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Research Article

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Multiplicity results for elliptic Kirchhoff-type problems

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Abstract: The aim of this paper is to establish the existence of multiple solutions for a perturbed Kirchhoff-type problem depending on two real parameters. More precisely, we show that an appropriate oscillating behaviour of the nonlinear part, even under small perturbations, ensures the existence of at least three nontrivial weak solutions. Our approach combines variational methods with properties of nonlocal fractional operators.

Keywords: Critical point, positive solution, Kirchhoff-type problems

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1 Introduction

Consider the elliptic problem

$$\begin{cases} -\left(a + b\int\limits_{\Omega} |\nabla u|^2\,dx\right)\Delta u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$K_{\lambda,\mu}^{f,g}$}$$

where a, b are two real positive constants, $\Omega \subset \mathbb{R}^N$ (with $N \geq 3$) is a nonempty bounded open subset with boundary $\partial\Omega$ of class C^1, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions having a suitable growth, and $(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}_0^+$.

In this paper, we establish sufficient conditions on the nonlinearities f and g under which the above problem possesses three nontrivial solutions for suitable values of the parameters λ and μ. The main tool in order to achieve our multiplicity result is a critical point theorem contained in [4] along with the technical approach developed in [5]; see Theorem 2.1 below.

In our setting, the presence of the term $a + b\int_\Omega |\nabla u|^2\,dx$ implies that the equation under consideration is of nonlocal type and this fact gives rise to some mathematical difficulties. In addition, these kinds of problems have motivations from physics. For instance, the above operator appears in the Kirchhoff equation which arises in nonlinear vibrations, namely,

$$\begin{cases} u_{tt} - \left(a + b\int\limits_{\Omega} |\nabla u|^2\,dx\right)\Delta u = h(x,u) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \end{cases}$$

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Hence, problem $(K^{f,g}_{\lambda,\mu})$ is related to the stationary counterpart of the above evolution equation. Such a hyperbolic equation is a general version of the relation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

which was proposed by Kirchhoff in 1883 (see [11]) as an extension of the D'Alembert wave equation for free vibrations of elastic strings. The parameters in (1.1) have the following meaning: L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

Many solvability conditions for Kirchhoff-type problems are known, such as the Yang index theory and via the use of invariant sets of descent flow; see [17, 21]. Moreover, there have been several multiplicity results for Kirchhoff problems by using variational methods; see, for instance, [1, 7, 10, 13, 15, 16]. In particular, Mao and Zhang studied in [14, Theorem 1.2] the existence of multiple solutions for problem $(K^{f,0}_{1,0})$. More precisely, under suitable conditions on the nonlinearity f, combining critical point theory, the invariant set of descent flow, and minimax methods, they proved the existence of three solutions: one positive, one negative, and one sign-changing. Among others, one of their key assumptions is the condition

$$\lim_{|t|\to 0}\frac{f(x,t)}{|t|} = 0$$

uniformly for every $x \in \Omega$.

Furthermore, very recently, Ricceri established in [19] the existence of at least three weak solutions to a class of Kirchhoff-type doubly eigenvalue boundary value problem. We point out that in the paper of Ricceri no estimate of the parameters μ is given and the condition

$$\limsup_{\xi\to 0}\frac{\sup_{x\in\Omega}\int_0^\xi f(x,t)\,dt}{\xi^2} < +\infty$$

is required; see [19, Theorem 1].

For completeness, we also observe that, very recently, the nonlocal fractional counterpart of Kirchhoff-type problems has been considered. In this order of ideas, a similar variational approach to the one adopted here can be used for proving the existence of weak solutions for the (doubly) nonlocal problem

$$\begin{cases} M\left(\displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)(-\Delta)^s u = f(x,u) & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}^n\setminus\Omega, \end{cases} \tag{$D_{M,f}$}$$

where Ω is a bounded domain in $(\mathbb{R}^n, |\cdot|)$ with smooth boundary $\partial\Omega$, $s \in (0,1)$ is fixed with $s < n/2$, and $(-\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) can be defined as

$$(-\Delta)^s u(x) := -\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy, \quad x \in \mathbb{R}^n.$$

Furthermore, $f : \bar{\Omega}\times\mathbb{R} \to \mathbb{R}$ and $M : [0,+\infty) \to [0,+\infty)$ are suitable continuous maps.

We explicitly observe that Theorem 3.1 below and the above cited results are mutually independent; see Remark 3.4 and Example 3.5. Finally, we emphasize that our novel approach adopted in the present paper yields the existence of at least three nontrivial solutions for the unperturbed problem as well; see Remark 3.3.

The structure of the paper is as follows. In Section 2, we recall our abstract framework. Then, Section 3 is devoted to the main theorem and, finally, we give some consequences and applications of the presented result.

2 Abstract framework

Let us put, as usual, $2^* := 2N/(N-2)$. The space $H_0^1(\Omega)$ indicates the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{1,2}(\Omega)$ with respect to the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^2\,dx\right)^{1/2}.$$

From the Sobolev embedding theorem (see, for instance, [18, Proposition B.7]), for every $q \in [1, 2^*]$, there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \le c_q \|u\|, \quad u \in H_0^1(\Omega), \tag{2.1}$$

and, in particular, the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact for every $q \in [1, 2^*[$. It follows that

$$c_q \le \frac{\operatorname{meas}(\Omega)^{(2^*-q)/(2^*q)}}{\sqrt{N(N-2)\pi}}\left(\frac{N!}{2\Gamma(N/2+1)}\right)^{1/N}, \tag{2.2}$$

where $\operatorname{meas}(\Omega)$ denotes the Lebesgue measure of the set Ω; see, for instance, [20].

From now on, we assume that there exist four nonnegative constants a_1, a_2, b_1, b_2 and $q, p \in]1, 2^*[$ such that

$$|f(x,t)| \le a_1 + a_2|t|^{q-1} \quad \text{and} \quad |g(x,t)| \le b_1 + b_2|t|^{p-1}$$

for every $(x,t) \in \Omega \times \mathbb{R}$.

Moreover, let us fix $(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}_0^+$. A function $u : \Omega \to \mathbb{R}$ is said to be a *weak solution* of problem $(K_{\lambda,\mu}^{f,g})$ if $u \in H_0^1(\Omega)$ and

$$(a + b\|u\|^2)\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx - \lambda\int_\Omega f(x,u(x))v(x)\,dx - \mu\int_\Omega g(x,u(x))v(x)\,dx = 0$$

for every $v \in H_0^1(\Omega)$. They are also classical solutions if f is locally Lipschitz continuous in $\Omega \times \mathbb{R}$; see, for instance, [14].

From a variational standpoint, the weak solutions of $(K_{\lambda,\mu}^{f,g})$ in $H_0^1(\Omega)$ are exactly the critical points of the C^1-functional given by

$$J_{\lambda,\mu}(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda\int_\Omega\left(\int_0^{u(x)} f(x,t)\,dt\right)dx - \mu\int_\Omega\left(\int_0^{u(x)} g(x,t)\,dt\right)dx$$

for every $u \in H_0^1(\Omega)$. Moreover, its Gâteaux derivative is given by

$$J_\lambda'(u)(v) = (a + b\|u\|^2)\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx - \lambda\int_\Omega f(x,u(x))v(x)\,dx - \mu\int_\Omega g(x,u(x))v(x)\,dx$$

for every $u, v \in H_0^1(\Omega)$.

Finally, we recall the following critical point theorem, obtained in [4].

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable, and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$ with $r < \Phi(\bar{x})$ such that

(a_1) *there holds*

$$\frac{\sup_{\Phi(x)\le r}\Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})};$$

(a_2) *for each*

$$\lambda \in \Lambda_r := \left] \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r} \Psi(x)} \right[,$$

the functional $J_\lambda := \Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$, *the functional* J_λ *has at least three distinct critical points in* X.

In the next section, adopting the technical approach developed in [5] and using the above abstract framework, we obtain Theorem 3.1.

3 Existence of three weak solutions

Put

$$F(x,\xi) := \int_0^\xi f(x,t)\,dt \quad \text{and} \quad G(x,\xi) := \int_0^\xi g(x,t)\,dt$$

for every $(x,\xi) \in \Omega \times \mathbb{R}$. Moreover, set

$$K := \left(\frac{2^N - 1}{2^{N-1}}\right)^{1/2} \frac{\pi^{N/4} D^{(N-2)/2}}{\Gamma(1 + N/2)} \left[a\Gamma(1 + N/2) + b\pi^{N/2} D^{N-2}\left(\frac{2^N - 1}{2^{N-1}}\right)\right]^{1/2},$$

where $D := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$.

Simple calculations show that there exists $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$, where $B(x_0, D)$ denotes the open ball of center x_0 and radius D. Finally, for every $q \in [1, 2^*[$, let

$$A_q := \frac{c_q^q}{q}\left(\frac{2}{a}\right)^{q/2},$$

where c_q is given by (2.1).

Our result reads as follows.

Theorem 3.1. *Let* $N \geq 4$ *and let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function such that*

(f_1) *there exist two nonnegative constants* a_1, a_2 *and* $q \in]1, 2^*[$ *such that*

$$|f(x,t)| \leq a_1 + a_2|t|^{q-1}.$$

Assume that

(f_2) $F(x,\xi) \geq 0$ *for every* $(x,\xi) \in \Omega \times \mathbb{R}^+$;

(f_3) *there exist two positive constants* γ *and* δ *with* $\gamma < K\min\{1,\delta^2\}$ *such that*

$$\frac{A_1 a_1}{\gamma} + A_q a_2 \gamma^{q-2} < \frac{1}{K^2} \frac{\int_{B(x_0,D/2)} F(x,\delta)\,dx}{\max\{1,\delta^4\}}.$$

Then, for each parameter

$$\lambda \in \Lambda_{(\gamma,\delta)} := \left] K^2 \frac{\max\{1,\delta^4\}}{\int_{B(x_0,D/2)} F(x,\delta)\,dx}, \frac{1}{A_1 a_1/\gamma + A_q a_2 \gamma^{q-2}} \right[$$

and for every Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfying that*

(g_1) *there are two nonnegative constants* b_1, b_2 *and* $p \in]1, 2^*[$ *such that*

$$|g(x,t)| \leq b_1 + b_2|t|^{p-1};$$

(g_2) $G(x,\xi) \geq 0$ *for every* $(x,\xi) \in \Omega \times \mathbb{R}^+$,

there exists a positive constant

$$\delta_{\lambda,g} := \frac{1 - \lambda(A_1 a_1/\gamma + A_q a_2 \gamma^{q-2})}{A_1 b_1/\gamma + A_p b_2 \gamma^{p-2}}$$

such that, for each $\mu \in [0, \delta_{\lambda,g}[$, *problem* $(K^{f,g}_{\lambda,\mu})$ *has at least three weak solutions in* $H^1_0(\Omega)$.

Proof. Fix λ, g, and μ as in the theorem's statement and take $X = H_0^1(\Omega)$ endowed with the usual norm

$$\|u\| := \Big(\int_\Omega |\nabla u(x)|^2\,dx\Big)^{1/2}$$

for each $u \in X$. Moreover, set

$$\Phi(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4$$

and

$$\Psi(u) := \int_\Omega \Big(F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))\Big)\,dx$$

for every $u \in X$.

Since the critical points of the functional $J_\lambda := \Phi - \lambda\Psi$ on X are exactly the weak solutions of $(K^{f,g}_{\lambda,\mu})$, our aim is to apply Theorem 2.1 to Φ and Ψ. To this end, we take into account that the regularity assumptions of Theorem 2.1 on Φ and Ψ are satisfied; see, for instance, [19]. Hence, we will verify (a_1) and (a_2). Owing to (f_1) and (g_1), one has

$$F(x,\xi) \le a_1|\xi| + a_2\frac{|\xi|^q}{q} \quad \text{and} \quad G(x,\xi) \le b_1|\xi| + b_2\frac{|\xi|^p}{p}$$

for every $(x,\xi) \in \Omega \times \mathbb{R}$.

Let $\eta \in\ ]0,+\infty[$ and consider the function

$$\chi(\eta) := \frac{\sup_{u\in\Phi^{-1}(]-\infty,\eta])}\Psi(u)}{\eta}.$$

Taking into account the above inequalities for F and G, it follows that

$$\Psi(u) \le a_1\|u\|_{L^1(\Omega)} + \frac{a_2}{q}\|u\|^q_{L^q(\Omega)} + \frac{\mu}{\lambda}\Big(b_1\|u\|_{L^1(\Omega)} + \frac{b_2}{p}\|u\|^p_{L^p(\Omega)}\Big).$$

Then, for every $u \in X$ such that $\Phi(u) \le \eta$, due to (2.1), we get

$$\Psi(u) \le \Big(\sqrt{\frac{2\eta}{a}}c_1a_1 + \frac{2^{q/2}c_q^qa_2}{qa^{q/2}}\eta^{q/2}\Big) + \frac{\mu}{\lambda}\Big(\sqrt{\frac{2\eta}{a}}c_1b_1 + \frac{2^{p/2}c_p^pb_2}{pa^{p/2}}\eta^{p/2}\Big).$$

Hence,

$$\sup_{u\in\Phi^{-1}(]-\infty,\eta])}\Psi(u) \le \Big(\sqrt{\frac{2\eta}{a}}c_1a_1 + \frac{2^{q/2}c_q^qa_2}{qa^{q/2}}\eta^{q/2}\Big) + \frac{\mu}{\lambda}\Big(\sqrt{\frac{2\eta}{a}}c_1b_1 + \frac{2^{p/2}c_p^pb_2}{pa^{p/2}}\eta^{p/2}\Big). \tag{3.1}$$

From (3.1), one has

$$\chi(\eta) \le \Big(\sqrt{\frac{2}{a\eta}}c_1a_1 + \frac{2^{q/2}c_q^qa_2}{qa^{q/2}}\eta^{q/2-1}\Big) + \frac{\mu}{\lambda}\Big(\sqrt{\frac{2}{a\eta}}c_1b_1 + \frac{2^{p/2}c_p^pb_2}{pa^{p/2}}\eta^{p/2-1}\Big) \tag{3.2}$$

for every $\eta > 0$. Then, by using our notation, the previous inequality can be written as

$$\chi(\eta) \le \Big(\frac{A_1a_1}{\sqrt{\eta}} + A_qa_q\eta^{(q-2)/2}\Big) + \frac{\mu}{\lambda}\Big(\frac{A_1b_1}{\sqrt{\eta}} + A_pb_p\eta^{(p-2)/2}\Big)$$

for every $\eta > 0$.

Hence, let $u_\delta \in H_0^1(\Omega)$ given by

$$u_\delta(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0,D),\\ \frac{2\delta}{D}(D - |x-x_0|) & \text{if } x \in B(x_0,D)\setminus B(x_0,D/2),\\ \delta & \text{if } x \in B(x_0,D/2).\end{cases}$$

Moreover, $\Phi(u_\delta) > \gamma^2$, indeed

$$\begin{aligned}\Phi(u_\delta) &= \frac{a}{2}\|u_\delta\|^2 + \frac{b}{4}\|u_\delta\|^4 \\ &= \frac{a}{2}\int_\Omega |\nabla u_\delta(x)|^2\,dx + \frac{b}{4}\Big(\int_\Omega |\nabla u_\delta(x)|^2\,dx\Big)^2 \\ &= \frac{a}{2}\int_{B(x_0,D)\setminus B(x_0,D/2)} \frac{(2\delta)^2}{D^2}\,dx + \frac{b}{4}\Big(\int_{B(x_0,D)\setminus B(x_0,D/2)} \frac{(2\delta)^2}{D^2}\,dx\Big)^2 \\ &= \frac{a}{2}\frac{(2\delta)^2}{D^2}[\mathrm{meas}(B(x_0,D)) - \mathrm{meas}(B(x_0,D/2))] + \frac{b}{4}\frac{(2\delta)^4}{D^4}[\mathrm{meas}(B(x_0,D)) - \mathrm{meas}(B(x_0,D/2))]^2 \\ &= \frac{2\delta^2\pi^{N/2}(D^N - (D/2)^N)}{D^4(\Gamma(1+N/2))^2}\Big[aD^2\Gamma(1+N/2) + 2b\delta^2\pi^{N/2}\Big(D^N - \frac{D^N}{2^N}\Big)\Big].\end{aligned}$$

Therefore, one has

$$\min\{1,\delta^4\}K^2 \le \Phi(u_\delta) \le \max\{1,\delta^4\}K^2. \tag{3.3}$$

Now, since $\gamma < K\min\{1,\delta^2\}$, it follows that $\Phi(u_\delta) > \gamma^2$.

Then,

$$\chi(\gamma^2) = \frac{\sup_{u\in\Phi^{-1}(]-\infty,\gamma^2])}\Psi(u)}{\gamma^2} \le \Big(\frac{A_1a_1}{\gamma} + A_qa_2\gamma^{q-2}\Big) + \frac{\mu}{\lambda}\Big(\frac{A_1b_1}{\gamma} + A_pb_2\gamma^{p-2}\Big). \tag{3.4}$$

Moreover, since $\mu < \delta_{\lambda,g}$, then

$$\mu < \frac{1 - \lambda(A_1a_1/\gamma + A_qa_2\gamma^{q-2})}{A_1b_1/\gamma + A_pb_2\gamma^{p-2}},$$

which means that

$$\Big(\frac{A_1a_1}{\gamma} + A_qa_2\gamma^{q-2}\Big) + \frac{\mu}{\lambda}\Big(\frac{A_1b_1}{\gamma} + A_pb_2\gamma^{p-2}\Big) < \frac{1}{\lambda}. \tag{3.5}$$

On the other hand, the conditions (f_2) and (g_2) yield

$$\Psi(u_\delta) = \int_\Omega F(x,u_\delta(x))\,dx + \frac{\mu}{\lambda}\int_\Omega G(x,u_\delta(x))\,dx \ge \int_{B(x_0,D/2)} F(x,\delta)\,dx.$$

Hence, from (3.3) and bearing in mind the above inequality, it follows that

$$\frac{\Psi(u_\delta)}{\Phi(u_\delta)} \ge \frac{1}{K^2}\frac{\int_{B(x_0,D/2)} F(x,\delta)\,dx}{\max\{1,\delta^4\}}. \tag{3.6}$$

So, owing to the choice of the parameter λ, that is,

$$\frac{1}{\lambda} < \frac{1}{K^2}\frac{\int_{B(x_0,D/2)} F(x,\delta)\,dx}{\max\{1,\delta^4\}},$$

one has

$$\frac{\Psi(u_\delta)}{\Phi(u_\delta)} > \frac{1}{\lambda}. \tag{3.7}$$

Then, from (3.4), (3.5), and (3.7), it follows that the condition (a_1) of Theorem 2.1 holds. Now, observe that from conditions (f_1) and (g_1), bearing in mind that $N \ge 4$, it automatically follows that

$$\limsup_{|\xi|\to\infty}\frac{\sup_{x\in\Omega} G(x,\xi)}{\xi^4} \le 0 \quad\text{and}\quad \limsup_{|\xi|\to\infty}\frac{\sup_{x\in\Omega} F(x,\xi)}{\xi^4} \le 0.$$

Moreover, we can find $l > 0$ such that

$$\limsup_{|\xi|\to\infty}\frac{\sup_{x\in\Omega} G(x,\xi)}{\xi^4} < l \quad\text{and}\quad \mu l < \frac{b}{4c_4^4}.$$

Therefore, there exists a function $h_l \in L^1(\Omega)$ such that

$$G(x,\xi) \leq l\xi^4 + h_l(x)$$

for each $(x,\xi) \in \Omega \times \mathbb{R}$. Now, fix

$$0 < \varepsilon < \frac{b - 4\mu l c_4^4}{4\lambda c_4^4}.$$

Furthermore, there exists a function $h_\varepsilon \in L^1(\Omega)$ such that

$$F(x,\xi) \leq \varepsilon\xi^4 + h_\varepsilon(x)$$

for each $(x,\xi) \in \Omega \times \mathbb{R}$. Then, for each $u \in X$,

$$\Phi(u) - \lambda\Psi(u) \geq \left(\frac{b}{4} - \lambda\varepsilon c_4^4 - \mu l c_4^4\right)\|u\|^4 - (\lambda\|h_\varepsilon\|_{L^1(\Omega)} + \mu\|h_l\|_{L^1(\Omega)}).$$

This leads to the coercivity of J_λ and the condition (a_2) of Theorem 2.1 is verified. Hence, Theorem 2.1 assures the existence of three critical points for the functional J_λ and the proof is complete. □

Remark 3.2. With the usual notation, let us put

$$B_q := \frac{c_q^q}{q}\left(\frac{4}{b}\right)^{q/4},$$

where $q \in [1, 2^*[$. In Theorem 3.1, instead of condition (f_3), assume that
(f_3') there exist two positive constants γ and δ with $\gamma < K^{1/2}\min\{1,\delta\}$ such that

$$\frac{B_1 a_1}{\gamma^3} + B_q a_2 \gamma^{q-4} < \frac{1}{K^2}\frac{\int_{B(x_0,D/2)} F(x,\delta)\,dx}{\max\{1,\delta^4\}}.$$

In this setting, our result guarantees that, for each parameter

$$\lambda \in \Lambda'_{(\gamma,\delta)} := \left]K^2\frac{\max\{1,\delta^4\}}{\int_{B(x_0,D/2)} F(x,\delta)\,dx}, \frac{1}{B_1 a_1/\gamma^3 + B_q a_2 \gamma^{q-4}}\right[$$

and for every Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying (g_1) and (g_2), there exists a positive constant

$$\delta'_{\lambda,g} := \frac{1 - \lambda(B_1 a_1/\gamma^3 + B_q a_2 \gamma^{q-4})}{B_1 b_1/\gamma^3 + B_p b_2 \gamma^{p-4}}$$

such that, for each $\mu \in [0, \delta'_{\lambda,g}[$, problem $(K_{\lambda,\mu}^{f,g})$ has at least three weak solutions in $H_0^1(\Omega)$.

Remark 3.3. We explicitly observe that if $f(x,0) \neq 0$ in a set $\Omega_0 \subseteq \Omega$ with positive Lebesgue measure, then, for every Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying (g_1) and (g_2), Theorem 3.1 ensures the existence of at least three nontrivial weak solutions for problem $(K_{\lambda,\mu}^{f,g})$. The attained result is also new for the unperturbed case, that is, for $\mu = 0$.

Now, if f and g are locally Lipschitz continuous, our result guarantees the existence of at least three classical solutions for problem $(K_{\lambda,\mu}^{f,g})$. Moreover, if in addition f and g are nonnegative functions, the strong maximum principle (see [9, Theorem 8.19]) guarantees that the (nontrivial) solutions are positive.

Indeed, let u_0 be a solution of $(K_{\lambda,\mu}^{f,g})$. Arguing by contradiction, assume that the set

$$A = \{x \in \Omega : u_0(x) < 0\}$$

is of positive measure. Put $\bar{v}(x) = \min\{0, u_0(x)\}$ for all $x \in \Omega$. Clearly, $\bar{v} \in H_0^1(\Omega)$ and one has

$$(a + b\|u_0\|^2)\int_\Omega \nabla u_0(x)\cdot\nabla\bar{v}(x)\,dx = \lambda\int_\Omega f(u_0(x))\bar{v}(x)\,dx + \mu\int_\Omega g(u_0(x))\bar{v}(x)\,dx,$$

that is,

$$(a+b\|u_0\|^2)\int_A |\nabla u_0(x)|^2\,dx = \lambda\int_A f(x,u_0(x))u_0(x)\,dx + \mu\int_A g(x,u_0(x))u_0(x)\,dx.$$

Therefore, it follows that

$$0 \le (a+b\|u_0\|^2)\int_A |\nabla u_0(x)|^2\,dx \le 0.$$

Hence, $u_0 = 0$ in A and this is absurd. Then, u_0 is nonnegative in Ω. From the strong maximum principle one has that either $u_0 \equiv 0$ or $u_0 > 0$ in Ω.

Remark 3.4. After a careful analysis of the proof of Theorem 3.1, one can see that it also works for $N = 3$, just requiring that the conditions (f_1) and (g_1) hold true for some constants $q, p \in]1,4[$. Moreover, for the sake of simplicity, let us consider the case $\mu = 0$ and take $f : \mathbb{R} \to \mathbb{R}$ to be a continuous function, nonnegative in $]0,+\infty[$, such that

$$|f(t)| \le a_2|t|^{q-1}$$

for all $t \in \mathbb{R}$ and for some $q \in]2,4[$.

Clearly, the above growth condition is a particular case of condition (f_1) and implies that $f(0) = 0$. In this setting, Theorem 3.1 ensures the existence of at least three (two nontrivial) solutions for every λ sufficiently large. In [19, Theorem 2], Ricceri obtained an analogous existence result for $N \ge 4$ and $q \in]1,2^*[$. In that paper, it was proposed to investigate if Theorem 2 also holds for $N = 3$. Anello, in [2], gave a negative answer to this question for every $q \in]4,6[$. We just point out that our approach produces a positive answer to the proposed problem whenever $q \in]2,4[$. Finally, as already pointed out in Section 1, [19, Theorem 1] and its consequences deal with elliptic Kirchhoff problems for which the assumption

$$\limsup_{\xi\to 0}\frac{\sup_{x\in\Omega}F(x,\xi)}{\xi^2} < +\infty$$

is required.

The cited result and Theorem 3.1 are mutually independent as the following example shows.

Example 3.5. Let $N = 3$ and assume that $K > 1$. Furthermore, fix $q \in]4,6[$ and consider the continuous and positive function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) := \begin{cases} 1+|t|^{q-1} & \text{if } t \le \delta,\\ \dfrac{(1+r^2)(1+r^{q-1})}{1+t^2} & \text{if } t > \delta,\end{cases}$$

where δ is a fixed constant such that

$$\delta > \max\left\{1, \left(\frac{qK^2}{\operatorname{meas}(B(x_0,D/2))}(A_1+A_q)\right)^{1/(q-4)}\right\}.$$

Clearly, the function h satisfies the conditions (f_1) and (f_2) of Theorem 3.1. Moreover, (f_3) is also verified taking into account the choice of δ. Since $\delta^2 > 1$, one has $1 < K\min\{1,\delta^2\}$. Furthermore, setting

$$H(\xi) := \int_0^\xi h(t)\,dt$$

for every $\xi \in \mathbb{R}$, it follows that

$$\frac{H(\delta)}{\delta^4} = \frac{\delta^{q-4}}{q} + \frac{1}{\delta^3} > \frac{K^2}{\operatorname{meas}(B(x_0,D/2))}(A_1+A_q).$$

Then, for each parameter λ belonging to

$$\left]\frac{K^2\delta^4}{\operatorname{meas}(B(x_0,D/2))H(\delta)}, \frac{1}{A_1+A_q}\right[$$

and for every μ such that

$$0 \leq \mu < \frac{1 - \lambda(A_1 + A_q)}{2A_1},$$

the perturbed problem

$$\begin{cases} -\left(a + b \int\limits_{\Omega} |\nabla u|^2 \, dx\right)\Delta u = \lambda h(u) + \mu(|\sin u| + 1) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{$K^h_{\lambda,\mu}$}$$

has at least three (positive) classical solutions in $H^1_0(\Omega)$.

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