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Well/ill-posedness for the dissipative Navier–Stokes system in generalized Carleson measure spaces

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Abstract: As an essential extension of the well known case $\beta \in (\frac{1}{2}, 1]$ to the hyper-dissipative case $\beta \in (1, \infty)$, this paper establishes both well-posedness and ill-posedness (not only norm inflation but also indistinguishability of the solution map) for the mild solutions of the incompressible Navier–Stokes system with dissipation $(-\Delta)^{\frac{\beta}{2}} \prec \infty$ through the generalized Carleson measure spaces of initial data that unify many diverse spaces, including the Q space $(Q_{s, \infty})^n$, the BMO-Sobolev space $((-\Delta)^{-\frac{s}{2}}BMO)^n$, the Lip-Sobolev space $((-\Delta)^{-\frac{s}{2}}Lip)^n$, and the Besov space $(\dot{B}_s^{\infty, \infty})^n$.

Keywords: Incompressible Navier–Stokes system with dissipation, well/ill-posedness for mild solutions, generalized Carleson measure spaces

MSC 2010: 30H25, 31C15, 35Q30, 42B37, 46E35

1 Introduction

Essentially, continuing from [47], we study the mild solutions (fluid velocities) of the so-called incompressible Navier–Stokes system with dissipation $(-\Delta)^{\beta}$, under the assumption $(\beta, x, t, \tau) \in (0, \infty) \times \mathbb{R}^n \times (0, \tau) \times (0, \infty]$ (cf. e.g., [8, 17, 28, 35, 39, 45, 52–54]), given by

\begin{align}
\begin{cases}
(\partial_t u + (-\Delta)^{\beta} u - u \cdot \nabla u - \nabla p)(x, t) = 0, \\
\nabla \cdot u(x, t) = 0, \\
\n\nabla \cdot u_0(x) = 0,
\end{cases}
\end{align}

with $p$ being the pressure of a fluid, i.e., the solutions $u$ that satisfy the following integral system arising from the initial data $u_0$:

\begin{align}
\begin{aligned}
u(x, t) &= e^{-t(-\Delta)^{\beta}} u_0(x) - \int_0^t e^{-(t-s)(-\Delta)^{\beta}} \mathcal{P} \nabla \cdot (u(x, s) \otimes u(x, s)) \, ds,
\end{aligned}
\end{align}

where $\mathcal{P} = \{\mathcal{P}_{jk}\}_{j,k=1,2,3} = \{\delta_{jk} + R_{jk}\}_{j,k=1,2,...,n}$, $\delta_{jk}$ is the Kronecker symbol, and $R_{jk} = \partial_j (-\Delta)^{-\frac{1}{2}}$ is the Riesz transform. In accordance with [38], the model (1.1) physically illustrates that the viscous stresses produce a dispersive momentum flow, which is determined by Darcy’s law of fractional order, while the dispersive flux divergence is the same as the change of momentum under Newton’s law and the assumption of fluid incom-
pressibility. Furthermore, as explained in [45], in order to overcome some obstacles coming from numerical simulations of turbulent fluids induced by system (1.1) with $\beta = 1$, we are suggested to handle system (1.1) with $\beta > 1$, through replacing $\Delta$ (responsible for dissipating energy from the system) with a higher order dissipation mechanism $(-\Delta)^{\beta - 1}$ (damping selectively the high wave numbers). Interestingly, upon taking the curl of the first equation of (1.1), setting $w = \nabla \wedge u$, and using the computation on [40, p. 25], we find that the first equation in (1.1) can be rewritten as the following heat-type equation:

$$\partial_t w + (-\Delta)^{\beta} w = (w \cdot \nabla)u - (u \cdot \nabla)w.$$  

Remarkably, the homogeneous form of the last system $\partial_t w + (-\Delta)^{\beta} w = 0$ (modelling anomalous diffusions) and its quasi-geophysical variant are of fundamental importance and interest in physics, probability and finance; see, e.g., [1, 4, 12, 13, 18, 23, 24, 29, 37, 46].

Here it is appropriate to mention three basic facts which reveal that the restriction $1/2 < \beta < \infty$ cannot be extended (to the challenging unsolved situation $0 < \beta < 1/2$) at least for our current casework regarding (1.1)–(1.2).

- System (1.1) with $\beta = 1$ goes back to the classical incompressible Navier–Stokes system, see [7, 36] for more details.
- System (1.1) has a scaling property. If $(u, p, u_0)$ solves (1.1), then so does

$$(u_\lambda(x, t) = \lambda^{2\beta - 1} u(\lambda x, \lambda^2 t), p_\lambda(x, t) = \lambda^{2\beta - 2} p(\lambda x, \lambda^2 t), (u_\lambda(0), \lambda) = \lambda^{2\beta - 1} u_0(\lambda)) \quad \text{for all } \lambda > 0.$$  

- System (1.1) is more meaningful in a critical space which is invariant under the scaling

$$f_\lambda(x) = \lambda^{2\beta - 1} f(\lambda x) \quad \text{for all } \lambda > 0. \quad (1.3)$$

In fact, the solutions of (1.2) with $\beta = 1$ in certain critical spaces have drawn a lot of attention since the pioneer work of Kato in [26], where he showed the global well-posedness with small data and the local well-posedness with large data in $(L^n)^n$ (cf. [20] for an earlier work). Some similar well-posedness results can be found in [22, 33, 44] for certain Morrey spaces, in [30] for the space $(\text{BMO}^{-1})^n$, and in [49] for the space $(Q_n^\alpha)^n$. Moreover, Li and Lin [31] showed global well-posedness in a subspace of $(\text{BMO}^{-1})^n$ with large initial data, and Bourgain and Pavlović [3] found the norm inflation in $(B_{1,\infty,\infty}^{-1/n})^n$, which is the largest critical space with respect to (1.3) with $\beta = 1$.

For $1/2 < \beta \neq 1$, a study of (1.2) has been carried out partially. Wu [48] got a well-posed result for (1.1) with $1 < \beta < 1/2$ in the space $(B_{1, \infty, \infty}^{1 - 2\beta})^n$. Li and Zhai [35] considered the fractional Navier–Stokes equation (1.1) with $1/2 < \beta < 1$, whence extending the above-mentioned well-posedness to $Q$-type spaces. Yu and Zhai [52] obtained a similar result in the largest critical space $(B_{1, \infty, \infty}^{-1})^n$. Cheskidov and Shvydkoy [11] discovered an ill-posed result in the largest critical space $(B_{1, \infty, \infty}^{1 - 2\beta})^n$ under assumption (1.3). Deng and Yao [15, 16] obtained a similar ill-posedness in certain Triebel–Lizorkin spaces, providing a connection between the well-posedness in [30] and the ill-posedness in [3]. Li, Xiao and Yang [34] found a global well-posedness in some Besov-Q type spaces. Cheskidov and Dai [8] revealed a norm inflation phenomenon in the largest critical space $(B_{1, \infty, \infty}^{1 - 2\beta})^n$, with respect to (1.3) with $1 \leq \beta < \infty$.

In this paper, partially motivated by [8, 25, 30, 35, 49, 50], under the natural constraint

$$1 < 2\beta < \infty \quad \text{and} \quad 1 - 2\beta < a < \infty,$$

we develop a uniform framework to deal with a dichotomy of the well/ill-posed results in the generalized Carleson measure spaces $(X^n_{\beta})^n$, which are critical with respect to (1.1) and, of course, contained in the homogeneous space $(B_{1, \infty, \infty}^{1 - 2\beta})^n$. In the above and below,

$$X^n_{\beta} = X^n_{\beta, \infty},$$

and for $0 < \tau \leq \infty$, the space $X^n_{\beta, \tau}$ is defined by the norm

$$\|f\|_{X^n_{\beta, \tau}} = \sup_{(x, t) \in \mathbb{R}^n \times (0, \tau)} \left( \int_0^\tau \left( \int_{B(x, r)} \left| e^{-t(-\Delta)^{\beta}} f(y) \right|^2 \, dy \right)^{\frac{1}{2}} r^{\frac{\beta}{\alpha} - 1} \, dt \right)^{\frac{1}{2}},$$
where $B(x, r)$ is the ball centered at $x$ with radius $r$. Meanwhile,

$$B^{1-2\beta}_{0,\infty, r} = B^{0,\infty}_{0,\infty, r},$$

and for $0 < r \leq \infty$, the space $B_{0,\infty, r}^{1-2\beta}$ is determined by the norm (cf. [36])

$$\|f\|_{B^{1-2\beta}_{0,\infty, r}} = \sup_{(x,t) \in \mathbb{R}^n \times (0,r)} t^{2\beta} |e^{-(\Delta)^{\beta}} f(x)|.$$

Clearly, $\| \cdot \|_{X^{s}_{\beta,0}}$ and $\| \cdot \|_{B^{1-2\beta}_{0,\infty, r}}$ are invariant under the scaling transform (1.3). Moreover,

$$\alpha \leq \alpha' \implies X^{\alpha}_{\beta,0} = X^{\alpha}_{\beta,0} \subseteq X^{\alpha'}_{\beta,0} = X^{\alpha'}_{\beta,0} \subseteq B^{1-2\beta}_{0,\infty,0} = B^{1-2\beta}_{0,\infty,0},$$

whose second inclusion becomes equality whenever $\alpha' > 0$. Accordingly,

$$Y^{\alpha}_{\beta} = Y^{\alpha}_{\beta,0},$$

and for $\tau \in (0, \infty)$, the associated solution space $Y^{\alpha}_{\beta,\tau}$ is decided by the norm

$$\|u\|_{Y^{\alpha}_{\beta,\tau}} = \sup_{(x,t) \in \mathbb{R}^n \times (0,\tau)} \left( (\int_{\mathbb{R}^n} |u(y,t)|^2 dy)^{\frac{1-\beta}{\beta}} + \int_{(x,t) \in \mathbb{R}^n \times (0,\tau)} |u(x,t)|^2 dt \right)^{\frac{1}{2}}.$$

The first theorem of this paper indicates that the well-posedness of (1.2) occurs only when $\alpha$ is relatively small.

**Theorem 1.1.** Suppose

\[
\begin{align*}
\beta &\in \left( \frac{1}{2}, 1 \right) \cup \left( 1 + \frac{n}{2}, \infty \right), \\
1 - 2\beta &< \alpha \leq 1 - \beta, \\
0 < \tau &\leq \infty,
\end{align*}
\]

or

\[
\begin{align*}
\beta &\in \left( \frac{1}{2}, \infty \right), \\
1 - 2\beta &< \alpha \leq 2 - 2\beta, \\
0 < \tau &\leq \infty.
\end{align*}
\]

Then (1.2) is well-posed in $(X^{\alpha}_{\beta,\tau})^n$ with sufficiently small norm

$$\|u_0 = (u_0) \|_{(X^{\alpha}_{\beta,\tau})^n} = \sum_{j=1}^{n} \|u_0 j\|_{X^{\alpha}_{\beta,\tau}}.$$ 

Furthermore, the solution $u = Y^{\alpha}_{\beta,\tau}$, and the solution map $T: u_0 \rightarrow u$ is analytic from a sufficient small neighborhood of origin of $(X^{\alpha}_{\beta,\tau})^n$ to $(Y^{\alpha}_{\beta,\tau})^n$.

Theorem 1.1 is essentially known for $\frac{1}{2} < \beta \leq 1$ and $\alpha \in (1 - 2\beta, 0)$, see [30, 35, 49, 50, 52] and the relevant references therein. Needless to say that for the hyper-dissipative case $1 < \beta < \infty$, Theorem 1.1 is new. In order to prove Theorem 1.1, we follow the method originated from [30] (which was developed in [32, 35, 49, 50]), but we have to find a new idea to treat the singularity, appearing in $(\Delta)^{1-\beta}$, on the integrability of the kernel of

$$(-\Delta)^{1-\beta} e^{-(\Delta)^{\beta}}$$

for $\frac{1}{2} < \beta \leq 1$.

To meet the case $1 < \beta < \infty$. However, when $\beta \in (1, 1 + \frac{n}{2})$, the singularity occurs both at the origin and at infinity, and so prevents us from getting the full range of $\alpha$, see Lemma 2.1 for more details. Here, it should be pointed out that the well-posedness is understood under Kato’s sense as in [26, 30, 32, 33, 35, 49, 50], i.e., both existence and uniqueness of a mild solution to (1.1) in the resolved space $(Y^{\alpha}_{\beta})^n$ are obtained by the standard fixed point theorem, which automatically ensures the analytic property of the solution map as stated above.

**Remark 1.2.** Remarkably, the restriction $1 - 2\beta < \alpha$ in Theorem 1.1 is natural – this can be seen from the following assertion (cf. [33, 45, 50] for $\beta = 1$). If $L_{2,n+2\alpha}$ (cf. [44]) stands for all real-valued $L^{2}_{loc}(\mathbb{R}^n)$-functions $f$ obeying

$$\|f\|_{L_{2,n+2\alpha}} = \sup_{(x,t) \in \mathbb{R}^n \times (0,\infty)} \left( (\int_{\mathbb{R}^n} |f(y)|^2 dy)^{\frac{1}{2}} < \infty, \right.$$
then
\[ L_{2,n+2(1-2\beta)} \subseteq X^\alpha_{\beta} \quad \text{when} \quad 1 - 2\beta < \alpha, \]
and hence (1.2) is well-posed in \((L_{2,n+2(1-2\beta)})^n\) with sufficiently small norm
\[
\|u_0\| = \|(u_0)\|_{(L_{2,n+2(1-2\beta)})^n} = \sum_{j=1}^n \|u_0\|_{L_{2,n+2(1-2\beta)}}.
\]

**Corollary 1.3.** If \(1/2 < \beta < 1\) and \(-1 < \alpha < \beta\), then (1.2) is well-posed in \((X^\alpha_{\beta})^n\) with sufficiently small norm
\[
\|u_0\| = \|(u_0)\|_{(X^\alpha_{\beta})^n} = \sum_{j=1}^n \|u_0\|_{X^\alpha_{\beta}}.
\]
Furthermore, the solution \(u \in (Y^\alpha_{\beta})^n\), and the solution map \(T : u_0 \rightarrow u\) is analytic from a sufficient small neighborhood of the origin of \((X^\alpha_{\beta})^n\) to \((Y^\alpha_{\beta})^n\).

Note that
\[
\beta \in \left(\frac{1}{2}, 1\right) \implies X^\alpha_{\beta} = \begin{cases} 
Q^{\beta-1}_{1-a-\beta,0} & \text{if} \; \alpha \in (-1, 0), \\
B^{1-2\beta}_{1-2\beta} & \text{if} \; \alpha = 0, \\
B^{1-2\beta}_{0,0} & \text{if} \; \alpha \in (0, \frac{1}{2}).
\end{cases}
\]

Thus, Corollary 1.3 extends and unifies partial well-posedness results in [35, 52, 53].

Upon taking into account \(1 - \beta < \alpha < \infty\), the second theorem of this paper is concerned with the ill-posedness of (1.2), illustrating that Theorem 1.1 is optimal under certain circumstance.

**Theorem 1.4.** Suppose
\[
1 \leq \beta < \infty \quad \text{and} \quad 1 - \beta < \alpha < \infty.
\]
Then there exist a smooth space periodic solution \(u(t)\) of (1.2) with period \(2\pi\), and initial data \(u_0\) such that the solution map \(T\) from \((X^\alpha_{\beta,1})^n\) to \((Y^\alpha_{\beta})^n\) is not differentiable at the origin of \((X^\alpha_{\beta,1})^n\). Furthermore, for sufficiently small \(\epsilon \in (0, 1)\), there exists a smooth space periodic solution \(u(t)\) of (1.2), with period \(2\pi\), such that
\[
\|u(0)\|_{(X^\alpha_{\beta,1})^n} \leq \epsilon \quad \text{and} \quad \|u(T)\|_{(X^\alpha_{\beta,1})^n} \geq \epsilon^{-1} \quad \text{for some} \; T \in (0, \epsilon).
\]
Additionally, the same assertion holds for \((X^\alpha_{\beta})^n\) and \((Y^\alpha_{\beta})^n\), provided that \(0 \leq \alpha < \infty\).

In order to verify Theorem 1.4, we suitably employ the counter-example constructed in [3, 8] to get such a smooth space-periodic mild solution (with an arbitrarily small initial data in \((X^\alpha_{\beta})^n\)) that becomes not only arbitrarily large in \((X^\alpha_{\beta})^n\) for an arbitrarily small time, but also relatively large in the resolution space \((Y^\alpha_{\beta})^n\).

Perhaps, it is appropriate to make two more comments on Theorems 1.1 and 1.4, and Corollary 1.3 as follows.

As described in Theorems 1.1 and 1.4, the well-posedness and the ill-posedness of (1.2) initialed in \((X^\alpha_{\beta})^n\) can be summarized in Figure 1. The well-posedness is set up for all parameter \((\alpha, \beta)\) in the region between the polyline \(\overline{ABC}\) and polyline \(\overline{DEF}\) but \(\Delta PQB\), while the ill-posed results are established for \((\alpha, \beta)\) above polyline \(\overline{DEF}\). It is most likely that system (1.2) is well-posed when \((\alpha, \beta)\) in the triangle \(\Delta PQB\) – unfortunately, we have failed to show this possible well-posedness because of Lemma 2.1(ii) (cf. Remark 2.2). It seems that a new method, such as the one in [2], is required to fill this unnatural gap.

As a direct consequence of Theorems 1.1 and 1.4, and Corollary 1.3 (whose argument ensures that \(X^0_{\beta,1} = \text{BMO}^{1-2\beta}\) and \(X^0_{\beta,1} = \overline{B^{1-2\beta}_{0,0}}\) with \(0 < \alpha < \frac{1}{2}\), we assert that (1.2) is:

- well-posed when \(\beta \in \left(\frac{1}{2}, 1\right]\), while ill-posed when \(\beta > 1\) initialed in \((\text{BMO}^{1-2\beta})^n\),
- well-posed when \(\beta \in \left(\frac{1}{2}, 1\right]\), while ill-posed when \(\beta \geq 1\) initialed in \((\overline{B^{1-2\beta}_{0,0}})^n\).

Although the well-posedness of this last assertion for \(\beta = 1\) and the ill-posedness for \(\beta = 1\) or \(\beta > 1\) reduce to the well-posedness in [30] for \(\beta = 1\) and the ill-posedness in [3, 8] (see, e.g., [9, 10, 14, 51] for more details) for \(\beta = 1\) or \(\beta > 1\), respectively, our ill-posedness in Theorem 1.4 cannot be implied by the results in [3, 8] at least because our space \(X^\alpha_{\beta}\) with \(\beta > 1 \geq 1 - \alpha\) behaves differently from their space \(B^{-\gamma}_{\gamma,0}\) with \((y, p) \in [1, \infty) \times (2, \infty)\), and yet includes non-differentiability of the solution map as an extra property.
If Corollary 1.5.

Interestingly, we have Table 1. Even more interestingly, we discover

Figure 1: Here $P = (-n, \frac{\alpha+2}{2})$ and $Q = (-\frac{n+2}{2}, \frac{\alpha+2}{2})$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\alpha+\beta=1$</th>
<th>Well-posed</th>
<th>Ill-posed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1 &lt; \alpha &lt; 1$</td>
<td>$\alpha = (-1, 0)$</td>
<td>$\alpha = 0$</td>
<td>$\alpha \in (0, 1)$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Spaces.

The preceding theorems can be straightforwardly applied to (1.2) initiated in the Campanato–Sobolev (CS) spaces explored in [47]:

$$L_{2,n+2a}^s = (-\Delta)^{-\frac{s}{2}} L_{2,n+2a}$$

for $-1 < \alpha < 1$ and $-\infty < s < \infty$,

where $(-\Delta)^{-\frac{s}{2}}$ is determined by the Fourier transforms

$$\hat{f}(\xi) = \int e^{-i\xi \cdot x} f(x) \, dx$$

and $(-\Delta)^{-\frac{s}{2}} f(\xi) = |\xi|^{-s} \hat{f}(\xi)$,

and $L_{2,n+2a}$ denotes the square Campanato space (cf. [5, 6, 41]) on $\mathbb{R}^n$ of all real-valued $L^2_{loc}(\mathbb{R}^n)$-functions $f$ satisfying

$$\|f\|_{L_{2,n+2a}} = \sup_{(x,r) \in \mathbb{R}^n \times (0, \infty)} \left( r^{-2a-n} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy \right)^{\frac{1}{2}} < \infty$$

and

$$\|f\|_{L_{2,n+2a}} = \|(-\Delta)^{\frac{s}{2}} f\|_{L_{2,n+2a}}$$

for all $f \in L_{2,n+2a}^s$.

Interestingly, we have Table 1. Even more interestingly, we discover

$$1 < \alpha < 1 \quad \text{and} \quad \frac{1}{2} < \beta < \frac{3}{2} \implies X^a_{\beta} = L_{2,n+2a}^{1-a-2\beta}.$$

This fundamental identification, along with Theorems 1.1 and 1.4, produces the following assertion of relatively independent interest.

**Corollary 1.5.** Suppose $\frac{1}{2} < \beta < \frac{3}{2}$ and $\max(-1, 1 - 2\beta) < \alpha < 1$.

(i) If $\frac{1}{2} < \beta < 1$, then (1.2) is well posed in $(L_{2,n+2a}^{1-a-2\beta})^n$ with sufficiently small norm

$$\|u_0\|_{(L_{2,n+2a}^{1-a-2\beta})^n} = \sum_{j=1}^{n} \|u_{0j}\|_{L_{2,n+2a}^{1-a-2\beta}}.$$

(ii) If $1 < \beta < \frac{3}{2}$, then (1.2) in $(L_{2,n+2a}^{1-a-2\beta})^n$, with sufficiently small norm $\|u_0\|_{(L_{2,n+2a}^{1-a-2\beta})^n}$, is well posed when $1 < \alpha < 2 - 2\beta$ but ill-posed, in the sense of Theorem 1.4, when $1 - \beta < \alpha < 1$.

The rest of the paper is organized as follows. In Section 2, we give an exposition of the details of the proofs of Theorem 1.1 and Remark 1.2. Section 3 provides a complete demonstration of Theorem 1.4. In Section 4, we check Corollary 1.5, using Theorems 1.1 and 1.4.
Notation. From now on, $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$. The symbol $A \lesssim B$ represents that there exists a positive constant $C$ satisfying $A \leq CB$, and thus $A \approx B$ represents the comparability of the quantities $A$ and $B$, i.e., $A \leq B$ and $B \leq A$.

2 Well-posedness in $(X^0_\beta)^n$

This section is devoted to a proof of Theorem 1.1 with $\tau = \infty$. The argument for Theorem 1.1 with $\tau < \infty$ is similar.

2.1 Estimation for some singular integrals

We need two technical results on some integrals of strong singularity.

Lemma 2.1. Let $s \in (0, 1)$ and $K_\beta^s(x)$ be the kernel of $(-\Delta)^{1-\beta}(e^{-(\Delta)^{\beta}} - e^{-s(\Delta)^{\beta}})$.

(i) If $\beta \in (0, 1)$, then

$$|K_\beta^s(x)| \leq (1 + |x|)^{-n-2+2\beta} + s\beta^{2n+2\beta} (1 + s^{1/\beta}|x|)^{-n-2+2\beta}. \quad (2.1)$$

(ii) If $\beta \in (1, 1 + \frac{n}{2})$, then

$$|K_\beta^s(x)| \leq (1 + |x|)^{-n-1} + s\beta^{2n+2\beta} (1 + s^{1/\beta}|x|)^{-n-2+2\beta}. \quad (2.2)$$

(iii) If $\beta = 1$ or $\beta \in (1 + \frac{n}{2}, \infty)$, then

$$|K_\beta^s(x)| \leq (1 + |x|)^{-n-1} + s\beta^{2n+2\beta} (1 + s^{1/\beta}|x|)^{-n-1}. \quad (2.3)$$

Proof. (i) Suppose $\beta \in (0, 1)$, thus $1 - \beta > 0$. The kernel $K(x)$ of $(-\Delta)^{1-\beta}e^{-(\Delta)^{\beta}}$ has the decay estimate

$$|K(x)| \leq (1 + |x|)^{\beta-1},$$

see [39]. Then (2.1) follows by a scaling argument.

(ii) Assume $1 < \beta < 1 + \frac{n}{2}$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\psi(\xi) = 1$ for $|\xi| < 1$, and denote by $m$ the symbol of the operator $(-\Delta)^{1-\beta}(e^{-(\Delta)^{\beta}} - e^{-s(\Delta)^{\beta}})$. Then, this symbol can be broken down into two terms:

$$m(\xi) = |\xi|^{2-2\beta}(e^{-|\xi|^{2\beta}} - e^{-s|\xi|^{2\beta}})\psi(s^{1/\beta} \xi) + |\xi|^{2-2\beta}(e^{-s|\xi|^{2\beta}} - e^{-|\xi|^{2\beta}})(1 - \psi(s^{1/\beta} \xi)) \equiv m_1(\xi) + m_2(\xi).$$

The first term $m_1$ is rewritten as

$$m_1(\xi) = |\xi|^{2-2\beta}(e^{-|\xi|^{2\beta}} - 1)\psi(s^{1/\beta} \xi) + |\xi|^{2-2\beta}(1 - e^{-s|\xi|^{2\beta}})\psi(s^{1/\beta} \xi) \equiv m_{11}(\xi) + m_{12}(\xi).$$

For $m_{12}$, by scaling, we only need to show

$$\left| \int_{\mathbb{R}^n} e^{ik\cdot\xi}|\xi|^{2-2\beta}(e^{-|\xi|^{2\beta}} - 1)\psi(\xi) \, d\xi \right| \leq (1 + |x|)^{-n-1},$$

which is obvious since the symbol $|\xi|^{2-2\beta}(e^{-|\xi|^{2\beta}} - 1)\psi(\xi)$ is compactly supported and has no singularity at the origin (cf. [39]). Note that the kernel of $m_{11}$ can be controlled similarly if $s > \frac{1}{\lambda}$. So, without loss of generality, we may assume $s \ll 1$ in the sequel. Write

$$m_{11}(\xi) = m_{111}(\xi) + m_{112}(\xi) + m_{113}(\xi),$$

where

$$m_{111}(\xi) \equiv |\xi|^{2-2\beta}(e^{-|\xi|^{2\beta}} - 1)\psi(\xi),$$

$$m_{112}(\xi) \equiv |\xi|^{2-2\beta}(1 - \psi(\xi))\psi(s^{1/\beta} \xi),$$

$$m_{111}(\xi) \equiv |\xi|^{2-2\beta}(1 - \psi(\xi))\psi(s^{1/\beta} \xi).$$
In view of the previous argument, only the kernel of the last term, denoted by $K_{113}$, needs a control. By a simple calculation, we get that
\[ \|m_{113}\|_{L^1} \lesssim s^{-\frac{n+2-2\beta}{2}}, \]
and so, if the multi-index $a$ satisfies $|a| = n + 1$, then
\[ \|\partial_x^a m_{113}\|_{L^1} \lesssim 1. \]
Thus, an integration by parts derives that the kernel $K_{113}$ of $m_{113}$ enjoys
\[ |K_{113}(x)| \lesssim \min\{s^{-\frac{n+2-2\beta}{2}}, |x|^{-n-1}\}. \]
In order to prove (2.2), an improvement must be made when $s^\frac{1}{n} \leq |x| \leq 1$. Now let $\delta \in (2, s^\frac{n-2}{2})$. Then
\[ K_{113}(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^{2-2\beta}(1 - \psi(\xi))\psi(s^\frac{1}{n}\xi)\,d\xi \]
\[ = \int_{\frac{1}{2} < |\xi| \leq \delta} e^{ix\cdot\xi}|\xi|^{2-2\beta}(1 - \psi(\xi))\,d\xi + \int_{\delta < |\xi| \leq s^\frac{1}{n}} e^{ix\cdot\xi}|\xi|^{2-2\beta}\psi(s^\frac{1}{n}\xi)\,d\xi \]
\[ \equiv A(\delta) + B(\delta). \]
It is easy to see that
\[ |A(\delta)| \lesssim \int_{\frac{1}{2} < |\xi| \leq \delta} |\xi|^{2-2\beta} \,d\xi \lesssim \delta^{n+2-2\beta} \quad \text{if } \beta < 1 + \frac{n}{2}. \]
Repeatedly using integration by parts, we obtain
\[ |B(\delta)| \lesssim |x|^{-n} \int_{\delta < |\xi| \leq s^\frac{1}{n}} |\xi|^{2-2\beta-n} \,d\xi \lesssim \delta^{2-2\beta}|x|^{-n}, \]
reaching the desired estimate (2.2) upon choosing $\delta = \frac{1}{|M|}$.

(iii) For $\beta = 1$, estimate (2.3) is obvious. So, it remains to treat $\beta > 1 + \frac{n}{2}$. In view of the argument in (ii), it is enough to handle $K_{113}$. Since
\[ |K_{113}(x)| \lesssim \int_{\mathbb{R}^n} |\xi|^{2-2\beta}(1 - \psi(\xi))\psi(s^\frac{1}{n}\xi)\,d\xi \lesssim \int_{|\xi| = \frac{1}{2}} |\xi|^{2-2\beta} \,d\xi \lesssim 1, \]
an integration by parts gives (as estimated in (ii))
\[ |K_{113}(x)| \lesssim |x|^{-n-1} \sum_{|a|=n+1} \|\partial_x^a m_{113}\|_{L^1} \lesssim |x|^{-n-1}, \]
and the desired result (2.3) follows.

\[ \square \]

**Remark 2.2.** It turns out that Lemma 2.1 (ii) is not sufficient for our purpose, since the decay in the second term of the right-hand side of (2.2) is not strong enough in small scale $|x| \leq 1$. This is the main reason why our well-posed results fail to cover the case $\beta \in (1, 1 + \frac{n}{2})$ and $2 - 2\beta < \alpha \leq 1 - \beta$ (the triangle $\Delta PQB$ in Figure 1). Note that $K^\beta_x(x)$ can be rewritten as $M(x) + E(x)$, where $E(x)$ is well-behaved as an error term, and $M(x)$ behaves like
\[ \int_{1 < |\xi| \leq s^\frac{1}{n}} e^{ix\cdot\xi}|\xi|^{2-2\beta} \,d\xi. \]
So, in view of the identity (for a dimensional constant $c_n$)
\[ \int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^{2-2\beta} \,d\xi = c_n|x|^{2\beta-n-2} \quad \text{for all } \beta \in (1, 1 + \frac{n}{2}), \]
it seems that (2.2) is the best expected decay in small scale as $s$ tends to zero.

As one of our new-discovered tools, Lemma 2.1 will be used to prove the following lemma.
Lemma 2.3. (i) If $\beta > \frac{1}{2}$, $1 - 2\beta < \alpha \leq 1 - \beta$ and
\[
C_{\beta}(f, t, x) = \int_0^t e^{-(t-s) (-\Delta)^{\beta}} (-\Delta)^{\beta} f(s, x) \, ds,
\]
then
\[
\int_0^\infty \|C_{\beta}(f, t, \cdot)\|_{L^2_x}^2 t^{1-\alpha/\beta} \, dt \leq \int_0^\infty \|f(t, \cdot)\|_{L^2_x}^2 t^{1-\alpha/\beta} \, dt.
\]

(ii) If $\beta > \frac{1}{2} + \frac{\alpha}{2}$ or $\beta = 1, 1 - 2\beta < \alpha \leq 1 - \beta$ and
\[
D_{\alpha, \beta}(g) = \sup_{(x, r) \in \mathbb{R}^n \times (0, 1)} r^{-(n+2\alpha)} \int_0^r |g(h, y)| \, dy \, dh,
\]
then
\[
\int_0^1 \left\| (-\Delta)^{\frac{1}{2}} e^{-\frac{t}{2} (-\Delta)^{\beta}} g(s, \cdot) \right\|_{L^2_x}^2 \, ds \leq D_{\alpha, \beta}(g) \int_0^1 \|g(s, \cdot)\|_{L^1_x} \, ds \, t^{\frac{1-\alpha}{\beta}} \cdot (2.4)
\]

(iii) If $\beta > \frac{1}{2}$ and $1 - 2\beta < \alpha \leq 2 - 2\beta$, then (2.4) still holds.

Proof. (i) Suppose $\beta > \frac{1}{2}$ and $1 - 2\beta < \alpha \leq 1 - \beta$. An application of the definition of $e^{-(t-s) (-\Delta)^{\beta}}$, Plancherel’s formula and Hölder’s inequality gives
\[
\int_0^\infty \|C_{\beta}(f, t, \cdot)\|_{L^2_x}^2 t^{1-\alpha/\beta} \, dt \leq \int_0^\infty \|\xi\|_{L^2_x}^2 e^{-(t-s) \xi^2 \int f(s, \xi)} \, ds \leq (2.4)
\]
as desired.

(ii) Suppose $\beta > \frac{1}{2} + \frac{\alpha}{2}$ or $\beta = 1$ and $1 - 2\beta < \alpha \leq 1 - \beta$. Using the inner-product $\langle \cdot, \cdot \rangle_{L^2}$ in $L^2$ with respect to the spatial variable $x \in \mathbb{R}^n$, we obtain
\[
\int_0^1 \left\| (-\Delta)^{\frac{1}{2}} e^{-\frac{t}{2} (-\Delta)^{\beta}} g(s, \cdot) \right\|_{L^2_x}^2 \, ds \leq \int_0^1 \left\langle (-\Delta)^{\frac{1}{2}} e^{-\frac{t}{2} (-\Delta)^{\beta}} g(s, \cdot), (-\Delta)^{\frac{1}{2}} e^{-\frac{t}{2} (-\Delta)^{\beta}} g(s, \cdot) \right\rangle_{L^2_x} \, ds \, t^{1-\alpha/\beta} \, dt
\]
\[
= 2R \int_0^1 \left\langle (-\Delta)^{\frac{1}{2}} e^{-\frac{t}{2} (-\Delta)^{\beta}} g(s, \cdot), \int_0^1 (-\Delta)^{\frac{1}{2}} e^{-\frac{t}{2} (-\Delta)^{\beta}} g(s, \cdot) \, ds \, dh \right\rangle_{L^2_x} \, ds \, dh.
\]
\[
\leq \int_0^1 \left\langle (\Delta)^{1-\beta} e^{-\frac{t}{2} (-\Delta)^{\beta}} g(s, \cdot), \int_0^1 (-\Delta)^{\frac{1}{2}} e^{-\frac{t}{2} (-\Delta)^{\beta}} g(s, \cdot) \, ds \, dh \right\rangle_{L^2_x} \, ds \, dh.
\]
If $K_\beta^g(x)$ is the kernel of $(-\Delta)^{1-\beta}(e^{-(\Delta)^\beta} - e^{-s(-\Delta)^\beta})$, then an application of Lemma 2.1 and Hölder’s inequality derives

$$\left| \int_0^s (-\Delta)^{1-\beta}(e^{-(\Delta)^\beta} - e^{-h(-\Delta)^\beta})g(h, x) \, dh \right| \leq \sup_{0 < s \leq 1} s^{2m-1-\beta} \int_0^s \int \frac{|g(h, y)| \, dy \, dh}{(1 + h^{-\frac{2}{n}}|x - y|)^{n+1}}$$

$$\leq \sup_{0 < s \leq 1} \sum_{k \in \mathbb{Z}^n} s^{2m-1-\beta} \int_0^s \int \frac{|g(h, y)| \, dy \, dh}{(1 + e^{-\frac{2}{n} |x - y|})^{n+1}}$$

$$\leq \sup_{t \in (0, 1), k \in \mathbb{Z}^n} \int_0^t \int \frac{|g(h, y)| \, dy \, dh}{(1 + e^{-\frac{2}{n} |x - y|})^{n+1}}$$

$$\leq \sup_{(x, t) \in \mathbb{R}^n \times (0, 1)} t^{-(2m+1)} \int_0^t |g(h, y)| \, dy \, dh \cdot \frac{1}{h^{\frac{2}{n} + \frac{1}{\beta}}}.$$}

This, along with another application of Hölder’s inequality, implies

$$\left\| (-\Delta)^{\frac{1}{2}} e^{-t(-\Delta)^\beta} \right\|_{L^2}^2 \left\| g(s, \cdot) \right\|_{L^2}^2 \leq \left\| g(s, \cdot) \right\|_{L^2} \left( \sup_{0 < s \leq 1} s^{1+\beta} \right)^{\frac{1}{2}}$$

(iii) Suppose $\beta > \frac{1}{2}$ and $1 - 2\beta < \alpha \leq 2(1 - \beta)$. In view of the argument used in (ii), we obtain

$$\left\| (-\Delta)^{\frac{1}{2}} e^{-\frac{t}{s}(-\Delta)^\beta} \right\|_{L^2}^2 \left\| g(s, \cdot) \right\|_{L^2}^2 \leq \left( \int_0^s \int \left\langle g(s, \cdot), \left( (-\Delta)^{\frac{1}{2}} e^{-\frac{t}{s}(-\Delta)^\beta} g(h, \cdot) \right) \frac{a(s, t)}{s} \right\rangle \, ds \, dh \right)$$

$$= 2\beta \left( \int_0^s \int \left\langle g(s, \cdot), \left( (-\Delta)^{\frac{1}{2}} e^{-\frac{t}{s}(-\Delta)^\beta} \frac{a(s, t)}{s} \right) g(h, \cdot) \right\rangle \, ds \, dh \right)$$

$$\leq \left\| g(s, \cdot), \left( \frac{a(s, t)}{s} \right) \right\|_{L^2}^2 \left\| (-\Delta)^{\frac{1}{2}} e^{-\frac{t}{s}(-\Delta)^\beta} \right\|_{L^2}^2 \left\| g(h, \cdot) \right\|_{L^2}^2 \left\| \frac{a(s, t)}{s} \right\|_{L^2}^2 \, ds \, dh.$$}

Denote by $\tilde{K}$ and $\tilde{m}$ the kernel and symbol of the differential operator

$$\int_{s^\frac{1}{\beta}}^1 (-\Delta)^{\frac{1}{2}} e^{-\frac{t}{s}(-\Delta)^\beta} \, dt.$$}

In view of the argument used in (ii), it suffices to prove

$$|\tilde{K}(x)| \leq (1 + |x|)^{-n-1} + s^{-\frac{2}{n}} (1 + s^{-\frac{1}{\beta}} |x|)^{-n-1}.$$ (2.5)

By a change of variables, we have

$$\tilde{m}(\xi) = \int_{s^\frac{1}{\beta}}^1 |\xi|^2 e^{-\frac{t}{s}(|\xi|^2)} \, dt = \int_{s^\frac{1}{\beta}|\xi|^2}^{|\xi|^2} e^{-t^\beta} \, dt$$

$$= F(s^\frac{1}{\beta} |\xi|^2) - F(|\xi|^2),$$

where

$$F(r) \equiv \int_r^\infty e^{-t^\beta} \, dt \quad \text{for all } r > 0.$$
It is clear that $F(|\xi|^2) \in L^1$ and $\partial^j_x F(|\xi|^2) \in L^1$ with $|j| = n + 1$. So, an integration by parts shows
\[
\left| \int_{\mathbb{R}^n} e^{i\xi \cdot x} F(|\xi|^2) \, d\xi \right| \leq (1 + |x|)^{-n+1}.
\]
Thus, (2.5) follows by a scaling argument thanks to $F(s^{\frac{1}{2}}|\xi|^2) = F(|\xi|^2)$.

\[\square\]

2.2 Proof of Theorem 1.1

The proof follows the idea originated from [30], see also [32, Chapter 16]. We rewrite (1.2) (cf. [22, 26, 27, 32, 44]) as
\[
\mathbf{u}(x, t) = e^{-t(-\Delta)^{\beta}} \mathbf{u}_0(x) - B(\mathbf{u}, \mathbf{u}),
\]
where $B(\cdot, \cdot)$ is the following bilinear form:
\[
B(\mathbf{u}, \mathbf{v}) = \int_0^t e^{-t(s-s')(-\Delta)^{\beta}} \mathcal{P} \nabla \cdot (\mathbf{u} \otimes \mathbf{v}) \, ds.
\]

Let $\alpha, \beta$ satisfy the conditions in Theorem 1.1. According to the standard fixed point argument, it suffices to prove that the integral equation (2.6) is solvable in a small neighborhood of the origin in $X^\beta_p$. Thanks to the definition, we have
\[
\|e^{-t(-\Delta)^{\beta}} \mathbf{u}_0\|_{Y^\alpha_p^n} \leq \|\mathbf{u}_0\|_{X^\alpha_p^n},
\]
thus it remains to verify that (2.7) is bounded from $(Y^\alpha_p^n \times Y^\alpha_p^n)$ to $(Y^\alpha_p^n \times Y^\alpha_p^n)$. Of course, it suffices to show both the $L^2$-bound
\[
\int_0^{r^{\beta}(2a+n)} \int_{|y-x|<r} |B(\mathbf{u}, \mathbf{v})|^2 \, dy \frac{dt}{t^{\frac{1}{2} + \beta}} \leq \|\mathbf{u}\|^2_{Y^\alpha_p^n} \|\mathbf{v}\|^2_{Y^\alpha_p^n},
\]
and the $L^\infty$-bound
\[
\|B(\mathbf{u}, \mathbf{v})\|_{L^\infty} \leq t^{\frac{1}{2} - 1} \|\mathbf{u}\|_{Y^\alpha_p^n} \|\mathbf{v}\|_{Y^\alpha_p^n},
\]
where
\[
\mathbf{u} = (u_1, u_2, \ldots, u_n), \quad \|\mathbf{u}\|_{Y^\alpha_p^n} = \sum_{j=1}^n \|u_j\|_{Y^\alpha_p},
\]
\[
\mathbf{v} = (v_1, v_2, \ldots, v_n), \quad \|\mathbf{v}\|_{Y^\alpha_p^n} = \sum_{j=1}^n \|v_j\|_{Y^\alpha_p}.
\]

Step 1: $L^2$-bound. Letting $1_{r, x}(y) = \chi_{B(x, 10r)}(y)$ be the characteristic function of $B(x, 10r)$ and $I$ the identity map, we divide $B(\mathbf{u}, \mathbf{v})$ into three parts:
\[
B(\mathbf{u}, \mathbf{v}) = B_1(\mathbf{u}, \mathbf{v}) + B_2(\mathbf{u}, \mathbf{v}) + B_3(\mathbf{u}, \mathbf{v}),
\]
where
\[
B_1(\mathbf{u}, \mathbf{v}) = (-\Delta)^{-\frac{1}{2}} \mathcal{P} \nabla \cdot \int_0^s e^{(-s-h)(-\Delta)^{\beta}} (-\Delta)^{-\frac{1}{2}} (1 - e^{-h(-\Delta)^{\beta}})(1_{r, x} \mathbf{u} \otimes \mathbf{v}) \, dh,
\]
\[
B_2(\mathbf{u}, \mathbf{v}) = (-\Delta)^{-\frac{1}{2}} \mathcal{P} \nabla \cdot (-\Delta)^{-\frac{1}{2}} e^{-s(-\Delta)^{\beta}} \int_0^s (1_{r, x} \mathbf{u} \otimes \mathbf{v}) \, dh,
\]
\[
B_3(\mathbf{u}, \mathbf{v}) = \int_0^s e^{(-s-h)(-\Delta)^{\beta}} \mathcal{P} \nabla \cdot ((1 - 1_{r, x}) \mathbf{u} \otimes \mathbf{v}) \, dh.
\]
For $B_1(u, v)$, we use the boundedness of the Riesz transform and Lemma 2.3 (i) to derive
\[
\int_0^{r^B} \left\| B_1(u, v) \right\|_{L^2}^2 \frac{dt}{t^{1-\frac{\beta}{p}}} \leq \int_0^{r^B} \left\| e^{-(s-h)(-\Delta)^\beta} (-\Delta)^{\frac{1}{2}}(I - e^{-t(-\Delta)^\beta})(1_{r^B}u \otimes v) \right\|_{L^2}^2 \frac{dt}{t^{1-\frac{\beta}{p}}}
\]
\[
\leq \int_0^{r^B} \left\| (-\Delta)^{\frac{1}{2}-\beta}(I - e^{-t(-\Delta)^\beta})(1_{r^B}u \otimes v) \right\|_{L^2}^2 \frac{dt}{t^{1-\frac{\beta}{p}}}.
\]
Notice that $(-\Delta)^{\frac{1}{2}-\beta}(I - e^{-t(-\Delta)^\beta})$ is bounded on $L^2$, provided $\frac{1}{2} < \beta < \infty$, with its operator norm $\leq t^{1-\frac{\beta}{2}}$. Thus, using the Cauchy–Schwarz inequality, we have
\[
\int_0^{r^B} \left\| B_1(u, v) \right\|_{L^2}^2 \frac{dt}{t^{1-\frac{\beta}{p}}} \leq \int_0^{r^B} t^{2-\frac{\beta}{2}} \left\| 1_{r^B}u \otimes v \right\|_{L^2}^2 \frac{dt}{t^{1-\frac{\beta}{p}}}
\]
\[
\leq \int_0^{r^B} t^{2-\frac{\beta}{2}} \int_{|y-x|<r} |u(y, t)|^2 |v(y, t)|^2 dy \frac{dt}{t^{1-\frac{\beta}{p}}}
\]
\[
\leq \left( \sup_{t \in (0, T)} \int_{|y-x|<r} |u(y, t)| \right) \left( \sup_{t \in (0, T)} \int_{|y-x|<r} |v(y, t)| \right)
\]
\[
\times \left( \int_0^{r^B} \int_{|y-x|<r} |u(y, t)|^2 dy \frac{dt}{t^{1-\frac{\beta}{p}}} \right)^{\frac{1}{2}} \left( \int_0^{r^B} \int_{|y-x|<r} |v(y, t)|^2 dy \frac{dt}{t^{1-\frac{\beta}{p}}} \right)^{\frac{1}{2}}.
\]
In view of the definition of $Y^\alpha_\beta$, we conclude
\[
\int_0^{r^B} \left\| B_1(u, v) \right\|_{L^2}^2 \frac{dt}{t^{1-\frac{\beta}{p}}} \leq r^{n+2\alpha} \left\| u \right\|_{Y^\alpha_\beta}^2 \left\| v \right\|_{Y^\alpha_\beta}^2. \tag{2.10}
\]

For $B_2(u, v)$, by the boundedness of the Riesz transform and Lemma 2.3 (ii), we have
\[
\int_0^{r^B} \left\| B_2(u, v) \right\|_{L^2}^2 \frac{dt}{t^{1-\frac{\beta}{p}}} \leq \int_0^{r^B} \left\| (-\Delta)^{\frac{1}{2}} e^{-t(-\Delta)^\beta} (1_{r^B}u \otimes v) \right\|_{L^2}^2 \frac{dt}{t^{1-\frac{\beta}{p}}}
\]
\[
\leq D_{\alpha, \beta}(1_{r^B}u \otimes v) \int_0^{r^B} \int_{\mathbb{R}^n} |1_{r^B}u \otimes v(x, s)| \frac{dx ds}{s^{\frac{1-\frac{\beta}{2}}{p}}}
\]
On the one hand, employing Hölder’s inequality to derive
\[
D_{\alpha, \beta}(1_{r^B}u \otimes v) \leq \sup_{(x,r) \in \mathbb{R}^n \times (0, r^B)} \int_0^{r^B} 1_{r^B}|u \otimes v(x, s)| \frac{dx ds}{s^{\frac{1-\frac{\beta}{2}}{p}}}
\]
\[
\leq \left\| u \right\|_{Y^\alpha_\beta} \left\| v \right\|_{Y^\alpha_\beta}.
\]
On the other hand, similarly have
\[
\int_0^{r^B} \int_{\mathbb{R}^n} |1_{r^B}u \otimes v(x, s)| \frac{dx ds}{s^{\frac{1-\frac{\beta}{2}}{p}}}
\]
\[
\leq \int_0^{r^B} \left\| u \right\|_{Y^\alpha_\beta}^2 \left\| v \right\|_{Y^\alpha_\beta}^2. \tag{2.11}
\]
For $B_3(u, v)$, by the decay property of the kernel of $e^{-(t-s)\Delta}Pv$ we get that if $|x-y| < r$ and $s < r^{2\beta}$, then

$$|B_3(u, v)| \leq \int_0^s e^{-(s-h)\Delta}Pv \cdot ((1 - 1_{r,x})u \otimes v) \, dh$$

\begin{align*}
&\leq \int_0^s \int_0^{r^{1/\beta}} \frac{|u(h, z)||v(h, z)|}{(s-h)^{\frac{n}{\beta}} + |z-y|^{n+1}} \, dz \, dh \\
&\leq \sum_{j=3}^{\infty} (2^j r)^{-n-1} \int_0^{r^{1/\beta}} \int_0^{r^{1/\beta}} |u(h, z)||v(h, z)| \, dz \, dh \\
&\leq \sum_{j=3}^{\infty} (2^j r)^{-n-1} r^{2-2a-2\beta} \int_0^{r^{1/\beta}} \left( \int_0^{r^{1/\beta}} |u(h, z)||v(h, z)| \, dz \right) h^{\frac{s-1}{r}} \, dh \\
&\leq r^{1-2\beta-n-2a} \int_0^{r^{1/\beta}} \left( \int_0^{r^{1/\beta}} |u(h, z)||v(h, z)| \, dz \right) h^{\frac{s-1}{r}} \, dh.
\end{align*}

Then, by Hölder’s inequality, we get

$$|B_3(u, v)| \leq r^{1-2\beta} \|u\|_{Y^{2,\beta}_{p,n}} \|v\|_{Y^{2,\beta}_{p,n}}.$$

Since $\alpha > 1 - 2\beta$, we have

$$\int_0^{r^{1/\beta}} \int_0^{r^{1/\beta}} |B_3(u, v)|^2 \, dy \, dt \leq r^{n+2-6\beta} \int_0^{r^{1/\beta}} dt \int_0^{r^{1/\beta}} \|u\|_{Y^{2,\beta}_{p,n}}^2 \|v\|_{Y^{2,\beta}_{p,n}}^2 \leq r^{n-2a} \|u\|_{Y^{2,\beta}_{p,n}}^2 \|v\|_{Y^{2,\beta}_{p,n}}^2. \quad (2.12)$$

Putting the estimates (2.10), (2.11) and (2.12) together, we reach (2.8).

**Step 2: $L^\infty$-bound.** Two situations are handled in the sequel.

If $\frac{1}{2} \leq s < t$, then

$$\|e^{-(t-s)\Delta}Pv \cdot (u \otimes v)\|_{L^\infty} \leq (t-s)^{-\frac{n}{\beta}} \|u\|_{L^\infty} \|v\|_{L^\infty} \leq (t-s)^{-\frac{n}{\beta}} s^{\frac{1}{2} - 2} \|u\|_{Y^{2,\beta}_{p,n}} \|v\|_{Y^{2,\beta}_{p,n}},$$

and hence, for $\beta > \frac{1}{2}$, we have

$$\int_0^{t-s} e^{-(t-s)\Delta}Pv \cdot (u \otimes v) \, ds \leq \int_0^{t-s} (t-s)^{\frac{1}{2} - 2} ds \|u\|_{Y^{2,\beta}_{p,n}} \|v\|_{Y^{2,\beta}_{p,n}} \leq t^{1-2\beta} \|u\|_{Y^{2,\beta}_{p,n}} \|v\|_{Y^{2,\beta}_{p,n}}. \quad (2.13)$$

If $0 < s < \frac{1}{2}$, then $t - s \approx t$, and hence

$$|e^{-(t-s)\Delta}Pv \cdot (u \otimes v)| \leq \int_{\mathbb{R}^n} \frac{|u(y, s)||v(y, s)|}{((t-s)^{\frac{1}{\beta}} + |y-x|)^{n+1}} \, dy$$

\begin{align*}
&\leq \int_{\mathbb{R}^n} \frac{|u(y, s)||v(y, s)|}{(t^{\frac{1}{\beta}} + |y-x|)^{n+1}} \, dy \\
&\leq t^{-\frac{n+1}{\beta}} \int_{|x-y| < 10r^{\frac{1}{\beta}}} |u(y, s)||v(y, s)| \, dy + \int_{|x-y| > 10r^{\frac{1}{\beta}}} \frac{|u(y, s)||v(y, s)|}{|y-x|^{n+1}} \, dy.
\end{align*}

Using the same calculation as in $B_3(u, v)$ with $r = t^{\frac{1}{\beta}}$, we obtain

$$\int_0^t \int_0^{\frac{1}{2}} \int_{|x-y| > 10r^{\frac{1}{\beta}}} \frac{|u(y, s)||v(y, s)|}{|x-y|^{n+1}} \, dy \, ds \leq t^{1-2\beta} \|u\|_{Y^{2,\beta}_{p,n}} \|v\|_{Y^{2,\beta}_{p,n}}.$$
Meanwhile, utilizing Hölder’s inequality, we derive
\[
\int_0^1 \left( \int_{|x-y| \leq 10^{1/3}} |u(y,s)||v(y,s)| \, dy \right) \, ds \leq t^{-\frac{2\beta-1}{\beta}} \int_0^1 \left( \int_{B(0,10^{1/3})} |u(z,s)||v(z,s)| \, dz \right) \, ds.
\]
Consequently,
\[
\int_0^1 e^{-(t-s)(-\Delta)^\beta} \mathbf{D} \cdot (u \otimes v) \, ds \leq t^{-\frac{1-2\beta}{\beta}} \|u\|_{(Y^\beta_p)^n} \|v\|_{(Y^\beta_p)^n}.
\]
Now, putting estimates (2.13) and (2.14) together yields the $L^\infty$-bound (2.9).

### 2.3 Proof of Remark 1.2

The argument is divided into two steps.

**Step 1.** Noting the following Minkowski-inequality-based estimates:
\[
\sup_{t \geq 0} t^{\frac{2\beta-1}{\beta}} \|e^{-t(-\Delta)^\beta} f\|_{L^\infty} \leq \|f\|_{L^2_{t,n+1-2\beta}}, \quad \sup_{t \geq 0} \|e^{-t(-\Delta)^\beta} f\|_{L^2_{t,n+1-2\beta}} \leq \|f\|_{L^2_{t,n+1-2\beta}},
\]
we get that if $\alpha > 1 - 2\beta$ and $(x, r) \in \mathbb{R}^n_+$, then
\[
r^{-2\alpha n} \left( \int_{B(x,r)} |e^{-t(-\Delta)^\beta} f(y)|^2 \, dy \right) t^{-1+\frac{\beta}{2}} \, dt \leq \left( \int_{L^2_{t,n+1-2\beta}} \|e^{-t(-\Delta)^\beta} f\|_{L^2_{t,n+1-2\beta}}^2 \, dt \right) t^{1-\frac{\beta}{2}} \leq \|f\|_{L^2_{t,n+1-2\beta}}^2,
\]
whence deriving
\[
L_{t,n+1-2\beta} \lesssim \mathbb{X}^\alpha_{\beta} \quad \text{when} \quad 1 - 2\beta < \alpha.
\]

**Step 2.** The desired well-posedness may be viewed as an extension of Kato’s $L^p$-theory, developed in [22, 26, 27, 33, 44, 45, 50], to (1.2). In order to deal with a mild solution of (1.1) initialized in $(L_{t,n+1-2\beta})^n$, we are required to control the boundedness of the initial data semi-group
\[
\mathbf{u}_0 \mapsto e^{t(-\Delta)^\beta} \mathbf{u}_0
\]
and the bilinear operator
\[
(\mathbf{u}, \mathbf{v}) \mapsto B(\mathbf{u}, \mathbf{v}) = \int_0^t e^{-(t-s)(-\Delta)^\beta} \mathbf{D} \cdot (\mathbf{u} \otimes \mathbf{v}) \, ds,
\]
acting on a suitable solution space. To see this, let us use the foregoing Minkowski-inequality-based estimates and the following Morrey norm:
\[
\|g\|_{L_{t,n+1-2\beta}} = \sup_{(x,r) \in \mathbb{R}^n_+} \left( \int_{B(x,r)} |g(y)|^6 \, dy \right)^{\frac{1}{6}},
\]
to derive
\[
\|e^{-t(-\Delta)^\beta} f\|_{L_{t,n+1-2\beta}} \leq \|e^{-t(-\Delta)^\beta} f\|_{L^2_{t,n+1-2\beta}} \|e^{-t(-\Delta)^\beta} f\|_{L^\infty} \leq t^{-\frac{2\beta-1}{\beta}} \|f\|_{L^2_{t,n+1-2\beta}},
\]
whence defining the solution space $(\mathbb{X}^\alpha_{\beta})^n$ of all vector-valued functions $\mathbf{u} = \{u_j\}_{j=1}^n$ with the norm
\[
\|\mathbf{u}\|_{(\mathbb{X}^\alpha_{\beta})^n} = \sum_{j=1}^n \|u_j\|_{(\mathbb{X}^\alpha_{\beta})^n} = \left( \sup_{t > 0} t^{\frac{2\beta-1}{\beta}} \|u_j(\cdot, t)\|_{L^2_{t,n+1-2\beta}} + \sup_{t > 0} t^{\frac{2\beta-1}{\beta}} \|u_j(\cdot, t)\|_{L^\infty} \right).
\]
On the one hand, for the initial data $u_0$ in (1.1), we have
\[ e^{-t\Delta^\beta}u_0 \in (X_\beta^a)^n \quad \text{with} \quad \|e^{-t\Delta^\beta}u_0\|_{(X_\beta^a)^n} \leq \|u_0\|_{(L^{2,1+\lambda\beta})^n}. \]

On the other hand, for the corresponding bilinear part, a direct computation as in [33] shows that if $t > s$, then
\[
\|e^{-(t-s)(-\Delta)^{\beta/2}}P\nabla \cdot (u \otimes v)\|_{(L^{2,1+\lambda\beta})^n} \leq \left( (t-s)^{\frac{1}{\beta}} s^{\frac{3(2\beta-1)}{2\beta}} \right)^{-1} \|u\|_{(L^{2,2+\lambda\beta})^n} S^{\frac{2\beta-1}{\beta}} \|v\|_{(L^\infty)^n}
\]
and
\[
(t-s)^{\frac{1}{\beta}} \|e^{-(t-s)(-\Delta)^{\beta/2}}P\nabla \cdot (u \otimes v)\|_{(L^\infty)^n}
\leq \min \left\{ \frac{S^{\frac{2\beta-1}{\beta}} \|u\|_{(L^{2,2+\lambda\beta})^n} S^{\frac{2\beta-1}{\beta}} \|v\|_{(L^\infty)^n}}{t-s} \right\}
\leq \min \left\{ (t-s)^{\frac{1}{\beta}} s^{\frac{3(2\beta-1)}{2\beta}}, s^{\frac{2\beta-1}{\beta}} \|u\|_{(X_\beta)^n} \|v\|_{(X_\beta)^n} \right\},
\]
and hence
\[
\left\| \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}}P\nabla \cdot (u \otimes v) \, ds \right\|_{(X_\beta^a)^n} \leq \|u\|_{(X_\beta)^n} \|v\|_{(X_\beta)^n}.
\]

This, along with the standard fixed-point argument, as in [33], completes the proof.

### 2.4 Proof of Corollary 1.3

In accordance with Theorem 1.1 and the well-posedness of (1.2) arising from $u_0 \in (B_{\infty,\infty}^{1-\lambda-\beta})^n$, obtained in [52], we are only required to prove that $X_\beta^a$ can be identified with $B_{\infty,\infty}^{1-\lambda-\beta}$ for $\frac{1}{2} > \alpha > 1 - \beta > 0$. On the one hand, if $f \in B_{\infty,\infty}^{1-\lambda-\beta}$, then
\[ \|f\|_{B_{\infty,\infty}^{1-\lambda-\beta}} \approx \sup_{(x,t) \in \mathbb{R}^{n+1}} t^{\frac{2\beta-1}{\beta}} \left| e^{-t(-\Delta)^{\beta/2}} f(x) \right| < \infty, \]
and hence
\[
\|f\|_{X_\beta^a} \approx \sup_{(x,t) \in \mathbb{R}^{n+1}} \left( r^{-(2\alpha+n)} \int_{B(x_0,t)} \int_0^r \left| e^{-t(-\Delta)^{\beta/2}} f(x) \right|^2 \frac{dt \, dx}{t^{1-\alpha-\beta}} \right)^{\frac{1}{2}}
\leq \|f\|_{B_{\infty,\infty}^{1-\lambda-\beta}} \sup_{(x,t) \in \mathbb{R}^{n+1}} \left( r^{-(2\alpha+n)} \int_{B(x_0,t)} \int_0^r \left( t^{\frac{2\beta-1}{\beta}} \right)^2 \frac{dt \, dx}{t^{1-\alpha-\beta}} \right)^{\frac{1}{2}}
\leq \|f\|_{B_{\infty,\infty}^{1-\lambda-\beta}},
\]
thanks to $\alpha > 1 - \beta > 0$. On the other hand, noting the following two facts:

- $B_{\infty,\infty}^{1-\lambda-\beta}$ is the largest space among all the Banach spaces that are translation-invariant and share the scaling (1.3) (cf. [7]),
- $X_\beta^a$ is translation-invariant and satisfies the scaling (1.3),
we achieve
\[ \|f\|_{B_{\infty,\infty}^{1-\lambda-\beta}} \leq \|f\|_{X_\beta^a} \quad \text{for all} \quad f \in X_\beta^a. \]
Thus, the desired identification follows.

### 3 Ill-posedness in $(X_\beta^a)^n$

This section is devoted to validating Theorem 1.4. The construction in the proof relies heavily on [3, 8].
### 3.1 Proof of Theorem 1.4 – Construction

To validate Theorem 1.4, we are required to find the initial data and its associated solution. Clearly, it is enough to handle the situation for \( n = 3 \). Referring to [3, 8], for a large integer \( l > 0 \), we choose the following initial data:

\[
\mathbf{u}_0(x) = t^{-\theta} \sum_{i=1}^{l} |k_i|^{\beta} (v \cos(k_i \cdot x) + v' \cos(k_i' \cdot x)),
\]

where \( \theta \in (0, \frac{1}{2}) \) and the vectors \( k_i \in \mathbb{Z}^n \) are parallel to \( \zeta = (1, 0, 0) \). For \( i = 1, 2, \ldots, l \) and a large integer \( N \) dependent on \( l \), let

\[
|k_i| = 2^{l-1} N, \quad k_i = k_i + \eta \in \mathbb{Z}^n, \quad \nu = (0, 0, 1), \quad \nu' = (0, 1, 0).
\]

For the initial data \( \mathbf{u}_0 \) (first constructed in [8] by an idea in [3]), we have

\[
\text{div} \mathbf{u}_0 = 0,
\]

\[
e^{-t(-\Delta)^{\beta}} \mathbf{u}_0 = t^{-\theta} \sum_{i=1}^{l} |k_i|^{\beta} (v \cos(k_i \cdot x) e^{-|k_i| t} + v' \cos(k_i' \cdot x) e^{-|k_i'| t}),
\]

\[
\|e^{-t(-\Delta)^{\beta}} \mathbf{u}_0\|_{L^\infty} \leq t^{-\theta - \frac{1}{2}} \quad \text{for all } t > 0.
\]

The following lemma is our main new tool, which asserts that the initial data \( \text{div} \mathbf{u}_0 \) constructed above is well behaved in our spaces \( X_\beta^a \).

**Lemma 3.1.** Suppose \( 1 \leq \beta < \infty \). If \( \mathbf{u}_0 \) is given in (3.1), then

\[
\Gamma^{-\theta} \begin{cases} 
\|\mathbf{u}_0\|_{(X_\beta^a)^n} & \text{if } \alpha > 1 - \beta, \\
\|\mathbf{u}_0\|_{(X_\beta^a)^n} & \text{if } \alpha \geq 0 \text{ and } \beta \neq 1.
\end{cases}
\]

**Proof.** In view of the definition of \((X_\beta^a)^n\), we have

\[
\|\mathbf{u}_0\|_{(X_\beta^a)^n}^2 = \sup_{0 < r < 1} r^{-(\pi + 2\alpha)} \int_0^{r^{\frac{1}{\beta}}} \int_{B(k_i, r)} |e^{-t(-\Delta)^{\beta}} \mathbf{u}_0|^2 t^{-1 - \frac{\alpha - \beta}{r}} dt \, dy
\]

\[
\leq t^{-2\alpha} \sup_{0 < r < 1} r^{-2\alpha} \int_0^{r^{\frac{1}{\beta}}} \left( \sum_{i=1}^{l} |k_i|^{\beta} (e^{-|k_i| t} + e^{-|k_i'| t}) \right)^2 t^{-1 - \frac{\alpha - \beta}{r}} dt.
\]

So, it remains to show that

\[
r^{-2\alpha} \int_0^{r^{\frac{1}{\beta}}} \left( \sum_{i=1}^{l} |k_i|^{\beta} (e^{-|k_i| t} + e^{-|k_i'| t}) \right)^2 t^{-1 - \frac{\alpha - \beta}{r}} dt \leq 1 \quad \text{for all } r \in (0, 1).
\]

Since

\[
\sum_{i=1}^{l} |k_i|^{\beta} e^{-|k_i| t} \leq t^{-\frac{1}{2}} \quad \text{for all } t \leq 1,
\]

we have

\[
r^{-2\alpha} \int_0^{r^{\frac{1}{\beta}}} \left( \sum_{i=1}^{l} |k_i|^{\beta} (e^{-|k_i| t} + e^{-|k_i'| t}) \right)^2 t^{-1 - \frac{\alpha - \beta}{r}} dt \leq r^{-2\alpha} \int_0^{r^{\frac{1}{\beta}}} \left( \sum_{i=1}^{l} |k_i|^{\beta} e^{-|k_i| t} \right)^2 t^{-1 - \frac{\alpha - \beta}{r}} dt \leq r^{\beta - 2}
\]

for any \( r \in (0, 1) \), provided that \( \alpha > 1 - \beta \), which is sufficient since \( \beta \geq 1 \).

Furthermore, if \( \alpha \geq 0 \), then the above estimate for \( r \in (0, 1) \) is still valid, and hence it remains to establish a similar estimate for \( 1 \leq r < \infty \). As a matter of fact, since

\[
\sum_{i=1}^{l} |k_i|^{\beta} e^{-|k_i| t} \leq e^{-N|k_i| t} \quad \text{for all } t \geq 2,
\]
we utilize $1 \ll N \leq |k|$, and $a \geq 0$, to obtain
\[
\begin{aligned}
    r^{-2a} \int_0^{r^\beta} \left( \sum_{i=1}^1 |k_i|^\beta \left( e^{-|k_i|^2 \beta t} + e^{-|k_i^1|^2 \beta t} \right) \right) t^{-\frac{2-\alpha-d}{2}} dt &\lesssim r^{-2a} \int_0^{r^\beta} \left( \sum_{i=1}^1 |k_i|^\beta \right) t^{-\frac{2-\alpha-d}{2}} dt \\
&\lesssim r^{-2a} \int_0^{r^\beta} t^{-\frac{2-\alpha-d}{2}} dt + r^{-2a} \int_0^{r^\beta} e^{-N|k^1|^2} t^{-\frac{2-\alpha-d}{2}} dt \\
&\lesssim r^{-2a} \quad \text{for all } r \in [1, \infty).
\end{aligned}
\]

The proof is completed. \hfill \square

Next, as in [8], we write
\[
\begin{aligned}
    u(t) &= e^{-(\Box)^{\beta}} u_0 - u_1(t) + y(t), \\
    u_1(t) &= B(e^{-(\Box)^{\beta}} u_0, e^{-(\Box)^{\beta}} u_0), \\
    y(t) &= \int_0^t e^{-(\Box)^{\beta}} [G_0(r) + G_1(r) + G_2(r)] \, dr,
\end{aligned}
\]

where
\[
\begin{aligned}
    G_0 &= \mathbb{P}[(e^{-(\Box)^{\beta}} u_0 \cdot \nabla) u_1 + (u_1 \cdot \nabla) e^{-(\Box)^{\beta}} u_0 + (u_1 \cdot \nabla) u_1], \\
    G_1 &= \mathbb{P}[(e^{-(\Box)^{\beta}} u_0 \cdot \nabla) y + (u_1 \cdot \nabla) y + (y \cdot \nabla) e^{-(\Box)^{\beta}} u_0 + (y \cdot \nabla) u_1], \\
    G_2 &= \mathbb{P}[(y \cdot \nabla) y].
\end{aligned}
\]

It turns out that $y$ gives no trouble as an error term. So, the main contribution comes from the bilinear term $u_1$. A straightforward calculation derives

\[
\begin{aligned}
    (e^{-(\Box)^{\beta}} u_0 \cdot \nabla) e^{-(\Box)^{\beta}} u_0 &= -r^{2\beta} \sum_{i=1}^1 \int_0^t \left( \sum_{j=1}^1 |k_i|^\beta |k_j|^\beta e^{-\|k_i\|^2 + |k_j|^2} t \right) v' \cos(k_j \cdot x) \sin(k_i \cdot x) \\
&= -\frac{r^{2\beta}}{2} \sum_{i=1}^1 \int_0^t \left( \sum_{j=1}^1 |k_i|^\beta |k_j|^\beta e^{-\|k_i\|^2 + |k_j|^2} t \right) \sin(\eta \cdot x) v' \\
&\quad - \frac{r^{2\beta}}{2} \sum_{i=1}^1 \int_0^t \left( \sum_{j=1}^1 |k_i|^\beta |k_j|^\beta e^{-\|k_i\|^2 + |k_j|^2} t \right) \sin(k_j - k_i) \cdot x) v' \\
&\quad - \frac{r^{2\beta}}{2} \sum_{i=1}^1 \int_0^t \left( \sum_{j=1}^1 |k_i|^\beta |k_j|^\beta e^{-\|k_i\|^2 + |k_j|^2} t \right) \sin((k_j + k_i) \cdot x) v' \\
&\equiv E_0 + E_1 + E_2. \quad (3.2)
\end{aligned}
\]

Then $u_1$ can be further decomposed according to
\[
\begin{aligned}
    u_1 &= \int_0^t e^{-(\Box)^{\beta}} E_0 \, d\tau + \int_0^t e^{-(\Box)^{\beta}} E_1 \, d\tau + \int_0^t e^{-(\Box)^{\beta}} E_2 \, d\tau \\
&\equiv u_{10} + u_{11} + u_{12}. \quad (3.3)
\end{aligned}
\]

This in turn gives
\[
\begin{aligned}
    u(x, t) &= e^{-(\Box)^{\beta}} u_0(x) - u_{10}(x, t) - u_{11}(x, t) - u_{12}(t) - y(x, t). \quad (3.4)
\end{aligned}
\]

It turns out that only $u_{10}$ matters, while other terms can be controlled easily under the $L^\infty$-norm. More precisely, we have the following two lemmas.
Lemma 3.2 (\(L^\infty\)-estimates from [8]). Let \(1 \leq \beta < \infty\) and \(0 < \theta < \frac{1}{2}\). Then
\[
\|e^{-t(-\Delta)^\beta} u_0(\cdot)\|_{L^\infty(y)} \leq \Gamma^\theta t^{-\frac{1}{2}},
\]
\[
\|u_{10}(\cdot, t)\|_{L^\infty(y)} \leq t^{1-2\theta},
\]
\[
\|u_{11}(\cdot, t)\|_{L^\infty(y)} \leq t^{-2\theta},
\]
\[
\|u_{12}(\cdot, t)\|_{L^\infty(y)} \leq t^{-2\theta},
\]
\[
\|y(\cdot, t)\|_{L^\infty(y)} \leq t^{1-3\theta} t^{\frac{1}{\beta}} + t^{2-\theta} t^{-\frac{\theta}{2}}
\]
for all \(t \in (0, T)\) when \(T\) is sufficiently small and \(l\) is sufficiently large. Actually, one can choose
\[
T = t^\gamma \text{ with } \gamma > \frac{1 - 2\theta}{1 - \frac{2}{2\beta}}.
\]

Lemma 3.3. Let \(u_{10}\) be defined as in (3.3). Then
\[
\|u_{10}(\cdot, t)\|_{(X^2_\beta)^n} \geq t^{1-2\theta} \text{ for all } t \in [N^{-2\beta}, 1].
\]

Furthermore, the solution \(u\) given by (3.4) is relatively large even in the resolution space:
\[
\|u\|_{(Y^\alpha_\gamma)^n} \geq \|u\|_{(Y^\alpha_{\beta, 1})^n} \geq \Gamma^\gamma.
\]

Proof. From (3.2)–(3.3) and a straightforward calculation, it follows that
\[
e^{-t(-\Delta)^\beta} u_{10} = -\frac{t^{2\beta} e^{-t|\eta|^2}}{2} \int \sum_{l=1}^l |k|^2|e^{-ik|\eta|^2 + |k|^2 - t} e^{-i|\eta|^2(1 - 1/\beta)} \sin(\eta \cdot x)|^2 t^{-1/\beta} dt
\]
\[
= -\frac{t^{2\beta} e^{-t|\eta|^2}}{2} \sin(\eta \cdot x) \int \sum_{l=1}^l |k|^2|e^{-i(|k|^2 + |k|^2 - t)} e^{-i|\eta|^2(1 - 1/\beta)}
\]
\[
\approx -t^{2\beta} e^{-t|\eta|^2} \sin(\eta \cdot x) \int \sum_{l=1}^l e^{-t(1 - |k|^2\beta)}
\]
\[
\approx -t^{2\beta} e^{-t|\eta|^2} \sin(\eta \cdot x)
\]

Consequently,
\[
\|u_{10}(\cdot, t)\|_{(X^2_\beta)^n} \geq \sup_{0 < r < 1} \int_0^{r^2} \int_0^{r^2} |e^{-t(-\Delta)^\beta} u_{10}|^2 t^{-1/\beta} dt \ dx \ dy
\]
\[
\geq t^{2-\beta} \sup_{0 < r < 1} \int_0^{r^2} (e^{-t|\eta|^2})^2 t^{-1/\beta} dt
\]
\[
\geq t^{2-\beta} \sup_{0 < r < 1} \int_0^{r^2} t^{-1/\beta} dt
\]
\[
\geq t^{2-\beta} \text{ when } \frac{1}{2} \leq \beta < \infty.
\]

Next we estimate \(u_{10}\) in \((Y^\alpha_{\beta, 1})^n\). In a similar calculation done as above, we have
\[
u_{10} = -t^{1-2\beta} \sin(\eta \cdot x)
\]
whence, in view of (3.4) and Lemma 3.2, getting
\[
\|u\|_{(Y^\alpha_\gamma)^n} \geq \|u\|_{(Y^\alpha_{\beta, 1})^n} \geq \sup_{(x, t) \in \mathbb{R}^n \times (0, 1)} t^{2\beta - 1} |u(x, t)|
\]
\[
\geq \sup_{(x, t) \in \mathbb{R}^n \times [N^{-2\beta}, T]} t^{2\beta - 1} \left( |u_{10}(x, t)| - |e^{-t(-\Delta)^\beta} u_0(x)| - |u_{11}(x, t)| - |u_{12}(x, t)| - |y(x, t)| \right)
\]
\[
\geq \sup_{(x, t) \in \mathbb{R}^n \times [N^{-2\beta}, T]} t^{2\beta - 1} \left( t^{1-2\beta} - t^{-\gamma} t^{-\frac{1}{\beta}} - t^{-2\beta} - t^{2-\beta} t^{1-\frac{1}{\beta}} + t^{2-\theta} t^{-\frac{\theta}{2}} \right)
\]
\[
\geq t^{1-2\beta} t^{1-\frac{1}{\beta}} - t^{-\gamma} t^{-\frac{1}{\beta}} - t^{-2\beta} t^{1-\frac{1}{\beta}} - t^{2-\beta} t^{1-\frac{1}{\beta}} - t^{2-\theta} t^{1-\frac{1}{\beta}}.
\]
Recall that $T = t^\gamma$ is as in Lemma 3.2 and $0 < \theta < \frac{1}{2}$. Then

$$
\|u\|_{\dot{Y}^\gamma_p} \geq t^{1 - 2\theta} t^{\gamma(1 - \frac{1}{2p})} - t^{\theta} t^{\gamma(1 - \frac{1}{2p})} - t^{2\theta} t^{\gamma(1 - \frac{1}{2p})} - t^{1 - 3\theta} t^{\gamma(1 - \frac{1}{2p})} - t^{2 - 4\theta} t^{\gamma(2 - \frac{1}{2p})}.
$$

If

$$
y = \frac{1 - 2\theta}{1 - \frac{1}{2p}} > \frac{1 - 2\theta}{1 - \frac{1}{2p}} > 0,
$$

then

$$
\|u\|_{\dot{Y}^\gamma_p} \geq t^{\frac{y}{2}} - t^{\theta} - t^{\gamma - 1} t^\frac{y}{2} - t^{\theta}.
$$

Since $\beta > 1$, we have $y < 2 - 3\theta$, whence getting

$$
\|u\|_{\dot{Y}^\gamma_p} \geq t^{\frac{y}{2}} - t^{\theta} - t^{\gamma - 1} t^\frac{y}{2} - t^{\theta} \geq t^{\frac{y}{2}},
$$

provided that $l$ is sufficiently large.

\[\square\]

### 3.2 Proof of Theorem 1.4 – Conclusion

The desired norm inflation part of Theorem 1.4 follows from Lemma 3.2 and (3.5) by a similar argument as that used in [8, Section 4.4]. It is only needed to disprove the differentiability of the associated solution map. In view of Lemma 3.1, we conclude that there exists a sequence $\{u_0\}$ with solution $\{u = T(u_0)\}$ such that

$$
\|u_0\|_{X^\alpha_{p,1}} \leq l^{-\theta} \text{ for } \alpha > 1 - \beta \text{ and } \beta \geq 1.
$$

However, using (3.6), we have

$$
\frac{\|T(u_0)\|_{Y^\gamma_p}}{\|u_0\|_{X^\alpha_{p,1}}} \geq \frac{\|u_1\|_{Y^\gamma_p}}{\|u_0\|_{X^\alpha_{p,1}}} \geq l^\frac{y}{2}
$$

for $0 < \theta < \frac{1}{2}$ and $l$ sufficiently large. Moreover, if $\alpha \geq 0$, then, by Lemma 3.1, $\|u_0\|_{X^\alpha_{p,1}} \leq l^{-\theta}$. Similarly, by applying (3.6), we obtain

$$
\frac{\|T(u_0)\|_{Y^\gamma_p}}{\|u_0\|_{X^\alpha_{p,1}}} \geq \frac{\|u_1\|_{Y^\gamma_p}}{\|u_0\|_{X^\alpha_{p,1}}} \geq l^\frac{y}{2}
$$

for $0 < \theta < \frac{1}{2}$ and $l$ sufficiently large. Thus, we finish the proof by letting $l \rightarrow \infty$.

### 4 Application to $\mathcal{L}^{1-\alpha-2\beta}_{2,n+2\alpha}$

In this section, we demonstrate Corollary 1.5.

#### 4.1 Characterization of CS functions

Given $\alpha \in (-1, 1)$. According to [49, Lemma 2.1] and [42, Theorem 2.5], each $f \in \mathcal{L}_{2,n+2\alpha}$ has an equivalent norm:

$$
\|f\|_{\mathcal{L}_{2,n+2\alpha}} \approx \sup_{(x,r) \in \mathbb{R}^n_+} \left( \int_0^r \left( \int_0^t \left( \int_0^u \phi_t * f(y) \right)^2 \frac{dt}{u} \right)^{\frac{1}{2}} \right),
$$

where $\phi$ is a radial function on $\mathbb{R}^n$ such that

- $\phi \in L^1$,
- $\phi_t(x) = t^{-\frac{n}{2}} \phi \left( \frac{x}{t} \right)$,
- $|\phi(x)| \leq (1 + |x|)^{-c-n}$ for some $c > 0$,
- $\int_{\mathbb{R}^n} \phi(x) \, dx = 0$, $0 < \int_0^\infty |\phi(t\xi)|^2 \frac{dt}{t} < \infty$. 

Since
\[ \psi_t * [(\sqrt{-\Delta})^3f](x) = (\sqrt{-\Delta})^3\psi_t * f(x) = t^{-\frac{3}{2}}[(\sqrt{-\Delta})^3\psi]_t * f(x), \]
we discover an equivalent norm for \( L^2_{2,n+2a} \):
\[
\|f\|_{L^2_{2,n+2a}} \approx \sup_{(x,t) \in \mathbb{R}^{n+1}} \left( r^{-2(2a+n)} \int_{B(x,t)} \left| [(\sqrt{-\Delta})^3\psi]_t * f(y) \right|^2 \frac{dt dy}{t^{1+2s}} \right)^{\frac{1}{2}},
\]
\[
\approx \sup_{(x,t) \in \mathbb{R}^{n+1}} \left( r^{-2(2a+n)} \int_{B(x,t)} \left| \psi_t * f(y) \right|^2 \frac{dt dy}{t^{1+2s}} \right)^{\frac{1}{2}},
\]
provided that \( \psi \) satisfies the above conditions on \( \phi \), where \( (\sqrt{-\Delta})^3\psi = \phi \).

Now, set \( \phi \) be the inverse Fourier transform of \( t^\xi \hat{\phi}(\xi)e^{-t(\xi^2/2)} \) and \( -\infty < s < 1 \). In view of the above analysis, we have a semi-group characterization for each Campanato–Sobolev (CS) function \( f \in L^2_{2,n+2a} \):
\[
\|f\|_{L^2_{2,n+2a}} \approx \sup_{(x,t) \in \mathbb{R}^{n+1}} \left( r^{-2(2a+n)} \int_{B(x,t)} \left| \nabla \left( t^\xi \hat{\phi}(\xi)e^{-t(\xi^2/2)} \right) \right|^2 \frac{dt dx}{t^{1+2s}} \right)^{\frac{1}{2}},
\]
\[
\approx \sup_{(x,t) \in \mathbb{R}^{n+1}} \left( r^{2\beta} \int_{B(x,t)} \left| \nabla e^{-t(\xi^2/2)} \hat{\phi}(\xi) \right|^2 t^{-\frac{1+2\beta}{2}} \frac{dt dy}{t^{1+2s}} \right)^{\frac{1}{2}},
\]
where \( \nabla \) stands for the spatial gradient.

### 4.2 Proof of Corollary 1.5

The preceding characterization leads to introducing the space \( (L^2_{2,n+2a})^{-1} \) of all functions \( f \in L^2_{loc} \) on \( \mathbb{R}^n \) with the norm
\[
\|f\|_{(L^2_{2,n+2a})^{-1}} = \sup_{(x,t) \in \mathbb{R}^{n+1}} \left( r^{-2(2a+n)} \int_{B(x,t)} \left| e^{-t(\xi^2/2)} \hat{\phi}(\xi) \right|^2 t^{-\frac{1+2\beta}{2}} \frac{dt dy}{t^{1+2s}} \right)^{\frac{1}{2}}.
\]
It is not hard to check the following implication:
\[
(\alpha, \beta, s) \in (-1, 1) \times [\frac{1}{2}, \infty) \times (-\infty, 1) \implies X^\alpha_{\beta} = (L^2_{2,n+2a})^{-1}.
\] (4.1)

Therefore, the assertions in Corollary 1.5 follow immediately from (4.1), Theorems 1.1 and 1.4, and the following lemma.

**Lemma 4.1.** Suppose \((\alpha, \beta, s) \in (-1, 1) \times [\frac{1}{2}, \frac{3}{2}) \times (-\infty, 1) \) and \( j = 1, 2, \ldots, n \). If \( R_j = \frac{\partial_j}{\sqrt{-\Delta}} \) is the \( j \)-th Riesz transform, then
\[
\|R_j f\|_{L^2_{2,n+2a}} \leq \|f\|_{L^2_{2,n+2a}} \quad \text{for all} \quad f \in L^2_{2,n+2a},
\]
and hence
\[
L^2_{2,n+2a} = (L^2_{2,n+2a})^{-1} = \nabla \cdot (L^2_{2,n+2a})^n.
\]

**Proof.** Assume \( f \in L^2_{2,n+2a} \). Since
\[
\|R_j f\|_{L^2_{2,n+2a}}^2 = \sup_{(x,t) \in \mathbb{R}^{n+1}} r^{-2(2a+n)} \int_{B(x,t)} t^{-\frac{1+2\beta}{2}} dt dy,
\]
we split \( R_j f \) into two pieces via the point-mass function \( \delta \):
\[
R_j f = \varphi_r * (R_j f) + (\delta - \varphi_r) * (R_j f),
\]
where
\[
\varphi \in C_0^\infty(\mathbb{R}^n), \quad \text{supp} \varphi \subset B(0, 1), \quad \int_{\mathbb{R}^n} \varphi(x) \, dx = 1, \quad \varphi_r(x) = r^{-n} \varphi \left( \frac{x}{r} \right).
\]
On the one hand, using the fact that the predual of $B^{s+a-1}_{c_0,c_0}$ is the homogeneous Besov space $B^{1-s-a}_{1,1}$, we estimate
\[ \int_{B(x,r)} \int_0^r \left\| \varphi_r \ast \nabla e^{-\frac{i}{\sqrt{-\Delta}} R_j(f(y))} \right\|^{2 + \frac{1}{r}} dt \, dr \lesssim r^{m-2s+2} \left\| \varphi_r \ast \nabla e^{-\frac{i}{\sqrt{-\Delta}} R_j(f(y))} \right\|_{L^{2s+1}(R^{n+1})}^2 \lesssim r^{m-2s} \left\| \varphi_r \right\|_{B^{1-s-a}_{1,1}}^2 \left\| \nabla e^{-\frac{i}{\sqrt{-\Delta}} R_j(f(y))} \right\|_{B^{s+a-1}_{c_0,c_0}}^2 \lesssim r^{m-2s} \| f \|_{L^{2s+1}(\mathbb{R}^{n+1})}^2 \lesssim r^{m-2s} \| f \|_{L^{s+a}_{1,1}}^2 \lesssim r^{m-2s} \| f \|_{L^{s+a}_{1,1}}^2. \]

On the other hand, noticing that 
\[ (\delta - \varphi_r) \ast R_j e^{-\frac{i}{\sqrt{-\Delta}} f(x)} (\delta - \varphi_r) \ast R_j e^{-\frac{i}{\sqrt{-\Delta}} f(x)} \]
and that $(\delta - \varphi_r) \ast R_j e^{-\frac{i}{\sqrt{-\Delta}} f(x)}$ is a convolution operator with its kernel $K_t(x)$ satisfying
\[ \sup_{t > 0} \int_{\mathbb{R}^n} |K_t(x)| \, dx \lesssim 1, \]
we get, by the argument used in the proof of [47, Lemma 3.1] and Hölder’s inequality,
\[ r^{-(2a+n)} \int_{B(x,r)} \int_0^r \left\| \left( \delta - \varphi_r \right) \ast \nabla e^{-\frac{i}{\sqrt{-\Delta}} R_j(f(x))} \right\|^{2 + \frac{1}{r}} t \, dt \, dr \lesssim r^{-(2a+n)} \int_{B(x,r)} \int \int_{\mathbb{R}^n} K_t(x-y) \left( \nabla e^{-\frac{i}{\sqrt{-\Delta}} f(y)} \right) \, dy \, t \, dt \, dx \lesssim \sup_{t > 0} \int_{\mathbb{R}^n} \left\| K_t(x) \right\| \, dx \| f \|_{L^{s+a}_{1,1}}^2 \lesssim \| f \|_{L^{s+a}_{1,1}}^2. \]

The above two-fold treatment yields
\[ \| R_j f \|_{L^{s+a}_{1,1}} \lesssim \| f \|_{L^{s+a}_{1,1}}. \]

To check the identification between those three spaces, we consider two inclusions.

On the one hand, if $f \in L^{s-1}_{2,m+2a}$, then $(\sqrt{-\Delta})^{-1} f \in L^s_{2,m+2a}$, by definition. An application of the estimate for the Riesz transform gives
\[ \frac{\partial_{x_j}}{\sqrt{-\Delta}} f = \left( \frac{\partial_{x_j}}{\sqrt{-\Delta}} \right) (\sqrt{-\Delta})^{-1} f \in L^s_{2,m+2a}, \]
and consequently $f \in \mathbf{V} \cdot (L^s_{2,m+2a})^n$. This in turn produces
\[ (f_1, \ldots, f_n) \in (L^s_{2,m+2a})^n \text{ such that } f = \sum_{j=1}^n \partial_{x_j} f_j. \]

An application of the triangle inequality implies $f \in (L^s_{2,m+2a})^{-1}$. On the other hand, if $f \in (L^s_{2,m+2a})^{-1}$, then choosing
\[ f_{j,k} = \partial_{x_j} \partial_{x_k} (\Delta)^{-1} f \text{ for all } j, k \in \{1, 2, \ldots, n\}, \]
one has $f_{j,k} \in (L^s_{2,m+2a})^{-1}$ due to the above-proved Riesz transform estimate. This further derives
\[ g_j = \partial_{x_j} (\Delta)^{-1} f \in L^2_{2,n+2a}, \]
So, there exist $f_j \in L^s_{2,m+2a}$ for $j = 1, \ldots, n$ such that $f = \sum_{j=1}^n \partial_{x_j} f_j$, and then
\[ \| f \|_{L^{s-1}_{1,1}} \lesssim \sum_{j=1}^n \| \partial_{x_j} f_j \|_{L^{s-1}_{1,1}} \lesssim \sum_{j=1}^n \| R_j f_j \|_{L^s_{1,1}} \lesssim \sum_{j=1}^n \| f_j \|_{L^s_{1,1}} \lesssim \sum_{j=1}^n \| f_j \|_{L^s_{2,m+2a}}, \]
as desired. \[ \Box \]
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