Research Article

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Ground state solutions for the Hénon prescribed mean curvature equation

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Abstract: In this paper, we consider the analogous of the Hénon equation for the prescribed mean curvature problem in $\mathbb{R}^N$, both in the Euclidean and in the Minkowski spaces. Motivated by the studies of Ni and Serrin [W. M. Ni and J. Serrin, Existence and non-existence theorems for ground states for quasilinear partial differential equations, *Att. Convegni Lincei* **77** (1985), 231–257], we have been interested in finding the relations between the growth of the potential and that of the local nonlinearity in order to prove the nonexistence of a radial ground state. We also present a partial result on the existence of a ground state solution in the Minkowski space.

Keywords: Quasilinear elliptic equations, mean curvature operator, ODEs techniques

MSC 2010: 35J62, 35J93, 35A24

1 Introduction

As is well known, the partial differential operator

$$\nabla \cdot \left[\frac{\nabla(\cdot)}{\sqrt{1 \pm |\nabla(\cdot)|^2}} \right]$$

is related with questions on mean curvature in Euclidean and Minkowski spaces, depending on whether the sign under the square root is + or −. For this reason, it is known as the *mean curvature operator*.

There is a wide literature about the general Dirichlet problem related with the mean curvature operator, which reads as follows:

$$\begin{cases} \nabla \cdot \left[\frac{\nabla u}{\sqrt{1 \pm |\nabla u|^2}} \right] = f(x, u) & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is assumed to be a bounded domain (see, for example, [4, 5, 9–11]).

When Ω is unbounded, the problem

$$\begin{cases} \nabla \cdot \left[\frac{\nabla u}{\sqrt{1 \pm |\nabla u|^2}} \right] = f(x, u) & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases} \quad (1.1)$$

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is definitely less studied. In particular, up to our knowledge, problem (1.1) has been dealt with only when $\Omega = \mathbb{R}^N$ and f does not depend on x , in [12, 15, 16] for the Euclidean case and in [1, 2, 8] for the Minkowski case, respectively.

Apart from some results in [7] (see [7, Theorems 6.2 and 6.4]), as far as we know the study of (1.1) turns out to be a completely new issue in the nonautonomous case.

To start with this new scope of investigation, in this paper we are interested in studying the nonautonomous prescribed mean curvature equation in a classical situation, namely when we introduce the power $|x|^\alpha$ as a potential in the equation before the nonlinearity u^p ; consider the following problem:

$$\begin{cases} \nabla \cdot \left[\frac{\nabla u}{\sqrt{1 \pm |\nabla u|^2}} \right] + |x|^\alpha u^p = 0, \\ u(x) > 0 \\ u(x) \rightarrow 0 \end{cases} \quad \begin{matrix} \text{in } \mathbb{R}^N, \\ \\ \text{as } |x| \rightarrow \infty, \end{matrix} \quad (\mathcal{P})$$

where $N \geq 3$, $\alpha > 0$ and $1 < p$.

In this situation, the PDE is presented as an Hénon-type equation where the difference with respect to the classical Hénon equation consists in the fact that the Laplacian is replaced by the mean curvature operator.

The Hénon equation drew the attention of the mathematicians in recent years because of some interesting questions related with the symmetry property of its solutions. In [18], for example, the equation on a ball with Dirichlet boundary conditions was showed to have nonradial solutions for large values of α in the subcritical case $p + 1 < 2^* = \frac{2N}{N-2}$. In [3] and in the papers referenced inside, one can find a number of results concerning Hénon and Hénon-type equations in bounded domains with specific attention on the conditions on α and p which guarantee the existence of nonradial solutions in spite of nice symmetry properties both of the domain and of the associated functional.

In this paper, we will focus our attention on the research of radial solutions for $(\mathcal{P})$ which, in accordance with a classical definition, we will call radial *ground states*. We will denote by $(\mathcal{P}_+)$ and $(\mathcal{P}_-)$ problem $(\mathcal{P})$ where respectively the sign “+” or the sign “−” appears in the differential operator.

The main result that we provide is presented in Section 2 and consists in proving the nonexistence of radial ground states to $(\mathcal{P})$ when p is less than a value $2'_\alpha = \frac{N+\alpha}{N-2}$. Ni and Serrin [15] treated a variety of problems including $(\mathcal{P}_+)$ with $\alpha = 0$, proving among other things a nonexistence result for $p \in]1, \frac{N}{N-2}]$. We emphasize the fact that the value $\frac{N}{N-2}$ corresponds exactly to the value $2'_\alpha$ when $\alpha = 0$. As the authors themselves say in [15, Remark on p. 247], in their proof they use a Pohozaev-type identity (see [15, (1.6)]) besides considerations on a-priori asymptotic decaying estimates on ground state solutions.

This Pohozaev identity is strictly related with that in [17, Proposition 1] which was used in [2] to prove nonexistence for $(\mathcal{P}_-)$ when $\alpha = 0$ and $p \in]1, \frac{N+2}{N-2}]$.

Differently from [2, 15], in our case the presence of a potential in the equation perturbs the identity in a way that makes it useless. We overcome this difficulty by means of a suitable exploitation of some arguments which are typical in the ODE theory, comparing the graphs of ground state and sign-changing solutions in order to achieve our conclusion by a contradiction argument. However, it is worthy of note that, even if our proof is based on different tools with respect to Ni and Serrin, we are able to obtain almost the same nonexistence result as that in [15] for $(\mathcal{P}_+)$.

In Section 3, we briefly discuss the problem of existence of ground state solutions. We point out that variational arguments as those in [8] work fine also for problem $(\mathcal{P})$ in the Minkowski space. This fact allows us to prove the existence of a radial ground state solution when $p + 1$ is larger than the α -critical exponent related to 2, that is,

$$2^*_\alpha := \frac{2(N + \alpha)}{N - 2}.$$

It is easily seen that, for every $\alpha > 0$, we always have the inequality $2'_\alpha < 2^*_\alpha - 1$. As a consequence, our study leaves as an open issue what happens for $p \in [2'_\alpha, 2^*_\alpha - 1]$. The problem of finding ground state solutions for the Hénon prescribed mean curvature equation in the Euclidean space seems to be more complex and, at the present, it is completely open.

2 Nonexistence

Our main result (compare with [2, 15]) is the following theorem.

Theorem 2.1. *Let $N \geq 3$, $\alpha > 0$ and $p \in]1, \frac{N+\alpha}{N-2}[$. Then there exists no radial solution to problem $(\mathcal{P})$.*

Radial solutions of $(\mathcal{P})$ can be found looking for global positive solutions of the Cauchy problem defined in $\mathbb{R}_+$ for some $\xi > 0$:

$$\begin{cases} \left(\frac{u'}{\sqrt{1 \pm (u')^2}} \right)' + \frac{N-1}{r} \frac{u'}{\sqrt{1 \pm (u')^2}} + r^\alpha |u|^{p-1} u = 0, \\ u'(0) = 0, \\ u(0) = \xi \end{cases} \quad (\mathcal{C}_\xi)$$

such that $\lim_{r \rightarrow +\infty} u(r) = 0$.

By standard arguments, it is easy to prove that problem $(\mathcal{C}_\xi)$ has a unique local solution in a right neighborhood of 0. In the sequel, every time we consider any Cauchy problem whose local solution u_ξ can be continued until it touches the line $u = 0$, we will denote by $R_0(u_\xi) > 0$ the point such that $u(R_0(u_\xi)) = 0$ and $u|_{[0, R_0(u_\xi)]} > 0$.

Assume the following notations: We set $\rho_u(r) = |u'(r)|$ and define the following functions:

$$\begin{aligned} A^\pm(q) &= \frac{1}{\sqrt{1 \pm q^2}}, \\ \Omega^\pm(q) &= qA^\pm(q) = \frac{q}{\sqrt{1 \pm q^2}}, \\ E^\pm(q) &= \frac{d}{dq}(\Omega^\pm(q)) = 1/(1 \pm q^2)^{\frac{3}{2}}, \\ H^\pm(q) &= \int_0^q tE^\pm(t) dt = \pm 1 \mp (1 \pm q^2)^{-\frac{1}{2}}, \end{aligned}$$

which are of class C^∞ (respectively in $\mathbb{R}$ and in $] -1, 1[$ according to whether the sign is + or -). We will use the notations A , Ω , E and H every time our arguments work fine regardless of the sign. Define also the following functions in $\mathbb{R}$:

$$f(t) = |t|^{p-1}t \quad \text{and} \quad F(t) = \int_0^t f(s) ds = \frac{1}{p+1} |t|^{p+1}.$$

We are going to present some preliminary lemmas.

Lemma 2.2. *Let $1 < p < \frac{N+2+2\alpha}{N-2}$. Then there exists $\bar{\xi} > 0$ such that for all $\delta > 0$ there exists $\bar{\varepsilon} > 0$ for which the local solution w_ε^δ of the problem*

$$\begin{cases} \left(\frac{u'}{\sqrt{1 \pm \varepsilon(u')^2}} \right)' + \frac{N-1}{r} \frac{u'}{\sqrt{1 \pm \varepsilon(u')^2}} + r^\alpha |u|^{p-1} u = 0, \\ u'(0) = 0, \\ u(0) = \bar{\xi} \end{cases} \quad (2.1)$$

corresponding to $\varepsilon \in]0, \bar{\varepsilon}]$ is sign-changing and $R_0(w_\varepsilon^\delta) \in [1 - \delta, 1 + \delta]$.

Proof. Consider the following boundary value problem set in the unit ball centered in 0, denoted by B_1 :

$$\begin{cases} \Delta u + |x|^\alpha u^p = 0 & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \quad (2.2)$$

By [14], we know that (2.2) possesses at least a radial solution w . Now pick $\delta > 0$ and consider the Cauchy problem (2.1) for $\xi = w(0)$. Since $w(r)$ solves

$$\begin{cases} u'' + \frac{N-1}{r}u' + r^\alpha |u|^{p-1}u = 0, \\ u'(0) = 0, \\ u(0) = \xi, \end{cases} \quad (2.3)$$

and the equation in (2.1) is a regular perturbation of the equation in (2.3), we can find $\bar{\varepsilon} > 0$ sufficiently small such that, for any $\varepsilon \in]0, \bar{\varepsilon}]$, the local solution w_ε of (2.1) touches the axis somewhere in $[1 - \delta, 1 + \delta]$. $\square$

Lemma 2.3. *Let $1 < p < \frac{N+2+2\alpha}{N-2}$ and $\bar{\delta} > 0$, and consider $\bar{\xi} > 0$ as in Lemma 2.2. If $(w_\varepsilon^\delta)_\varepsilon$ is the family of sign-changing solutions of (2.1) built by means of Lemma 2.2, the family $(u_\varepsilon^\delta)_\varepsilon$ whose elements are defined for any $r > 0$ by the relation*

$$u_\varepsilon^\delta(r) = \varepsilon^{\frac{2+\alpha}{2(p+\alpha+1)}} w_\varepsilon^\delta(\varepsilon^{\frac{p-1}{2(p+\alpha+1)}} r)$$

is such that, for any $\varepsilon \in]0, \bar{\varepsilon}]$, the function u_ε^δ is a local sign-changing solution of $(\mathcal{C}_\xi)$ with

$$\xi = \varepsilon^{\frac{2+\alpha}{2(p+\alpha+1)}} \bar{\xi} \quad \text{and} \quad R_0(u_\varepsilon^\delta) \in [\varepsilon^{-\frac{p-1}{2(p+\alpha+1)}}(1 - \bar{\delta}), \varepsilon^{-\frac{p-1}{2(p+\alpha+1)}}(1 + \bar{\delta})].$$

Proof. The proof follows from Lemma 2.2 by straightforward computations. $\square$

Lemma 2.4. *If v is a radial solution of $(\mathcal{P})$, then the following assertions hold:*

- (i) $\liminf_{r \rightarrow +\infty} \rho_v(r) = 0$.
- (ii) *There exist $C > 0$ and $\tilde{R} > 0$ such that $v(r) > C/r^{N-2}$ for $\tilde{R} < r$.*

Proof. (i) By contradiction, let η be a positive number such that for some $R > 0$ and for any $r > R$ we have $\rho_v(r) \geq \eta$. Then, since $v'(r) = -\rho_v(r)$, for any $r > R$ we have

$$v(r) = v(R) - \int_R^r \rho_v(s) ds \leq v(R) + \eta(R - r).$$

Of course the inequality is false for r large enough.

(ii) We know that, for every $r > 0$,

$$\frac{d}{dr}[r^{N-1}\Omega(\rho_v(r))] = r^{\alpha+N-1}f(v(r)) > 0.$$

Then there exist $C > 0$ and $\tilde{R} > 0$ such that, for all $r > \tilde{R}$,

$$r^{N-1}\Omega(\rho_v(r)) > C.$$

Now we distinguish two cases:

- If we are considering Ω^+ , then we trivially deduce $\rho_v(r) > Cr^{1-N}$.
- If we are considering Ω^- , then either $\rho_v(r) \leq 1/\sqrt{2}$, and then, since $A^-(\rho_v(r)) \leq \sqrt{2}$, we have

$$\rho_v(r) > \frac{C}{\sqrt{2}}r^{1-N},$$

or $\rho_v(r) > 1/\sqrt{2} \geq Cr^{1-N}$ just, if needed, redefining $\tilde{R}$.

In any case we can assume that for suitable positive constants C and $\tilde{R}$ we have

$$\rho_v(r) > Cr^{1-N} \quad \text{for } \tilde{R} < r.$$

Integrating in $]r, +\infty[$, we get the conclusion. $\square$

Lemma 2.5. *Assume v is a ground state solution of $(\mathcal{C}_\xi)$. Then, if w is a sign-changing solution of $(\mathcal{C}_\xi)$ such that $w(0) < v(0)$, the graphs of w and v must intersect somewhere in $[0, R_0(w)] \times]0, +\infty[$.*

Proof. Assume by contradiction that w is a solution as in the statement of the lemma, and the set of points in $[0, R_0(w)] \times]0, +\infty[$ where the graphs of w and v intersect is empty.

Set $\xi_v = v(0)$ and $\xi_w = w(0)$. Since v and w are decreasing in $[0, R_0(u)]$ and in $\mathbb{R}^+$, respectively, we can define the functions $s(u)$ and $t(u)$, the inverses of $v(r)$ and $w(r)$, respectively, the first defined into $]0, \xi_v]$, the second into $[0, \xi_w]$.

Observe that, since $w'(R_0(w)) < 0$ and $\limsup_{r \rightarrow +\infty} v'(r) = 0^-$ by Lemma 2.4, we obtain

$$\lim_{u \rightarrow 0^+} t'(u) = \lim_{u \rightarrow 0^+} \frac{1}{w'(t(u))} = \frac{1}{w'(R_0(w))} > -\infty,$$

$$\liminf_{u \rightarrow 0^+} s'(u) = \liminf_{u \rightarrow 0^+} \frac{1}{v'(s(u))} = -\infty.$$

Then, for every $\eta > 0$, there exist $u_\eta \in]0, \eta]$ and $\mu_\eta \in]0, u_\eta[$ such that we have $(s - t)'(u) < 0$ for any $u \in]u_\eta - \mu_\eta, u_\eta + \mu_\eta[$.

On the other hand, since $v'(s(\xi_w)) < 0$ and $\lim_{u \rightarrow \xi_w^-} w'(t(u)) = 0^-$, we have

$$\lim_{u \rightarrow \xi_w^-} s'(u) = \lim_{u \rightarrow \xi_w^-} \frac{1}{v'(s(u))} = \frac{1}{v'(s(\xi_w))} > -\infty,$$

$$\lim_{u \rightarrow \xi_w^-} t'(u) = \lim_{u \rightarrow \xi_w^-} \frac{1}{w'(t(u))} = -\infty,$$

and then, if we have taken $\eta > 0$ sufficiently small, we have $(s - t)'(u) > 0$ for $u \in]\xi_w - \eta, \xi_w[$.

We deduce that, necessarily, the function $s - t$ has a local minimum in the interval $]u_\eta - \mu_\eta, \xi_w[$.

On the other hand, by arguments analogous to those in the proof of [13, Lemma 3.3.1], we have that the following equations hold in the interval $]0, \xi_w[$:

$$E\left(\left|\frac{1}{s_u}\right|\right)s_{uu} - \frac{N-1}{s}A\left(\left|\frac{1}{s_u}\right|\right)s_u^2 - s^\alpha s_u^3 u^p = 0,$$

$$E\left(\left|\frac{1}{t_u}\right|\right)t_{uu} - \frac{N-1}{t}A\left(\left|\frac{1}{t_u}\right|\right)t_u^2 - t^\alpha t_u^3 u^p = 0.$$

If $\tilde{u} \in]0, \xi_w[$ is a critical point of $s - t$, computing the previous two equations in $\tilde{u}$ and subtracting one from the other, we should obtain the following relation (observe that $s_u(\tilde{u}) = t_u(\tilde{u})$):

$$(s - t)_{uu}(\tilde{u}) = (N-1)Q\left(\left|\frac{1}{s_u(\tilde{u})}\right|\right)\left(\frac{1}{s(\tilde{u})} - \frac{1}{t(\tilde{u})}\right)s_u^2(\tilde{u}) + \frac{\tilde{u}^p}{E(|t_u(\tilde{u})|^{-1})}(s^\alpha(\tilde{u}) - t^\alpha(\tilde{u}))s_u^3(\tilde{u}) \quad (2.4)$$

where we have assumed the notation $Q = \frac{A}{E}$. From (2.4) we deduce that, since $s_u(\tilde{u}) < 0$ and by our contradiction assumption we know that $s(\tilde{u}) > t(\tilde{u})$, the critical point must be a maximum. $\square$

Proof of Theorem 2.1. Assume v is a radial solution of $(\mathcal{P})$ and consider the family $(u_\varepsilon^\delta)_\varepsilon$ from Lemma 2.3 (which exists since $p < \frac{N+a}{N-2}$).

Set

$$R_\varepsilon := R_0(u_\varepsilon^\delta).$$

By Lemma 2.3, Lemma 2.4 and since v is decreasing, we have that, for $\varepsilon > 0$ sufficiently small and all $r \in]0, R_\varepsilon]$,

$$v(r) > v(R_\varepsilon) > v\left(\varepsilon^{-\frac{p-1}{2(p+a+1)}}(1 + \delta)\right) \geq C\varepsilon^{\frac{(N-2)(p-1)}{2(p+a+1)}}. \quad (2.5)$$

On the other hand, for any $\varepsilon \in]0, \bar{\varepsilon}]$ and every $r \in]0, R_\varepsilon]$,

$$u_\varepsilon^\delta(r) < u_\varepsilon^\delta(0) = \bar{\xi}\varepsilon^{\frac{2+a}{2(p+a+1)}}. \quad (2.6)$$

By our assumption on α and taking ε small enough, from (2.5) and (2.6) we find that in $]0, R_\varepsilon]$,

$$u_\varepsilon^\delta(r) < \bar{\xi}\varepsilon^{\frac{2+a}{2(p+a+1)}} < C\varepsilon^{\frac{(N-2)(p-1)}{2(p+a+1)}} < v(r),$$

contradicting Lemma 2.5. $\square$

3 Remark on the existence in Minkowski spaces

When $\alpha = 0$, the existence of solutions for $(\mathcal{P}_-)$ has been considered in [2, 8] by different approaches.

In [2], the ODE radial formulation of the problem has permitted to find infinitely many infinite energy solutions by using the Erbe–Tang identity as stated in [17, Proposition 1]. Actually, this approach does not seem so suitable when $\alpha > 0$ because the Erbe–Tang identity becomes apparently more involved and its application not so immediate.

On the other hand, the variational approach used in [8] to find infinitely many finite energy solutions when $\alpha = 0$ seems to be fitting also for the Hénon mean curvature equation. Indeed, as far as we can see, exactly the same arguments presented in [8] can be repeated for $(\mathcal{P}_-)$ by simply adapting the variational setting, replacing the Lebesgue spaces $L^s(\mathbb{R}^N, \mathbb{R})$ with the weighted Lebesgue spaces

$$L^s_\alpha(\mathbb{R}^N) := L^s((\mathbb{R}^N, |x|^\alpha dx), \mathbb{R}).$$

We just wish to show how to prove the compact embedding theorem in our situation (compare with [8, Lemma 2.4]). We point out that we believe that the following theorem is an already known result of the theory of weighted Sobolev and Lebesgue spaces, and actually our proof uses standard arguments.

We will assume the following notation: for any $k \in [1, N[$, we define the α -critical exponent related to k by $k_\alpha^* := \frac{k(N+\alpha)}{N-k}$.

Theorem 3.1. *Let $q \in [2, N[$ and define the Sobolev spaces $\mathcal{D}_{\text{rad}}^{1,q}(\mathbb{R}^N)$ and $\mathcal{D}_{\text{rad}}^{1;(2,q)}(\mathbb{R}^N)$ obtained as the completion of smooth compactly supported radially symmetric functions with respect to the norms*

$$\|u\|_{\mathcal{D}^{1,q}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^q dx \right)^{\frac{1}{q}}$$

and

$$\|u\|_{\mathcal{D}^{1;(2,q)}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^N} |\nabla u|^q dx \right)^{\frac{1}{q}}.$$

Then for every $q \in]2, N[$ we have that $\mathcal{D}_{\text{rad}}^{1;(2,q)}(\mathbb{R}^N)$ is embedded compactly in $L^s_\alpha(\mathbb{R}^N)$ for any $s \in]2_\alpha^*, q_\alpha^*]$.

Proof. We will use a well-known estimate related with radial functions in Sobolev spaces, using it in the dual role of a decaying at infinity and of a growing at zero estimate.

As is well known, since

$$\mathcal{D}_{\text{rad}}^{1;(2,q)}(\mathbb{R}^N) \subset \mathcal{D}_{\text{rad}}^{1,2}(\mathbb{R}^N) \cap \mathcal{D}_{\text{rad}}^{1,q}(\mathbb{R}^N),$$

we have that there exists $C > 0$ such that, for $|x| \neq 0$, we have

$$|u(x)| \leq C \frac{\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}}{|x|^{\frac{N-2}{2}}} \quad (3.1)$$

and

$$|u(x)| \leq C \frac{\|u\|_{\mathcal{D}^{1,q}(\mathbb{R}^N)}}{|x|^{\frac{N-q}{q}}} \quad (3.2)$$

for all $u \in \mathcal{D}_{\text{rad}}^{1;(2,q)}(\mathbb{R}^N)$.

As a consequence, denoting by B_1 the unit ball centered at zero and by B_1^c its complement, if $s > 2_\alpha^*$ by (3.1), we have (hereunder the constant $C > 0$ changes from line to line)

$$\begin{aligned} \int_{B_1^c} |x|^\alpha |u(x)|^s dx &\leq C \int_{B_1^c} |x|^{\alpha - \frac{(N-2)s}{2}} dx \|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^s \\ &\leq C \int_1^{+\infty} |t|^{\alpha + N - 1 - \frac{(N-2)s}{2}} dt \|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^s \\ &\leq C \|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^s, \end{aligned} \quad (3.3)$$

and if $s < q_\alpha^*$ by (3.2), we have

$$\begin{aligned} \int_{B_1} |x|^\alpha |u(x)|^s dx &\leq C \int_{B_1} |x|^{\alpha - \frac{(N-q)s}{q}} dx \|u\|_{\mathcal{D}^{1,q}(\mathbb{R}^N)}^s \\ &\leq C \int_0^1 |t|^{\alpha + N - 1 - \frac{(N-q)s}{q}} dt \|u\|_{\mathcal{D}^{1,q}(\mathbb{R}^N)}^s \\ &\leq C \|u\|_{\mathcal{D}^{1,q}(\mathbb{R}^N)}^s. \end{aligned} \quad (3.4)$$

Then, if $s \in]2^*, q^*[$, inequalities (3.3) and (3.4) imply

$$\int_{\mathbb{R}^N} |x|^\alpha |u(x)|^s dx \leq C (\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^s + \|u\|_{\mathcal{D}^{1,q}(\mathbb{R}^N)}^s) \leq C \|u\|_{\mathcal{D}^{1,(2,q)}(\mathbb{R}^N)}^s,$$

from which we deduce the continuous inclusion of $\mathcal{D}_{\text{rad}}^{1,(2,q)}(\mathbb{R}^N)$ in $L_\alpha^s(\mathbb{R}^N)$.

Now, by standard arguments based on the application of the previous uniform decaying estimates (see [6, Theorem A.I]), we deduce that the embedding is compact. $\square$

Repeating the arguments in [8], we can prove the following result.

Theorem 3.2. *If $p > 2_\alpha^* - 1$, then problem $(\mathcal{P}_-)$ has a solution belonging to $\mathcal{D}_{\text{rad}}^{1,2}(\mathbb{R}^N)$.*

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