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A new extension and applications of Caputo fractional derivative operator

<https://doi.org/10.1515/anly-2019-0005>

Received January 12, 2019; revised September 5, 2019; accepted July 27, 2020

Abstract: The main objective of this paper is to introduce a further extension of the extended Caputo fractional derivative operator and establish the extension of an extended fractional derivative of some known elementary functions. Additionally, we investigate the extended fractional derivative of some familiar special functions, the Mellin transform of the newly defined Caputo fractional derivative operator and generating relations for the extensions of extended hypergeometric functions.

Keywords: Hypergeometric function, beta function, extended hypergeometric function, Mellin transform, fractional derivative, Caputo fractional derivative, Appell's function, generating relation

MSC 2010: 26A33, 33C20, 33C15, 33C05, 33C65

1 Introduction and preliminaries

Recently, researchers (see, e.g., [1, 4–6, 11, 13, 14]) introduced various extensions and generalizations of various special functions. In [4], Chaudhry, Qadir, Rafique and Zubair introduced the extended beta function defined by

$$B(\tau_1, \tau_2; p) = B_p(\tau_1, \tau_2) = \int_0^1 t^{\tau_1-1} (1-t)^{\tau_2-1} e^{-\frac{p}{t(1-t)}} dt, \quad (1.1)$$

where $\Re(p) > 0$, $\Re(\tau_1) > 0$, $\Re(\tau_2) > 0$. When $p = 0$, then $B(\tau_1, \tau_2; 0) = B(\tau_1, \tau_2)$.

Also, the extended hypergeometric and confluent hypergeometric functions are given in [5] by using the definition of the extended beta function $B_p(\tau_1, \tau_2)$ as follows:

$$F_p(\tau_1, \tau_2; \tau_3; z) = \sum_{n=0}^{\infty} \frac{B_p(\tau_2 + n, \tau_3 - \tau_2)}{B(\tau_2, \tau_3 - \tau_2)} (\tau_1)_n \frac{z^n}{n!}, \quad (1.2)$$

where $p \geq 0$, and

$$\Phi_p(\tau_2; \tau_3; z) = \sum_{n=0}^{\infty} \frac{B_p(\tau_2 + n, \tau_3 - \tau_2)}{B(\tau_2, \tau_3 - \tau_2)} \frac{z^n}{n!}, \quad (1.3)$$

where $p \geq 0$.

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In the same paper [5], Chaudhry, Qadir, Srivastava and Paris defined the following integral representations of extended hypergeometric and confluent hypergeometric functions by

$$\begin{cases} F_p(\tau_1, \tau_2; \tau_3; z) = \frac{1}{B(\tau_2, \tau_3 - \tau_2)} \int_0^1 t^{\tau_2-1} (1-t)^{\tau_3-\tau_2-1} (1-zt)^{-\tau_1} \exp\left(\frac{-p}{t(1-t)}\right) dt, \\ p \geq 0, \quad \Re(\tau_3) > \Re(\tau_2) > 0, \quad |\arg(1-z)| < \pi, \end{cases} \quad (1.4)$$

and

$$\begin{cases} \Phi_p(\tau_2, \tau_3; \gamma; z) = \frac{1}{B(\tau_2, \tau_3 - \tau_2)} \int_0^1 t^{\tau_2-1} (1-t)^{\tau_3-\tau_2-1} \exp\left(zt - \frac{p}{t(1-t)}\right) dt, \\ p \geq 0, \quad \Re(\tau_3) > \Re(\tau_2) > 0. \end{cases} \quad (1.5)$$

The extended Appell's function is defined by (see [1])

$$F_1(\tau_1, \tau_2, \tau_3; \tau_4; x, y; p) = \sum_{m,n=0}^{\infty} \frac{B_p(\tau_1 + m + n, \tau_4 - \tau_1)}{B(\tau_1, \tau_4 - \tau_1)} (\tau_2)_m (\tau_3)_n \frac{x^m y^n}{m!n!}, \quad (1.6)$$

where $p \geq 0$, and its integral representation by

$$\begin{cases} F_1(\tau_1, \tau_2, \tau_3; \tau_4; x, y; p) \\ = \frac{1}{B(\tau_1, \tau_4 - \tau_1)} \int_0^1 t^{\tau_1-1} (1-t)^{\tau_4-\tau_1-1} (1-xt)^{-\tau_2} (1-yt)^{-\tau_3} \exp\left(\frac{-p}{t(1-t)}\right) dt, \\ p \geq 0, \quad \Re(\tau_4) > \Re(\tau_1) > 0, \quad |\arg(1-x)| < \pi, \quad |\arg(1-y)| < \pi. \end{cases} \quad (1.7)$$

It is clear that when $p = 0$, then equations (1.2)–(1.7) reduce to the well-known hypergeometric, confluent hypergeometric and Appell's series and their integral representation, respectively (see [18]). Parmar, Chopra and Paris [15] introduced the following extended beta function as

$$B_v(\tau_1, \tau_2; p) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{\tau_1-\frac{3}{2}} (1-t)^{\tau_2-\frac{3}{2}} K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt, \quad (1.8)$$

where $K_{v+1/2}(\cdot)$ is the modified Bessel function of order $v + \frac{1}{2}$. Clearly, when $v = 0$, then (1.8) reduces to (1.1) by using the fact that

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$$

In the same paper, Parmar et al. [15] defined the following extended hypergeometric and confluent hypergeometric functions and their integral representation respectively by

$$\begin{cases} F_{p,v}(\tau_1, \tau_2; \tau_3; z) = \sum_{n=0}^{\infty} (\tau_1)_n \frac{B_v(\tau_2 + n, \tau_3 - \tau_2; p)}{B(\tau_2, \tau_3 - \tau_2)} \frac{z^n}{n!}, \\ p, v \geq 0, \quad \Re(\tau_3) > \Re(\tau_2) > 0, \quad |z| < 1, \end{cases} \quad (1.9)$$

$$\begin{cases} \Phi_{p,v}(\tau_2; \tau_3; z) = \sum_{n=0}^{\infty} \frac{B_v(\tau_2 + n, \tau_3 - \tau_2; p)}{B(\tau_2, \tau_3 - \tau_2)} \frac{z^n}{n!}, \\ p, v \geq 0, \quad \Re(\tau_3) > \Re(\tau_2) > 0, \end{cases} \quad (1.10)$$

$$\begin{cases} F_{p,v}(\tau_1, \tau_2; \tau_3; z) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(\tau_2, \tau_3 - \tau_2)} \int_0^1 t^{\tau_2-\frac{3}{2}} (1-t)^{\tau_3-\tau_2-\frac{3}{2}} (1-zt)^{-\tau_1} K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt, \\ p, v \geq 0, \quad \Re(\tau_3) > \Re(\tau_2) > 0, \quad |\arg(1-z)| < \pi, \end{cases} \quad (1.11)$$

$$\begin{cases} \Phi_{p,v}(\tau_2; \tau_3; z) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(\tau_2, \tau_3 - \tau_2)} \int_0^1 t^{\tau_2-\frac{3}{2}} (1-t)^{\tau_3-\tau_2-\frac{3}{2}} \exp(zt) K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt, \\ p, v \geq 0, \quad \Re(\tau_3) > \Re(\tau_2) > 0. \end{cases} \quad (1.12)$$

It is clear that when $\nu = 0$, then equations (1.9)–(1.12) reduce to the extended hypergeometric, confluent hypergeometric functions and their integral representations, respectively, defined in (1.2)–(1.5), by using the fact that

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$$

Recently, Dar and Paris [7] have introduced the following Appell's hypergeometric function by

$$\begin{aligned} F_{1,p,\nu}(\tau_1, \tau_2, \tau_3; \tau_4; x, y) &= F_{1,p,\nu}(\tau_1, \tau_2, \tau_3; \tau_4; x, y; p) \\ &= \sum_{m,n=0}^{\infty} (\tau_2)_m (\tau_3)_n \frac{B_{p,\nu}(\tau_1 + m + n, \tau_4 - \tau_1)}{B(\tau_1, \tau_4 - \tau_1)} \frac{x^m y^n}{n! m!}, \end{aligned} \quad (1.13)$$

where $|x| < 1$, $|y| < 1$, $\tau_1, \tau_2, \tau_3, \tau_4 \in \mathbb{C}$ and $\tau_4 \neq 0, -1, -2, -3, \dots$. In the same paper [7], the integral representation is given as

$$\begin{aligned} F_{1,p,\nu}(\tau_1, \tau_2, \tau_3; \tau_4; x, y) \\ = \sqrt{\frac{2p}{\pi}} \frac{1}{B(\tau_1, \tau_4 - \tau_1)} \int_0^1 t^{\tau_1 - \frac{3}{2}} (1-t)^{\tau_4 - \tau_1 - \frac{3}{2}} (1-tx)^{-\tau_2} (1-ty)^{-\tau_3} K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt, \end{aligned} \quad (1.14)$$

where $\Re(p) \geq 0$, $\nu \geq 0$, $\Re(\tau_4) > \Re(\tau_1) > 0$, $|\arg(1-x)| < \pi$ and $|\arg(1-y)| < \pi$. Obviously, when we have $\nu = 0$ in (1.13) and (1.14), then we get the extended Appell function and its integral representation (see, e.g., (1.6) and (1.7)) by using the fact that

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$$

Very recently, Bohner, Rahman, Mubeen and Nisar [3] introduced the extension of the extended fractional derivative operator of Riemann–Liouville as

$$\mathfrak{D}_x^\mu \{f(x); p, \nu\} = \sqrt{\frac{2px^2}{\pi}} \frac{1}{\Gamma(-\mu)} \int_0^x f(t) t^{-\frac{1}{2}} (x-t)^{-\mu-\frac{3}{2}} K_{\nu+\frac{1}{2}}\left(\frac{px^2}{t(x-t)}\right) dt, \quad (1.15)$$

where $\Re(\mu) < 0$, $\Re(p) > 0$ and $\nu \geq 0$. It is clear that if $\nu = 0$, then (1.15) reduces to the extended fractional derivative defined in [1] by using the fact that

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$$

The Caputo derivative and its applications can be found in [19]. For some recent application of the time fractional derivative and fractional order initial value equations, we refer the interested reader to [20, 21].

2 Extension of hypergeometric functions

In this section, we define the further extensions of hypergeometric and Appell's hypergeometric functions.

Definition 2.1. The extension of the extended hypergeometric function is defined by

$${}_2F_{1,\nu}(\beta, \eta; \mu; z; p) = \sum_{n=0}^{\infty} \frac{(\beta)_n (\eta)_n}{(\eta - m)_n} \frac{B_\nu(\eta - m + n, m - \eta + \mu; p)}{B(\eta - m, \mu - \eta + m)} \frac{z^n}{n!}, \quad (2.1)$$

where $m - 1 < \Re(\eta - \mu) < m < \Re(\eta) > 0$, $\Re(p) > 0$, $\nu \geq 0$ and $|z| < 1$.

Definition 2.2. The extension of Appell's hypergeometric function is defined by

$$F_{1,\nu}(\eta; \alpha, \beta; \mu; z; z; p) = \sum_{n,k=0}^{\infty} \frac{(\alpha)_n (\beta)_n (\eta)_{n+k}}{(\eta - m)_{n+k}} \frac{(z)^n}{n!} \frac{(z)^k}{k!} \frac{B_\nu(\eta + m + n + k, \mu - \eta; p)}{B(\eta - m, m + \mu - \eta)}, \quad (2.2)$$

where $m - 1 < \Re(\eta - \mu) < m < \Re(\eta)$, $\Re(p) > 0$, $\nu \geq 0$.

The integral representations of (2.1) and (2.2) are defined respectively by

$${}_2F_{1,v}(\tau_1, \tau_2; \tau_3; z; p) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(\tau_2 - m, \tau_3 - \tau_2 + m)} \int_0^1 t^{\tau_2 - m - \frac{3}{2}} (1-t)^{\tau_3 - \tau_2 + m - \frac{3}{2}} \\ \times K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) {}_2F_1(\tau_1, \tau_2; \tau_2 - m; zt) dt \quad (2.3)$$

and

$$F_{1,v}(\tau_1, \tau_2, \tau_3; \tau_4; x, y; p) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(\tau_1 - m, \tau_4 - \tau_1 + m)} \int_0^1 t^{\tau_1 - m - \frac{3}{2}} (1-t)^{\tau_4 - \tau_1 + m - \frac{3}{2}} K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) \\ \times F_1(\tau_1, \tau_2; \tau_3; \tau_1 - m; xt; yt) dt. \quad (2.4)$$

Remark 2.3. If we let $v = 0$, then equations (2.1)–(2.4) respectively reduce to the extended hypergeometric functions ${}_2F_1$ and F_1 and their integral representations (see [9]).

3 Extension of fractional derivative operator

Recently, the application and importance of fractional calculus has attracted more attention. In the field of mathematical analysis, the fractional calculus is a more helpful tool to find out differentials and integrals with the real numbers or with the complex number powers of the fractional calculus. Various extensions and generalizations of fractional derivative operators were recently investigated by researchers (see [2, 10, 12, 17, 22]). In this section, we define the further extension of the extended Caputo fractional derivative operator. We recall the classical Caputo fractional derivative operator as follows.

Definition 3.1 (see [8]). Define

$$\mathfrak{D}_z^\mu \{f(z)\} = \frac{1}{\Gamma(m - \mu)} \int_0^z (z-t)^{m-\mu-1} \frac{d^m}{dt^m} f(t) dt,$$

where $m - 1 < \Re(\mu) < m$ with $m = 1, 2, \dots$.

Recently, Kiyamaz, Çetinkaya and Agarwal [9] introduced the extended Caputo fractional derivative operator as follows.

Definition 3.2. Define

$$\mathfrak{D}_z^\mu \{f(z)\} = \frac{1}{\Gamma(m - \mu)} \int_0^z (z-t)^{m-\mu-1} \exp\left(-\frac{pz^2}{t(z-t)}\right) \frac{d^m}{dt^m} f(t) dt, \quad (3.1)$$

where $m - 1 < \Re(\mu) < m$ with $m = 1, 2, \dots$ and $\Re(p) > 0$.

In view of [3], we introduce the further extension of the extended Caputo fractional derivative operator.

Definition 3.3. The extension of the extended Caputo fractional derivative operator is defined by

$$\mathfrak{D}_z^\mu \{f(z); p, v\} = \mathfrak{D}_{z;v}^{\mu;p} \{f(z)\} = \sqrt{\frac{2pz^2}{\pi}} \frac{1}{\Gamma(m - \mu)} \int_0^z t^{-\frac{1}{2}} (z-t)^{m-\mu-\frac{3}{2}} K_{v+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right) \frac{d^m}{dt^m} f(t) dt \quad (3.2)$$

for the case $m - 1 < \Re(\mu) < m$ where $m = 1, 2, \dots$, $\Re(p) > 0$ and $v \geq 0$.

Remark 3.4. Obviously, if $v = 0$, then (3.2) reduces to the extended Caputo fractional derivative defined in (3.1) (see [9]) by using the fact that

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$$

Now, we prove some theorems involving the modified extension of the fractional derivative operator.

Theorem 3.5. *The following formula holds true:*

$$\mathfrak{D}_z^\mu \{z^\eta; p, v\} = \frac{\Gamma(\eta + 1)B_v(\eta - m + 1, m - \mu; p)}{\Gamma(\eta - \mu + 1)B(\eta - m + 1, m - \mu)} z^{\eta - \mu},$$

where $m - 1 < \Re(\mu) < m$ and $\Re(\mu) < \Re(\eta)$.

Proof. From (3.2) we have

$$\begin{aligned} \mathfrak{D}_z^\mu \{z^\eta; p, v\} &= \frac{\sqrt{\frac{2pz^2}{\pi}}}{\Gamma(m - \mu)} \int_0^z t^{-\frac{1}{2}} (z - t)^{m - \mu - \frac{3}{2}} K_{v + \frac{1}{2}} \left(\frac{pz^2}{t(z - t)} \right) \frac{d^m}{dt^m} t^\eta dt \\ &= \frac{\sqrt{\frac{2pz^2}{\pi}}}{\Gamma(m - \mu)} [\eta(\eta - 1) \cdots (\eta - m + 1)] \int_0^z (z - t)^{m - \mu - \frac{3}{2}} K_{v + \frac{1}{2}} \left(\frac{pz^2}{t(z - t)} \right) t^{\eta - m - \frac{1}{2}} dt \\ &= \frac{\sqrt{\frac{2pz^2}{\pi}} \Gamma(\eta + 1)}{\Gamma(\eta - m + 1) \Gamma(m - \mu)} \int_0^z (z - t)^{m - \mu - \frac{3}{2}} K_{v + \frac{1}{2}} \left(\frac{pz^2}{t(z - t)} \right) t^{\eta - m - \frac{1}{2}} dt. \end{aligned} \quad (3.3)$$

Substituting $t = uz$ in (3.3), we have

$$\mathfrak{D}_z^\mu \{z^\eta; p, v\} = \frac{\sqrt{\frac{2p}{\pi}} \Gamma(\eta + 1)}{\Gamma(\eta - m + 1) \Gamma(m - \mu)} z^{\eta - \mu} \int_0^1 u^{\eta - m - \frac{1}{2}} (1 - u)^{m - \mu - \frac{3}{2}} K_{v + \frac{1}{2}} \left(\frac{p}{u(1 - u)} \right) du.$$

Using (1.8) in the above equation, we get

$$\begin{aligned} \mathfrak{D}_z^\mu \{z^\eta; p, v\} &= \frac{\Gamma(\eta + 1)}{\Gamma(\eta - m + 1) \Gamma(m - \mu)} z^{\eta - \mu} B_v(\eta - m + 1, m - \mu; p) \\ &= \frac{\Gamma(\eta + 1)B_v(\eta - m + 1, m - \mu; p)}{\Gamma(\eta - \mu + 1)B(\eta - m + 1, m - \mu)} z^{\eta - \mu}, \end{aligned}$$

which is the required result. $\square$

Theorem 3.6. *Let $m - 1 < \Re(\mu) < m$ and suppose that the function $f(z)$ is analytic on the disk $|z| < r$ for some $r \in \mathbb{R}^+$ and with its power series expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then*

$$\mathfrak{D}_z^\mu \{f(z); p, v\} = \sum_{n=0}^{\infty} a_n \mathfrak{D}_z^\mu \{z^n; p, v\}.$$

Proof. Using the series expansion of the function $f(z)$ in (3.2) gives

$$\mathfrak{D}_z^\mu \{f(z); p, v\} = \frac{\sqrt{\frac{2pz^2}{\pi}}}{\Gamma(m - \mu)} \int_0^z t^{-\frac{1}{2}} (z - t)^{m - \mu - \frac{3}{2}} K_{v + \frac{1}{2}} \left(\frac{pz^2}{t(z - t)} \right) \sum_{n=0}^{\infty} a_n \frac{d^m}{dt^m} t^n dt.$$

As the series is uniformly convergent and the integrand is absolutely convergent, interchanging the order of summation and integration gives

$$\begin{aligned} \mathfrak{D}_z^\mu \{f(z); p, v\} &= \sum_{n=0}^{\infty} a_n \left\{ \frac{\sqrt{\frac{2pz^2}{\pi}}}{\Gamma(m - \mu)} \int_0^z t^{-\frac{1}{2}} (z - t)^{m - \mu - \frac{3}{2}} K_{v + \frac{1}{2}} \left(\frac{pz^2}{t(z - t)} \right) \frac{d^m}{dt^m} t^n dt \right\} \\ &= \sum_{n=0}^{\infty} a_n \mathfrak{D}_z^\mu \{z^n; p, v\}, \end{aligned}$$

which is the required result. $\square$

Theorem 3.7. *Let $m - 1 < \Re(\mu) < m$ and suppose that the function $f(z)$ is analytic on the disk $|z| < r$ for some $r \in \mathbb{R}^+$ and with its power series expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then*

$$\mathfrak{D}_z^\mu \{z^{\eta-1} f(z); p, v\} = \frac{\Gamma(\eta) z^{\eta-1}}{\Gamma(\eta - \mu)} \sum_{n=0}^{\infty} a_n \frac{(\eta)_n}{(\eta - m)_n} \frac{B_v(\eta - m + n, m - \mu; p)}{B(\eta - m, m - \mu)} z^n.$$

Proof. By applying Theorems (3.6) and (3.5), we have

$$\begin{aligned}\mathfrak{D}_z^\mu \{z^{\eta-1} f(z); p, v\} &= \sum_{n=0}^{\infty} a_n \mathfrak{D}_z^\mu \{z^{\eta+n-1}; p, v\} \\ &= \frac{\Gamma(\eta) z^{\eta-\mu-1}}{\Gamma(\eta-\mu)} \sum_{n=0}^{\infty} a_n \frac{(\eta)_n}{(\eta-\mu)_n} \frac{B_v(\eta-m+n, m-\mu; p)}{B(\eta-m+n, m-\mu)} z^n \\ &= \frac{\Gamma(\eta) z^{\eta-\mu-1}}{\Gamma(\eta-\mu)} \sum_{n=0}^{\infty} a_n \frac{(\eta)_n}{(\eta-m)_n} \frac{B_v(\eta-m+n, m-\mu; p)}{B(\eta-m, m-\mu)} z^n,\end{aligned}$$

which is the desired result. $\square$

Theorem 3.8. *The following result holds true:*

$$\begin{aligned}\mathfrak{D}_z^{\eta-\mu} \{z^{\eta-1} (1-z)^{-\beta}; p, v\} &= \frac{\Gamma(\eta)}{\Gamma(\mu)} z^{\mu-1} \sum_{n=0}^{\infty} \frac{(\beta)_n (\eta)_n}{(\eta-m)_n} \frac{B_v(\eta-m+n, m-\eta+\mu; p)}{B(\eta-m, \mu-\eta+m)} \frac{z^n}{n!}, \\ &= \frac{\Gamma(\eta)}{\Gamma(\mu)} z^{\mu-1} {}_2F_{1,v}(\beta, \eta; \mu; z; p).\end{aligned}$$

where $m-1 < \Re(\eta-\mu) < m < \Re(\eta) > 0$, $\Re(p) > 0$ and $v \geq 0$.

Proof. Using the power series of $(1-z)^{-\beta}$ and applying Theorem 3.5, we have

$$\begin{aligned}\mathfrak{D}_z^{\eta-\mu} \{z^{\eta-1} (1-z)^{-\beta}; p, v\} &= \mathfrak{D}_z^{\eta-\mu} \left\{ z^{\eta-1} \sum_{n=0}^{\infty} (\beta)_n \frac{z^n}{n!} \right\} \\ &= \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \mathfrak{D}_z^{\eta-\mu} \{z^{\eta+n-1}; p, v\} \\ &= \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \frac{\Gamma(\eta+n)}{\Gamma(\eta+n-m)\Gamma(m-\eta+\mu)} B_v(\eta-m+n, m-\eta+\mu; p) z^{\mu+n-1} \\ &= \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \frac{\Gamma(\eta+n)}{\Gamma(\mu+n)} \frac{B_v(\eta-m+n, m-\eta+\mu; p)}{B(\eta-m+n, m-\eta+\mu)} z^{\mu+n-1} \\ &= \frac{\Gamma(\eta)}{\Gamma(\mu)} z^{\mu-1} \sum_{n=0}^{\infty} \frac{(\beta)_n (\eta)_n}{(\mu)_n} \frac{B_v(\eta-m+n+\frac{1}{2}, m-\eta+\mu; p)}{B(\eta-m+n, m-\eta+\mu)} \frac{z^n}{n!} \\ &= \frac{\Gamma(\eta)}{\Gamma(\mu)} z^{\mu-1} \sum_{n=0}^{\infty} \frac{(\beta)_n (\eta)_n}{(\eta-m)_n} \frac{B_v(\eta-m+n, m-\eta+\mu; p)}{B(\eta-m, \mu-\eta+m)} \frac{z^n}{n!}.\end{aligned}$$

With the aid of (2.1), we get the required result. $\square$

Theorem 3.9. *The following result holds true:*

$$\begin{aligned}\mathfrak{D}_z^{\eta-\mu} \{z^{\eta-1} (1-az)^{-\alpha} (1-bz)^{-\beta}; p, v\} &= \frac{\Gamma(\eta)}{\Gamma(\mu)} z^{\mu-1} \sum_{n,k=0}^{\infty} \frac{(\alpha)_n (\beta)_n (\eta)_{n+k}}{(\eta-m)_{n+k}} \frac{(az)^n}{n!} \frac{(bz)^k}{k!} \frac{B_v(\eta+m+n, \mu-\eta; p)}{B(\eta-m, m+\mu-\eta)}, \\ &= \frac{\Gamma(\eta)}{\Gamma(\mu)} z^{\mu-1} F_{1,v}(\eta; \alpha, \beta; \mu; az; bz; p),\end{aligned}\tag{3.4}$$

where $m-1 < \Re(\eta-\mu) < m < \Re(\eta)$, $\Re(p) > 0$ and $v \geq 0$.

Proof. To prove (3.4), we use the following power series expansion:

$$(1-az)^{-\alpha} (1-bz)^{-\beta} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (\alpha)_n (\beta)_k \frac{(az)^n}{n!} \frac{(bz)^k}{k!}.$$

Now, applying Theorem 3.8, we obtain

$$\mathfrak{D}_z^{\eta-\mu} \{z^{\eta-1} (1-az)^{-\alpha} (1-bz)^{-\beta}; p, \alpha\} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (\alpha)_n (\beta)_k \frac{(a)^n}{n!} \frac{(b)^k}{k!} \mathfrak{D}_z^{\eta-\mu} \{z^{\eta+n+k-1}; p, v\}.$$

Using Theorem 3.5, we have

$$\begin{aligned}\mathfrak{D}_z^{\eta-\mu}\{z^{\eta-1}(1-az)^{-\alpha}(1-bz)^{-\beta}; p, v\} &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (\alpha)_n (\beta)_k \frac{(a)^n (b)^k}{n! k!} \frac{\Gamma(\eta+n+k) B_v^\alpha(\eta+m+n, \mu-\eta; p)}{\Gamma(m+\mu-\eta) \Gamma(\eta-m+n+k)} z^{\mu+n+k-1} \\ &= \frac{\Gamma(\eta)}{\Gamma(\mu)} z^{\mu-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha)_n (\beta)_k (\eta)_{n+k}}{(\eta-m)_{n+k}} \frac{(az)^n (bz)^k}{n! k!} \frac{B_v(\eta+m+n, \mu-\eta; p)}{B(\eta-m, m+\mu-\eta)}.\end{aligned}$$

With the aid of (2.2), we get the required result. $\square$

4 Further results of extended Caputo fractional derivative operator

In this section, we apply the extension of the Caputo fractional derivative operator (3.2) to some known functions. Also, we investigate the Mellin transforms of the extension of the Caputo fractional derivative operator.

Theorem 4.1. *The following result holds true:*

$$\mathfrak{D}_z^\mu\{e^z; p, v\} = \frac{z^{m-\mu}}{\Gamma(m-\mu)} \sum_{n=0}^{\infty} \frac{z^n}{n!} B_v(n+1, m-\mu; p)$$

for all z .

Proof. Using the power series of e^z and applying Theorems 3.6 and 3.5, we have

$$\begin{aligned}\mathfrak{D}_z^\mu\{e^z; p, v\} &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \mathfrak{D}_z^\mu\{z^n; p, v\} \\ &= \sum_{n=m}^{\infty} \frac{\Gamma(n+1) B_v(n-m+1, m-\mu; p)}{\Gamma(n-\mu+1) B(n-m+1, m-\mu)} \frac{z^{n-\mu}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+m+1) B_v(n+1, m-\mu; p)}{\Gamma(n+m-\mu+1) B(n+1, m-\mu)} \frac{z^{n+m-\mu}}{(n+m)!} \\ &= \frac{z^{m-\mu}}{\Gamma(m-\mu)} \sum_{n=0}^{\infty} \frac{z^n}{n!} B_v(n+1, m-\mu; p).\end{aligned}$$

$\square$

Theorem 4.2. *The following result holds true:*

$$\begin{aligned}\mathfrak{D}_z^\mu\{{}_2F_1(\tau_1, \tau_2; \tau_3; z); p, v\} \\ = \frac{(\tau_1)_m (\tau_2)_m}{(\tau_3)_m} \frac{z^{m-\mu}}{\Gamma(1-\mu+m)} \sum_{n=0}^{\infty} \frac{(\tau_1+m)_n (\tau_2+m)_n}{(\tau_3+m)_n (1-\mu+m)_n} \frac{B_v(n+1, m-\mu; p) z^n}{B(n+1, m-\mu)}\end{aligned}$$

for all $|z| < 1$.

Proof. Using the power series of the Gauss hypergeometric function ${}_2F_1(\cdot)$ and applying Theorems 3.6 and 3.5, we have

$$\begin{aligned}\mathfrak{D}_z^\mu\{{}_2F_1(\tau_1, \tau_2; \tau_3; z); p, v\} \\ = \sum_{n=0}^{\infty} \frac{(\tau_1)_n (\tau_2)_n}{(\tau_3)_n n!} \mathfrak{D}_z^\mu\{z^n; p, v\} \\ = \sum_{n=m}^{\infty} \frac{(\tau_1)_n (\tau_2)_n}{(\tau_3)_n n!} \frac{\Gamma(n+1) B_v(n-m+1, m-\mu; p)}{\Gamma(n-\mu+1) B(n-m+1, m-\mu)} \frac{z^{n-\mu}}{n!} \\ = \sum_{n=0}^{\infty} \frac{(\tau_1)_{n+m} (\tau_2)_{n+m}}{(\tau_3)_{n+m} (n+m)!} \frac{\Gamma(n+m+1) B_v(n+1, m-\mu; p)}{\Gamma(n+m-\mu+1) B(n+1, m-\mu)} \frac{z^{n+m-\mu}}{(n+m)!} \\ = \frac{(\tau_1)_m (\tau_2)_m}{(\tau_3)_m} \frac{z^{m-\mu}}{\Gamma(1-\mu+m)} \sum_{n=0}^{\infty} \frac{(\tau_1+m)_n (\tau_2+m)_n}{(\tau_3+m)_n (1-\mu+m)_n} \frac{B_v(n+1, m-\mu; p) z^n}{B(n+1, m-\mu)}.\end{aligned}$$

$\square$

Theorem 4.3. *The following result holds true:*

$$\mathfrak{D}_z^\mu [E_{\gamma,\delta}^\tau(z); p, v] = \frac{z^{m-\mu}(\tau)_m}{\Gamma(m-\mu)} \sum_{n=0}^{\infty} \frac{(\tau)_n}{\Gamma(\gamma n + \gamma m + \delta)} B_v(n+1, \mu - \eta; p) \frac{z^n}{n!}, \quad (4.1)$$

where $\gamma, \delta, \tau, \in \mathbb{C}$, $\Re(p) > 0$ and $E_{\gamma,\delta}^\tau(z)$ is the Mittag-Leffler function (see [16]) defined by

$$E_{\gamma,\delta}^\tau(z) = \sum_{n=0}^{\infty} \frac{(\tau)_n}{\Gamma(\gamma n + \delta)} \frac{z^n}{n!}. \quad (4.2)$$

Proof. Using (4.2) in (4.1), we have

$$\mathfrak{D}_z^\mu [E_{\gamma,\delta}^\tau(z); p, v] = \mathfrak{D}_z^\mu \left\{ \sum_{n=0}^{\infty} \frac{(\tau)_n}{\Gamma(\gamma n + \delta)} \frac{z^n}{n!}; p, v \right\}.$$

By Theorem 3.6, we have

$$\mathfrak{D}_z^\mu \{E_{\gamma,\delta}^\tau(z); p, v\} = \sum_{n=0}^{\infty} \frac{(\tau)_n}{\Gamma(\gamma n + \delta)n!} \{\mathfrak{D}_z^\mu \{z^n; p, v\}\}.$$

Applying Theorem 3.5, we get

$$\begin{aligned} \mathfrak{D}_z^\mu \{E_{\gamma,\delta}^\tau(z); p, v\} &= \sum_{n=m}^{\infty} \frac{(\tau)_n}{\Gamma(\gamma n + \delta)} \frac{\Gamma(n+1)B_v(n-m+1, m-\mu; p)}{\Gamma(n-\mu+1)B(n-m+1, m-\mu)} \frac{z^{n-\mu}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\tau)_{n+m}}{\Gamma(\gamma n + \gamma m + \delta)} \frac{\Gamma(n+m+1)B_v(n+1, m-\mu; p)}{\Gamma(n+m-\mu+1)B(n+1, m-\mu)} \frac{z^{n+m-\mu}}{(n+m)!} \\ &= \frac{z^{m-\mu}(\tau)_m}{\Gamma(m-\mu)} \sum_{n=0}^{\infty} \frac{(\tau+m)_n}{\Gamma(\gamma n + \gamma m + \delta)} \frac{z^n}{n!} B_v(n+1, m-\mu; p). \end{aligned} \quad \square$$

Theorem 4.4. *The following result holds true:*

$$\mathfrak{D}_z^\mu \left\{ {}_m\Psi_n \left[\begin{matrix} (\alpha_i, A_i)_{1,m}; \\ (\beta_j, B_j)_{1,n}; \end{matrix} \middle| z \right]; p, v \right\} = \frac{z^{m-\mu}}{\Gamma(m-\mu)} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^m \Gamma(\alpha_i + A_i k)}{\prod_{j=1}^n \Gamma(\beta_j + B_j k)} B_v(n+1, m-\mu; p) \frac{z^n}{n!},$$

where $\Re(p) > 0$, $v \geq 0$ and ${}_m\Psi_n(z)$ denotes the Fox–Wright function defined by (see [8, pp. 56–58])

$${}_m\Psi_n(z) = {}_m\Psi_n \left[\begin{matrix} (\alpha_i, A_i)_{1,m}; \\ (\beta_j, B_j)_{1,n}; \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^m \Gamma(\alpha_i + A_i k)}{\prod_{j=1}^n \Gamma(\beta_j + B_j k)} \frac{z^k}{k!}.$$

Proof. Applying Theorem 3.5 and following the same procedure used in Theorem 4.3, we get the desired result. $\square$

Theorem 4.5. *The following Mellin transform formula holds true:*

$$M\{\mathfrak{D}_{z;p}^{\mu;v}(z^\eta); p \rightarrow r\} = \frac{2^{1-r}\Gamma(\eta+1)\Gamma(\frac{r-v}{2})\Gamma(\frac{r+v+1}{2})}{\sqrt{\pi}\Gamma(m-\mu)\Gamma(\eta-m+1)} B(\eta-m+r+1, m-\mu+r) z^{\eta-\mu},$$

where $\Re(\eta) > m-1$ and $\Re(r) > 0$.

Proof. Applying the Mellin transform on (3.2) and by Theorem 3.5, we have

$$\begin{aligned} M\{\mathfrak{D}_{z;p}^{\mu;v}(z^\eta); p \rightarrow r\} &= \int_0^\infty p^{r-1} \mathfrak{D}_{z;p}^{\mu;v}(z^\eta) dp \\ &= \int_0^\infty p^{r-1} \frac{\Gamma(\eta+1)B_v(\eta-m+1, m-\mu; p)}{\Gamma(\eta-\mu+1)} z^{\eta-\mu} dp \\ &= \frac{\Gamma(\eta+1)}{\Gamma(\eta-m+1)\Gamma(m-\mu)} z^{\eta-\mu} \int_0^\infty p^{r-1} B_v(\eta-m+1, m-\mu; p) dp. \end{aligned} \quad (4.3)$$

By using the following Mellin transform of $B_\nu(x, y; p)$ in (4.3) (see [15]), we get the required result:

$$\int_0^\infty p^{r-1} B_\nu(\eta - m + 1, m - \mu; p) dp = \frac{2^{r-1}}{\sqrt{\pi}} \Gamma\left(\frac{r-\nu}{2}\right) \Gamma\left(\frac{r+\nu+1}{2}\right) B(\eta - m + 1 + r, m - \mu + r). \quad \square$$

Theorem 4.6. *The following Mellin transform formula holds true:*

$$M\{\mathfrak{D}_{z;p}^{\mu;\nu}((1-z)^{-\alpha}); p \rightarrow r\} = \frac{2^{1-r} \Gamma(\frac{r-\nu}{2}) \Gamma(\frac{r+\nu+1}{2}) (\alpha)_m}{\sqrt{\pi} \Gamma(m-\mu)} z^{m-\mu} \sum_{n=0}^{\infty} (\alpha+m)_n \frac{z^n}{n!} B(n+r+1, m-\mu+r),$$

where $\Re(p) > 0$, $\Re(r) > 0$ and $|z| < 1$.

Proof. Applying Theorem 4.5 with $\eta = n$ and using the power series extension of $(1-z)^{-\alpha}$, we can write

$$\begin{aligned} M\{\mathfrak{D}_{z;p}^{\mu;\nu}((1-z)^{-\alpha}); p \rightarrow r\} &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} M\{\mathfrak{D}_{z;p}^{\mu;\alpha}(z^n); p \rightarrow r\} \\ &= \frac{2^{1-r} \Gamma(\frac{r-\nu}{2}) \Gamma(\frac{r+\nu+1}{2})}{\sqrt{\pi} \Gamma(m-\mu)} \sum_{n=m}^{\infty} \frac{(\alpha)_n \Gamma(n+1)}{n! \Gamma(n-m+1)} B(n-m+r+1, m-\mu+r) z^{n-\mu} \\ &= \frac{2^{1-r} \Gamma(\frac{r-\nu}{2}) \Gamma(\frac{r+\nu+1}{2})}{\sqrt{\pi} \Gamma(m-\mu)} z^{m-\mu} \sum_{n=0}^{\infty} (\alpha)_{n+m} \frac{z^n}{n!} B(n+r+1, m-\mu+r) \\ &= \frac{2^{1-r} \Gamma(\frac{r-\nu}{2}) \Gamma(\frac{r+\nu+1}{2}) (\alpha)_m}{\sqrt{\pi} \Gamma(m-\mu)} z^{m-\mu} \sum_{n=0}^{\infty} (\alpha+m)_n \frac{z^n}{n!} B(n+r+1, m-\mu+r), \end{aligned}$$

which is the required result. $\square$

5 Generating relations

In this section, we applying Theorems 3.8 and 3.9 and obtain generating relations for the extensions of the extended hypergeometric functions ${}_2F_{1,\nu}$ and $F_{1,\nu}$.

Theorem 5.1. *Assume that $m-1 < \Re(\eta-\mu) < m < \Re(\eta)$ and $|z| < \min\{1, |1-t|\}$. Then*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_{1,\nu}(\lambda+n, \eta; \mu; z; p) t^n = (1-t)^{-\lambda} {}_2F_{1,\nu}\left(\lambda, \eta; \mu; \frac{z}{1-t}; p\right).$$

Proof. Consider the following series identity:

$$[(1-z)-t]^{-\lambda} = (1-t)^{-\lambda} \left(1 - \frac{z}{1-t}\right)^{-\lambda}$$

Thus, the power series expansion yields

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1-z)^{-\lambda} \left(\frac{t}{1-z}\right)^n = (1-t)^{-\lambda} \left[1 - \frac{z}{1-t}\right]^{-\lambda}. \quad (5.1)$$

Multiplying both sides of (5.1) by $z^{\eta-1}$ and then applying the operator $\mathfrak{D}_{z;\nu}^{\eta-\mu;p}$ on both sides, we have

$$\mathfrak{D}_{z;\nu}^{\eta-\mu;p} \left[\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1-z)^{-\lambda} \left(\frac{t}{1-z}\right)^n z^{\eta-1} \right] = (1-t)^{-\lambda} \mathfrak{D}_{z;\nu}^{\eta-\mu;p} \left[z^{\eta-1} \left(1 - \frac{z}{1-t}\right)^{-\lambda} \right].$$

Interchanging the order of summation and the operator $\mathfrak{D}_{z;\nu}^{\eta-\mu;p}$, we have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \mathfrak{D}_{z;\nu}^{\eta-\mu;p} [z^{\eta-1} (1-z)^{-\lambda-n}] t^n = (1-t)^{-\lambda} \mathfrak{D}_{z;\nu}^{\eta-\mu;p} \left[z^{\eta-1} \left(1 - \frac{z}{1-t}\right)^{-\lambda} \right].$$

Thus by applying Theorem 3.8, we obtain the required result. $\square$

Theorem 5.2. *The following generating relation holds true:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_{1;\nu}(\delta - n, \eta; \mu; z; p) t^n = (1-t)^{-\beta} F_1\left(\delta, \lambda, \eta; \mu; -\frac{zt}{1-t}; p\right),$$

where $|t| < \frac{1}{1+|t|}$ and $m < \Re(\eta - \mu) < m < \Re(\eta)$.

Proof. Consider the series identity

$$[1 - (1-z)t]^{-\lambda} = (1-t)^{-\beta} \left[1 + \frac{zt}{1-t}\right]^{-\lambda}.$$

Using the power series expansion on the left-hand side, we have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1-z)^n t^n = (1-t)^{-\lambda} \left[1 - \frac{zt}{1-t}\right]^{-\lambda}. \quad (5.2)$$

Multiplying both sides of (5.2) by $z^{\eta-1}(1-z)^{-\delta}$ and applying the operator $\mathfrak{D}_{z;\nu}^{\eta-\mu;p}$ on both sides, we have

$$\mathfrak{D}_{z;\nu}^{\eta-\mu;p} \left[\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^{\eta-1} (1-z)^{-\delta+n} t^n \right] = (1-t)^{-\lambda} \mathfrak{D}_{z;\nu}^{\eta-\mu;p} \left[z^{\eta-1} (1-z)^{-\delta} \left(1 - \frac{zt}{1-t}\right)^{-\lambda} \right],$$

where $\Re(\eta) > \Re(\mu) > 0$ and $|zt| < |1-t|$. Thus by Theorem 3.6, we have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \mathfrak{D}_{z;\nu}^{\eta-\mu;p} [z^{\eta-1} (1-z)^{-\delta+n}] t^n = (1-t)^{-\lambda} \mathfrak{D}_{z;\nu}^{\eta-\mu;p} \left[z^{\eta-1} (1-z)^{-\delta} \left(1 - \frac{zt}{1-t}\right)^{-\lambda} \right].$$

Applying Theorem 3.9 on both sides, we get the desired result. $\square$

6 Concluding remarks

Recently, Kiyimaz et al. [9] introduced the extended Caputo fractional derivative operator and they obtained many interesting results related to some familiar special functions by applying the mentioned operator. In this paper, we established the further extension of the Caputo fractional derivative operator and obtained many results related to some known special functions which are extensions of the work of Kiyimaz et al. [9]. We conclude that when $\nu = 0$, then all results established in this paper will reduce to the results associated with the extended Caputo fractional derivative operator by using the term

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$$

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