEXT # 4: I say that there does not arise an infinite continuous quantity (*kam muttaṣil* [= *miqdār* = magnitude]) that exists essentially possessing a position (*waḍ'*); there is also no ordered infinite number that exists all together.⁵⁴

It is particularly noteworthy that, in the first text, he distinguishes number (*'adad*) from numbered things (*ma'dūdāt*) and claims that number and magnitude (i. e., the subject matters of mathematics) cannot be infinite.⁵⁵ He does not, therefore, restrict himself just to those physical objects upon which numbers and magnitude are predicated. This shows that, in his discussions of *The Mapping Argument*, Avicenna 'deliberately' considers not only physical but also mathematical infinity – even if we suppose that the two aforementioned arguments are exclusively targeted at physical infinity. We should not be misled, therefore, by the fact that this argument appears mostly in the *Physics* parts of his works.⁵⁶ Nonetheless, there might still be some remaining concerns. In his discussions of TEXT # 4 McGinnis argues that:

Avicenna consistently uses 'position' (*waḍ'*) as one of the concomitants that follows upon matter, and in fact in the version of the proof as it appears in *al-Ishārāt wa al-Tanbihāt*, Avicenna make[s] clear that the argument merely proves that "corporeal extension (*al-imtidād al-jismānī*) must be finite." In it[s] simplest terms the mapping argument, Avicenna seems to think, merely shows that there can be no material instantiation of an actual infinite.⁵⁷

53 Avicenna 2009, III.8 [1]. I have slightly revised McGinnis's translation. He has translated '*tartīb*' into 'ordered position', but I prefer to translate it simply as 'order'. The translation should not incautiously induce that there is a necessary connection between *being ordered* and *having a position* (*waḍ'*); especially if, as McGinnis does (McGinnis 2010, 217), one interprets *position* as one of the concomitants that follow only upon matter.

54 Avicenna 1985, IV.2, 244; my translation. He repeats the same claim at the beginning of his discussions on this argument in *The Metaphysics of 'Alā'ī Encyclopedia* (Avicenna 2004, Ch. 16, 58 f.) and *The Physics of Fountains of Wisdom* (Avicenna 1980, Ch. 3, 19).

55 See also Avicenna's *The Notes* (Avicenna 1973, 38) for the claim that numbers are not actually infinite, though they are potentially infinite.

56 Its appearance in *The Metaphysics of 'Alā'ī Encyclopedia* is an exception.

57 McGinnis 2010, 217. For the sake of consistency with my other transliterations in this paper, I have revised McGinnis's transliterations.

By appealing to this line of reasoning, one might claim that *The Mapping Argument* has nothing to do with numerical infinity. I disagree however. Avicenna's argument in *al-Ishārāt wa al-Tanbīhāt* is definitely not a version of *The Mapping Argument*. As I mentioned above, it is a version of *The Ladder Argument*. Therefore, even if we accept that Avicenna's argument in that book can be applied merely to corporeal extension, this does not entail that the target of *The Mapping Argument* is restricted to the same thing, and thus it is not a direct argument against the actual infinity of numbers. TEXT # 3 and TEXT # 4, which appear in the introductions to Avicenna's discussions of *The Mapping Argument*, explicitly show that he believes that this argument can be applied not only to magnitudes but also to numbers.⁵⁸ Therefore, one of the main functions of *The Mapping Argument* is to reject the actuality of mathematical infinity – in its general sense – in a direct way.

The two aforementioned texts reveal some other important points about the function of *The Mapping Argument* to which we shall shortly return. Before touching on these points we should first clarify the structure of the argument.

4.1 The Structure of the Mapping Argument

Consider the straight line AB which starts from the point A and extends infinitely in the direction of B . AB represents a one-dimensional magnitude infinitely extended in one direction, or an infinite set of numbered objects possessing an order, the first element of which is placed on A while the other elements are successively lined up on some discrete points on the rest of AB .⁵⁹ Suppose that we take the finite part AC from AB . Avicenna argues that:

⁵⁸ As Euclid has shown in books 7–9 of his *Elements*, numbers can, in a sense, be constructed from magnitudes. They can be treated as the sets of units of magnitudes. Therefore, not only *The Mapping Argument*, but every argument against the actual infinity of magnitudes can, in principle, be considered as an *indirect* argument against the actual infinity of numbers. However, Avicenna seems to believe that, contrary to the other arguments he discusses, *The Mapping Argument* is applicable to the case of numbers in a more direct manner. This is why although he is silent about the applicability of the other arguments to the case of numerical infinities, he explicitly mentions that *The Mapping Argument* is applicable to the case of numbers and numbered things. I am thankful to an anonymous reviewer for pushing me to clarify this point.

⁵⁹ Before proposing the details of *The Mapping Argument* in the *Physics* part of *The Healing* (Avicenna 2009, III.8 [1]) and *Fountains of Wisdom* (Avicenna 1980, Ch. 3, 19), he briefly argues that if something is infinite in more than one dimension or direction, then we can restrict ourselves to the first dimension and consider a single direction in which that thing is infinite. It seems that he wants to justify why *The Mapping Argument* is applied only to either one-dimensional

TEXT # 5: [I]f some amount equal to CB were mapped on or parallel to AB (or you were to consider some other analogous relation between them), then either $[CB]$ will proceed infinitely in the way AB does, or it will fall short of AB by an amount equal to AC . If, on the one hand, AB corresponds with CB [in proceeding] infinitely, and CB is a part or portion of AB , then the part and the whole correspond [with one another], which is a contradiction. If, on the other hand, CB falls short of AB in the direction of B and is less than it, then CB is finite and AB exceeds it by the finite [amount] AC , in which case AB is finite; but it was infinite. So it becomes evidently clear from this that the existence of an actual infinite in magnitudes and ordered numbers is impossible.⁶⁰

Here, Avicenna argues that after taking the finite part AC from AB , we can compare the sizes of CB and AB by mapping (*aṭbaqa*) something equal (*musāw*) to the former on the latter, i. e., by mapping a copy of CB on AB in a way that the first point of the copy of CB corresponds with A . Avicenna uses the term ' CB ' (*jīm bā'*) equivocally in referring to both CB and its copy (i. e., the thing equal to CB). For clarity, we use the term C^*B^* to refer to the copy of CB . Therefore, Avicenna believes that we can compare the sizes of CB and AB , by mapping C^*B^* onto AB in a way that C^* corresponds with A . After this mapping, either C^*B^* extends infinitely in the direction AB does (Fig. 3a) or C^*B^* falls short of AB (Fig. 3b). In the former case, C^*B^* corresponds with AB . But C^*B^* is equal to CB . Therefore, CB corresponds with AB . This entails that a whole (i. e., AB) totally corresponds with its part (i. e., CB). Avicenna believes that this is a contradiction. Now we should check the other horn. Suppose that C^*B^* falls short of AB . This means that C^*B^* corresponds with a part of AB which starts at A and terminates at a determinate point on AB . Call this latter point D (Fig. 3b). AD is an interval which terminates at two determinate points A and D ; therefore, it has determinate limits and is finite. Now, since B^*C^* corresponds with AD , B^*C^* is finite too. On the other hand, the amount by which C^*B^* falls short of AB is equal to AC , for the fact that $C^*B^* = CB$ implies that $AB - C^*B^* = AB - CB = AC$. This means that the sum of C^*B^* and AC is equal to AB . But C^*B^* and AC are both finite. Therefore, their sum, which is equal to AB , is finite. As a result, we should accept that AB is finite. This contradicts our first supposition. Avicenna concludes that an infinity like AB cannot actually exist.

magnitudes or numbers (or numbered things) ordered on a line and infinitely extended in one direction. If we show that nothing can be actually infinite in the chosen direction, then by generalization of this result we can claim that there is no direction in which something can be actually infinite. Unless otherwise mentioned, by 'magnitude', I mean only *one-dimensional* magnitude.

60 Avicenna 2009, III.8 [1]. I have slightly revised McGinnis's translation.

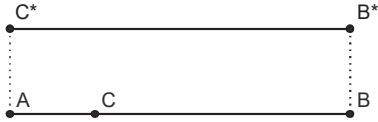


Fig. 3a

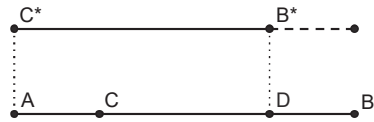


Fig. 3b

This clarification of *The Mapping Argument*, which brings some geometrical diagrams (such as Fig. 3) to mind, shows how this argument is intended to be applied to magnitudes. But the last sentence of TEXT # 5, in addition to the aforementioned evidence, confirms that Avicenna believes that this argument can also work perfectly against numerical infinity and show “that the existence of an actual infinite in [...] ordered numbers is impossible.” Therefore, one might ask how this argument works in the case of numbers or numbered things. To the best of my knowledge, Avicenna has not explicitly replied to this question, at least not in his major works. He has astutely realized that the mapping technique can be employed against the actual infinity of numbers and numbered things, but it seems that his own *explanations* of the application of this technique are more perfectly matched to the case of magnitudes, rather than that of numbers. Fortunately, it is not very difficult to guess what he had in mind for the case of numbers.

If we consider AB as an infinite set of numbers or numbered objects possessing an order the first element of which is placed on A and the other elements are successively lined up on some discrete points of the rest of AB , then AC can be considered as the finite subset of the initial elements of AB placed from A to C . Accordingly, mapping C^*B^* (i. e., a copy of CB) onto AB pairs the first element of C^*B^* with the first element of AB , the second element of C^*B^* with the second element of AB , and so on (i. e., for every natural number n , pairing the n^{th} element of C^*B^* with the n^{th} element of AB). If this pairing procedure ends at some finite stage by pairing the last element of C^*B^* with an element of AB , this means that C^*B^* and consequently CB are finite. Therefore AB , which is the union of AC and CB , would be finite too (Fig. 4b). On the other hand, if the elements of C^*B^* extend infinitely, then it is possible to set a one-to-one correspondence between C^*B^* and AB by pairing every n^{th} element of the former with the n^{th} element of the latter. Therefore, AB corresponds with B^*C^* and, consequently, with BC (Fig. 4a). This means that a whole (i. e., AB) corresponds with one of its proper parts (i. e., BC). Since Avicenna sees the correspondence between a whole and its proper part as a contradiction, this line of argument can establish for him that there cannot be any actually infinite set of numbers or numbered things possessing an order.

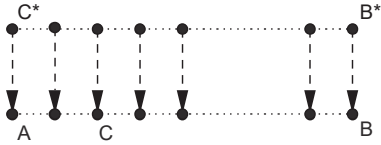


Fig. 4a

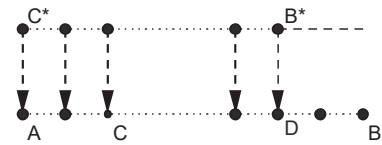


Fig. 4b

However, as far as I know, Avicenna himself has nowhere clarified the details of the application of *The Mapping Argument* to the case of numerical infinity. His explanations, as I discussed, highlight only the application of this argument to the case of geometrical infinity. He clearly claims that this argument rejects the actuality of numerical infinity, but he does not say how it really works in that case. Nonetheless, it does not seem implausible to accept that Avicenna had in mind something very similar to what is set out in the above paragraph. At least four different considerations support this claim. The first consideration is that the only natural way to develop the notion of mapping from the context of continuous geometrical magnitudes to the context of sets containing discrete elements seems to be interpreting mapping between these sets as one-to-one correspondence between their elements. Since Avicenna believes that the mapping technique can successfully work in the case of numbers and numbered things, it is highly probable that he has such a natural understanding of the notion of mapping in the context of numbers. The second consideration comes from an abstruse passage in *The Discussions* presenting a brief version of *The Mapping Argument* in response to a question probably raised by Ibn Zayla (c. 983–1048). It is argued there that it is possible to have two potential infinities of different sizes (i. e., one of them is bigger than the other). But it is impossible to have such infinities in actuality, because:

TEXT # 6: When it [i. e., an infinity] became concurrent with and parallel to it [i. e., another infinity] in terms of connectedness or in terms of orderedness, or when it [i. e., one of those infinities] became a part of the other, then one of them would come to an end in one side and a remnant of one of them would remain at the other side. Therefore, the finitude of [... the shorter infinity] is necessary [and this is absurd].⁶¹

⁶¹ Avicenna 1992, VI, § 588. Avicenna accepts that potential infinities can be of different sizes. For example, time is potentially infinite and the time passed until last year is a potential infinity smaller than the time passed this year. See section 4.3 for why *The Mapping Argument* is not applicable to time.

The passage seems to be saying that after mapping one of those infinities on the other, they become parallel to each other in terms of connectedness *or* in terms of orderedness (*muḥādḥāt fi ittiṣāl aw fi tartīb*). This disjunctive phrase suggests that being parallel in terms of connectedness/continuity differs from being parallel in terms of order/orderedness. If the mapping technique was applicable only to continuous magnitudes, then we would have only one kind of parallelism. All continuous magnitudes are isomorphic to each other. Consequently, their being parallel to each other cannot be of more than one kind. Even if Avicenna thought that all parallelisms of magnitudes are of both mentioned kinds, he would have to use a conjunctive phrase rather than a disjunctive one. This he does not do. It indicates that the mapping technique is applicable to not only continuous magnitudes but also numerical infinities. The distinction between these two kinds of parallelism brings to mind the two different ways of the application of the mapping technique I explained above. By applying this technique to infinite magnitudes they become parallel to each other in terms of connectedness and continuity. By contrast, the application of this technique to the case of numerical infinities makes them parallel to each other in terms of order and orderedness. Therefore, ‘parallelism in terms of orderedness’ could be interpreted as Avicenna’s term for the notion of *one-to-one correspondence*.

The third consideration supporting that Avicenna was aware of the application of the mapping technique to the case of numerical infinities is this: even early commentators on Avicenna (who are obviously closer than we are to the historical context of Avicenna’s discussions) have had the same interpretation of the intended function of Avicenna’s mapping argument in the case of numbers. For example, Fakhr Al-Din Al-Rāzī, in his commentary on Avicenna’s *Fountains of Wisdom*, explains this function of *The Mapping Argument* along exactly the same lines as the interpretation mentioned above. Consider two sets of numbers: (1) A proper subset of natural numbers, for example, the set of natural numbers bigger than 10, and (2) the set of all natural numbers from one to infinity. Al-Rāzī writes that one can argue, based on the mapping technique, that:

TEXT # 7: We put the first position of this [latter] set in front of the first position of that [former] set, and the second of this in front of the second of that, and so on successively. So, if the remainder does not appear [i. e., if by following this procedure nothing of the latter set remains unpaired], then the more is identical to the less. And, if the remainder appears at the end of the positions [of the former set, and some positions of the latter set remains unpaired], then it entails the finitude of number in the direction of its increase; and it is self-evident for the intellect that this is impossible.⁶²

62 Al-Rāzī 1994, 53.

The text evidently shows that Al-Rāzī understands Avicenna's mapping notion to be one-to-one correspondence in the case of numbers and numbered things.⁶³ The plausibility of the attribution of this position to Avicenna is reinforced by a fourth consideration: the explanatory power of this interpretation as to why Avicenna emphasizes that possessing an order, in either nature or position, is a necessary condition for the applicability of *The Mapping Argument* in the case of numbers and numbered things. If we understand Avicenna as believing that correspondence between sets of discrete elements is one-to-one correspondence between their elements, then we have a very convincing explanation of why non-ordered sets of discrete objects cannot be the subject of *The Mapping Argument*. I will come back to this issue in section 4.3 below.⁶⁴ Now it is time to discuss the significance of the insights behind the surface structure of *The Mapping Argument*.

4.2 Insightful Ideas behind the Surface Structure

An exhaustive analysis of Avicenna's explanation of *The Mapping Argument* reveals some remarkably interesting aspects of his treatment of the concept of infinity. The general structure of this argument is as follows: Avicenna argues that the actual existence of an infinity entails the equality of a whole to some of its

63 It should be mentioned, however, that Al-Rāzī does not believe that *The Mapping Argument* can successfully prove that the infinite set of natural numbers cannot actually exist. He thinks that although this argument works against the actual infinity of numbered things, it does not succeed in showing the non-actuality of the infinite set of natural numbers. He does so because he believes that (1) numbers are merely mental and completely separated from matter, and (2) *The Mapping Argument* has nothing to do with immaterial entities. It seems that Al-Rāzī attributes (wrongly, as I believe) this position to Avicenna too. Therefore, he does not interpret Avicenna as believing that *The Mapping Argument* can successfully reject the non-actuality of the infinity of numbers. In fact, Al-Rāzī considers what is expressed by TEXT # 7 as an objection to the soundness of *The Mapping Argument*. According to this objection, if the form of *The Mapping Argument* is valid then one might reject the actual infinity of numbers by appealing to the lines of argument expressed by TEXT # 7. Al-Rāzī discusses this objection as the 11th question he proposes about *The Mapping Argument*. In his reply to this question, he says that this argument, though successful against numbered material things such as causes, does not work against the numbers themselves which are, according to him, completely mental and immaterial (Al-Rāzī 1994, 57). My concern, however, is not Al-Rāzī's view about the nature of mathematical objects and the soundness of *The Mapping Argument* in the case of numbers. I merely want to highlight his understanding of how Avicenna's notion of *correspondence* should be interpreted in the case of numbers or numbered things.

64 See note 93.

parts. He denies the equality of whole and part, at least for physical objects and mathematical objects which are, as he believes, properties of physical objects. He concludes therefore that the infinite cannot exist in actuality. Before Avicenna, some other philosophers had correctly noticed that the existence of an actual infinity entails the equality of a whole to some of its parts;⁶⁵ and indeed, some of them had appealed to this fact to reject the actuality of infinity. For example, Al-Kindī followed such a line of reasoning in parts of his previously mentioned argument against infinity, from which *The Mapping Argument* was inspired.⁶⁶ Both Al-Kindī and Avicenna employed the mapping technique to compare the sizes of infinities. They however had two different understandings of the notion of *equality* (*tasāwī*). According to Al-Kindī, things with an equal distance between their limits (*ḥudūd*), or things whose dimensions between their limits are the same are equal. This understanding is however controversial for three reasons.

⁶⁵ John E. Murdoch 1982, 569 n13, refers to some ancient sources (by philosophers such as Plutarch, Philoponus, Alexander of Aphrodisias, Proclus and Lucretius) in which this issue is discussed. However, I have suspicions that at least some of the cited arguments – for example, Philoponus’s argument in his *De aeternitate mundi contra proclum* (2004, I.3, 24 f.) – are not explicitly grounded on the absurdity of the whole-part equality. Philoponus believed that the Aristotelian doctrine that the universe has no beginning entails that the size of some infinities can (and indeed do) increase. The number of years until this year is an infinity smaller than the infinite number of years until next year. As a result, there is an infinity such that there exists something larger than it. Since Philoponus believed that the existence of something larger than infinity is absurd, he concludes that Aristotle is wrong about the infinity of the past. It may seem, at first glimpse, that there are some commonalities between the general structure of Philoponus’s argument and that of *The Mapping Argument*. If the claim that there is nothing larger than an infinity can be construed as the claim that all infinities are of the same size, then we can understand Philoponus as arguing as follows: (1) if the past is infinite, then the set of the years until this present year and the set of the years until the next year are both infinite. (2) Since all infinities are of the same size, both of these sets must be of the same size. Nonetheless, (3) the former set is a proper subset of the latter; i. e., the latter set is a whole and the former is one of its parts. Therefore, (4) infinity of the past entails the equality of a whole to some of its parts. Finally, (5) since the equality of a whole to one of its parts is absurd, the infinity of the past is unjustified. This construal might lead one to think that the general structure of Philoponus’s argument is, by and large, similar to those of Al-Kindī and Avicenna. But this view is not, in my idea, compelling. There is nothing in Philoponus’s argument to show that he is attacking the equality of a whole and its part. He just wants to rule out the possibility of infinities of different lengths or of something outside an infinity. The sophisticated connection between this goal and the rejection of the equality of whole and part is not something made by Philoponus himself. He does not see the issue under this latter guise. For a detailed discussion of Philoponus’s views against the Aristotelian doctrine of infinity and eternity, see Sorabji 2010. See particularly 213 f. for the argument I have discussed here.

⁶⁶ See Al-Kindī 2012, 20 f.

First, Al-Kindī defines equality in terms of itself. It seems therefore that the definition is question-begging and somehow uninformative. One can take the question of equality one step further and legitimately ask how the equality of the dimensions between the limits of things can be investigated. Al-Kindī seems not to have a non-circular convincing answer for this question. Second, it is not clear how this notion of equality can be applied to infinite magnitudes. If a magnitude is infinite, then it has no limit, at least in one direction. Therefore, the sizes of infinite magnitudes cannot be compared based on the equality of the dimensions between their limits. Third, even if we accept that the sizes of infinite magnitudes can be compared based on this understanding of the notion of equality, it is far from clear how this understanding can be applied to numerical infinities. Talking about the dimensions between the limits of infinite sets of numbers or discrete objects does not seem to be meaningful.⁶⁷ The latter point is one of the reasons why Al-Kindī's argument, despite having the same general structure as that of Avicenna, should not be interpreted as directly applicable to the case of numerical infinities.

Avicenna, on the other hand, employs the notion of *mapping* or *correspondence* (*taṭābuq*) to compare the sizes of infinities. The idea of understanding the notion of equality in terms of a *mapping* or *correspondence* relation goes back to Euclid. According to his fourth common notion in the first book of *The Elements*, “things which fit onto/coincide/correspond with (*ta epharmozonta*) one another are equal to one another.”⁶⁸ This common notion is translated into Arabic in the so-called Ishāq–Thābit version as “*allatī lā yufaḍḍal 'aḥaduha 'alā al-ākhar idha inṭabaq ba'duhā 'alā ba'd fahiya mutasāwiya*.”⁶⁹ “Things which none of them exceeds another, when some of them are mapped on some others, are equal.” Avicenna appeals to this simple principle to compare the sizes of different infinite magnitudes or different infinite sets of numbers or numbered things.⁷⁰ He elegantly develops the application of this principle to the context of infinities. The significance of this strategy becomes more evident when we consider that most

⁶⁷ See McGinnis 2010 for a discussion on the shortcomings of Al-Kindī's definition of equality.

⁶⁸ In his translation of *The Elements*, Heath prefers to use the notion of *coincidence*. See Euclid 1908, Vol. 1, 155.

⁶⁹ See a copy of the Ishāq–Thābit version in the codex now numbered 6581/1 of the Majlis Shawrā Library in Tehran. The PDF version of this manuscript can be found here (accessed 19 October 2017): http://dlib.ical.ir/faces/search/bibliographic/biblioFullView.jspx?_afPfm=uzzsmt9nt The essential features of this manuscript has been described by De Young 2015.

⁷⁰ It does not mean, however, that Al-Kindī's definition of equality does not originate from the same principle. It could be an imprecise construal of this principle, perhaps based on an inaccurate Arabic translation of the Greek.

of the tools we usually use to compare the sizes of finite things are ineffective in the case of infinities.⁷¹

The equality of the sizes of two different sets of objects is usually understood as the equality of the numbers of their elements. Since these sets are finite, we can count and determine the number of their elements. If these numbers, which are obtained by two distinct counting procedures, are equal to each other, then we can say that those sets are of the same size; otherwise, they are of different sizes and the larger set is the one whose number of elements is larger. Similarly, we can measure the lengths of two different finite magnitudes separately and one by one, then we can compare the numbers obtained to decide whether they are equal or not. But it is obviously impossible to follow this approach in the case of infinities. Enumerating the elements of an infinite set or measuring the length of an infinite magnitude is impossible.⁷² By definition, the sizes of infinities cannot be described by a finite number. Therefore, comparison between the sizes of different infinities must be grounded on something else: Avicenna's creative suggestion is returning to Euclid and borrowing his notion of *correspondence*. Avicenna shows that appropriate interpretations of this notion can be successfully applied to the case of infinities. I will return to this issue in section 4.3.

The Mapping Argument indicates, therefore, that the existence of an actual infinity entails the correspondence between that infinity and some of its infinite parts – which is, according to Avicenna, simply contradictory.⁷³ A careful investigation of the notion of *correspondence*, especially when it applies to the infinite sets of numbers or numbered things, makes it clear that Avicenna's comprehension of the notion of infinity is much more modern than we might expect. In the previous section I argued that what Avicenna means by correspondence between two sets of numbers or numbered things is one-to-one correspondence between their elements, in such a way that every element of each of those sets corresponds with one and only one element of the other set. So, if I am right, Avicenna believes that on the one hand, (1) correspondence between sets is corre-

71 I am extremely thankful to an anonymous reviewer for this journal for drawing my attention to the possible link between Euclid's *ta epharmozonta* and Avicenna's *taṭābuq*.

72 There are some studies – originating from Piaget 1952 – showing that the ability to put different sets of objects in one-to-one correspondence with each other is more fundamental than the ability to count. This means that we cannot count the number of the elements of even a finite set of objects without understanding the notion of one-to-one correspondence and without having the ability to put the elements of that set into a one-to-one correspondence with numerals. I am thankful to Amir Asghari for drawing my attention to Piaget's work.

73 See TEXT # 5 above. See also *The Physics of the Salvation* (Avicenna 1985, IV.2, 245) where he, discussing *The Mapping Argument*, says that if the whole and the part “correspond [with each other] in extension, then the greater and the lesser are equal; while it is absurd.”

spondence between their elements, and on the other hand, (2) the existence of an infinite set entails its correspondence with some of its proper subsets. The moral is that Avicenna sees infinite sets of numbers or numbered things as sets which are in one-to-one correspondence with some of their proper subsets. But this is exactly what Dedekind proposed in 1888 as a definition for infinite sets. Erich Reck's reading of Dedekind's definition is as follows:

A set of objects is infinite – “Dedekind-infinite”, as we now say – if it can be mapped one-to-one onto a proper subset of itself. (A set can then be defined to be finite if it is not infinite in this sense.)⁷⁴

It is definitely surprising that more than eight centuries earlier, Avicenna had had a similar conception of the notion of infinity. This does not mean, however, that the above definition is Avicenna's own *definition* of infinity. He was aware, if I am right, that every infinite set of numbers or numbered things has the property of being in one-to-one correspondence with some of its proper subsets (borrowing the first letters of the names of Avicenna and Dedekind, I would like to call this property ‘AD-property’), but he never explicitly proposed having this property as a definition for being infinite.⁷⁵ Another important dissimilarity between Avicenna and Dedekind's views concerning infinity is that Dedekind, contrary to Avicenna, does accept the actual existence of infinite sets.⁷⁶ In other words, for Avicenna no actual thing can *instantiate* or *exemplify* the AD-property.⁷⁷ So, I do not want to exaggerate the commonalities between Avicenna's view and our post-Dedekindian, post-Cantorian understanding of infinity. My claim is merely that Avicenna had correctly noticed that every infinite set of objects has the AD-property, and this is enough to show that there is a strong connection between his views about

74 Reck 2016, Sec. 2.2. Dedekind's original definition is this 1963, 63: “A system *S* is said to be *infinite* when it is similar to a proper part of itself [...]; in the contrary case *S* is said to be a *finite* system” (emphasis in the original). Before proposing this definition he clarifies, by some consecutive definitions and theorems, that what he means by “similarity” between systems is exactly the existence of a one-to-one mapping between the elements of those two systems.

75 As I mentioned at the beginning of the second section, Avicenna's definition of infinity is an Aristotelian definition according to which infinity is something that whatever we take from it, we always find something outside of it.

76 Dedekind claims and tries to prove that his “own realm of thoughts, i. e., the totality *S* of all things, which can be objects of [his] thought, is infinite” (Dedekind 1963, 64).

77 One might therefore suggest that AD-property is not a *real* property for Avicenna; it is like the property *being a partner of God* that can never actually be exemplified.

infinity and ours. I do not know of any philosopher before Avicenna who was aware of this property of infinite sets.⁷⁸

Jon McGinnis believes that, before Avicenna, the Ṣābīan mathematician Thābit Ibn Qurra al-Ḥarrānī (d. 901) recognized the “modern definition of infinity as a set capable of being put into one-to-one correspondence with a proper subset of itself.”⁷⁹ I disagree. To justify his position, McGinnis refers to two sections of a series of questions addressed by Abū Mūsā ‘Īsā Ibn Usayyid to Thābit Ibn Qurra in which Thābit argues that there is an infinite number of different sizes that an infinite set may have.⁸⁰ I believe, however, that a careful analysis of the relevant passages to which McGinnis has referred reveals nothing confirming that Thābit recognized the idea of one-to-one correspondence and the AD-property of infinite sets. This quotation expresses the core of Thābit’s claim and clarifies the structure of his argument:

TEXT # 8: We [i. e., Ibn Usayyid and his friends] questioned him [i. e., Thābit] regarding a proposition put into service by many revered commentators, namely that an infinite cannot be greater than an infinite. – He pointed out to us the falsity of this (proposition) also by reference to numbers. For (the totality of) numbers itself is infinite, and the even numbers alone are infinite, and so are the odd numbers, and these two classes are equal, and each is half the totality of numbers. That they are equal is manifest from the fact that in every two consecutive numbers one will be even and the other odd; that the (totality of) numbers is twice each of the two [other classes] is due to their equality and the fact that they (together) exhaust (that totality), leaving out no other division in it, and therefore each of them is half (the totality) of numbers. – It is also clear that an infinite is one third, or a quarter, or a fifth, or any assumed part of one and the same (totality of) numbers. For the numbers divisible by three are infinite, and they are one third of the totality of numbers; [...] and so on for other parts of (the totality of numbers). For we find in every three consecutive numbers one that is divisible by three, [...] and in every multitude of consecutive numbers, whatever the multitude’s number, one number that has a part named after this multitude’s number.⁸¹

78 It has been argued by Netz *et al* 2001 that Archimedes was aware of the one-to-one mapping technique and implicitly employed it, in his *Method*, to show that two infinite sets of objects are *equal in multitude*. I have some doubts about the plausibility of this claim which cannot be discussed here. But even if this claim is true, it does not entail that Archimedes was aware – even implicitly – that infinite sets can be put in one-to-one correspondence with some of their subsets. Moreover, no evidence has been presented to show that his method is grounded on an explicitly intentional application of the notion of correspondence, as proposed by Euclid and consciously and deliberately employed by Avicenna.

79 McGinnis 2010, 211 n32.

80 McGinnis refers to Sections 13 and 14 from Sabra’s translation of those questions (McGinnis 1997, 24 f.). For an earlier discussion of Thābit’s views on numbers and mathematical infinity, see Pines 1968.

81 Sabra 1997, 24 f.

Here, Thābit is providing an elegant argument that there are infinities of different sizes. He argues that from every two consecutive numbers one is even and the other odd. Therefore, the totality of even numbers is equal to the totality of odd numbers; since the union of these totalities is equal to the totality of numbers, each of the totalities of even or odd numbers is equal to a half of the totality of numbers. By similar lines of argument, Thābit shows that the totality of numbers divisible by three is equal to one third the totality of numbers, and the totality of numbers divisible by four is equal to a quarter the totality of numbers, and so on. But he does not say anything confirming that any of those proper subsets of numbers are equal to the totality of numbers; nor does he mention the idea of one-to-one correspondence. Even his argument for the equality of the set of even numbers to the set of odd numbers is not grounded on the one-to-one correspondence of the elements of these two sets. Paolo Mancosu is right to say:

When ibn Qurra states that odd numbers and even numbers have the same size one should be careful not to immediately read his argument as being the standard one based on one-to-one correspondence, for the motivation adduced does not generalize to other arbitrary infinite sets. Rather, it would seem that some informal notion of frequency (how often do even numbers (respectively odd numbers) show up?) is in the background of ibn Qurra's conception of infinite sizes ("we find in every three consecutive numbers one that is divisible by 3").

What would he have replied to the possible objection that there are as many even numbers as natural numbers based on a one-to-one correspondence between the two collections? *The text is silent on this issue.*⁸²

Without doubt, Thābit's discussion has genuinely innovative aspects.⁸³ But he aims only at proving the existence of different sizes of infinite sets, not at showing the equality of infinite sets to some of their proper subsets by appealing to the notion of one-to-one correspondence. We do not have enough evidence, therefore, to claim that Thābit had recognized that infinite sets of numbers have the AD-property.⁸⁴ At least, the passages cited by McGinnis do not provide such evidence.

⁸² Mancosu 2009, 615. My emphasis.

⁸³ Mancosu says that "[t]he first occurrence I know of a defense of the existence of different sizes of infinity given in terms of collections of natural numbers comes from the Islamic philosopher and mathematician Thābit ibn Qurra", and then he quotes TEXT # 8 (Mancosu 2009, 614).

⁸⁴ Despite this deficiency, Thābit's account of infinity has an important advantage over Avicenna's. By contrast to Avicenna who does not provide any explicit numerical example, Thābit grounds his discussions on some concrete examples of numerical sets.

It is worth mentioning that neither Thābit nor Avicenna nor any other scholar before Georg Cantor recognized the (either actual or potential) existence of infinite sets of different *cardinalities*. From the standpoint of modern mathematics, two sets are of the same cardinality if there is a one-to-one correspondence between their elements. Therefore, the existence of infinite sets of different cardinalities means the existence of infinite sets which cannot be put in one-to-one correspondence with each other. Avicenna, as I argued, was aware that infinite sets of numbers can be put in one-to-one correspondence with some of their proper subsets, but he did not know that there are some infinite sets that cannot be put in one-to-one correspondence with the set of natural numbers.⁸⁵ So he did not know about infinite sets of different cardinalities. *A fortiori*, Thābit (who, as I showed, was not familiar with the notion of one-to-one correspondence) did not know about the different cardinalities of infinite sets. This means that Thābit's claim that the totality of even numbers is equal to the half of the totality of all numbers should not be understood as the claim that the cardinality of the former set is less than the latter's.⁸⁶

In sum, Avicenna employs the notion of *mapping* or *correspondence* as a tool for the comparison of the different sizes of infinite magnitudes or infinite sets of numbers or numbered things possessing an order. By *The Mapping Argument* he shows that an infinite magnitude is necessarily in correspondence with some of its proper parts. He believes and indeed explicitly states that the argument is appropriate to the case of numerical infinities, but he himself does not discuss the details of this application. By (1) appealing to the most natural development of the notion of correspondence to the case of numbers and numbered objects, and (2) relying on some of Avicenna's own texts and some early commentaries on his works which confirm that Avicenna had such a development in mind, we can say that, according to Avicenna, correspondence between two sets of numbers or numbered things means nothing other than one-to-one correspondence between

85 The fact that some infinite sets cannot be put in one-to-one correspondence with the set of natural numbers was proved by Georg Cantor in 1891. His proof is based on the so-called *diagonalisation* technique. The complexity of this technique was far beyond the boundaries of mathematical knowledge in the time of Avicenna.

86 Jon McGinnis 2010, 221, in explaining Philoponus's criticism of the Aristotelian doctrine of the eternity of the world, says: "Such a thesis, argued Philoponus, violated a number of Aristotelian dicta concerning infinity, as, for example, the impossibility of an actual infinity being realized, an infinite's being traversed, the ability to increase an infinite as well resulting in infinities of varying *cardinality* [...]" (My emphasis. See also page 205 of the same paper). I think, however, that this interpretation suffers from imprecision. As I mentioned in note 65, Philoponus did argue about the different sizes of infinity, but what he means by 'different sizes of infinities' is by no means the same as 'infinities with different cardinalities' in its modern sense.

the elements of these sets. Coupling this fact with Avicenna's insistence on the applicability of *The Mapping Argument* to the case of numbers and ordered objects, we can conclude that he was aware, like Dedekind, that every infinite set of objects possessing an order has the AD-property. However, by contrast to Dedekind, he did not propose the having of this property as a definition for being infinite. Moreover, he did not know anything about infinities with different cardinalities, in the modern sense of the term 'cardinality'. Finally, contrary to Dedekind and most other modern set-theorists, Avicenna rejects the actual existence of infinite sets of numbers or numbered objects. He believes that, on the one hand, (I) every mathematical infinity corresponds with some of its proper subsets, and on the other hand, (II) it is impossible for a set of *actually existent* mathematical objects to correspond with some of its proper subsets. Therefore, he concludes that it is impossible for a magnitude or a set of discrete mathematical objects to be infinite. His justification for (I) comes from *The Mapping Argument*, but he himself does not provide any justification for (II); he simply accepts it. I will argue, in the next two sections, that Avicenna's position regarding the ontology of mathematics justifies (II).

There is still another problem that we have not yet touched on. According to TEXT # 3 and TEXT # 4, *The Mapping Argument* works against the actual existence of infinite magnitudes and infinite sets of numbers or numbered things *having an order*. But we have not yet clarified why having an order is a necessary condition for the elements of an infinite set of objects to be the subject of *The Mapping Argument*. In the following section, I discuss this issue and show how it is related to the possibility of having an actual infinity of immaterial objects.

4.3 Non-Ordered Can Be Actually Infinite

In the following passage, from *The Physics of the Salvation*, Avicenna explicitly says that *The Mapping Argument* does not work against infinities such that either their parts do not exist totally together at the same time (e. g., the infinite time line and infinite motions) or they cannot be ordered (e. g., immaterial objects such as angels and devils):

TEXT # 9: [There are two situations in which we may have infinities:] either when [(1) the totality of] the parts are infinite and do not exist all together – therefore, it is not impossible for them to exist one before or after another, but not all together – or when [(2)] the number [i. e., the set of numbered things] itself is not ordered in either nature or position – therefore, there is nothing preventing it [i. e., the non-ordered set of numbered things] from existing [with] all [its members] together. There is no demonstration for its impossibility, rather there is a demonstration for its [actual] existence. As for the first kind [of these infinities], time

and motion are proven to be such. As for the second kind, [the existence of] a multiplicity of angels and devils – that are infinite with respect to number – is proven to us, as will become clear to you [too]. All of this [i. e., the set of angels and devils] is susceptible to increase, but this susceptibility does not make the [the application of] the mapping [technique] permissible; for what has no order in either nature or position is not susceptible to [the use] of the mapping [technique]; and [the use of this technique] in the case of what does not exist all together is even more impermissible.⁸⁷

In this passage Avicenna discusses two conditions for the applicability of *The Mapping Argument*. Following McGinnis, I call these conditions respectively (1) the ‘wholeness condition’ and (2) the ‘ordering condition’. According to the wholeness condition, *The Mapping Argument* is applicable to a set or totality only if its elements exist all together at the same time. According to the ordering condition, the argument is applicable to a set or totality only if its elements have an order in either nature or position. Avicenna seems to believe that both magnitudes and sets of numbers (or numbered things) must satisfy the wholeness condition to be the subject of *The Mapping Argument*. For him it is nonsense to speak about the (non-)equality of things – whether magnitudes or numbers or numbered things – that do not actually exist. This is why time cannot be the subject of the mapping argument. Time does not satisfy the wholeness condition because temporal moments do not exist simultaneously and all together. In each moment there actually exists only one point of the temporal line. Therefore, the infinity of the temporal line and the eternity of the world do not entail the *actual* existence of an infinity. Since temporal moments do not exist all together at the same time, it is nonsense for Avicenna to say that the whole of the temporal line corresponds with – and is therefore equal to – some of its parts. As a result, *The Mapping Argument* is not applicable to time and cannot reject its potential infinity.⁸⁸

The case of the ordering condition is more complicated. Is having an order a necessary condition for both numbers and magnitudes to be the subject of *The Mapping Argument*? In his discussions of *The Mapping Argument* Avicenna

⁸⁷ Avicenna 1985, IV.2, 245f. My translation. At least at first glance, there seems to be a tension between Avicenna’s claim in this text that “[a]ll of this [i. e., the set of angels and devils] is susceptible to increase” and his view in *The Metaphysics of the Healing* that “[n]umber whose existence is in things separate [from matter] cannot become subject to any relation of increase or decrease that may occur but will only remain as it is” (Avicenna 2005, I.3 [17]). However, discussing this issue would take us too far afield from our main concerns. See also note 94 for a related issue.

⁸⁸ Since Al-Kindi does not consider the wholeness condition as a necessary condition for the applicability of the mapping technique, he rejects the eternity of the world based on the very argument from which Avicenna’s mapping argument is inspired.

repeatedly says that this argument rejects the actual infinity of magnitudes and numbers or numbered things having an *order* in either nature or position (*tartīb fi al-ṭab‘ aw al-waḍ‘*). Accordingly, in the above passage, he says that no argument, including *The Mapping Argument*, can reject the actual existence of an infinite set of non-ordered numbered things. It is therefore undisputable that, for Avicenna, the ordering condition is a necessary condition for numbers and numbered things to be the subject of the mapping argument. Is it also a necessary condition for the case of magnitudes? McGinnis argues for a positive answer to this question. He believes that (1) there is a sense in which magnitudes are (or at least can be) ordered and (2) having an order is a necessary condition for magnitudes to be the subject of *The Mapping Argument*.⁸⁹ The former claim seems to be defensible. However, I think that there are reasons to be suspicious of the latter. To justify these claims, we should first have a clear understanding of what exactly Avicenna means by having an order. In the introduction to his discussion of *The Mapping Argument* in the *Metaphysics* part of *Alā‘ī Encyclopedia* Avicenna writes:

TEXT # 10: Beforeness and afterness (*pīshī wa sipasī*) is either by nature – as in number (*shumār*) – or by supposition (*farḍ*) – as in measures (*andāza-hā*) [i. e., one-dimensional magnitudes] – that you can start from any direction you may want. And, everything that either there is beforeness and afterness in it by nature or it is a magnitude (*miqdār*) which has parts that exist all together is finite.⁹⁰

A comparison of this passage with TEXT # 3 and TEXT # 4 shows that Avicenna uses the Arabic phrase ‘*tartīb*’ as an equivalent for the Persian phrase ‘*pīshī wa sipasī*’. It indicates that, for Avicenna, a set or totality of things have an order only if for every two members of this totality one of them is, in a sense, before (or after) the other. As Avicenna explicitly confirms in the above passage, numbers have a natural and essential order. This is because for every two numbers one of them is before or less than the other. One-dimensional magnitudes (i. e., lines), on the other hand, do not have a natural order. This is because they lack essential directionality. They can be traversed in two opposite directions. But as soon as we fix one of these directions, we have a beforeness/afterness (or priority/posteriority) relation between the parts of this one-dimensional magnitude. Therefore, every one-dimensional magnitude can in principle have two different suppositional orderings, depending on the direction we consider for it. This demonstrate (1); at least in the case of one-dimensional magnitudes. Nonetheless, it does not show that satisfaction of this condition is necessary for magnitudes to be the subject

⁸⁹ McGinnis 2010, 218.

⁹⁰ Avicenna 2004, Ch. 16, 58 f. My Translation.

of *The Mapping Argument*. Indeed, although they are ordered, it is not in virtue of this orderedness that *The Mapping Argument* is applicable to them. Let me explain why.

According to Avicenna, correspondence between two geometrical magnitudes L and L' is nothing more than projection and coverage: L corresponds with L' if and only if it is possible to map L onto L' in such a way that no part of either L or L' remains uncovered. It seems, therefore, that the notion of correspondence could be understood by appealing merely to our geometrical intuitions without the aid of the notion of ordering (or, equivalently, beforeness/afterness). It is in virtue of the continuity and connectedness of geometrical magnitudes – as suggested by TEXT # 6 – that the mapping technique and the notion of correspondence in the sense of coverage and projection are applicable to them. Considering the case of two-dimensional magnitudes shows us that the application of the mapping technique to the case of magnitudes is not in virtue of their orderedness. The sizes of two-dimensional figures can be compared by defining equality in terms of coverage. Consider two triangles drawn on a paper. By mapping one of them on the other and checking whether or not they completely cover each other, we can compare their areas. But there is no obvious/natural order among the points of two-dimensional magnitudes/shapes. More precisely: the natural positions of the points on a two-dimensional magnitude do not impose a beforeness/afterness relation on them. Those points are not ordered by their natural positions. It shows that the applicability of the mapping technique in the sense of coverage and projection is something completely independent from orderedness in the sense of having beforeness/afterness. That is why Avicenna mentions the ordering condition only for the case of numbers. Orderedness does not play any crucial role in the applicability of *The Mapping Argument* to the case of magnitudes.

Quite differently, correspondence between two sets of discrete objects A and B is one-to-one correspondence between their elements.⁹¹ But the notion of one-to-one correspondence cannot be reduced to geometrical projection and coverage; for the elements of A and B may be of different sizes, different shapes and have different places (if they have these properties at all). If A and B are ordered then we can compare their sizes by pairing the first element of A with the first element of B , the second element of A with the second element of B , and so on (i. e., for every natural number n , pairing the n^{th} element of A with the n^{th} element of B). In fact, Avicenna seems to believe that the possibility of being ordered is a necessary condition for the possibility of being put in a one-to-one correspond-

⁹¹ As TEXT # 6 suggests, correspondence in the case of numerical infinities is being parallel in terms of orderedness.

ence. He thinks that if a set of objects cannot be ordered, it cannot be put in a one-to-one correspondence with other sets.⁹² In other words, the possibility of picking elements of a set one-by-one to put them in a one-to-one correspondence with the elements of another set is equivalent to the possibility of putting an order on the elements of that set. Therefore, if it is impossible for a set to be ordered, then it is equivalently impossible for it to be put in a one-to-one correspondence. Accordingly, the size of such a set – that cannot be ordered – cannot be compared to the sizes of other sets, including its own proper subsets. As a result, it cannot be shown that that set corresponds with some of its subsets. So, no contradiction arises and *The Mapping Argument* does not work in such a case.⁹³

Apparently, it is by following this line of argument that Avicenna claims that *The Mapping Argument* is not applicable to the case of immaterial objects such as angels and devils; for he believes that the set of those objects cannot be ordered. More precisely, they do not have a natural and essential order, nor it is possible to posit a conventional order for them (at least it seems inconceivable how they might have such an order). The actual infinity of such sets cannot be rejected by appealing to *The Mapping Argument*. Avicenna seems to believe that (1) other arguments against the actual infinity are of no use in the case of sets of discrete elements, and (2) there are some arguments for the actual infinity of the set of angels and demons. Consequently, he believes that such infinite sets are uncontroversially actual.⁹⁴

But what will happen if – by contrast to the case of angels and devils – it is possible to put a conventional order on a set of immaterial objects? Can we

⁹² By generalizing this idea we may arrive at the modern idea of the priority of the notion of *ordinality* to the notion of *cardinality*.

⁹³ On the one hand, the above explanation can precisely clarify why non-ordered sets cannot be the subject of *The Mapping Argument*. On the other hand, the plausibility of this explanation is itself dependent on the interpretation of the correspondence between sets of discrete elements as one-to-one correspondence between their elements. Since there seems to be no defensible rival for the above explanation, the explanatory power of this scenario about the role of ordering in *The Mapping Arguments* can itself be considered as evidence for the claim that, according to Avicenna's philosophy, the correspondence between two sets of discrete elements should be understood as a one-to-one correspondence between their elements.

⁹⁴ It should be noted that here we can find a serious problem for Avicenna's philosophy. The infinity of a set of immaterial objects depends on the numerical individuality of its elements. If its elements are not individuated, then it seems nonsense to believe in their numerosity and, *a fortiori*, their infinity. Therefore, if Avicenna accepts the actual existence of an infinity of immaterial individuals of the same kind, then he has no way out but to reject the Aristotelian principle that objects of the same kind are numerically individuated by their matters. This is one of Averroes's (1126–1198) criticisms of Avicenna. See Marmura 1960, Sec. II, 173 f.

still argue, based on the mapping technique, that such a set cannot be actually infinite?

For example, Al-Ghazālī (1058–1111) argued that the eternity of the world implies the eternity of the species human being. This means that every moment of time is preceded by an infinite number of people who have died. Therefore every moment of time is preceded by an infinity of human souls who have been separated from matter but still actually exist. Moreover, Al-Ghazālī believes that we can put a conventional order on (at least a subset of) the set of the souls who have been separated from matter until now.⁹⁵ Suppose that the first element of the set *A* is the last soul who was separated from her body before the present time, the second element of *A* is the last soul who was separated from her body before the end of yesterday, the third one is the soul who was separated from its body before the end of the day before yesterday, and so on.⁹⁶ The eternity of the world and human beings implies the actual existence of and the infinity of the set *A*. But, since *A* is ordered, its actual infinity can be rejected by *The Mapping Argument*. This is simply a contradiction. Based on this line of argument, Al-Ghazālī concludes that Avicenna's doctrine of the eternity of the world and human being contradicts his rejection of the actual infinity.⁹⁷

I think that we have a strong strategy to rebut this objection on behalf of Avicenna. Having an order guarantees that we can employ the mapping technique to argue that an infinite set of discrete objects corresponds with some of its proper subsets. But it cannot itself guarantee that such a correspondence is a contradictory conclusion. Avicenna can say that although the correspondence of a whole to its proper parts is a contradiction in the case of material objects and their properties (e. g., mathematical objects), such a correspondence is not controversial in the case of immaterial objects. In other words, the equality of the whole and its part is unacceptable for physical and mathematical objects, but it is acceptable for immaterial objects, because the mereology of the realm of materiality is not similar to that of the realm of immateriality. The principle of the impossibility of the equality of the whole and its proper part is not necessarily valid in the latter

⁹⁵ The core of Al-Ghazālī's idea is to produce an order based on temporal successions and regressions. Nonetheless, the order I will propose, though faithful to Al-Ghazālī's general approach, is not identical to his proposal.

⁹⁶ For the sake of simplicity, I have assumed that (1) for every day there has been at least one soul who has been separated from her body, and (2) in every temporal moment at most only one soul has been separated from her body. Even if we reject these assumptions, we can propose some more sophisticated conventional orders for some infinite subsets of the set of immaterial souls who have been separated from their matter until now.

⁹⁷ For more discussion on the problem of the infinity of souls, see Marmura 1960.

realm. Therefore, it is in principle possible for some ordered infinite set of objects to exist actually; but they cannot be material or dependent on materiality.⁹⁸

According to this solution, there are two crucial steps in the application of *The Mapping Argument*: (1) employing the mapping technique to show that a whole corresponds with some of its proper subsets, and (2) extracting a contradiction from such a correspondence. Having an order, either natural or conventional, guarantees that the first goal can be accomplished. The second one depends on the ontology of the objects under discussion. If they are material, such a correspondence is absolutely unacceptable. But if they are immaterial, it *may* be justified. It is worth emphasizing, however, that I do not claim that there is no such immaterial whole-part correspondence that is contradictory.

In sum, Avicenna believes that *The Mapping Argument* (his only argument against the actuality of discrete infinity) does not work in the case of those sets of immaterial objects which cannot be ordered. It is therefore possible, in principle, that some such sets are actually infinite. However, such an infinite immaterial numerosity cannot be the subject of mathematical studies, because Avicenna believes that numerosity completely separated from matter should be studied by metaphysics, not mathematics. Moreover, it is not conceivable, at least from Avicenna's standpoint, that one could undertake mathematical studies on a set that is not ordered and cannot be put in a simple one-to-one correspondence. It is also worth mentioning that there is no immaterial infinite *continuity*; there is no immaterial infinite magnitude. This is because magnitudes, according to Avicenna's ontology of mathematics, by contrast with numbers, cannot be separated from matter. They have an ontological dependency on matter. Therefore, there is neither finite nor infinite immaterial continuity.⁹⁹

5 Conclusion

Avicenna endorses some sort of mathematical finitism by rejecting the actual existence of infinite magnitudes and infinite sets of ordered numbers and numbered things. His main argument against the actuality of mathematical infinity, *The Mapping Argument*, is grounded on the whole-part inequality axiom. He

⁹⁸ To show that *The Mapping Argument* cannot reject the actual infinity of numbers, Al-Rāzī follows this approach. See note 63 above.

⁹⁹ Remember that number can be separated from matter; but if it is, it cannot be the subject of mathematical studies, but rather of metaphysical studies. By contrast, magnitudes cannot be detached from materiality. See Zarepour 2016 and section 2 above.

shows that the existence of an actual infinity implies the equality of that infinity with some of its proper parts. Since Avicenna believes that such an equality is absurd, at least in the case of mathematical objects, he concludes that no mathematical infinity can actually exist. Therefore, his main argument for mathematical finitism has two principal elements: (1) the whole-part equality in the case of mathematical infinities, and (2) the absurdity of the whole-part equality in the case of mathematical objects.

To argue for (1) Avicenna employs the notion of correspondence as a tool for comparing the size of different mathematical infinities and determining whether they are equal. The concept of correspondence, in turn, is reduced to the concept of geometrical projection and coverage in the case of magnitudes and to the concept of one-to-one correspondence in the case of numbers and numbered things. The sizes of all one-dimensional magnitudes can be compared to each other using the notions of projection and coverage. It is in virtue of their continuity and connectedness that they can be compared in this way. But sets of discrete objects, e. g., numbers and numbered things, can be put in one-to-one correspondences only if they can be ordered.

I have argued that (2) can be justified by Avicenna's preferred ontology for mathematical objects. He believes that mathematical objects are properties of physical objects and, consequently, have some sort of dependency on materiality. It seems that this dependency on materiality – which ties the mereology of mathematical objects to that of material objects – is what renders the whole-part equality absurd in the case of mathematical infinities. This shows that Avicenna's mathematical finitism is heavily founded on his views about the nature of mathematical objects. The map shown in Fig. 5 is the Avicennan strategy which leads to mathematical finitism.¹⁰⁰

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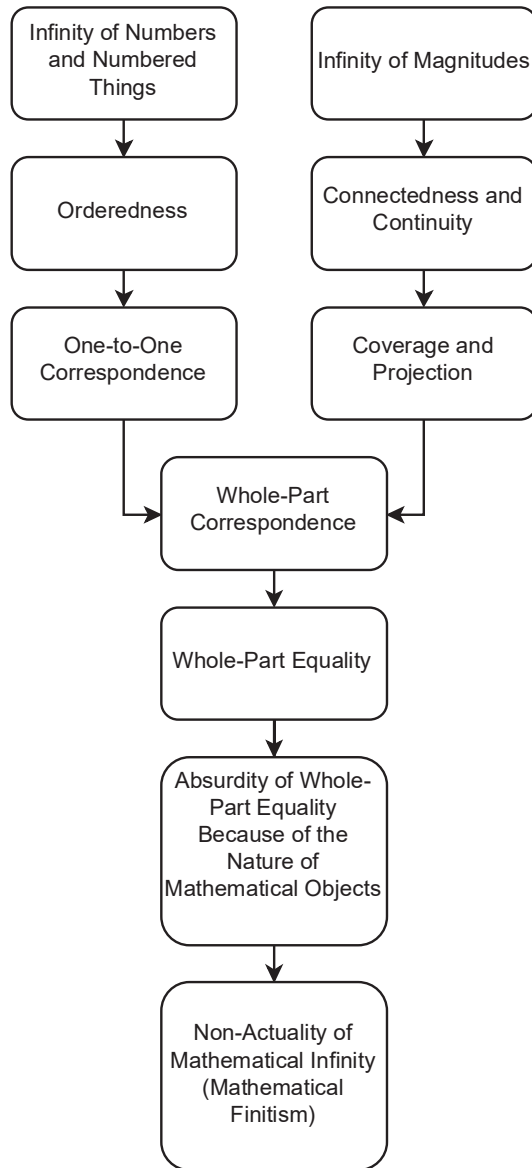


Fig. 5

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