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A new extension of Srivastava's triple hypergeometric functions and their associated properties

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Abstract: In this paper, we define a new extension of Srivastava's triple hypergeometric functions by using a new extension of Pochhammer's symbol that was recently proposed by Srivastava, Rahman and Nisar [H. M. Srivastava, G. Rahman and K. S. Nisar, Some extensions of the Pochhammer symbol and the associated hypergeometric functions, *Iran. J. Sci. Technol. Trans. A Sci.* **43** (2019), no. 5, 2601–2606]. We present their certain basic properties such as integral representations, derivative formulas, and recurrence relations. Also, certain new special cases have been identified and some known results are recovered from main results.

Keywords: Pochhammer's symbols, Gamma function, hypergeometric functions, Bessel and modified Bessel functions, Appell functions, Whittaker function, Srivastava's triple hypergeometric functions

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1 Introduction

In recent years, many researchers have presented and investigated various extensions of the well-known special functions. We refer the interested readers to the works [1, 2, 6–8, 11, 13, 14, 16, 27, 30]. In 1994, Chaudhry and Zubair [3] (see also [4]) have proposed an extension of the classical gamma function $\Gamma(z)$ (see [24]) by employing the regularization term $\exp(-\frac{p}{t})$ as follows:

$$\Gamma_p(\zeta) = \begin{cases} \int_0^\infty \theta^{\zeta-1} \exp\left(-\theta - \frac{p}{\theta}\right) d\theta & (\Re(p) > 0, \zeta \in \mathbb{C}), \\ \Gamma(\zeta) & (p = 0, \Re(\zeta) > 0). \end{cases}$$

Throughout this paper, we denote the sets of positives integers, negative integers and complex numbers by $\mathbb{N}$, $\mathbb{Z}^-$ and $\mathbb{C}$, respectively and we let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$.

In [28], Srivastava, Parmar and Joshi proposed the extension of three Lauricella functions $F_A^{(n)}$, $F_B^{(n)}$ and $F_C^{(n)}$ of n variables which are the generalized form of the corresponding Appell functions F_2 , F_4 and F_3 , respectively.

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Recall the following extension of the second Appell function F_2 (see [28]) and the two confluent Appell series [25] defined respectively by

$$F_2[(\sigma, p), b_1, b_2; c_1, c_2; \theta_1, \theta_2] = \frac{\sum_{m_1, m_2=0}^{\infty} (\sigma, p)_{m_1+m_2} (b_1)_{m_1} (b_2)_{m_2} \frac{\theta_1^{m_1}}{m_1!} \frac{\theta_2^{m_2}}{m_2!}}{(c_1)_{m_1} (c_2)_{m_2}},$$

where $\Re(p) \geq 0$, $|\theta_1| + |\theta_2| < 1$ when $p = 0$, and

$$\psi_2[\sigma; c_1, c_2; \theta_1, \theta_2] = \sum_{m, n=0}^{\infty} \frac{(\sigma)_{m+n}}{(c_1)_m (c_2)_n} \frac{\theta_1^m}{m!} \frac{\theta_2^n}{n!} \quad (|\theta_1| < \infty, |\theta_2| < \infty) \quad (1.1)$$

and

$$\phi_3[\sigma; c; \theta_1, \theta_2] = \sum_{m, n=0}^{\infty} \frac{(\sigma)_m}{(c)_{m+n}} \frac{\theta_1^m}{m!} \frac{\theta_2^n}{n!} \quad (|\theta_1| < \infty, |\theta_2| < \infty). \quad (1.2)$$

In [29], Srivastava, Rahman and Nisar recently defined the following extension of the generalized hypergeometric function:

$${}_pF_q[(\eta_1; p, \nu), \dots, (\eta_r); (\varrho'_1), \dots, (\varrho'_s); z] = \sum_{m=0}^{\infty} \frac{(\eta_1; p, \nu)_m \cdots (\eta_r)_m}{(\varrho'_1)_m \cdots (\varrho'_s)_m} \frac{z^m}{m!},$$

where $\eta_i \in \mathbb{C}$ for $i = 1, 2, \dots, r$, $\varrho'_j \in \mathbb{C}$ for $j = 1, 2, \dots, s$, $\varrho'_j \neq 0, -1, -2, \dots$, and $(\varrho; p, \nu)_\mu$ is the new extension of the generalized Pochhammer symbol defined by (see [29])

$$(\varrho; p, \nu)_\mu = \begin{cases} \frac{\Gamma_\nu(\varrho + \mu; p)}{\Gamma(\varrho)}, & (\Re(p), \Re(\nu) > 0, \varrho, \mu \in \mathbb{C}), \\ (\varrho; p)_\mu & (\nu = 0, \varrho, \mu \in \mathbb{C} \setminus \{0\}). \end{cases} \quad (1.3)$$

The integral representation of $(\varrho; \sigma, \nu)_\mu$ is defined by

$$(\varrho; p, \nu)_\mu = \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\varrho)} \int_0^\infty t^{\varrho+\mu-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) dt. \quad (1.4)$$

where $K_\nu(\cdot)$ is the well-known modified Bessel function of order ν . When we consider $\nu = 0$ in (1.4), by substituting the result

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z},$$

we get the following formula (see [23]):

$$(\varrho; p, 0)_\mu = (\varrho; p)_\mu = \frac{1}{\Gamma(\varrho)} \int_0^\infty t^{\varrho+\mu-1} e^{-t} dt.$$

Motivated by the above investigations, we propose a new extension of the generalized Srivastava's triple hypergeometric functions H_A , H_B and H_C (see [5]) and investigate the various basic properties of the said functions.

2 Extension of the generalized Srivastava's triple hypergeometric functions

The following new extension of the generalized Srivastava's triple hypergeometric functions H_A , H_B and H_C (see [5]) in terms of the new extended Pochhammer's symbol in (1.3) recently proposed by Srivastava et al. [29] are presented in this section:

$$H_A[(\sigma; p, \nu), b_1, b_2; c_1, c_2; \theta_1, \theta_2, \theta_3] = \frac{\sum_{l_1, l_2, l_3=0}^{\infty} (\sigma; p, \nu)_{l_1+l_3} (b_1)_{l_1+l_2} (b_2)_{l_2+l_3} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!}}{(c_1)_{l_1} (c_2)_{l_2+l_3}}, \quad (2.1)$$

where $p \geq 0$, $|\theta_1| < r_1$, $|\theta_2| < s_1$, $|\theta_3| < t_1$ and $r_1 + s_1 + t_1 = 1 + s_1 t_1$,

$$H_B[(\sigma; p, \nu), b_1, b_2; c_1, c_2, c_3; \theta_1, \theta_2, \theta_3] = \frac{\sum_{l_1, l_2, l_3=0}^{\infty} (\sigma; p, \nu)_{l_1+l_3} (b_1)_{l_1+l_2} (b_2)_{l_2+l_3} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!}}{(c_1)_l (c_2)_m (c_3)_{l_3}}, \quad (2.2)$$

where $p \geq 0$, $|\theta_1| < r_1$, $|\theta_2| < s_1$, $|\theta_3| < t_1$ and $r_1 + s_1 + t_1 + 2\sqrt{r_1 s_1 t_1} = 1$, and

$$H_C[(\sigma; p, \nu), b_1, b_2; \gamma; \theta_1, \theta_2, \theta_3] = \frac{\sum_{l_1, l_2, l_3=0}^{\infty} (\sigma; p, \nu)_{l_1+l_3} (b_1)_{l_1+l_2} (b_2)_{l_2+l_3} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!}}{(\gamma)_{l_1+l_2+l_3}}, \quad (2.3)$$

where $p \geq 0$, $|\theta_1| < r_1$, $|\theta_2| < s_1$, $|\theta_3| < t_1$, $r_1 + s_1 + t_1 - 2\sqrt{(1-r_1)(1-s_1)(1-t_1)} < 2$ and $\sigma, b_1, b_2 \in \mathbb{C}$ and $\gamma, c_1, c_2, c_3 \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}$.

Remark 2.1. If we let $\nu = 0$ in (2.1)–(2.3), then we get the generalized Srivastava's triple hypergeometric functions H_A , H_B and H_C proposed earlier by Choi and Parmar [5], respectively.

Remark 2.2. When setting $\theta_2 = 0$ in (2.1) and (2.2), we get the following new extension of the second Appell hypergeometric function F_2 defined by

$$F_2[(\sigma; p, \nu), b_1, b_2; c_1, c_2; \theta_1, \theta_3] = \frac{\sum_{l_1, l_2=0}^{\infty} (\sigma; p, \nu)_{l_1+l_2} (b_1)_{l_1} (b_2)_{l_2} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!}}{(c_1)_l (c_2)_{l_2}}$$

where $\Re(p) \geq 0$, $|\theta_1| + |\theta_2| < 1$ when $p = 0$.

Similarly, when we set $\theta_2 = 0$, we get the following new extension of the first Appell hypergeometric function F_1 defined by

$$F_1[(\sigma; p, \nu), b_1, b_2; \gamma; \theta_1, \theta_2] = \frac{\sum_{l_1, l_2=0}^{\infty} (\sigma; p, \nu)_{l_1+l_2} (b_1)_{l_1} (b_2)_{l_2} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!}}{(c)_{l_1+l_2}}$$

where $\Re(p) \geq 0$, $|\theta_1| < 1$, $|\theta_2| < 1$ when $p = 0$.

3 Integral representations

Certain various integral representations of the newly defined extension of Srivastava's triple hypergeometric functions (2.1), (2.2) and (2.3) are presented in this section. An integral representation of Srivastava's hypergeometric function H_A in (2.1) is presented in the following theorem.

Theorem 3.1. For $\Re(p) > 0$, $\max\{\Re(\theta_2), \Re(\theta_3)\} < 1$ and $\min\{\Re(\sigma), \Re(b_1)\} > 0$, the integral

$$H_A[(\sigma; p, \nu), b_1, b_2; c_1, c_2; \theta_1, \theta_2, \theta_3] = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) \times {}_0F_1(-; c_1; \theta_1 s t) {}_1F_1(b_2; ; c_2; \theta_2 s + \theta_3 t) ds dt \quad (3.1)$$

holds for the hypergeometric function H_A defined in (2.1).

Proof. We have

$$(\sigma; p, \nu)_{l_1+l_3} = \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) dt$$

and

$$(b_1)_{l_1+l_2} = \frac{1}{\Gamma(b_1)} \int_0^\infty s^{b_1+l_1+l_2-1} e^{-s} ds.$$

Using these both equations in (2.1), we obtain

$$\begin{aligned}
 H_A[(\sigma; p, \nu), b_1, b_2; c_1, c_2; \theta_1, \theta_2, \theta_3] &= \frac{\sum_{l_1, l_2, l_3=0}^{\infty} \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\sigma)} \int_0^{\infty} t^{\sigma+l_1+l_3-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) dt \frac{1}{\Gamma(b_1)} \int_0^{\infty} s^{b_1+l_1+l_2-1} e^{-s} ds (b_2)_{l_2+l_3} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!}}{(c_1)_l (c_2)_{l_2+l_3}} \\
 &= \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\sigma)} \int_0^{\infty} t^{\sigma+l_1+l_3-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) dt \frac{1}{\Gamma(b_1)} \int_0^{\infty} s^{b_1+l_1+l_2-1} e^{-s} ds \sum_{l_1, l_2, l_3=0}^{\infty} \frac{(b_2)_{l_2+l_3}}{(c_1)_l (c_2)_{l_2+l_3}} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!} \\
 &= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)} \int_0^{\infty} \int_0^{\infty} t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} e^{-(s+t)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) ds dt \sum_{l_1, l_2, l_3=0}^{\infty} \frac{1}{(c_1)_l} \frac{(b_2)_{l_2+l_3}}{(c_2)_{l_2+l_3}} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!}.
 \end{aligned}$$

Since

$$\sum_{l=0}^{\infty} \frac{1}{(c_1)_l} \frac{\theta_1^{l_1}}{l_1!} = {}_0F_1(-; c_1; \theta_1 st)$$

and

$$\sum_{m, n=0}^{\infty} \frac{(b_2)_{l_2+l_3}}{(c_2)_{l_2+l_3}} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!} = {}_1F_1(b_2, ; c_2; \theta_2 s + \theta_3 t).$$

we obtain

$$\begin{aligned}
 H_A[(\sigma; p, \nu), b_1, b_2; c_1, c_2; \theta_1, \theta_2, \theta_3] &= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)} \int_0^{\infty} \int_0^{\infty} t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) {}_0F_1(-; c_1; \theta_1 st) {}_1F_1(b_2, ; c_2; \theta_2 s + \theta_3 t) ds dt,
 \end{aligned}$$

as desired. $\square$

Theorem 3.2. For $\Re(p) > 0$, $\Re(\theta_2) < 1$, $\Re(\theta_3) < 1$, $\Re(\sigma) > 0$ and $\Re(b_1) > 0$, the integral

$$\begin{aligned}
 H_B[(\sigma; p, \nu), b_1, b_2; c_1, c_2; c_3; \theta_1, \theta_2, \theta_3] &= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)} \int_0^{\infty} \int_0^{\infty} t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) \\
 &\quad \times {}_0F_1(-; c_1; \theta_1 st) \psi_2(b_2, ; c_2; c_3; \theta_2 s, \theta_3 t) ds dt \quad (3.2)
 \end{aligned}$$

holds for the Srivastava hypergeometric function H_B defined in (2.2).

Proof. We have

$$(\sigma; p, \nu)_{l_1+l_3} = \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\sigma)} \int_0^{\infty} t^{\sigma+l_1+l_3-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) dt$$

and

$$(b_1)_{l_1+l_2} = \frac{1}{\Gamma(b_1)} \int_0^{\infty} s^{b_1+l_1+l_2-1} e^{-s} ds.$$

Using these both equations in (2.2), we obtain

$$\begin{aligned}
 H_B[(\sigma; p, \nu), b_1, b_2; c_1, c_2; c_3; \theta_1, \theta_2, \theta_3] &= \frac{\sum_{l_1, l_2, l_3=0}^{\infty} \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\sigma)} \int_0^{\infty} t^{\sigma+l_1+l_3-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) dt \frac{1}{\Gamma(b_1)} \int_0^{\infty} s^{b_1+l_1+l_2-1} e^{-s} ds (b_2)_{l_2+l_3} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!}}{(c_1)_l (c_2)_{l_2} (c_3)_{l_3}} \\
 &= \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\sigma)} \int_0^{\infty} t^{\sigma+l_1+l_3-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) dt \frac{1}{\Gamma(b_1)} \int_0^{\infty} s^{b_1+l_1+l_2-1} e^{-s} ds \sum_{l_1, l_2, l_3=0}^{\infty} \frac{(b_2)_{l_2+l_3}}{(c_1)_l (c_2)_{l_2} (c_3)_{l_3}} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!} \\
 &= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)} \int_0^{\infty} \int_0^{\infty} t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} e^{-(s+t)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) ds dt \sum_{l_1, l_2, l_3=0}^{\infty} \frac{1}{(c_1)_l} \frac{(b_2)_{l_2+l_3}}{(c_2)_{l_2} (c_3)_{l_3}} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!}.
 \end{aligned}$$

Hence

$$H_B[(\sigma; p, \nu), b_1, b_2; c_1, c_2; c_3; \theta_1, \theta_2, \theta_3] \\ = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) {}_0F_1(-; c_1; \theta_1 st) \psi_2(b_2, ; c_2; c_3; \theta_2 s \theta_3 t) ds dt.$$

Using the identity in (1.1), we obtain

$$\psi_2[\sigma; c_1, c_2; \theta_1, \theta_2] = \sum_{m,n=0}^{\infty} \frac{(\sigma)_{l_2+l_3}}{(c_1)_{l_2}(c_2)_{l_3}} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!}. \quad \square$$

Theorem 3.3. For $\Re(p) > 0$, and $\min\{\Re(\sigma), \Re(b_1), \Re(b_2)\} > 0$, the triple integral

$$H_B[(\sigma; p, \nu), b_1, b_2; c_1, c_2; c_3; \theta_1, \theta_2, \theta_3] \\ = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)\Gamma(b_2)} \int_0^\infty \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} u^{b_2+l_2+l_3-1} \\ \times e^{-(s+t+u)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) {}_0F_1(-; c_1; \theta_1 st) \times {}_0F_1(-; c_2; \theta_2 us) {}_0F_1(-; c_3; \theta_3 ut) dt ds du \quad (3.3)$$

holds for the Srivastava hypergeometric function H_B defined in (2.2).

Proof. We have

$$(\sigma; p, \nu)_{l_1+l_3} = \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) dt, \\ (b_1)_{l_1+l_2} = \frac{1}{\Gamma(b_1)} \int_0^\infty s^{b_1+l_1+l_2-1} e^{-s} ds, \\ (b_2)_{l_2+l_3} = \frac{1}{\Gamma(b_2)} \int_0^\infty u^{b_2+l_2+l_3-1} e^{-u} du.$$

Using these above equations in (2.2), we obtain

$$H_B[(\sigma; p, \nu), b_1, b_2; c_1, c_2, c_3; \theta_1, \theta_2, \theta_3] \\ = \frac{\sum_{l_1, l_2, l_3=0}^{\infty} (\sigma; p, \nu)_{l_1+l_3} (b_1)_{l_1+l_2} (b_2)_{l_2+l_3} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!}}{(c_1)_{l_1} (c_2)_{l_2} (c_3)_{l_3}} \\ = \frac{\sum_{l_1, l_2, l_3=0}^{\infty} \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) dt \frac{1}{\Gamma(b_1)} \int_0^\infty s^{b_1+l_1+l_2-1} e^{-s} ds \frac{1}{\Gamma(b_2)} \int_0^\infty u^{b_2+l_2+l_3-1} e^{-u} du}}{(c_1)_{l_1} (c_2)_{l_2} (c_3)_{l_3}} \\ \times \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!} \\ = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)\Gamma(b_2)} \int_0^\infty \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} u^{b_2+l_2+l_3-1} e^{-(s+t+u)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) \\ \times \sum_{l_1, l_2, l_3=0}^{\infty} \frac{1}{(c_1)_{l_1} (c_2)_{l_2} (c_3)_{l_3}} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!} ds dt du.$$

Hence

$$H_B[(\sigma; p, \nu), b_1, b_2; c_1, c_2; c_3; \theta_1, \theta_2, \theta_3] \\ = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)\Gamma(b_2)} \int_0^\infty \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} u^{b_2+l_2+l_3-1} e^{-(s+t+u)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) {}_0F_1(-; c_1; \theta_1 st) \\ \times {}_0F_1(-; c_2; \theta_2 us) {}_0F_1(-; c_3; \theta_3 ut) ds dt du,$$

as desired. □

Theorem 3.4. For $\Re(p) > 0$, $\Re(\theta_2) < 1$, $\Re(\theta_3) < 1$, $\Re(\sigma) > 0$ and $\Re(b_1) > 0$, the integral

$$H_C[(\sigma; p, \nu), b_1, b_2; \gamma; \theta_1, \theta_2, \theta_3] \\ = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) \phi_3(b_2; \gamma; \theta_2 s + \theta_3 t, \theta_1 st) dt ds$$

holds for the Srivastava hypergeometric function H_C defined in (2.3).

Proof. We have

$$(\sigma; p, \nu)_{l_1+l_3} = \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) dt$$

and

$$(b_1)_{l_1+l_2} = \frac{1}{\Gamma(b_1)} \int_0^\infty s^{b_1+l_1+l_2-1} e^{-s} ds.$$

Using these both equations in (2.3), we obtain

$$H_C[(\sigma; p, \nu), b_1, b_2; \gamma; \theta_1, \theta_2, \theta_3] \\ = \frac{\sum_{l_1, l_2, l_3=0}^\infty (\sigma; p, \nu)_{l_1+l_3} (b_1)_{l_1+l_2} (b_2)_{l_2+l_3} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!}}{(\gamma)_{l_1+l_2+l_3}} \\ = \frac{\sum_{l_1, l_2, l_3=0}^\infty \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) dt \frac{1}{\Gamma(b_1)} \int_0^\infty s^{b_1+l_1+l_2-1} e^{-s} ds (b_2)_{l_2+l_3} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!}}{(\gamma)_{l_1+l_2+l_3}} \\ = \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) dt \frac{1}{\Gamma(b_1)} \int_0^\infty s^{b_1+l_1+l_2-1} e^{-s} ds \sum_{l_1, l_2, l_3=0}^\infty \frac{(b_2)_{l_2+l_3}}{(\gamma)_{l_1+l_2+l_3}} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!} \\ = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) dt ds \sum_{l_1, l_2, l_3=0}^\infty \frac{(b_2)_{l_2+l_3}}{(\gamma)_{l_1+l_2+l_3}} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!}.$$

Hence

$$H_C[(\sigma; p, \nu), b_1, b_2; \gamma; \theta_1, \theta_2, \theta_3] \\ = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) \phi_3(b_2; \gamma; \theta_2 s + \theta_3 t, \theta_1 st) dt ds.$$

By using the identity in (1.2), we obtain

$$\phi_3[\sigma; c; \theta_1, \theta_2] = \sum_{m, n=0}^\infty \frac{(\sigma)_{l_2}}{(c)_{l_2+l_3}} \frac{\theta_1^{l_2}}{l_2!} \frac{\theta_2^{l_3}}{l_3!}.$$

□

Theorem 3.5. For $\Re(p) > 0$ and $\min\{\Re(\sigma), \Re(b_1), \Re(b_2)\} > 0$, the triple integral

$$H_C[(\sigma; p, \nu), b_1, b_2; \gamma; \theta_1, \theta_2, \theta_3] = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)\Gamma(b_2)} \int_0^\infty \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} u^{b_2+l_2+l_3-1} e^{-(t+s+u)} \\ \times K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) F_1(-; \gamma; \theta_1 st + \theta_2 us + \theta_3 ut) dt ds du \quad (3.4)$$

holds for the Srivastava hypergeometric function H_C defined in (2.3).

Proof. We have

$$\begin{aligned}
 (\sigma; p, v)_{l_1+l_3} &= \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} e^{-t} K_{v+\frac{1}{2}}\left(\frac{p}{t}\right) dt, \\
 (b_1)_{l_1+l_2} &= \frac{1}{\Gamma(b_1)} \int_0^\infty s^{b_1+l_1+l_2-1} e^{-s} ds, \\
 (b_2)_{l_2+l_3} &= \frac{1}{\Gamma(b_2)} \int_0^\infty u^{b_2+l_2+l_3-1} e^{-u} du.
 \end{aligned}$$

Using these three equations in (2.3), we obtain

$$\begin{aligned}
 &H_C[(\sigma; p, v), b_1, b_2; \gamma; \theta_1, \theta_2, \theta_3] \\
 &= \frac{\sum_{l_1, l_2, l_3=0}^\infty (\sigma; p, v)_{l_1+l_3} (b_1)_{l_1+l_2} (b_2)_{l_2+l_3} \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!}}{(\gamma)_{l_1+l_2+l_3}} \\
 &= \frac{\sum_{l_1, l_2, l_3=0}^\infty \sqrt{\frac{2p}{\pi}} \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} e^{-t} K_{v+\frac{1}{2}}\left(\frac{p}{t}\right) dt \frac{1}{\Gamma(b_1)} \int_0^\infty s^{b_1+l_1+l_2-1} e^{-s} ds \frac{1}{\Gamma(b_2)} \int_0^\infty u^{b_2+l_2+l_3-1} e^{-u} du}{(\gamma)_{l_1+l_2+l_3}} \\
 &\quad \times \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!} \\
 &= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)\Gamma(b_2)} \int_0^\infty \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} u^{b_2+l_2+l_3-1} e^{-(s+t+u)} K_{v+\frac{1}{2}}\left(\frac{p}{t}\right) \sum_{l_1, l_2, l_3=0}^\infty \frac{1}{(\gamma)_{l_1+l_2+l_3}} \\
 &\quad \times \frac{\theta_1^{l_1}}{l_1!} \frac{\theta_2^{l_2}}{l_2!} \frac{\theta_3^{l_3}}{l_3!} ds dt du.
 \end{aligned}$$

Hence

$$\begin{aligned}
 H_C[(\sigma; p, v), b_1, b_2; \gamma; \theta_1, \theta_2, \theta_3] &= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)\Gamma(b_2)} \int_0^\infty \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} u^{b_2+l_2+l_3-1} \\
 &\quad \times e^{-(t+s+u)} K_{v+\frac{1}{2}}\left(\frac{p}{t}\right) {}_0F_1(-; \gamma; \theta_1 st + \theta_2 us + \theta_3 ut) dt ds du,
 \end{aligned}$$

as desired. $\square$

Next, we consider the Laguerre polynomial $L_n^{(\sigma)}(x)$ (see [15]) of order (index) a and degree n in x , the Whittaker function $M_{k,m,n}(x, y)$ of two variables [22], the Bessel function $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$ (see [9, 10, 12, 15]), which can be represented in terms of the confluent function ψ_2 and the confluent hypergeometric functions ${}_0F_1$, respectively given by

$$L_n^{(\sigma)}(x) = \frac{(\sigma+1)_n}{l_3!} {}_1F_1(-n; \sigma+1; x), \quad (3.5)$$

$$M_{k,m,n}(x, y) = x^{m+\frac{1}{2}} y^{n+\frac{1}{2}} e^{-\frac{1}{2}(x+y)} \psi_2(l_2+l_3-k+1; 2m+1, 2n+1; x, y), \quad (3.6)$$

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)} {}_0F_1(-; \nu+1; -\frac{1}{4}z^2) \quad (\nu \in \mathbb{C} \setminus \mathbb{Z}^-), \quad (3.7)$$

$$I_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)} {}_0F_1(-; \nu+1; \frac{1}{4}z^2) \quad (\nu \in \mathbb{C} \setminus \mathbb{Z}^-). \quad (3.8)$$

The following certain integral representations for the new extension of Srivastava's hypergeometric function H_A in (2.1) can be deduced by applying the relations (3.5) to (3.1), (3.7) and (3.8) to (3.1), and (3.5), (3.7) and (3.8) to (3.1), respectively, which are asserted by corollaries.

Corollary 3.6. *The integral*

$$H_A[(\sigma; x; \nu), b_1, -m; c_1, c_2 + 1; \theta_1, \theta_2, \theta_3] \\ = \frac{\sqrt{\frac{2x}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)} \frac{l_2!}{(c_2 + 1)l_2} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{3}{2}} s^{b_1+l_1+l_2-1} e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) {}_0F_1(-; c_1; \theta_1 st) L_m^{(c_2)}(\theta_2 s + \theta_3 t) dt ds$$

holds for H_A in (2.1) such that the integral involved is convergent.

Corollary 3.7. *The integrals*

$$H_A[(\sigma; p, \nu), b_1, b_2; c_1 + 1, c_2; -\theta_1, \theta_2, \theta_3] = \sqrt{\frac{2p}{\pi}} \frac{\Gamma(c_1 + 1)}{\Gamma(\sigma)\Gamma(b_1)} \theta_1^{-\frac{c_1}{2}} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{c_1}{2}-\frac{3}{2}} s^{b_1+l_1+l_2-\frac{c_1}{2}-1} \\ \times e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) J_{c_1}(2\sqrt{\theta_1 st}) {}_1F_1(b_2; c_2; \theta_2 s + \theta_3 t) dt ds$$

and

$$H_A[(\sigma; p, \nu), b_1, b_2; c_1 + 1, c_2; \theta_1, \theta_2, \theta_3] = \sqrt{\frac{2p}{\pi}} \frac{\Gamma(c_1 + 1)}{\Gamma(\sigma)\Gamma(b_1)} \theta_1^{-\frac{c_1}{2}} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{c_1}{2}-\frac{3}{2}} s^{b_1+l_1+l_2-\frac{c_1}{2}-1} \\ \times e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) I_{c_1}(2\sqrt{\theta_1 st}) {}_1F_1(b_2; c_2; \theta_2 s + \theta_3 t) dt ds$$

hold for H_A in (2.1) such that the integrals involved are convergent.

Corollary 3.8. *The integrals*

$$H_A[(\sigma; p, \nu), b_1, -m, c_1 + 1, c_2 + 1; -\theta_1, \theta_2, \theta_3] \\ = \frac{\sqrt{\frac{2x}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)} \frac{l_2!}{(c_2 + 1)l_2} \Gamma(c_1 + 1) \theta_1^{-\frac{c_1}{2}} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{c_1}{2}-\frac{3}{2}} \\ \times s^{b_1+l_1+l_2-\frac{c_1}{2}-1} e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) J_{c_1}(2\sqrt{\theta_1 st}) L_m^{(c_2)}(\theta_2 s + \theta_3 t) dt ds$$

and

$$H_A[(\sigma; p, \nu), b_1, -m, c_1 + 1, c_2 + 1; \theta_1, \theta_2, \theta_3] \\ = \frac{\sqrt{\frac{2x}{\pi}}}{\Gamma(\sigma)\Gamma(b_1)} \frac{l_2!}{(c_2 + 1)l_2} \Gamma(c_1 + 1) \theta_1^{-\frac{c_1}{2}} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{c_1}{2}-\frac{3}{2}} \\ \times s^{b_1+l_1+l_2-\frac{c_1}{2}-1} e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) I_{c_1}(2\sqrt{\theta_1 st}) L_m^{(c_2)}(\theta_2 s + \theta_3 t) dt ds$$

hold for H_A in (2.1) such that the integrals involved are convergent.

Further, we deduce the following certain integral representations for the new extension of Srivastava's hypergeometric function H_B in (2.2) by applying the relations (3.6) to (3.2), (3.7) and (3.8) to (3.2), (3.6), (3.7) and (3.8) to (3.2), and (3.7) and (3.8) to (3.3), respectively, which are asserted by corollaries.

Corollary 3.9. *The integrals*

$$H_B[(\sigma; p, \nu), b_1, b_2; c_1 + 1, c_2; c_3; -\theta_1, \theta_2, \theta_3] \\ = \sqrt{\frac{2p}{\pi}} \frac{\Gamma(c_1 + 1)}{\Gamma(\sigma)\Gamma(b_1)} \theta_1^{-\frac{c_1}{2}} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{c_1}{2}-\frac{3}{2}} s^{b_1+l_1+l_2-\frac{c_1}{2}-1} \\ \times e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) J_{c_1}(2\sqrt{\theta_1 st}) \psi_2(b_2; c_2, c_3; \theta_2 s, \theta_3 t) dt ds$$

and

$$\begin{aligned} & H_B[(\sigma; p, \nu), b_1, b_2; c_1 + 1, c_2; \theta_1, \theta_2, \theta_3] \\ &= \sqrt{\frac{2p}{\pi}} \frac{\Gamma(c_1 + 1)}{\Gamma(\sigma)\Gamma(b_1)} \theta_1^{-\frac{c_1}{2}} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{c_1}{2}-\frac{3}{2}} s^{b_1+l_1+l_2-\frac{c_1}{2}-1} \\ & \quad \times e^{-(t+s)} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) I_{c_1}(2\sqrt{\theta_1 st}) \psi_2(b_2; c_2, c_3; \theta_2 s, \theta_3 t) dt ds \end{aligned}$$

hold for H_B in (2.2) such that the integrals involved are convergent.

Corollary 3.10. *The integral*

$$\begin{aligned} & H_B[(\sigma; p, \nu), \mu, \sigma + p - k + 1; z, 2\sigma + 1, 2p + 1; \theta_1, \theta_2, \theta_3] \\ &= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(\mu)} \theta_2^{-\sigma-\frac{1}{2}} \theta_3^{-p-\frac{1}{2}} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-p-\frac{3}{2}} s^{\mu+l_1+l_2-\sigma-\frac{3}{2}} e^{-(1-\frac{1}{2}\theta_2)s-(1-\frac{1}{2}\theta_3)t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) \\ & \quad \times {}_0F_1(-; z; \theta_1 st) M_{k,\sigma,p}(\theta_2 s, \theta_3 t) dt ds \end{aligned}$$

holds for H_B in (2.2) such that given integral is convergent.

Corollary 3.11. *The double integrals*

$$\begin{aligned} & H_B[(\sigma; p, \nu), \mu, \sigma + p - k + 1; z + 1, 2\sigma + 1, 2p + 1; -\theta_1, \theta_2, \theta_3] \\ &= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(\mu)} \theta_1^{-\frac{1}{2}z} \theta_2^{-\sigma-\frac{1}{2}} \theta_3^{-p-\frac{1}{2}} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-p-\frac{1}{2}z-\frac{3}{2}} s^{\mu+l_1+l_2-\sigma-\frac{1}{2}z-\frac{3}{2}} \\ & \quad \times e^{-(1-\frac{1}{2}\theta_2)s-(1-\frac{1}{2}\theta_3)t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) I(2\sqrt{\theta_1 st}) M_{k,\sigma,p}(\theta_2 s, \theta_3 t) dt ds \end{aligned}$$

and

$$\begin{aligned} & H_B[(\sigma; p, \nu), \mu, \sigma + p - k + 1; z + 1, 2\sigma + 1, 2p + 1; \theta_1, \theta_2, \theta_3] \\ &= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\sigma)\Gamma(\mu)} \theta_1^{-\frac{1}{2}z} \theta_2^{-\sigma-\frac{1}{2}} \theta_3^{-p-\frac{1}{2}} \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-p-\frac{1}{2}z-\frac{3}{2}} s^{\mu+l_1+l_2-\sigma-\frac{1}{2}z-\frac{3}{2}} \\ & \quad \times e^{-(1-\frac{1}{2}\theta_2)s-(1-\frac{1}{2}\theta_3)t} K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) I(2\sqrt{\theta_1 st}) M_{k,\sigma,p}(\theta_2 s, \theta_3 t) dt ds \end{aligned}$$

hold for H_B in (2.2) such that the integrals involved are convergent.

Corollary 3.12. *The triple integrals*

$$\begin{aligned} & H_B[(\sigma; p, \nu), b_1, b_2; c_1 + 1, c_2 + 1, c_3 + 1; -\theta_1, -\theta_2, -\theta_3] \\ &= \frac{\sqrt{\frac{2p}{\pi}} \Gamma(c_1 + 1) \Gamma(c_2 + 1) \Gamma(c_3 + 1)}{\Gamma(\sigma)\Gamma(b_1)\Gamma(b_2)} \theta_1^{-\frac{c_1}{2}} \theta_2^{-\frac{c_2}{2}} \theta_3^{-\frac{c_3}{2}} \\ & \quad \times \int_0^\infty \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{c_1}{2}-\frac{c_2}{2}-\frac{3}{2}} s^{b_1+l_1+l_2-\frac{c_1}{2}-\frac{c_2}{2}-1} u^{b_2+l_2+l_3-\frac{c_2}{2}-\frac{c_3}{2}-1} e^{-(s+t+u)} \\ & \quad \times K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) J_{c_1}(2\sqrt{\theta_1 st}) J_{c_2}(2\sqrt{\theta_2 us}) J_{c_3}(2\sqrt{\theta_3 ut}) dt ds du \end{aligned}$$

and

$$\begin{aligned}
 & H_B[(\sigma; p, \nu), b_1, b_2; c_1 + 1, c_2 + 1, c_3 + 1; \theta_1, \theta_2, \theta_3] \\
 &= \frac{\sqrt{\frac{2p}{\pi}} \Gamma(c_1 + 1) \Gamma(c_2 + 1) \Gamma(c_3 + 1)}{\Gamma(\sigma) \Gamma(b_1) \Gamma(b_2)} \theta_1^{-\frac{c_1}{2}} \theta_2^{-\frac{c_2}{2}} \theta_3^{-\frac{c_3}{2}} \\
 &\quad \times \int_0^\infty \int_0^\infty \int_0^\infty t^{\sigma+l_1+l_3-\frac{c_1}{2}-\frac{c_2}{2}-\frac{3}{2}} s^{b_1+l_1+l_2-\frac{c_1}{2}-\frac{c_2}{2}-1} u^{b_2+l_2+l_3-\frac{c_2}{2}-\frac{c_3}{2}-1} e^{-(s+t+u)} \\
 &\quad \times K_{\nu+\frac{1}{2}}\left(\frac{p}{t}\right) I_{c_1}(2\sqrt{\theta_1 st}) I_{c_2}(2\sqrt{\theta_2 us}) I_{c_3}(2\sqrt{\theta_3 ut}) dt ds du
 \end{aligned}$$

hold for H_B in (2.2) such that the integrals involved are convergent.

4 Derivatives and recurrence relation formulas

Certain derivative and recurrence relation formulas for the newly defined functions H_A , H_B and H_C in (2.1), (2.2) and (2.3) are derived in this section.

4.1 Derivative formulas

Differentiating both sides of (2.1), (2.2) and (2.3) partially with respect to the variables θ_1, θ_2 and θ_3 , l , m and n times, respectively, we get the derivative formulas for the functions H_A, H_B and H_C in (2.1), (2.2) and (2.3), which are derived in the following theorem.

Theorem 4.1. *The derivative formulas*

$$\begin{aligned}
 & \frac{\partial^{l_1+l_2+l_3}}{\partial \theta_1^{l_1} \partial \theta_2^{l_2} \partial \theta_3^{l_3}} H_A[(\sigma; p, \nu), b_1, b_2; c_1, c_2; \theta_1, \theta_2, \theta_3] \\
 &= \frac{(\sigma)_{l_1+l_3} (b_1)_{l_1+l_2} (b_2)_{l_2+l_3}}{(c_1)_{l_1} (c_2)_{l_2+l_3}} \\
 &\quad \times H_A[(\sigma + l_1 + l_3; p, \nu), b_1 + l_1 + l_2, b_2 + l_2 + l_3; c_1 + l, c_2 + l_2 + l_3; \theta_1, \theta_2, \theta_3], \\
 & \frac{\partial^{l_1+l_2+l_3}}{\partial \theta_1^{l_1} \partial \theta_2^{l_2} \partial \theta_3^{l_3}} H_B[(\sigma; p, \nu), b_1, b_2; c_1, c_2, c_3; \theta_1, \theta_2, \theta_3] \\
 &= \frac{(\sigma)_{l_1+l_3} (b_1)_{l_1+l_2} (b_2)_{l_2+l_3}}{(c_1)_{l_1} (c_2)_{l_2} (c_3)_{l_3}} \\
 &\quad \times H_B[(\sigma + l_1 + l_3; p, \nu), b_1 + l_1 + l_2, b_2 + l_2 + l_3; c_1 + l, c_2 + m, c_3 + n; \theta_1, \theta_2, \theta_3], \\
 & \frac{\partial^{l_1+l_2+l_3}}{\partial \theta_1^{l_1} \partial \theta_2^{l_2} \partial \theta_3^{l_3}} H_C[(\sigma; p, \nu), b_1, b_2; \gamma; \theta_1, \theta_2, \theta_3] \\
 &= \frac{(\sigma)_{l_1+l_3} (b_1)_{l_1+l_2} (b_2)_{l_2+l_3}}{(\gamma)_{l_1+l_2+l_3}} H_C[(\sigma + l_1 + l_3; p, \nu), b_1 + l_1 + l_2, b_2 + l_2 + l_3; \gamma + l_1 + l_2 + l_3; \theta_1, \theta_2, \theta_3]
 \end{aligned}$$

hold for H_A , H_B and H_C .

4.2 Recurrence relations

We derive recurrence relations for the functions H_A , H_B and H_C in (2.1), (2.2) and (2.3), which are derived in the following theorems.

Theorem 4.2. *The relation*

$$H_A[(\sigma; p, \nu), b_1, b_2; c_1, c_2; \theta_1, \theta_2, \theta_3] = H_A[(\sigma; p, \nu), b_1, b_2; c_1 - 1, c_2; \theta_1, \theta_2, \theta_3] \\ + \frac{\sigma b_1 \theta_1}{c_1(1 - c_1)} H_A[(\sigma + 1; p, \nu), b_1 + 1, b_2; c_1 + 1, c_2; \theta_1, \theta_2, \theta_3],$$

holds for the function H_A .

Proof. In the integral representation (3.1), we employ the following relation of the function ${}_0F_1$ (see [17]):

$${}_0F_1(-; \gamma - 1; x) - {}_0F_1(-; \gamma; x) - \frac{x}{\gamma(\gamma - 1)} {}_0F_1(-; \gamma + 1; x) = 0.$$

Hence, we get the desired result. $\square$

Theorem 4.3. *The relation*

$$H_B[(\sigma; p, \nu), b_1, b_2; c_1, c_2, c_3; \theta_1, \theta_2, \theta_3] \\ = H_B[(\sigma; p, \nu), b_1, b_2; c_1 - 1, c_2, c_3; \theta_1, \theta_2, \theta_3] \\ + \frac{\sigma b_1 \theta_1}{\gamma(1 - \gamma)} H_B[(\sigma + 1; p, \nu), b_1 + 1, b_2; c_1 + 1, c_2, c_3; \theta_1, \theta_2, \theta_3]$$

holds for the function H_B .

Proof. Applying the similar procedure used in Theorem 4.2 together with (3.2) will give the desired result. $\square$

Theorem 4.4. *The relation*

$$H_C[(\sigma; p, \nu), b_1, b_2; \gamma; \theta_1, \theta_2, \theta_3] = H_C[(\sigma; p, \nu), b_1, b_2; \gamma - 1; \theta_1, \theta_2, \theta_3] \\ + \frac{\sigma b_1 \theta_1}{\gamma(1 - \gamma)} H_C[(\sigma + 1; p, \nu), b_1 + 1, b_2; \gamma + 1, \gamma + 1; \theta_1, \theta_2, \theta_3] \\ + \frac{b_1 b_2 \theta_2}{\gamma(1 - \gamma)} H_C[(\sigma + 1; p, \nu), b_1 + 1, b_2 + 1; \gamma + 1, \gamma + 1; \theta_1, \theta_2, \theta_3] \\ + \frac{\sigma b_2 \theta_3}{\gamma(1 - \gamma)} H_C[(\sigma + 1; p, \nu), b_1, b_2 + 1; \gamma + 1, \gamma + 1; \theta_1, \theta_2, \theta_3]$$

holds for the function H_C .

Proof. Applying the similar procedure used in Theorem 4.2 together with (3.4) gives the desired result. $\square$

5 Concluding remarks

In this present paper, we investigated a new extension of the generalized Srivastava's triple hypergeometric functions H_A , H_B and H_C in (2.1), (2.2) and (2.3) by using the extended Pochhammer symbol $(\rho; p, \nu)_\mu$ recently defined by Srivastava et al. [29]. Also, we investigated certain special cases when $\theta_2 = 0$ in (2.1), (2.2) and (2.3) are respectively reduced to a new extension of the Appell functions F_2 and F_1 of two variables. The special cases of our results by setting $\nu = 0$ can be found in the works [18–22, 25, 26].

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