Research Article

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Ground state solutions for a semilinear elliptic problem with critical-subcritical growth

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Abstract: We prove the existence of at least one ground state solution for the semilinear elliptic problem

$$\begin{cases} -\Delta u = u^{p(x)-1}, & u > 0, \quad \text{in } G \subseteq \mathbb{R}^N, \quad N \geq 3, \\ u \in D_0^{1,2}(G), \end{cases}$$

where G is either $\mathbb{R}^N$ or a bounded domain, and $p: G \rightarrow \mathbb{R}$ is a continuous function assuming critical and subcritical values.

Keywords: Variational methods, positive solutions, critical growth

MSC 2010: 35J20, 35J61, 35B33

1 Introduction

In this paper we deal with the existence of a ground state solution for the semilinear elliptic problem

$$\begin{cases} -\Delta u = u^{p(x)-1}, & u > 0, \quad \text{in } G, \\ u \in D_0^{1,2}(G), \end{cases} \quad (\text{P})$$

where either $G = \mathbb{R}^N$ and $D_0^{1,2}(G) = D^{1,2}(\mathbb{R}^N)$ or G is a bounded domain in $\mathbb{R}^N$ and $D_0^{1,2}(G) = H_0^1(G)$. In both cases, $N \geq 3$ and $p: G \rightarrow \mathbb{R}$ is a continuous function satisfying the following condition:

(H₁) There exist a bounded set $\Omega \subset G$, with positive N -dimensional Lebesgue measure, and positive constants p^- , p^+ and δ such that

$$2 < p^- \leq p(x) \leq p^+ < 2^* \quad \text{for all } x \in \Omega, \quad (\text{H}_1\text{a})$$

$$p(x) \equiv 2^* \quad \text{for all } x \in G \setminus \Omega_\delta, \quad (\text{H}_1\text{b})$$

$$2 < p^- \leq p(x) < 2^* \quad \text{for all } x \in \Omega_\delta, \quad (\text{H}_1\text{c})$$

where

$$\Omega_\delta := \{x \in G : \text{dist}(x, \overline{\Omega}) \leq \delta\}$$

and $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent.

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The energy functional associated with (P) is given by

$$I(u) := \frac{1}{2} \int_G |\nabla u|^2 \, dx - \int_G \frac{1}{p(x)} (u^+)^{p(x)} \, dx$$

and the corresponding Nehari manifold is defined by

$$\mathcal{N} := \{u \in D_0^{1,2}(G) \setminus \{0\} : I'(u)(u) = 0\}.$$

Our goal in this paper is to obtain a critical point u of I satisfying $u > 0$ in G and

$$I(u) = \inf_{v \in \mathcal{N}} I(v).$$

We refer to such a critical point as a ground state solution of (P).

There are several works in the literature dealing with semilinear problems in the particular case $p(x) \equiv 2^*$. Let us mention some of them.

In [19], Pohozaev showed that the problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u, & u > 0, \quad \text{in } G, \\ u \in H_0^1(G) \end{cases} \quad (P_1)$$

does not admit a nontrivial solution if $\lambda \leq 0$, provided that the bounded domain G is strictly star-shaped with respect to the origin in $\mathbb{R}^N$, $N \geq 3$.

In [7], Brezis and Nirenberg proved the following results: if $N \geq 4$, problem (P_1) has a positive solution for every $\lambda \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of $(-\Delta, H_0^1(\Omega))$; if $N = 3$, there exists $\lambda_* \in [0, \lambda_1)$ such that for any $\lambda \in (\lambda_*, \lambda_1)$ problem (P_1) admits a positive solution. In the particular case where G is a ball, they proved that a positive solution exists if, and only if, $\lambda \in (\lambda_1/4, \lambda_1)$ and also that when $N = 3$ there exists $\lambda_* > 0$ such that (P_1) does not have a solution for $\lambda \leq \lambda_*$.

In [9], Coron proved that if there exist $R, r > 0$ such that

$$G \supset \{x \in \mathbb{R}^N : r < |x| < R\} \quad \text{and} \quad \overline{G} \not\supset \{x \in \mathbb{R}^N : |x| < r\}$$

and the ratio R/r is sufficiently large, then problem (P_1) with $\lambda = 0$ has a positive solution in $H_0^1(G)$.

In [6], Bahri and Coron showed that if $\lambda = 0$ and $\mathcal{H}_i(G; \mathbb{Z}/2) \neq 0$ (i th homology group) for some $i > 0$, then problem (P_1) has at least one positive solution. (When $\mathcal{H}_i(G; \mathbb{Z}/2) \neq 0$ the boundary ∂G is not connected.)

Existence results for (P_1) related to the topology of G were also obtained by Bahri, in [5]. In [8], Carpio, Comte and Lewandowski obtained nonexistence results for (P_1) , with $\lambda = 0$, in contractible non-star-shaped domains.

On the other hand, the subcritical problem

$$\begin{cases} -\Delta u = |u|^{q-2}u & \text{in } G, \quad 2 < q < 2^*, \\ u = 0 & \text{on } \partial G \end{cases} \quad (P_2)$$

has an unbounded set of solutions in $H_0^1(G)$ (see [13]).

Problem (P_2) with $q = 2^* - \epsilon$ ($\epsilon > 0$) was studied in the papers [3] and [14]. In the former, Atkinson and Peletier considered G a ball and determined the exact asymptotic behavior of the corresponding (radial) solutions u_ϵ , as $\epsilon \rightarrow 0$. In [14], where a general bounded domain G was considered, Garcia Azorero and Peral Alonso provided an alternative for the asymptotic behavior of $J_\epsilon(u_\epsilon)$, as $\epsilon \rightarrow 0$, where J_ϵ denotes the energy functional associated with problem (P_2) and $q = 2^* - \epsilon$. More precisely, they showed that if

$$\frac{1}{N} S^{\frac{N}{2}} < \lim_{\epsilon \rightarrow 0} J_\epsilon(u_\epsilon) < \frac{2}{N} S^{\frac{N}{2}},$$

where S denotes the Sobolev constant, then u_ϵ converges to either a Dirac mass or a solution of the critical problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } G, \\ u = 0 & \text{on } \partial G. \end{cases}$$

We recall that the Sobolev constant is defined by

$$S := \inf \left\{ \frac{\|u\|_{1,2}^2}{\|u\|_{2^*}^2} : u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\} \right\} \quad (1.1)$$

and given explicitly by the expression

$$S := \pi N(N-2) \left(\frac{\Gamma(N/2)}{\Gamma(N)} \right)^{\frac{2}{N}},$$

where $\Gamma(t) = \int_0^\infty s^{t-1} e^{-s} ds$ is the Gamma function (see the papers [4] and [20] by Aubin and Talenti, respectively). Furthermore, S is achieved in (1.1) by the Aubin–Talenti function

$$w(x) = \frac{[N(N-2)]^{(N-2)/4}}{(1+|x|^2)^{(n-2)/2}} \quad (1.2)$$

(and also by translations, rescalings and scalar multiples of it). This function also satisfies

$$\int_{\mathbb{R}^N} |\nabla w|^2 dx = \int_{\mathbb{R}^N} |w|^{2^*} dx = S^{\frac{N}{2}}$$

and is a ground state solution of problem (P) with $G = \mathbb{R}^N$ and $p(x) \equiv 2^*$.

In [16], Kurata and Shioji studied the compactness of the embedding $H_0^1(G) \hookrightarrow L^{p(x)}(G)$ for a bounded domain G and a variable exponent $1 \leq p(x) \leq 2^*$. (For the definition and properties of $L^{p(x)}(G)$ see [12]). They showed the existence of a positive solution of (P) under the hypothesis of existence of a point $x_0 \in G$, a small $\eta > 0$, $0 < l < 1$ and $c_0 > 0$ such that $p(x_0) = 2^*$ and

$$p(x) \leq 2^* - \frac{c_0}{(\log(1/|x - x_0|))^l}, \quad |x - x_0| \leq \eta.$$

In [1], Alves and Souto studied the existence of nonnegative solutions for the equation

$$-\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = u^{q(x)-1} \quad \text{in } \mathbb{R}^N,$$

where the variable exponents $p(x)$ and $q(x)$ are radially symmetric functions satisfying $1 < \operatorname{ess\,inf}_{\mathbb{R}^N} p(x) \leq \operatorname{ess\,sup}_{\mathbb{R}^N} p(x) < N$, $p(x) \leq q(x) \leq 2^*$ and

$$p(x) = 2, \quad q(x) = 2^* \quad \text{if either } |x| \leq \delta \text{ or } |x| \geq R,$$

for constants $0 < \delta < R$.

Finally, in [18], Liu, Liao and Tang proved the existence of a ground state solution for (P) with $G = \mathbb{R}^N$ and

$$p(x) = \begin{cases} p & \text{if } x \in \Omega, \\ 2^* & \text{if } x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where the constant p belongs to $(2, 2^*)$ and $\Omega \subset \mathbb{R}^N$ has nonempty interior.

In Section 2, motivated by the results of [18], we use the concentration-compactness lemma by P. L. Lions and properties of the Nehari manifold $\mathcal{N}$ to prove the existence of at least one ground state for problem (P) when $G = \mathbb{R}^N$ and $p \in C(\mathbb{R}^N, \mathbb{R})$ is a function satisfying condition (H₁). A key point in the proof of our existence result is the achievement of the strict inequality

$$\inf_{v \in \mathcal{N}} I(v) < \frac{S^{\frac{N}{2}}}{N} \quad (1.3)$$

and we get this by exploring the “projection” on the Nehari manifold of the sequence (w_k) , where $w_k(x) = w(x + ke_N)$ and $e_N = (0, 0, \dots, 1)$ is the N th coordinate vector.

In Section 3, we study the case where G is a bounded domain in $\mathbb{R}^N$. In this case, the argument based on the sequences of translations of the Aubin–Talenti function is not applicable. Thus, in order to achieve the inequality (1.3) we assume an additional hypothesis (H_2) that is stated in terms of a subdomain U of Ω and the value

$$\bar{q} := \min \left\{ q \in (2, 2^*) : g(q) = \frac{1}{N} S^{\frac{N}{2}} \right\}, \quad (1.4)$$

where the function $g: (2, 2^*) \rightarrow (0, \infty)$ is given by

$$g(q) = \left(\frac{1}{2} - \frac{1}{q} \right) S_q(U)^{\frac{q}{q-2}}$$

and $S_q(U)$ denotes the best constant of the embedding $H_0^1(U) \hookrightarrow L^q(U)$, that is,

$$S_q(U) := \inf \left\{ \frac{\|\nabla u\|_{L^2(U)}^2}{\|u\|_{L^q(U)}^2} : u \in H_0^1(U) \setminus \{0\} \right\}.$$

More precisely, we assume that the function $p \in C(G, \mathbb{R})$, satisfying (H_1) , also verifies the following hypothesis, where Ω , p^- and p^+ are defined in (H_1) and $\bar{q}$ is defined by (1.4):

(H_2) There exists a subdomain U of Ω such that

$$S_2(U) \leq 1 \quad \text{and} \quad p^- \leq p(x) \equiv q < \min\{\bar{q}, p^+\} \quad \text{for all } x \in U.$$

Under the hypotheses (H_1) and (H_2) , we show that problem (P) has at least one ground state solution.

We remark that the constant q in the statement of (H_2) can be any lower bound for $\bar{q}$ in the interval (p^-, p^+) . Considering that the particular value $S_2(U)$ is available for several domains (especially those with some kind of symmetry), we derive two lower bounds $q_1 > q_2$ for $\bar{q}$ in terms of $S_2(U)$, S and $|U|$ and a third, q_3 , depending only on S and $|U|$. Moreover, we present sufficient conditions for $S_2(U) \leq 1$ to hold, when the subdomain U is either a ball B_R or an annular-shaped domain $B_R \setminus \overline{B_r}$, with $\overline{B_r} \subset B_R$. We also show that if R and $R - r$ are sufficiently large, then $S_2(U) < 1$ for $U = B_R$ and $U = B_R \setminus \overline{B_r}$, respectively.

2 The semilinear elliptic problem in $\mathbb{R}^N$

In this section, we consider the semilinear elliptic problem with variable exponent

$$\begin{cases} -\Delta u = u^{p(x)-1}, & u > 0, \quad \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases} \quad (2.1)$$

where $N \geq 3$ and $p: \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous function verifying hypothesis (H_1) .

We recall that the space $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{1,2} := \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$

The dual space of $D^{1,2}(\mathbb{R}^N)$ will be denoted by D^{-1} .

The energy functional $I: D^{1,2}(\mathbb{R}^N) \rightarrow \mathbb{R}$ associated with (2.1) is given by

$$I(u) = \frac{1}{2} \|u\|_{1,2}^2 - \int_{\mathbb{R}^N} \frac{1}{p(x)} (u^+)^{p(x)} \, dx,$$

where $u^+(x) = \max\{u(x), 0\}$. Hence, under hypothesis (H_1) , we can write

$$I(u) = \frac{1}{2} \|u\|_{1,2}^2 - \int_{\Omega_\delta} \frac{1}{p(x)} (u^+)^{p(x)} \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N \setminus \Omega_\delta} (u^+)^{2^*} \, dx.$$

For a posterior use, let us estimate the second term in the above expression. For this, let $u \in D^{1,2}(\mathbb{R}^N)$ and consider the set $E = \{x \in \Omega_\delta : |u(x)| < 1\}$. Then

$$\begin{aligned} \int_{\Omega_\delta} \frac{1}{p(x)} (u^+)^{p(x)} dx &\leq \frac{1}{p^-} \int_E (u^+)^{p^-} dx + \frac{1}{p^-} \int_{\Omega_\delta \setminus E} (u^+)^{2^*} dx \\ &\leq \frac{1}{p^-} \int_{\Omega_\delta} |u|^{p^-} dx + \frac{1}{p^-} \|u\|_{2^*}^{2^*} \\ &\leq \frac{1}{p^-} \left(\int_{\Omega_\delta} |u|^{2^*} dx \right)^{\frac{p^-}{2^*}} |\Omega_\delta|^{\frac{2^*-p^-}{2^*}} + \frac{1}{p^-} \|u\|_{2^*}^{2^*} \\ &\leq \frac{1}{p^-} |\Omega_\delta|^{\frac{2^*-p^-}{2^*}} \|u\|_{2^*}^{p^-} + \frac{1}{p^-} \|u\|_{2^*}^{2^*}, \end{aligned}$$

where we have used (H_1) and Hölder's inequality. Hence, it follows from (1.1) and (H_1c) that

$$\frac{1}{2^*} \int_{\Omega_\delta} (u^+)^{p(x)} dx \leq \int_{\Omega_\delta} \frac{1}{p(x)} (u^+)^{p(x)} dx \leq a \|u\|_{1,2}^{p^-} + b \|u\|_{1,2}^{2^*}, \quad (2.2)$$

where

$$a = \frac{1}{p^-} |\Omega_\delta|^{\frac{2^*-p^-}{2^*}} S^{-\frac{p^-}{2}} \quad \text{and} \quad b = \frac{S^{-\frac{2^*}{2}}}{p^-}.$$

We observe from (2.2) that the functional I is well defined.

The next lemma establishes that I is of class C^1 . Since its proof is standard, it will be omitted.

Lemma 2.1. *Let $p \in C(\mathbb{R}^N, \mathbb{R})$ a function satisfying (H_1a) . Then $I \in C^1(D^{1,2}(\mathbb{R}^N), \mathbb{R})$ and*

$$I'(u)(v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v dx - \int_{\mathbb{R}^N} (u^+)^{p(x)-1} v dx \quad \text{for all } u, v \in D^{1,2}(\mathbb{R}^N).$$

Remark 2.2. The previous lemma ensures that $u \in D^{1,2}(\mathbb{R}^N)$ is a weak solution of (2.1) if, and only if, u is a critical point of I (i.e. $I'(u) = 0$). We remark that a critical point u of I is nonnegative, since

$$0 = I'(u)(u^-) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla u^- dx - \int_{\mathbb{R}^N} (u^+)^{p(x)-1} u^- dx = \|u^-\|_{1,2}^2,$$

where $u^-(x) = \min\{u(x), 0\}$. Consequently, according to the Strong Maximum Principle, if $u \neq 0$ is a critical point of I , then $u > 0$ in $\mathbb{R}^N$.

2.1 The Nehari manifold

In this subsection we prove some properties of the Nehari manifold associated with (2.1), which is defined by

$$\mathcal{N} := \{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\} : J(u) = 0\},$$

where

$$J(u) := I'(u)(u) = \|u\|_{1,2}^2 - \int_{\mathbb{R}^N} (u^+)^{p(x)} dx.$$

Of course, critical points of I belong to $\mathcal{N}$.

Definition 2.3. We say that $u \in \mathcal{N}$ is a *ground state solution* for (2.1) if $I'(u) = 0$ and $I(u) = m$, where

$$m := \inf_{u \in \mathcal{N}} I(u).$$

In the sequel we show important properties involving the Nehari manifold, which are crucial in our approach.

Proposition 2.4. Assume that (H_1) holds. Then $m > 0$.

Proof. For an arbitrary $u \in \mathcal{N}$ we have

$$\|u\|_{1,2}^2 = \int_{\mathbb{R}^N} (u^+)^{p(x)} dx = \int_{\Omega_\delta} (u^+)^{p(x)} dx + \int_{\mathbb{R}^N \setminus \Omega_\delta} (u^+)^{p(x)} dx \leq \int_{\Omega_\delta} (u^+)^{p(x)} dx + S^{-\frac{2^*}{2}} \|u\|_{1,2}^{2^*}.$$

Thus, it follows from (2.2) that

$$\|u\|_{1,2}^2 \leq C_1 \|u\|_{1,2}^{p^-} + C_2 \|u\|_{1,2}^{2^*},$$

where C_1 and C_2 denote positive constants that do not depend on u . Consequently,

$$1 \leq C_1 \|u\|_{1,2}^{p^- - 2} + C_2 \|u\|_{1,2}^{2^* - 2},$$

from which we conclude that there exists $\eta > 0$ such that

$$\|u\|_{1,2} \geq \eta \quad \text{for all } u \in \mathcal{N}. \quad (2.3)$$

Therefore,

$$I(u) = I(u) - \frac{1}{p^-} I'(u)(u) = \left(\frac{1}{2} - \frac{1}{p^-} \right) \|u\|_{1,2}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{p^-} - \frac{1}{p(x)} \right) (u^+)^{p(x)} dx \geq \left(\frac{1}{2} - \frac{1}{p^-} \right) \|u\|_{1,2}^2. \quad (2.4)$$

In view of (2.3), this implies that $m \geq \left(\frac{1}{2} - \frac{1}{p^-} \right) \eta^2 > 0$. $\square$

Proposition 2.5. Assume (H_1) . Then, for each $u \in D^{1,2}(\mathbb{R}^N)$ with $u^+ \not\equiv 0$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$.

Proof. Let

$$f(t) := I(tu) = \frac{t^2}{2} \|u\|_{1,2}^2 - \int_{\mathbb{R}^N} \frac{t^{p(x)}}{p(x)} (u^+)^{p(x)} dx, \quad t \in (0, +\infty).$$

We note that

$$f'(t) = I'(tu)(u) = t \|u\|_{1,2}^2 - \int_{\mathbb{R}^N} t^{p(x)-1} (u^+)^{p(x)} dx = \frac{1}{t} J(tu) \quad \text{for all } t \in (0, +\infty).$$

Since $1 < p^- - 1 \leq p(x) - 1$, we have

$$\begin{aligned} f'(t) &\geq t \left(\|u\|_{1,2}^2 - t^{p^- - 2} \int_{\mathbb{R}^N} (u^+)^{p(x)} dx \right) \quad \text{for all } t \in (0, 1), \\ f'(t) &\leq t \left(\|u\|_{1,2}^2 - t^{p^- - 2} \int_{\mathbb{R}^N} (u^+)^{p(x)} dx \right) \quad \text{for all } t \geq 1. \end{aligned}$$

Thus, we can see that $f'(t) > 0$ for all $t > 0$ sufficiently small and also that $f'(t) < 0$ for all $t \geq 1$ sufficiently large. Therefore, there exists $t_u > 0$ such that

$$f'(t_u) = \frac{1}{t_u} J(t_u u) = 0,$$

showing that $t_u u \in \mathcal{N}$.

In order to prove the uniqueness of t_u , let us assume that $0 < t_1 < t_2$ satisfy $f'(t_1) = f'(t_2) = 0$. Then

$$\|u\|_{1,2}^2 = \int_{\mathbb{R}^N} t_1^{p(x)-2} (u^+)^{p(x)} dx = \int_{\mathbb{R}^N} t_2^{p(x)-2} (u^+)^{p(x)} dx.$$

Hence,

$$\int_{\mathbb{R}^N} (t_1^{p(x)-2} - t_2^{p(x)-2}) (u^+)^{p(x)} dx = 0.$$

Since $t_1^{p(x)-2} < t_2^{p(x)-2}$ for all $x \in \mathbb{R}^N$, the above equality leads to the contradiction $u^+ \equiv 0$. $\square$

Proposition 2.6. Assume that (H_1) holds. Then

$$J'(u)(u) \leq (2 - p^-)\eta^2 < 0 \quad \text{for all } u \in \mathcal{N},$$

where η was given in (2.3). Hence, $J'(u) \neq 0$ for all $u \in \mathcal{N}$.

Proof. For $u \in \mathcal{N}$ we have

$$\begin{aligned} J'(u)(u) &= 2\|u\|_{1,2}^2 - \int_{\mathbb{R}^N} p(x)(u^+)^{p(x)} dx \\ &\leq 2\|u\|_{1,2}^2 - p^- \int_{\mathbb{R}^N} (u^+)^{p(x)} dx = (2 - p^-)\|u\|_{1,2}^2 \leq (2 - p^-)\eta^2 < 0, \end{aligned}$$

according to (2.3). $\square$

Proposition 2.7. Assume (H_1) and that there exists $u_0 \in \mathcal{N}$ such that $I(u_0) = m$. Then u_0 is ground state solution for (2.1) and $u_0 > 0$ in $\mathbb{R}^N$.

Proof. Since m is the minimum of I on $\mathcal{N}$, Lagrange multiplier theorem implies that there exists $\lambda \in \mathbb{R}$ such that $I'(u_0) = \lambda J'(u_0)$. Thus

$$\lambda J'(u_0)(u_0) = I'(u_0)(u_0) = J(u_0) = 0.$$

According to the previous proposition, $\lambda = 0$, and so, $I'(u_0) = 0$.

Proposition 2.4 implies that $u_0 \neq 0$. Therefore, $u_0 > 0$ in $\mathbb{R}^N$ (see Remark 2.2). $\square$

The next proposition shows that, under (H_1) , there exists a Palais–Smale sequence for I associated with the minimum m .

Proposition 2.8. Assume (H_1) . There exists a sequence $(u_n) \subset \mathcal{N}$ such that: $u_n \geq 0$ in $\mathbb{R}^N$, $I(u_n) \rightarrow m$ and $I'(u_n) \rightarrow 0$ in D^{-1} .

Proof. According to the Ekeland variational principle (see [21, Theorem 8.5]), there exist $(v_n) \subset \mathcal{N}$ and $(\lambda_n) \subset \mathbb{R}$ such that

$$I(v_n) \rightarrow m \quad \text{and} \quad I'(v_n) - \lambda_n J'(v_n) \rightarrow 0 \quad \text{in } D^{-1}.$$

It follows from (2.4) that

$$\left(\frac{1}{2} - \frac{1}{p^-} \right) \|v_n\|_{1,2}^2 \leq I(v_n).$$

This implies that (v_n) is bounded in $D^{1,2}(\mathbb{R}^N)$. Hence, taking into account that

$$|I'(v_n)(v_n) - \lambda_n J'(v_n)(v_n)| \leq \|I'(v_n) - \lambda_n J'(v_n)\|_{D^{-1}} \|v_n\|_{1,2},$$

we have

$$I'(v_n)(v_n) - \lambda_n J'(v_n)(v_n) \rightarrow 0.$$

Using the fact that $I'(v_n)(v_n) = 0$, we conclude from Proposition 2.6 that $\lambda_n \rightarrow 0$. Consequently, $I'(v_n) \rightarrow 0$ in D^{-1} .

We affirm that the sequence (v_n^+) satisfies $I(v_n^+) \rightarrow m$ and $I'(v_n^+) \rightarrow 0$ in D^{-1} . Indeed, since

$$\|v_n^-\|_{1,2} = I'(v_n)(v_n^-) \rightarrow 0,$$

we derive

$$I(v_n^+) = I(v_n) - \frac{1}{2} \|v_n^-\|_{1,2}^2 \rightarrow m.$$

Moreover,

$$|I'(v_n^+)|_{D^{-1}} = \sup_{|\phi| \leq 1} \|I'(v_n^+)(\phi)\| = \sup_{|\phi| \leq 1} \|I'(v_n)(\phi) - \int_{\mathbb{R}^N} \nabla(v_n^-) \cdot \nabla \phi dx\| \leq \|I'(v_n)\|_{D^{-1}} + \|v_n^-\|_{1,2} \rightarrow 0.$$

Now, let us fix $t_n > 0$ such that $u_n := t_n v_n^+ \in \mathcal{N}$. Using the fact that $I'(u_n)u_n = 0$ and $I'(v_n^+)v_n^+ = o_n(1)$, a simple computation gives

$$t_n \rightarrow 1,$$

so that

$$I(u_n) - I(v_n^+) = o_n(1) \quad \text{and} \quad I'(u_n) - I'(v_n^+) = o_n(1).$$

Hence,

$$u_n \in \mathcal{N}, \quad u_n \geq 0, \quad I(u_n) \rightarrow m \quad \text{and} \quad I'(u_n) \rightarrow 0.$$

This proves the proposition. $\square$

The next proposition provides a special upper bound for m .

Proposition 2.9. Assume (H_1) . Then $m < \frac{1}{N} S^{\frac{N}{2}}$, where S denotes the Sobolev constant defined by (1.1).

Proof. Let

$$w_k(x) := w(x + k e_N), \quad e_N = (0, 0, \dots, 0, 1),$$

where $w: \mathbb{R}^N \rightarrow \mathbb{R}$ is the Aubin–Talenti function given by (1.2), which satisfies

$$\|w\|_{1,2}^2 = \|w\|_{2^*}^{2^*} = S^{\frac{N}{2}}.$$

A direct computation shows that $\|w_k\|_{2^*} = \|w\|_{2^*}$ and $\|w_k\|_{1,2} = \|w\|_{1,2}$. Moreover, exploring the expression of w , we can easily check that $w_k \rightarrow 0$ uniformly in bounded sets and, therefore,

$$\lim_{k \rightarrow \infty} \int_{\Omega_\delta} |w_k|^\alpha dx = 0 \quad (2.5)$$

for any $\alpha > 0$.

By Proposition 2.5, there exists $t_k > 0$ such that $t_k w_k \in \mathcal{N}$, which means that

$$t_k^2 \|w_k\|_{1,2}^2 = \int_{\Omega_\delta} (t_k w_k)^{p(x)} dx + \int_{\mathbb{R}^N \setminus \Omega_\delta} (t_k w_k)^{2^*} dx.$$

Hence,

$$\|w\|_{1,2}^2 = \|w_k\|_{1,2}^2 \geq t_k^{2^*-2} \int_{\mathbb{R}^N \setminus \Omega_\delta} |w_k|^{2^*} dx$$

and then, by using (2.5) for $\alpha = 2^*$, we can verify that the sequence (t_k) is bounded,

$$\limsup_{k \rightarrow \infty} t_k \leq \limsup_{k \rightarrow \infty} \left(\frac{\|w\|_{1,2}^2}{\int_{\mathbb{R}^N \setminus \Omega_\delta} (w_k)^{2^*} dx} \right)^{\frac{1}{2^*-2}} = \left(\frac{\|w\|_{1,2}^2}{\|w\|_{2^*}^{2^*}} \right)^{\frac{1}{2^*-2}} = 1.$$

Moreover, since $t_k w_k \in \mathcal{N}$,

$$\begin{aligned} m &\leq I(t_k w_k) \\ &= \frac{t_k^2}{2} \|w_k\|_{1,2}^2 - \int_{\mathbb{R}^N \setminus \Omega_\delta} \frac{(t_k w_k)^{2^*}}{2^*} dx - \int_{\Omega_\delta} \frac{(t_k w_k)^{p(x)}}{p(x)} dx \\ &= \frac{t_k^2}{2} S^{\frac{N}{2}} - \int_{\mathbb{R}^N} \frac{(t_k w_k)^{2^*}}{2^*} dx + \int_{\Omega_\delta} \frac{(t_k w_k)^{2^*}}{2^*} dx - \int_{\Omega_\delta} \frac{(t_k w_k)^{p(x)}}{p(x)} dx \\ &= S^{\frac{N}{2}} \left(\frac{t_k^2}{2} - \frac{t_k^{2^*}}{2^*} \right) + \int_{\Omega_\delta} \left(\frac{(t_k w_k)^{2^*}}{2^*} - \frac{(t_k w_k)^{p(x)}}{p(x)} \right) dx \\ &\leq \frac{S^{\frac{N}{2}}}{N} + \int_{\Omega_\delta} \left(\frac{(t_k w_k)^{2^*}}{2^*} - \frac{(t_k w_k)^{p(x)}}{p(x)} \right) dx, \end{aligned}$$

where we have used that the maximum value of the function $t \in [0, \infty) \mapsto \frac{t^2}{2} - \frac{t^{2^*}}{2^*}$ is $\frac{1}{N}$.

Combining the boundedness of the sequence (t_k) with the fact that $w_k \rightarrow 0$ uniformly in Ω_δ , we can select k sufficiently large, such that $t_k w_k \leq 1$ in Ω_δ . Therefore, for this k ,

$$\begin{aligned} m &\leq \frac{S^{\frac{N}{2}}}{N} + \int_{\Omega_\delta} \left(\frac{(t_k w_k)^{2^*}}{2^*} - \frac{(t_k w_k)^{2^*}}{p(x)} \right) dx \\ &= \frac{S^{\frac{N}{2}}}{N} + t_k^{2^*} \int_{\Omega_\delta} (w_k)^{2^*} \left(\frac{1}{2^*} - \frac{1}{p(x)} \right) dx < \frac{S^{\frac{N}{2}}}{N}, \end{aligned}$$

since the latter integrand is strictly negative in Ω , which has positive N -dimensional Lebesgue measure. $\square$

2.2 Existence of a ground state solution

Our main result in this section is the following.

Theorem 2.10. *Assume that (H_1) holds. Then problem (2.1) has at least one ground state solution.*

We prove this theorem throughout this subsection by using the following well-known result.

Lemma 2.11 (Lions' lemma [17]). *Let (u_n) be a sequence in $D^{1,2}(\mathbb{R}^N)$, $N > 2$, satisfying*

- $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$,
- $|\nabla u_n|^2 \rightharpoonup \mu$ in $\mathcal{M}(\mathbb{R}^N)$,
- $|u_n|^{2^*} \rightharpoonup \nu$ in $\mathcal{M}(\mathbb{R}^N)$.

Then there exist an at most countable set of indices $\mathcal{J}$, points $(x_i)_{i \in \mathcal{J}}$ and positive numbers $(v_i)_{i \in \mathcal{J}}$ such that

- (i) $\nu = |u|^{2^*} + \sum_{i \in \mathcal{J}} v_i \delta_{x_i}$,
- (ii) $\mu(\{x_i\}) \geq v_i^{2/2^*} S$ for any $i \in \mathcal{J}$,

where δ_{x_i} denotes the Dirac measure supported at x_i .

We know from Proposition 2.8 that there exists a sequence $(u_n) \subset \mathcal{N}$ satisfying $u_n \geq 0$ in $\mathbb{R}^N$, $I(u_n) \rightarrow m$ and $I'(u_n) \rightarrow 0$ in D^{-1} . Since (u_n) is bounded in $D^{1,2}(\mathbb{R}^N)$, we can assume (by passing to a subsequence) that there exists $u \in D^{1,2}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$, $u_n \rightarrow u$ in $L^s_{\text{loc}}(\mathbb{R}^N)$ for $1 \leq s < 2^*$ and $u_n(x) \rightarrow u(x)$ a.e. in $\mathbb{R}^N$. Moreover, $|\nabla u_n|^2 \rightharpoonup \mu$ and $|u_n|^{2^*} \rightharpoonup \nu$ in $\mathcal{M}(\mathbb{R}^N)$.

We claim that $u \neq 0$. Indeed, let us suppose, by contradiction, that $u \equiv 0$. We affirm that this assumption implies that the set $\mathcal{J}$ given by Lions' lemma is empty. Otherwise, let us fix $i \in \mathcal{J}$, $x_i \in \mathbb{R}^N$ and $v_i > 0$ as in Lions' lemma. Let $\phi \in C_c^\infty(\mathbb{R}^N)$ such that

$$\phi(x) = \begin{cases} 1, & x \in B_1(0), \\ 0, & x \notin B_2(0), \end{cases}$$

and $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}^N$, where B_1 and B_2 denotes the balls centered at the origin, with radius 1 and 2, respectively.

For $\epsilon > 0$ fixed, define

$$\phi_\epsilon(x) = \phi\left(\frac{x - x_i}{\epsilon}\right).$$

Since (u_n) is bounded in $D^{1,2}(\mathbb{R}^N)$, the same holds for the sequence $(\phi_\epsilon u_n)$. Thus,

$$|I'(u_n)(\phi_\epsilon u_n)| \leq \|I'(u_n)\|_{D^{-1}} \|\phi_\epsilon u_n\|_{1,2} = o_n(1),$$

so that

$$\int_{\mathbb{R}^N} \nabla u_n \cdot \nabla(\phi_\epsilon u_n) dx = \int_{\mathbb{R}^N} (u_n)^{p(x)} \phi_\epsilon dx + o_n(1).$$

Consequently,

$$\int_{\mathbb{R}^N} \phi_\epsilon |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} u_n \nabla u_n \cdot \nabla \phi_\epsilon dx \leq \int_{\mathbb{R}^N} |u_n|^{p^-} \phi_\epsilon dx + \int_{\mathbb{R}^N} |u_n|^{2^*} \phi_\epsilon dx + o_n(1). \quad (2.6)$$

According to Lions' lemma,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \phi_\epsilon \, dx \rightarrow \int_{\mathbb{R}^N} \phi_\epsilon \, d\mu \quad \text{and} \quad \int_{\mathbb{R}^N} |u_n|^{2^*} \phi_\epsilon \, dx \rightarrow \int_{\mathbb{R}^N} \phi_\epsilon \, dv.$$

Since

$$\left| \int_{\mathbb{R}^N} u_n \nabla u_n \cdot \nabla \phi_\epsilon \, dx \right| \leq \|\nabla \phi_\epsilon\|_\infty \left(\int_{B_{2\epsilon}(x_i)} |u_n|^2 \, dx \right)^{\frac{1}{2}} \|u_n\|_{1,2} \rightarrow 0$$

and

$$\int_{\mathbb{R}^N} |u_n|^{p^-} \phi_\epsilon \, dx \rightarrow 0,$$

it follows from (2.6) that

$$\int_{\mathbb{R}^N} \phi_\epsilon \, d\mu \leq \int_{\mathbb{R}^N} \phi_\epsilon \, dv \quad \text{for all } \epsilon > 0.$$

Now, making $\epsilon \rightarrow 0$, we get

$$\mu(\{x_i\}) \leq v_i.$$

Combining this inequality with part (ii) of Lions' lemma, we obtain $v_i \geq S^{\frac{N}{2}}$. It follows that

$$S^{\frac{N}{2}} \leq S v_i^{2/2^*} \leq \mu(\{x_i\}) \leq v_i.$$

Let $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $\phi(x_i) = 1$ and $0 \leq \phi(x) \leq 1$, for any $x \in \mathbb{R}^N$. Recalling that

$$I(u_n) = I(u_n) - \frac{1}{2^*} I'(u_n)(u_n) = \frac{1}{N} \|u_n\|_{1,2}^2 + \int_{\Omega_\delta} \left(\frac{1}{2^*} - \frac{1}{p(x)} \right) |u_n|^{p(x)} \, dx,$$

we have

$$I(u_n) \geq \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_n|^2 \phi \, dx + \int_{\Omega_\delta} \left(\frac{1}{2^*} - \frac{1}{p(x)} \right) |u_n|^{p(x)} \, dx. \quad (2.7)$$

Since $p: \mathbb{R} \rightarrow \mathbb{R}$ is continuous, for each $\epsilon > 0$, there exists $\Omega_{\delta,\epsilon} \subset \Omega_\delta$ such that

$$\left| \frac{1}{2^*} - \frac{1}{p(x)} \right| < \frac{\epsilon}{2M}, \quad x \in \Omega_\delta \setminus \Omega_{\delta,\epsilon},$$

where $M = \sup_{n \in \mathbb{N}} (\int_{\Omega_\delta} |u_n|^{p^-} + |u_n|^{2^*} \, dx)$. Thus,

$$\begin{aligned} \left| \int_{\Omega_\delta} \left(\frac{1}{2^*} - \frac{1}{p(x)} \right) |u_n|^{p(x)} \, dx \right| &\leq \frac{\epsilon}{2M} \int_{\Omega_\delta \setminus \Omega_{\delta,\epsilon}} |u_n|^{p(x)} \, dx + \left(\frac{1}{p^-} - \frac{1}{2^*} \right) \int_{\Omega_{\delta,\epsilon}} |u_n|^{p(x)} \, dx \\ &\leq \frac{\epsilon}{2M} \int_{\Omega_\delta} (|u_n|^{p^-} + |u_n|^{2^*}) \, dx + \left(\frac{1}{p^-} - \frac{1}{2^*} \right) \int_{\Omega_{\delta,\epsilon}} (|u_n|^{p^-} + |u_n|^q) \, dx \\ &\leq \frac{\epsilon}{2} + \left(\frac{1}{p^-} - \frac{1}{2^*} \right) \int_{\Omega_\delta} (|u_n|^{p^-} + |u_n|^q) \, dx, \end{aligned}$$

where $2 < p^- \leq p(x) \leq q < 2^*$, for $x \in \Omega_{\delta,\epsilon}$. Then, since $u_n \rightarrow 0$ in $L_{\text{loc}}^s(\mathbb{R}^N)$, for $s \in [1, 2^*)$, and ϵ is arbitrary, we conclude that

$$\lim_{n \rightarrow \infty} \int_{\Omega_\delta} \left(\frac{1}{2^*} - \frac{1}{p(x)} \right) |u_n|^{p(x)} \, dx = 0.$$

Therefore, by making $n \rightarrow \infty$ in (2.7), we obtain

$$m \geq \frac{1}{N} \int_{\mathbb{R}^N} \phi \, d\mu \geq \frac{1}{N} \int_{\{x_i\}} \phi \, d\mu = \frac{1}{N} \mu(\{x_i\}) \geq \frac{1}{N} S^{\frac{N}{2}}$$

which contradicts Proposition 2.9, showing that $\mathcal{J} = \emptyset$. Hence, it follows from Lions' lemma that

$$u_n \rightarrow 0 \text{ in } L_{\text{loc}}^{2^*}(\mathbb{R}^N).$$

In particular, $u_n \rightarrow 0$ in $L^{2^*}(\Omega_\delta)$, so that

$$0 \leq \int_{\Omega_\delta} |u_n|^{p(x)} dx \leq \int_{\Omega_\delta} |u_n|^{p^-} dx + \int_{\Omega_\delta} |u_n|^{2^*} dx \rightarrow 0.$$

Since $(u_n) \subset \mathcal{N}$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|u_n\|_{1,2}^2 &= \lim_{n \rightarrow \infty} \left(\int_{\Omega_\delta} |u_n|^{p(x)} dx + \int_{\mathbb{R}^N \setminus \Omega_\delta} |u_n|^{2^*} dx \right) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus \Omega_\delta} |u_n|^{2^*} dx =: L. \end{aligned}$$

Thus, by making $n \rightarrow \infty$ in the equality

$$I(u_n) = \frac{1}{2} \|u_n\|_{1,2}^2 - \int_{\Omega_\delta} \frac{|u_n|^{p(x)}}{p(x)} dx - \frac{1}{2^*} \int_{\mathbb{R}^N \setminus \Omega_\delta} |u_n|^{2^*} dx,$$

we obtain

$$m = \frac{1}{2}L - \frac{1}{2^*}L = \frac{1}{N}L.$$

Since

$$S \leq \frac{\|u_n\|_{1,2}^2}{\|u_n\|_{2^*}^2} \leq \frac{\|u_n\|_{1,2}^2}{\left(\int_{\mathbb{R}^N \setminus \Omega_\delta} |u_n|^{2^*} dx \right)^{2/2^*}},$$

we obtain $m = \frac{L}{N} \geq \frac{1}{N}S^{\frac{N}{2}}$, which contradicts Proposition 2.9 and proves that $u \neq 0$.

Now, combining the weak convergence

$$u_n \rightharpoonup u \text{ in } D^{1,2}(\mathbb{R}^N)$$

with the fact that $I'(u_n) \rightarrow 0$ in D^{-1} , we conclude that

$$I'(u)(v) = 0 \quad \text{for all } v \in D^{1,2}(\mathbb{R}^N),$$

meaning that u is a nontrivial critical point of I .

Thus, taking into account Proposition 2.7, in order to complete the proof that u is a ground state solution for (2.1) we need to verify that $I(u) = m$. Indeed, since

$$I(u_n) = \left(\frac{1}{2} - \frac{1}{p^-} \right) \|u_n\|_{1,2}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{p^-} - \frac{1}{p(x)} \right) u_n^{p(x)} dx,$$

the weak convergence $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$ and Fatou's lemma imply that

$$\begin{aligned} m &\geq \left(\frac{1}{2} - \frac{1}{p^-} \right) \liminf_{n \rightarrow \infty} \|u_n\|_{1,2}^2 + \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left(\frac{1}{p^-} - \frac{1}{p(x)} \right) u_n^{p(x)} dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p^-} \right) \|u\|_{1,2}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{p^-} - \frac{1}{p(x)} \right) u^{p(x)} dx \\ &= I(u) - \frac{1}{p^-} I'(u)(u) = I(u) \geq m, \end{aligned}$$

showing that $I(u) = m$.

3 The semilinear elliptic problem in a bounded domain

In this section we consider the elliptic problem

$$\begin{cases} -\Delta u = u^{p(x)-1}, & u > 0, \quad \text{in } G \\ u = 0 & \text{on } \partial G, \end{cases} \quad (3.1)$$

where G is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 3$, and $p: G \rightarrow \mathbb{R}$ is a continuous function verifying (H_1) and an additional hypothesis (H_2) , which is stated in the sequel.

We recall that the usual norm in $H_0^1(G)$ is given by

$$\|u\| := \|\nabla u\|_2 = \left(\int_G |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$

We denote the dual space of $H_0^1(G)$ by H^{-1} .

The energy functional $I: H_0^1(G) \rightarrow \mathbb{R}$ associated with problem (3.1) is defined by

$$I(u) := \frac{1}{2} \int_G |\nabla u|^2 \, dx - \int_G \frac{1}{p(x)} (u^+)^{p(x)} \, dx.$$

It belongs to $C^1(H_0^1(G), \mathbb{R})$ and its derivative is given by

$$I'(u)(v) = \int_G \nabla u \cdot \nabla v \, dx - \int_G (u^+)^{p(x)-1} v \, dx \quad \text{for all } u, v \in H_0^1(G).$$

Thus, a function $u \in H_0^1(G)$ is a weak solution of (3.1) if, and only if, u is a critical point of I . Moreover, as in Section 2, the nontrivial critical points of I are positive in G (a consequence of the Strong Maximum Principle).

We maintain the notation of Section 2. Thus,

$$J(u) := I'(u)(u) = \|u\|^2 - \int_G (u^+)^{p(x)} \, dx,$$

the Nehari manifold associated with (3.1) is defined by

$$\mathcal{N} := \{u \in H_0^1(G) \setminus \{0\} : J(u) = 0\}$$

and $m := \inf_{u \in \mathcal{N}} I(u)$.

Definition 3.1. We say that $u \in \mathcal{N}$ is a *ground state solution* for (3.1) if $I'(u) = 0$ and $I(u) = m$.

We gather in the next lemma some results that can be proved as in Section 2.

Lemma 3.2. Assume (H_1) . We claim that:

- (i) $m > 0$.
- (ii) $J'(u)(u) < 0$ for all $u \in \mathcal{N}$. (Thus, $J'(u) \neq 0$ for all $u \in \mathcal{N}$.)
- (iii) If $I(u_0) = m$, then $I'(u_0) = 0$. (Thus, u_0 is a positive weak solution of (3.1).)
- (iv) There exists a sequence $(u_n) \subset \mathcal{N}$ such that $u_n \geq 0$ in G , $I(u_n) \rightarrow m$ and $I'(u_n) \rightarrow 0$ in H^{-1} .

Unfortunately, hypothesis (H_1) by itself is not sufficient to guarantee that $m < \frac{S_N^{\frac{N}{2}}}{N}$ as in Proposition 2.9. The reason is that the translation argument used in the proof of that Proposition does not apply to a bounded domain. So we assume an additional assumption (H_2) . In order to properly state such an assumption, we need some background information.

Let $U \subset \mathbb{R}^N$ be a bounded domain and define

$$S_q(U) := \inf \left\{ \frac{\|\nabla v\|_{L^2(U)}^2}{\|v\|_{L^q(U)}^2} : v \in H_0^1(U) \setminus \{0\} \right\}, \quad 1 \leq q \leq 2^*. \quad (3.2)$$

It is well known that if $1 \leq q < 2^*$, then the infimum in (3.2) is attained by a positive function ϕ_q in $H_0^1(U)$. Actually, this follows from the compactness of the embedding $H_0^1(U) \hookrightarrow L^q(U)$.

Another well-known fact is that in the case $q = 2^*$ the infimum in (3.2) coincides with the best Sobolev constant, i.e.

$$S_{2^*}(U) = S := \inf \left\{ \frac{\|\nabla v\|_{L^2(\mathbb{R}^N)}^2}{\|v\|_{L^{2^*}(\mathbb{R}^N)}^2} : v \in D^{1,2}(\mathbb{R}^N) \setminus \{0\} \right\}. \quad (3.3)$$

Moreover, in this case the infimum (3.2) is not attained if U is a proper subset of $\mathbb{R}^N$.

In the sequence we make use of the function

$$g(q) := \left(\frac{1}{2} - \frac{1}{q} \right) S_q(U)^{\frac{q}{q-2}}, \quad q \in (2, 2^*].$$

Lemma 3.3. *If $S_2(U) \leq 1$, then there exists $\bar{q} \in (2, 2^*]$ such that*

$$g(q) < g(\bar{q}) = \frac{1}{N} S^{\frac{N}{2}} \quad \text{for all } q \in (2, \bar{q}). \quad (3.4)$$

Proof. The following facts are known, where $|U|$ denotes the volume of U (see [2, 10]): the function

$$q \in [1, 2^*] \rightarrow S_q(U)$$

is continuous (in fact it is α -Hölder continuous, for any $0 < \alpha < 1$, as proved in [11]) and the function

$$q \in [1, 2^*] \rightarrow |U|^{\frac{2}{q}} S_q(U)$$

is strictly decreasing. It follows that

$$\lim_{q \rightarrow 2^+} S_q(U) = S_2(U)$$

and

$$|U|^{\frac{2}{q}} S_q(U) < |U| S_2(U), \quad q \in (2, 2^*].$$

This latter inequality implies that

$$g(q) < g_1(q) := |U| \left(\frac{1}{2} - \frac{1}{q} \right) S_2(U)^{\frac{q}{q-2}}, \quad q \in (2, 2^*]. \quad (3.5)$$

Hence, using that $S_2(U) \leq 1$, we see that

$$\lim_{q \rightarrow 2^+} g(q) = 0.$$

Taking into account that $S_{2^*}(U) = S$, we can easily check that $g(2^*) = \frac{1}{N} S^{\frac{N}{2}}$. Thus, defining

$$\bar{q} := \min \left\{ q \in (2, 2^*] : g(q) = \frac{1}{N} S^{\frac{N}{2}} \right\}, \quad (3.6)$$

we arrive at (3.4). $\square$

The additional hypothesis (H_2) is stated as follows, where Ω , p^- and p^+ are defined in (H_1) and $\bar{q}$ is given by (3.6):

(H_2) There exists a subdomain U of Ω such that

$$S_2(U) \leq 1 \quad \text{and} \quad p^- \leq p(x) \equiv q < \min\{\bar{q}, p^+\} \quad \text{for all } x \in U.$$

Lemma 3.4. *Assume that (H_1) and (H_2) hold. Then $m < \frac{1}{N} S^{\frac{N}{2}}$.*

Proof. Let $\phi_q \in H_0^1(U)$ denote a positive extremal function of $S_q(U)$. Thus, $\phi_q > 0$ in U and

$$S_q(U) = \frac{\|\nabla \phi_q\|_{L^2(U)}^2}{\|\phi_q\|_{L^q(U)}^2}.$$

Let us define the function $\tilde{\phi}_q \in H_0^1(G)$ by

$$\tilde{\phi}_q(x) := \begin{cases} \phi_q(x) & \text{if } x \in U, \\ 0 & \text{if } x \in G \setminus U. \end{cases}$$

For each $t > 0$ we have

$$I(t\tilde{\phi}_q) = \frac{t^2}{2} \int_G |\nabla \tilde{\phi}_q|^2 dx - \int_G \frac{t^{p(x)}}{p(x)} (\tilde{\phi}_q)^{p(x)} dx = \frac{\alpha}{2} t^2 - \frac{\beta}{q} t^q,$$

where

$$\alpha := \int_U |\nabla \phi_q|^2 dx \quad \text{and} \quad \beta := \int_U (\phi_q)^q dx.$$

Taking

$$t_q := \left(\frac{\alpha}{\beta} \right)^{\frac{1}{q-2}},$$

it is easy to see that $t_q \tilde{\phi} \in \mathcal{N}$ and

$$I(t_q \tilde{\phi}_q) = \left(\frac{1}{2} - \frac{1}{q} \right) \left(\frac{\alpha^q}{\beta^2} \right)^{\frac{1}{q-2}} = \left(\frac{1}{2} - \frac{1}{q} \right) S_q(U)^{\frac{q}{q-2}}.$$

Since $S_2(U) \leq 1$ and $p^- \leq q < \min\{p^+, \bar{q}\}$, it follows from Lemma 3.3 that

$$g(q) = I(t_q \tilde{\phi}_q) < \frac{1}{N} S^{\frac{N}{2}}.$$

This implies that $m < \frac{1}{N} S^{\frac{N}{2}}$. □

The main result in this section is the following.

Theorem 3.5. Assume (H_1) and (H_2) . Then problem (3.1) has at least one ground state solution.

Proof. According to item (iv) of Lemma 3.2, there exists a sequence $(u_n) \subset \mathcal{N}$ satisfying $I(u_n) \rightarrow m$ and $I'(u_n) \rightarrow 0$ in H^{-1} . Since (u_n) is bounded in $H_0^1(G)$, there exist $u \in H_0^1(G)$ and a subsequence, still denoted by (u_n) , such that $u_n \rightharpoonup u$ in $H_0^1(G)$, $u_n \rightarrow u$ in $L^p(G)$, for $1 \leq p < 2^*$, and $u_n(x) \rightarrow u(x)$ a.e. in G . Arguing as in Section 2, we can combine Lions' lemma and Lemma 3.4 to prove that $u \neq 0$, $I'(u) = 0$ and $I(u) = m$, showing thus that u is a ground state solution of (3.1). □

3.1 On hypothesis (H_2)

In this subsection we present some lower bounds for the value of $\bar{q}$, defined by (3.6), which can be used as the value constant for $p(x)$ in hypothesis (H_2) . Moreover, we give some examples of simple bounded domains U such that $S_2(U) \leq 1$.

The value of $\bar{q}$ depends on the function $q \mapsto g(q)$, which in turn depends on the function $q \mapsto S_q(U)$. It is well known that $S_q(U)$ is the least value of λ for which the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda \|u\|_{L^2(U)}^{2-q} |u|^{q-2} u & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$$

has a nontrivial weak solution. When $p = 2$, this is the well-studied eigenvalue problem for $(-\Delta, H_0^1(U))$ and $S_2(U)$ is its first eigenvalue. It follows that $S_2(U)$ can be found analytically for some simple domains as balls, rectangles and other domains enjoying some kind of symmetry. For instance, if U is a ball of radius R , then

$$S_2(U) = \left(\frac{j_{\frac{N}{2}-1,1}}{R} \right)^2, \quad (3.7)$$

where $j_{\alpha,1}$ denotes the first positive root of the first kind Bessel function of order α .

When $q \neq 2$ the above problem is no longer linear and, consequently, it is more difficult to be solved analytically, even for simple domains. For this reason, determining an analytical expression for the function g on the interval $(2, 2^*)$ is a hard task and we do not know the exact value of $\bar{q}$ given by (3.6). However, the inequality (3.5) allows us to derive lower bounds for $\bar{q}$, in terms of $S_2(U)$, $|U|$ and S , which can be used as the value constant for $p(x)$ in hypothesis (H_2) . In fact, assuming $S_2(U) \leq 1$, we can easily verify that $g'_1(q) > 0$ for all $q \in (2, 2^*)$, where the function $q \mapsto g_1(q)$ is defined in (3.5). Therefore, taking into account that $\lim_{q \rightarrow 2^+} g_1(2) = 0$ and $g_1(\bar{q}) > g(\bar{q}) = \frac{1}{N} S^{\frac{N}{2}}$, there exists a unique value $q_1 \in (2, \bar{q})$ such that $g_1(q_1) = \frac{1}{N} S^{\frac{N}{2}}$, that is

$$|U| \left(\frac{1}{2} - \frac{1}{q_1} \right) S_2(U)^{\frac{q_1}{q_1-2}} = \frac{1}{N} S^{\frac{N}{2}},$$

an equation that can be solved at least numerically.

A rougher but explicit lower bound q_2 for $\bar{q}$ follows from the inequality

$$g_1(q) < g_2(q) := \frac{|U|}{N} S_2(U)^{\frac{q}{q-2}}, \quad q \in (2, 2^*),$$

which is obtained from (3.5) by observing that $\frac{1}{2} - \frac{1}{q} \leq \frac{1}{2} - \frac{1}{2^*} = \frac{1}{N}$. Indeed, since the function g_2 enjoys the same properties as g_1 , there exists a unique point $q_2 \in (2, \bar{q})$ satisfying $g_2(q_2) = \frac{1}{N} S^{\frac{N}{2}}$. A simple calculation yields

$$q_2 := \frac{2 \log\left(\frac{|U|}{S^{\frac{N}{2}}}\right)}{\log\left(\frac{S_2(U)|U|}{S^{\frac{N}{2}}}\right)}.$$

Of course, $2 < q_2 < q_1 < \bar{q}$.

A third lower bound q_3 for $\bar{q}$ also follows from (3.5). Indeed, by using that $S_2(U) \leq 1$ in (3.5), we obtain

$$g(q) < g_3(q) := |U| \left(\frac{1}{2} - \frac{1}{q} \right), \quad q \in (2, 2^*].$$

Hence, since $g'_3 > 0$ and $g_3(2) = 0$, there exists a unique point $q_3 \in (2, \bar{q})$ satisfying $g_3(q_3) = \frac{S^{\frac{N}{2}}}{N}$. Such a point is given explicitly by

$$q_3 := \frac{2N|U|}{N|U| - 2S^{\frac{N}{2}}}.$$

Another conclusion that follows easily from the monotonicity of the function $q \mapsto |U|^{\frac{2}{q}} S_q(U)$ combined with (3.3) is that if $S_2(U) \leq 1$, then $|U| > S^{\frac{N}{2}}$.

In the sequel, we present sufficient conditions for the inequality $S_2(U) \leq 1$ to hold when U is either a ball or an annulus. We will denote by $B_R(y)$ the ball centered at y with radius $R > 0$. When $y = 0$, we will write simply B_R .

Example 3.6. Let $U = B_R(y) \subset \Omega$. Since the Laplacian operator is invariant under translations,

$$S_2(B_R) = S_2(B_R(y)).$$

Moreover, a simple scaling argument (or (3.7)) yields

$$S_2(B_R) = R^{-2} S_2(B_1).$$

So, if $R \geq S_2(B_1)^{\frac{1}{2}}$, then $S_2(U) = S_2(B_R(y)) \leq 1$.

Example 3.7. Let $U = B_R(y) \setminus \overline{B_r(z)} \subset \Omega$, with $\overline{B_r(z)} \subset B_R(y)$, for some $y, z \in \Omega$ and $R > r > 0$. Since the Laplacian operator is invariant under orthogonal transformations, we can see that

$$S_2(B_R \setminus \overline{B_r(se_1)}) = S_2(U)$$

for some $s \in [0, R - r]$, where e_1 denotes the first coordinate vector. According to [15, Proposition 3.2], the function $t \mapsto S_2(B_R \setminus \overline{B_r(te_1)})$ is strictly decreasing for $t \in [0, R - r]$. Therefore,

$$S_2(B_R \setminus \overline{B_r}) \geq S_2(U). \quad (3.8)$$

Since $B_{(R-r)/2}$ is the largest ball contained in $B_R \setminus \overline{B_r}$, we have

$$S_2(B_R \setminus \overline{B_r}) < S_2(B_{(R-r)/2}) = \left(\frac{R-r}{2}\right)^{-2} S_2(B_1). \quad (3.9)$$

Hence, if $R-r \geq 2S_2(B_1)^{\frac{1}{2}}$, then (3.8) and (3.9) imply that $S_2(U) < 1$.

Thus, we can replace the condition $S(U) \leq 1$ in (H_2) by either $R \leq S_2(B_1)^{\frac{1}{2}}$ when $U = B_R(y)$ or $R-r \geq 2S_2(B_1)^{\frac{1}{2}}$ when $U = B_R(y) \setminus \overline{B_r(z)}$.

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