Research Article

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Clustered solutions for supercritical elliptic equations on Riemannian manifolds

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Abstract: Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$. We are concerned with the following elliptic problem:

$$-\Delta_g u + a(x)u = u^{\frac{n+2}{n-2}+\varepsilon}, \quad u > 0 \text{ in } M,$$

where $\Delta_g = \operatorname{div}_g(\nabla)$ is the Laplace–Beltrami operator on M , $a(x)$ is a C^2 function on M such that the operator $-\Delta_g + a$ is coercive, and $\varepsilon > 0$ is a small real parameter. Using the Lyapunov–Schmidt reduction procedure, we obtain that the problem under consideration has a k -peaks solution for positive integer $k \geq 2$, which blow up and concentrate at one point in M .

Keywords: Clustered solutions, supercritical elliptic equation on manifolds, Lyapunov–Schmidt reduction procedure

MSC 2010: 58G03, 58E30

1 Introduction

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, where g denotes the metric tensor. We are interested in the following supercritical elliptic problem:

$$-\Delta_g u + a(x)u = u^{\frac{n+2}{n-2}+\varepsilon}, \quad u > 0 \text{ in } M, \quad (1.1)$$

where $\Delta_g = \operatorname{div}_g(\nabla)$ is the Laplace–Beltrami operator on M , $a(x)$ is a C^2 function on M , and $\varepsilon > 0$ is a real parameter with $\varepsilon \rightarrow 0$.

There are many results about the existence and properties of solutions for nonlinear elliptic equations on compact Riemannian manifolds. Let us mention the following problem:

$$-\varepsilon^2 \Delta_g u + u = |u|^{p-2}u \quad \text{in } M, \quad (1.2)$$

where (M, g) is a compact, connected, Riemannian manifold of class C^∞ with Riemannian metric g , with dimension $n \geq 3$, $2 < p < \frac{2n}{n-2}$ and ε is a positive parameter. The existence and multiplicity of solutions to problem (1.2) was considered in [2, 4, 25]. Moreover, the existence of peak solutions for (1.2) was obtained by Dancer, Micheletti and Pistoia [6, 14, 15].

The asymptotically critical case on a Riemannian manifold was studied by Micheletti, Pistoia and Vétois in [16]. They proved that problem (1.1) has blowing-up families of positive solutions, provided the graph of $a(x)$ is distinct at some point from the graph of $\frac{n-2}{4(n-1)} \operatorname{Scal}_g$. Moreover, the existence of multi-peak solutions that are *separate from each other* for (1.1) was considered by Deng in [7]. Pistoia and Vétois [18] discovered the existence of sign-changing bubble towers for (1.1).

In the case $a \equiv \frac{n-2}{4(n-1)} \operatorname{Scal}_g$, equation (1.1) is intensively studied as the Yamabe equation whose positive solutions u are such that the scalar curvature of the conformal metric $u^{2^*-2}g$ is constant (see [1, 21, 24]).

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It is important to recall some results for the following linear perturbation of the Yamabe problem:

$$-\Delta_g u + \left(\frac{n-2}{4(n-1)} \text{Scal}_g + \varepsilon \right) u = u^{\frac{n+2}{n-2}} \quad \text{in } (M, g), \quad (1.3)$$

where (M, g) is a non-locally conformally flat compact Riemannian manifold. Druet in [8] proved that problem (1.3) does not have any blowing-up solution when $\varepsilon < 0$ and the dimension of the manifold is $n = 3, 4, 5$ (except when the manifold is conformally equivalent to the round sphere). In case $\varepsilon > 0$, if $n = 3$, there is no blowing-up solutions to problem (1.3) as proved by Li and Zhu [13]. Esposito, Pistoia and Vétois in [9] showed that there exist blowing-up solutions for $n \geq 6$, and they built solutions which blow up at non-vanishing stable critical points ξ_0 of the Weyl tensor, i.e., $|\text{Weyl}_g(\xi_0)|_g \neq 0$. Recently, Robert and Vétois [20], Pistoia and Vaira [17], Thizy and Vétois [23] provided several geometric and analytic settings in which linear perturbations to (1.3) have positive clustered bubbles that are non-isolated blowing-up solutions. In particular, Pistoia and Vaira in [17] investigated the existence of cluster solutions for problem (1.3). More precisely, they proved that for any point $\xi_0 \in M$, which is non-degenerate and a non-vanishing minimum point of the Weyl tensor, and for any integer k , there exists a family of solutions developing k peaks collapsing to ξ_0 as ε goes to zero. Moreover, Thizy and Vétois [23] constructed clustering positive solutions for a perturbed critical elliptic equation on a closed manifold of dimension four and five.

Motivated by the previous consideration, in the present paper, we construct a family of cluster solutions for equation (1.1) with ε small enough.

Let L^q be the Banach space $L^q(M)$ with the norm

$$|u|_q = \left(\int_M |u|^q d\mu_g \right)^{\frac{1}{q}}.$$

We assume that the operator $-\Delta_g + a$ is coercive, and the Sobolev space $H_g^1(M)$ is endowed with the scalar product $\langle \cdot, \cdot \rangle_a$, defined by

$$\langle u, v \rangle_a = \int_M (\langle \nabla u, \nabla v \rangle_g + a uv) d\mu_g$$

for all $u, v \in H_g^1(M)$. We let $\|\cdot\|_a$ be the norm induced by $\langle \cdot, \cdot \rangle_a$. This norm is equivalent to the standard norm on $H_g^1(M)$. Let u_ε be a family of solutions of (1.1). We say that u_ε blows up at k points which collapse to ξ_0 as $\varepsilon \rightarrow 0$ if there exists $\xi_{1,\varepsilon}, \xi_{2,\varepsilon}, \dots, \xi_{k,\varepsilon} \in M$ and $\lambda_{1,\varepsilon}, \lambda_{2,\varepsilon}, \dots, \lambda_{k,\varepsilon} \in \mathbb{R}^+$ such that $\xi_{j,\varepsilon} \rightarrow \xi_0$, $\lambda_{j,\varepsilon} \rightarrow 0$, $j = 1, 2, \dots, k$, and

$$\left\| u_\varepsilon(x) - \sum_{j=1}^k \lambda_{j,\varepsilon}^{-\frac{n-2}{2}} U\left(\frac{\exp^{-1}(x)}{\lambda_{j,\varepsilon}}\right) \right\|_a \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

where the function

$$U(z) = U(|z|) = \alpha_n (1 + |z|^2)^{-\frac{n-2}{2}}, \quad \alpha_n = (n(n-2))^{\frac{n-2}{4}},$$

is the solution of the limit equation

$$-\Delta U = U^{2^*-1} \quad \text{in } \mathbb{R}^n, \quad (1.4)$$

It is known that [1, 22] the functions $\lambda^{\frac{2-n}{2}} U(\lambda^{-1}z)$ satisfy equation (1.4).

Our main result can be stated as follows.

Theorem 1.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, let $a(x)$ be a C^2 function on M such that the operator $-\Delta_g + a$ is coercive, and let ξ_0 be a nondegenerate maximum point of the function*

$$\varphi(\xi) := a(\xi) - \frac{n-1}{4(n-2)} \text{Scal}_g(\xi), \quad (1.5)$$

with $\varphi(\xi_0) < 0$. For any given integer $k \geq 2$, if $\varepsilon > 0$ is small enough, then problem (1.1) has a solution u_ε blowing up at k points which collapse to ξ_0 as $\varepsilon \rightarrow 0$.

Remark 1.2. We can get the same result for the subcritical case. Precisely, if (M, g) is a smooth compact Riemannian manifold of dimension $n \geq 5$, $a(x)$ is a C^2 function on M such that the operator $-\Delta_g + a$ is coercive, and ξ_0 is a nondegenerate minimum point of the function $\varphi(\xi)$ with $\varphi(\xi_0) > 0$, then for any $\varepsilon < 0$ small enough, problem (1.1) has a clustered solution u_ε , which blows up at k points which collapse to ξ_0 as $\varepsilon \rightarrow 0$. We will not give the details of the proof in this case.

The proof of our result relies on a very well known finite dimensional Lyapunov–Schmidt reduction procedure, introduced in [10, 19] and used in many of the quoted papers. In particular, we refer to [6, 14, 15] for nonlinear elliptic problems on Riemannian manifolds, [7, 16] for asymptotically critical elliptic equations on Riemannian manifolds, [9, 17] for linear perturbations on the Yamabe problems, and recently this method has been used to study the fractional Yamabe problem by Choi and Kim in [5], and Kim, Musso and Wei in [12].

This paper is organized as follows. In Section 2, we introduce the framework and present some preliminary results. The proof of the main result is given in Section 3. Section 4 contains the asymptotic expansion of the energy functional.

2 The framework and preliminary results

Let r be a positive real number less than i_M , where i_M is the injectivity radius of M , and χ_r be a smooth cut-off function such that $0 \leq \chi_r \leq 1$ in $\mathbb{R}^n$, $\chi_r(z) = 1$ if $z \in B(0, \frac{r}{2})$, $\chi_r(z) = 0$ if $z \in \mathbb{R}^n \setminus B(0, r)$, and $|\nabla \chi_r(z)| \leq \frac{2}{r}$, $|\nabla^2 \chi_r(z)| \leq \frac{2}{r^2}$, where $B(0, r)$ denotes the ball in $T_\xi M$ centered at 0 with radius r . For any point ξ in M and any positive real number λ , we define the function $W_{\lambda, \xi}$ on M by

$$W_{\lambda, \xi}(x) := \begin{cases} \chi_r(\exp_\xi^{-1}(x)) \lambda^{\frac{2-n}{2}} U(\lambda^{-1} \exp_\xi^{-1}(x)) & \text{if } x \in B_g(\xi, r), \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

It is clear that the embedding $i: H_g^1(M) \hookrightarrow L^{\frac{2n}{n-2}}(M)$ is a continuous map. Let $i^*: L^{\frac{2n}{n+2}}(M) \hookrightarrow H_g^1(M)$ be the adjoint operator of the embedding i . Then the embedding i^* is a continuous map such that for any $w \in L^{\frac{2n}{n+2}}(M)$, the function $u = i^*(w)$ in $H_g^1(M)$ is the unique solution of the equation $-\Delta_g u + au = w$ in M . By the continuity of the embedding $H_g^1(M)$ into $L^{\frac{2n}{n-2}}(M)$, we have

$$\|i^*(w)\|_a \leq C|w|_{\frac{2n}{n+2}}$$

for some positive constant C independent of w .

By standard elliptic estimates [11], given a real number $s > \frac{2n}{n-2}$, that is, $\frac{ns}{n+2s} > \frac{2n}{n+2}$, for any $w \in L^{\frac{ns}{n+2s}}(M)$, the function $i^*(w)$ belongs to $L^s(M)$ and satisfies

$$|i^*(w)|_s \leq C|w|_{\frac{ns}{n+2s}} \quad (2.2)$$

for some positive constant C independent of w . For ε small, we set

$$s_\varepsilon := \frac{2n}{n-2} + \frac{n}{2}\varepsilon,$$

and let $\mathcal{H}_\varepsilon = H_g^1(M) \cap L^{s_\varepsilon}(M)$ be the Banach space equipped with the norm

$$\|u\|_{a, s_\varepsilon} = \|u\|_a + |u|_{s_\varepsilon}.$$

Taking into account that

$$\frac{ns_\varepsilon}{n+2s_\varepsilon} = \frac{s_\varepsilon}{2^* - 1 + \varepsilon},$$

and by (2.2), we can write problem (1.1) as

$$u = i^*(f_\varepsilon(u)), \quad u \in \mathcal{H}_\varepsilon, \quad (2.3)$$

where $f_\varepsilon(u) = u_+^{\frac{n+2}{n-2} + \varepsilon}$ and $u_+ = \max\{u, 0\}$.

It is known, see [3, 19], that every solution of the linear equation

$$-\Delta v = \frac{n+2}{n-2} U^{\frac{4}{n-2}} v, \quad v \in \mathcal{D}_0^{1,2}(\mathbb{R}^n),$$

is a linear combination of the functions

$$V_0(z) = \frac{d(\lambda^{\frac{2-n}{2}} U(\lambda^{-1}z))}{d\lambda} \Big|_{\lambda=1} = \frac{1}{2} \alpha_n (n-2) \frac{|z|^2 - 1}{(1+|z|^2)^{\frac{n}{2}}}$$

and

$$V_i(z) = -\frac{\partial U}{\partial z_i}(z) = \alpha_n (n-2) \frac{z_i}{(1+|z|^2)^{\frac{n}{2}}} \quad \text{for } i = 1, 2, \dots, n.$$

Let us define on M the functions

$$Z_{\lambda, \xi}^i(x) := \begin{cases} \chi_r(\exp_{\xi}^{-1}(x)) \lambda^{\frac{2-n}{2}} V_i(\lambda^{-1} \exp_{\xi}^{-1}(x)) & \text{if } x \in B_g(\xi, r), \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 0, 1, 2, \dots, n$.

Let ξ_0 be a nondegenerate local maximum point of

$$\varphi(\xi) = a(\xi) - \frac{n-1}{4(n-2)} \text{Scal}_g(\xi),$$

and $\bar{\xi} = (\xi_1, \xi_2, \dots, \xi_k) \in M^k$. Given $\eta > 0$, $R > 0$, we define the open set

$$\mathcal{O}_{\eta, R}^k = \{(\bar{d}, \bar{\tau}) = (d_1, \dots, d_k, \tau_1, \dots, \tau_k) \in [\eta^{-1}, \eta]^k \times \overline{(B^n(0, R))^k} : |\tau_i - \tau_j| \geq R^{-1} \text{ for all } i \neq j\} \quad (2.4)$$

and the parameters

$$\lambda_j = \varepsilon^\alpha d_j > 0 \quad \text{and} \quad \xi_j = \exp_{\xi_0}(\varepsilon^\beta \tau_j) \in M \quad (2.5)$$

for each $j = 1, \dots, k$, where $\exp_{\xi_0} : B^n(0, i_M) \rightarrow M$ is the exponential map whose base point is ξ_0 and

$$\alpha = \frac{1}{2} \quad \text{and} \quad \beta = \frac{n-4}{2n}.$$

Let

$$K_{\bar{\lambda}, \bar{\xi}} = \text{Span}\{Z_{\lambda_j, \xi_j}^i : i = 0, 1, 2, \dots, n, j = 1, 2, \dots, k\}$$

and

$$K_{\bar{\lambda}, \bar{\xi}}^\perp = \{\phi \in \mathcal{H}_\varepsilon : \langle \phi, Z_{\lambda_j, \xi_j}^i \rangle_a = 0, i = 0, 1, 2, \dots, n, j = 1, 2, \dots, k\}.$$

3 Scheme of the proof of Theorem 1.1

We will look for a solution to (2.3), or equivalently to (1.1), of the form

$$u_\varepsilon = \mathcal{W}_{\bar{d}, \bar{\tau}} + \phi_{\varepsilon, \bar{d}, \bar{\tau}}, \quad \text{with } \mathcal{W}_{\bar{d}, \bar{\tau}} = \sum_{j=1}^k W_{\lambda_j, \xi_j}, \quad (3.1)$$

for $(\bar{d}, \bar{\tau}) \in \mathcal{O}_{\eta, R}^k$, where the rest term $\phi_{\varepsilon, \bar{d}, \bar{\tau}}$ belongs to the space $K_{\bar{d}, \bar{\tau}}^\perp$ and the functions W_{λ_j, ξ_j} are defined in (2.1).

Let $\Pi_{\bar{d}, \bar{\tau}} : \mathcal{H}_\varepsilon \rightarrow K_{\bar{d}, \bar{\tau}}$ and $\Pi_{\bar{d}, \bar{\tau}}^\perp : \mathcal{H}_\varepsilon \rightarrow K_{\bar{d}, \bar{\tau}}^\perp$ be the orthogonal projections. In order to solve problem (2.3), we will solve the system

$$\Pi_{\bar{d}, \bar{\tau}}^\perp \{\mathcal{W}_{\bar{d}, \bar{\tau}} + \phi_{\varepsilon, \bar{d}, \bar{\tau}} - i^*[f_\varepsilon(\mathcal{W}_{\bar{d}, \bar{\tau}} + \phi_{\varepsilon, \bar{d}, \bar{\tau}})]\} = 0, \quad (3.2)$$

$$\Pi_{\bar{d}, \bar{\tau}} \{\mathcal{W}_{\bar{d}, \bar{\tau}} + \phi_{\varepsilon, \bar{d}, \bar{\tau}} - i^*[f_\varepsilon(\mathcal{W}_{\bar{d}, \bar{\tau}} + \phi_{\varepsilon, \bar{d}, \bar{\tau}})]\} = 0. \quad (3.3)$$

The first step in the proof consists in solving equation (3.2). This requires Proposition 3.1 below. We skip the proof of this result which is rather standard in the literature on Lyapunov–Schmidt reduction; we refer to [16] and [7].

Proposition 3.1. For any $(\bar{d}, \bar{\tau}) \in \mathcal{O}_{\eta, R}^k$, there exists a positive constant C such that for ε small enough, equation (3.2) admits a unique solution $\phi_{\varepsilon, \bar{d}, \bar{\tau}}$ in $K_{\bar{\lambda}, \bar{\tau}}^\perp$ satisfying

$$\|\phi_{\varepsilon, \bar{d}, \bar{\tau}}\|_{a, s_\varepsilon} \leq C \|i^*(f_\varepsilon(\mathcal{W}_{\bar{d}, \bar{\tau}})) - \mathcal{W}_{\bar{d}, \bar{\tau}}\|_{a, s_\varepsilon} \leq C\varepsilon |\ln \varepsilon|. \quad (3.4)$$

Moreover, $\phi_{\varepsilon, \bar{d}, \bar{\tau}}$ is continuously differentiable with respect to $\bar{\tau}$ and $\bar{d}$.

We introduce the functional $J_\varepsilon: H_g^1(M) \rightarrow \mathbb{R}$, defined by

$$J_\varepsilon(u) = \frac{1}{2} \int_M |\nabla_g u|^2 d\mu_g + \frac{1}{2} \int_M a(x) u^2 d\mu_g - \frac{1}{2^* + \varepsilon} \int_M u^{2^* + \varepsilon} d\mu_g,$$

where $2^* = \frac{2n}{n-2}$ denotes the Sobolev critical exponent. It is well known that any critical point of J_ε is a solution to problem (1.1). We define the functional $\tilde{J}_\varepsilon: (\mathbb{R}^+)^k \times (\mathbb{R}^n)^k \rightarrow \mathbb{R}$ by

$$\tilde{J}_\varepsilon(\bar{d}, \bar{\tau}) = J_\varepsilon(\mathcal{W}_{\bar{d}, \bar{\tau}} + \phi_{\varepsilon, \bar{d}, \bar{\tau}}),$$

where $\mathcal{W}_{\bar{d}, \bar{\tau}}$ is as (3.1) and $\phi_{\varepsilon, \bar{d}, \bar{\tau}}$ is given by Proposition 3.1.

The next result, whose proof is postponed to Section 4, allows us to solve equation (3.3), by reducing the problem to a finite dimensional one.

Proposition 3.2. (i) For ε small, if $(\bar{d}, \bar{\tau})$ is a critical point of the functional $\tilde{J}_\varepsilon$, then $\mathcal{W}_{\bar{d}, \bar{\tau}} + \phi_{\varepsilon, \bar{d}, \bar{\tau}}$ is a solution of (2.3), or equivalently of problem (1.1).

(ii) If $n \geq 5$ and $(\bar{\lambda}, \bar{\tau}) \in \mathcal{O}_{\eta, R}^k$ satisfies (2.4), then

$$\begin{aligned} \tilde{J}_\varepsilon(\bar{d}, \bar{\tau}) = & kc_0 + \varepsilon k(c_1 + c_2 \ln \varepsilon) + \sum_{j=1}^k [c_3 \ln(d_j) + c_4 d_j^2 \varphi(\xi_0)] \varepsilon \\ & + \varepsilon^{\frac{2(n-2)}{n}} \left[\frac{c_4}{2} \sum_{j=1}^k d_j^2 D^2 \varphi(\xi_0) [\tau_j, \tau_j] - c_5 \sum_{j < l} \frac{d_l^{\frac{n-2}{2}} d_j^{\frac{n-2}{2}}}{|\tau_l - \tau_j|^{n-2}} \right] + o(\varepsilon^{\frac{2(n-2)}{n}}) \end{aligned} \quad (3.5)$$

as $\varepsilon \rightarrow 0$, C^0 -uniformly with respect to $\bar{\tau}$ in $(\mathbb{R}^n)^k$ and to $\bar{d}$ in compact subsets of $(\mathbb{R}^+)^k$, where c_i , for $i = 0, 1, \dots, 5$, are positive constants and φ is defined in (1.5).

Proof of Theorem 1.1. From Proposition 3.2 (i), it follows that $\mathcal{W}_{\bar{d}, \bar{\tau}} + \phi_{\varepsilon, \bar{d}, \bar{\tau}}$, where $\mathcal{W}_{\bar{d}, \bar{\tau}}$ is defined in (3.1) and $\phi_{\varepsilon, \bar{d}, \bar{\tau}}$, whose existence is guaranteed by Proposition 3.1, is a solution of (1.1) if $(\bar{d}, \bar{\tau})$ is a critical point of the functional $\tilde{J}_\varepsilon$, which is a consequence of finding a maximum point of

$$\mathcal{F}_\varepsilon(\bar{d}, \bar{\tau}) = \sum_{j=1}^k [c_3 \ln(d_j) + c_4 d_j^2 \varphi(\xi_0)] + \varepsilon^{\frac{n-4}{n}} \left[\frac{c_4}{2} \sum_{j=1}^k d_j^2 D^2 \varphi(\xi_0) [\tau_j, \tau_j] - c_5 \sum_{j < l} \frac{d_l^{\frac{n-2}{2}} d_j^{\frac{n-2}{2}}}{|\tau_l - \tau_j|^{n-2}} \right] + o(\varepsilon^{\frac{2(n-2)}{n}})$$

in the interior of $\mathcal{O}_{\eta, R}^k$.

To the contrary, assume that the maximum of $\mathcal{F}_\varepsilon$ is achieved only on the boundary $\partial \mathcal{O}_{\eta, R}^k$. There are three possibilities, namely,

- (1) one of $d_1, \dots, d_k$ is equal to either η^{-1} or η ,
- (2) one of $\tau_1, \dots, \tau_k$ is located on $\partial B^n(0, R)$,
- (3) there exist $1 \leq i \neq j \leq k$ such that $|\tau_i - \tau_j| = R^{-1}$.

We will exclude each of them, thereby showing the existence of the maximum point in the interior.

Suppose first that (1) occurs. Define the function

$$h(d) := c_3 \ln d + c_4 d^2 \varphi(\xi_0)$$

By the assumption $\varphi(\xi_0) < 0$, we have that h attains the maximum M_0 at the point $d_{\max} = (\frac{-c_3}{2c_4 \varphi(\xi_0)})^{1/2}$. Let us take $\eta > 0$ so large that $\max\{h(\eta), h(\eta^{-1})\} \leq \min\{-1, 2M_0\}$ and $d_0 \in (\eta^{-1}, \eta)$. Then we obtain

$$\begin{aligned} \min\{-1, 2M_0\} + (k-1)M_0 + o(1) &\geq \max_{(\bar{d}, \bar{\tau}) \in \mathcal{O}_{\eta, R}^k} \mathcal{F}_\varepsilon(\bar{d}, \bar{\tau}) \\ &\geq \max\{\mathcal{F}_\varepsilon(d_0, \dots, d_0, \bar{\tau}) : \bar{\tau} \in \overline{(B^n(0, R))^k}, |\tau_i - \tau_j| \geq R^{-1} \text{ for } 1 \leq i \neq j \leq k\} \\ &= kM_0 + o(1), \end{aligned}$$

which gives that $\min\{-1, 2M_0\} \geq M_0 + o(1)$. This in turn implies that $0 \leq M_0 < -1$. Hence, a contradiction arises and (1) cannot occur.

We next prove that cases (2) and (3) never take place, provided that $R > 0$ is sufficiently large. Because $D_g^2\varphi(\xi_0)$ is negative definite, for any $M_1 > 0$, one can choose $R \gg 1$ so large that

$$D_g^2\varphi(\xi_0)(\sigma, \sigma) \leq -M_1 \quad \text{on } \{\sigma \in \mathbb{R}^n : |\sigma| = R\} \quad (3.6)$$

and

$$-\frac{1}{|\tau_i - \tau_j|^{n-2}} \leq -M_1 \quad \text{for } \tau_i, \tau_j \in \mathbb{R}^n, \text{ with } |\tau_i - \tau_j| = R^{-1}. \quad (3.7)$$

Let $\bar{\tau}_0 = (\tau_{01}, \dots, \tau_{0k})$ be any point in $(\mathbb{R}^n)^k$ which constitutes of vertices of a k -regular polygon whose center and circumradius are 0 and $1/\sin(\pi/k)$, respectively. Then it is easy to see that

$$|\tau_{01}| = \dots = |\tau_{0k}| = 1/\sin(\pi/k) \quad \text{and} \quad |\tau_{0i} - \tau_{0j}| = 2 \text{ for each } 1 \leq i \neq j \leq k.$$

For each small $\varepsilon > 0$, let also $\bar{d}_\varepsilon = (d_{\varepsilon 1}, \dots, d_{\varepsilon k}) \in (\eta^{-1}, \eta)^k$ be an element attaining the maximum value of the map $\sum_{j=1}^k h(d_j)$. Observe that $d_{\varepsilon i} \rightarrow d_0$ as $\varepsilon \rightarrow 0$ for all $i = 1, \dots, k$. From (3.6) and (3.7), we get an upper bound of the maximum of $\mathcal{F}_\varepsilon$ as follows:

$$\begin{aligned} \max_{(\bar{d}, \bar{\tau}) \in \mathcal{O}_{\eta, R}^k} \mathcal{F}_\varepsilon(\bar{d}, \bar{\tau}) &\leq \sum_{j=1}^k h(d_{j\varepsilon}) + \varepsilon^{\frac{n-4}{n}} \max_{(\bar{d}, \bar{\tau}) \in \mathcal{O}_{\eta, R}^k} \left[\frac{c_4}{2} \sum_{j=1}^k d_j^2 D^2\varphi(\xi_0)[\tau_j, \tau_j] - c_5 \sum_{j < l} \frac{d_l^{\frac{n-2}{2}} d_j^{\frac{n-2}{2}}}{|\tau_l - \tau_j|^{n-2}} \right] \\ &\leq \sum_{j=1}^k h(d_{j\varepsilon}) - \varepsilon^{\frac{n-4}{n}} k M_1 \left[\frac{c_4}{2} \eta^{-2} + \frac{k-1}{2} c_5 \eta^{-(n-2)} \right]. \end{aligned}$$

However, given $M_1 > 0$ large enough, the above estimate does not make sense, since we can derive

$$\max_{(\bar{d}, \bar{\tau}) \in \mathcal{O}_{\eta, R}^k} \mathcal{F}_\varepsilon(\bar{d}, \bar{\tau}) \geq \mathcal{F}_\varepsilon(\bar{d}_\varepsilon, \bar{\tau}_0) \geq \sum_{j=1}^k h(d_{j\varepsilon}) - \varepsilon^{\frac{n-4}{n}} k \left[\frac{|D_g^2\varphi(\xi_0)|c_4}{2\sin(\pi/k)^2} d_0^2 + \frac{k-1}{2^{n-2+1}} c_5 d_0^{n-2} + o(1) \right]$$

at the same time.

Thus, $(\bar{d}, \bar{\tau}) \in \mathcal{O}_{\eta, R}^k$. Then problem (1.1) has a solution of the form $u_\varepsilon = \mathcal{W}_{\bar{d}, \bar{\tau}} + \phi_{\varepsilon, \bar{d}, \bar{\tau}}$ for $\varepsilon > 0$ sufficiently small, which we call a solution blowing-up at k points that collapse to ξ_0 as $\varepsilon \rightarrow 0$. By taking into account the definition of the approximate solution $\phi_{\varepsilon, \bar{d}, \bar{\tau}}$ and (3.4), the proof is completed. $\square$

4 Proof of Proposition 3.2: Expansion of the energy

This section is devoted to the proof of Proposition 3.2. As a first step, we need the following lemma.

Lemma 4.1. *For ε small, if $(\bar{d}, \bar{\tau})$ is a critical point of the functional $\tilde{J}_\varepsilon$, then $\mathcal{W}_{\bar{d}, \bar{\tau}} + \phi_{\varepsilon, \bar{d}, \bar{\tau}}$ is a solution of (2.3), or equivalently of problem (1.1).*

Proof. The proof is the same as that of [7, Lemma 5.1]. $\square$

Lemma 4.2. *If $n \geq 5$ and $(\bar{\lambda}, \bar{\tau}) \in \mathcal{O}_{\eta, R}^k$ satisfies (2.4), then*

$$\tilde{J}_\varepsilon(\bar{d}, \bar{\tau}) = J_\varepsilon(\mathcal{W}_{\bar{d}, \bar{\tau}}) + o(\varepsilon^{\frac{2(n-2)}{n}})$$

as $\varepsilon \rightarrow 0$, C^0 -uniformly with respect to $\bar{\tau}$ in $(\mathbb{R}^n)^k$ and to $\bar{d}$ in compact subsets of $(\mathbb{R}^+)^k$.

Proof. It can be proved by the same argument used in the proof of [16, Lemma 4.2]. $\square$

We next give the expansion of $J_\varepsilon(\mathcal{W}_{\bar{d}, \bar{\tau}})$ for $(\bar{d}, \bar{\tau}) \in \mathcal{O}_{\eta, R}^k$ and $\varepsilon \rightarrow 0$. We denote K_n the sharp constant for the embedding of $\mathcal{D}^{1,2}(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$, that is,

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{1/n}}},$$

where ω_n is the volume of the unit n -sphere. We have

$$\begin{aligned} J_\varepsilon(\mathcal{W}_{\vec{d}, \vec{\tau}}) &= \frac{1}{2} \int_M \left| \nabla_g \left(\sum_{j=1}^k W_{\lambda_j, \xi_j} \right) \right|^2 d\mu_g + \frac{1}{2} \int_M a(x) \left(\sum_{j=1}^k W_{\lambda_j, \xi_j} \right)^2 d\mu_g - \frac{1}{2^* + \varepsilon} \int_M \left(\sum_{j=1}^k W_{\lambda_j, \xi_j} \right)_+^{2^* + \varepsilon} d\mu_g \\ &= \sum_{j=1}^k J_\varepsilon(W_{\lambda_j, \xi_j}) - \sum_{j < l} \int_M W_{\lambda_l, \xi_l}^{2^* - 1 + \varepsilon} W_{\lambda_j, \xi_j} d\mu_g \\ &\quad + \sum_{l < j} \int_M [\nabla_g W_{\lambda_j, \xi_j} \nabla_g W_{\lambda_l, \xi_l} + a(x) W_{\lambda_j, \xi_j} W_{\lambda_l, \xi_l} - W_{\lambda_l, \xi_l}^{2^* - 1 + \varepsilon} W_{\lambda_j, \xi_j}] d\mu_g \\ &\quad - \frac{1}{2^* + \varepsilon} \int_M \left[\left(\sum_{j=1}^k W_{\lambda_j, \xi_j} \right)^{2^* + \varepsilon} - \sum_{j=1}^k W_{\lambda_j, \xi_j}^{2^* + \varepsilon} - (2^* + \varepsilon) \sum_{j \neq l} W_{\lambda_l, \xi_l}^{2^* - 1 + \varepsilon} W_{\lambda_j, \xi_j} \right] d\mu_g. \end{aligned} \quad (4.1)$$

We will estimate each term in the following lemmas.

Lemma 4.3 ([16, Lemma 4.1]). *For any $j = 1, 2, \dots, k$, we have*

$$\begin{aligned} J_\varepsilon(W_{\lambda_j, \xi_j}) &= \frac{K_n^{-n}}{n} \left(1 + \frac{(n-2)^2}{8} \varepsilon \ln \varepsilon + \frac{(n-2)^2}{4} \varepsilon \ln(d_j) + C_n \varepsilon \right) \\ &\quad + \frac{K_n^{-n}}{n} \frac{2(n-1)}{(n-2)(n-4)} d_j^2 \left(a(\xi_j) - \frac{n-2}{4(n-1)} \text{Scal}_g(\xi_j) \right) \varepsilon + o(\varepsilon) \end{aligned}$$

as $\varepsilon \rightarrow 0$, C^0 -uniformly with respect to $\vec{\tau}$ in $(\mathbb{R}^n)^k$ and to $\vec{d}$ in compact subsets of $(\mathbb{R}^+)^k$, where C_n is the positive constant

$$C_n = 2^{n-3} (n-2)^2 \frac{\omega_{n-1}}{\omega_n} \int_0^{+\infty} \frac{t^{\frac{n-2}{2}} \ln(1+t)}{(1+t)^n} dt + \frac{(n-2)^2}{4} (1 - n \ln \sqrt{n(n-2)}). \quad (4.2)$$

Lemma 4.4. *We have*

$$- \sum_{j \neq l} \int_M W_{\lambda_l, \xi_l}^{2^* - 1 + \varepsilon} W_{\lambda_j, \xi_j} d\mu_g = -\varepsilon^{(n-2)(\alpha-\beta)} \left(\sum_{j \neq l} \frac{d_l^{\frac{n-2}{2}} d_j^{\frac{n-2}{2}}}{|\tau_l - \tau_j|^{n-2}} \alpha_n \int_{\mathbb{R}^n} U(z)^{2^*-1} dz + o(1) \right) \quad (4.3)$$

as $\varepsilon \rightarrow 0$, C^0 -uniformly with respect to $\vec{\tau}$ in $(\mathbb{R}^n)^k$ and to $\vec{d}$ in compact subsets of $(\mathbb{R}^+)^k$.

Proof. By the definition (2.1) of the functions W_{λ_l, ξ_l} and W_{λ_j, ξ_j} , for $l \neq j$, we have

$$\begin{aligned} \int_M W_{\lambda_l, \xi_l}^{2^* - 1 + \varepsilon}(x) W_{\lambda_j, \xi_j}(x) d\mu_g &= \int_M \left[\chi_r(\exp_{\xi_l}^{-1}(x)) \lambda_l^{\frac{2-n}{2}} U\left(\frac{\exp_{\xi_l}^{-1}(x)}{\lambda_l}\right) \right]^{2^* - 1 + \varepsilon} \chi_r(\exp_{\xi_j}^{-1}(x)) \lambda_j^{\frac{2-n}{2}} U\left(\frac{\exp_{\xi_j}^{-1}(x)}{\lambda_j}\right) d\mu_g \\ &= \int_{B(0, r)} \left[\chi_r(y) \lambda_l^{\frac{2-n}{2}} U\left(\frac{y}{\lambda_l}\right) \right]^{2^* - 1 + \varepsilon} \chi_r(y) \lambda_j^{\frac{2-n}{2}} U\left(\frac{\exp_{\xi_j}^{-1}(\exp_{\xi_l}(y))}{\lambda_j}\right) \sqrt{|g(\exp_{\xi_l}(y))|} dy, \end{aligned}$$

where we set $\exp_{\xi_l}^{-1}(x) = y$. Let θ be a positive constant, satisfying $\beta < \theta < \alpha$, and we divide the domain $B(0, r)$ into two parts, i.e., $B(0, r) = B(0, \varepsilon^\theta) \cup (B(0, r) \setminus B(0, \varepsilon^\theta))$. Then we have

$$\begin{aligned} &\int_{B(0, r)} W_{\lambda_l, \xi_l}^{2^* - 1 + \varepsilon}(x) W_{\lambda_j, \xi_j}(x) d\mu_g \\ &= \int_{B(0, \varepsilon^\theta)} \left[\chi_r(y) \lambda_l^{\frac{2-n}{2}} U\left(\frac{y}{\lambda_l}\right) \right]^{2^* - 1 + \varepsilon} \chi_r(y) \lambda_j^{\frac{2-n}{2}} U\left(\frac{\exp_{\xi_j}^{-1}(\exp_{\xi_l}(y))}{\lambda_j}\right) \sqrt{|g(\exp_{\xi_l}(y))|} dy \\ &\quad + \int_{B(0, r) \setminus B(0, \varepsilon^\theta)} \left[\chi_r(y) \lambda_l^{\frac{2-n}{2}} U\left(\frac{y}{\lambda_l}\right) \right]^{2^* - 1 + \varepsilon} \chi_r(y) \lambda_j^{\frac{2-n}{2}} U\left(\frac{\exp_{\xi_j}^{-1}(\exp_{\xi_l}(y))}{\lambda_j}\right) \sqrt{|g(\exp_{\xi_l}(y))|} dy \\ &=: L_1 + L_2. \end{aligned} \quad (4.4)$$

We estimate each term as follows. Let us introduce the transition map $m_{lj}: B^n(0, i_M) \rightarrow \mathbb{R}^n$, defined as $m_{lj}(z) = \exp_{\xi_j}^{-1}(\exp_{\xi_l}(z))$, so as to estimate the interaction between W_{λ_l, ξ_l} and W_{λ_j, ξ_j} with $l \neq j$. We point out that

$$|m_{lj}(0)| = |\exp_{\xi_j}^{-1}(\xi_l)| = d_g(\xi_l, \xi_j) = \varepsilon^\beta(|\tau_l - \tau_j| + o(1)), \quad (4.5)$$

where $o(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$, and the last equality comes from (2.5). For L_1 , setting $y = \lambda_l z$, we have

$$\begin{aligned} L_1 &= \lambda_l^{\frac{n-2}{2}(1+\varepsilon)} \int_{B(0, \varepsilon^\theta \lambda_l^{-1})} [\chi_r(\lambda_l z) U(z)]^{2^*-1+\varepsilon} \chi_r(m_{lj}(\lambda_l z)) \lambda_j^{\frac{2-n}{2}} U\left(\frac{m_{lj}(\lambda_l z)}{\lambda_j}\right) \sqrt{|g(\exp_{\xi_l}(\lambda_l z))|} dz \\ &= \lambda_l^{\frac{n-2}{2}(1+\varepsilon)} \lambda_j^{\frac{n-2}{2}} \int_{B(0, \varepsilon^\theta \lambda_l^{-1})} [\chi_r(\lambda_l z) U(z)]^{2^*-1+\varepsilon} \chi_r(m_{lj}(\lambda_l z)) \frac{\alpha_n}{(\lambda_j^2 + |m_{lj}(\lambda_l z)|^2)^{\frac{n-2}{2}}} \sqrt{|g(\exp_{\xi_l}(\lambda_l z))|} dz \\ &= \alpha_n \frac{\lambda_l^{\frac{n-2}{2}(1+\varepsilon)} \lambda_j^{\frac{n-2}{2}}}{(\lambda_j^2 + |m_{lj}(0)|^2)^{\frac{n-2}{2}}} \int_{B(0, \varepsilon^\theta \lambda_l^{-1})} [\chi_r(\lambda_l z) U(z)]^{2^*-1+\varepsilon} \chi_r(m_{lj}(\lambda_l z)) \left(\frac{\lambda_j^2 + |m_{lj}(0)|^2}{\lambda_j^2 + |m_{lj}(\lambda_l z)|^2} \right)^{\frac{n-2}{2}} \sqrt{|g(\exp_{\xi_l}(\lambda_l z))|} dz, \end{aligned}$$

where $\alpha_n = (n(n-2))^{\frac{n-2}{4}}$. Using (2.5) and (4.5), it follows that

$$\begin{aligned} \frac{\lambda_l^{\frac{n-2}{2}(1+\varepsilon)} \lambda_j^{\frac{n-2}{2}}}{(\lambda_j^2 + |m_{lj}(0)|^2)^{\frac{n-2}{2}}} &= \left(\frac{\lambda_l \lambda_j}{\lambda_j^2 + |m_{lj}(0)|^2} \right)^{\frac{n-2}{2}} \lambda_l^{\frac{n-2}{2} \varepsilon} \\ &= \left(\frac{\varepsilon^{2\alpha} d_l d_j}{\varepsilon^{2\alpha} d_j^2 + \varepsilon^{2\beta}(|\tau_l - \tau_j|^2 + o(1))} \right)^{\frac{n-2}{2}} (1 + O(\varepsilon |\ln \varepsilon|)) \\ &= \varepsilon^{(n-2)(\alpha-\beta)} \frac{d_l^{\frac{n-2}{2}} d_j^{\frac{n-2}{2}}}{|\tau_l - \tau_j|^{n-2}} (1 + o(1)). \end{aligned}$$

Note that

$$\lim_{r+|\xi_l-\xi_j| \rightarrow 0} \sup_{z \in B(0, r) \setminus \{0\}} \frac{|m_{lj}(z) - m_{lj}(0)|}{|z|} = 1. \quad (4.6)$$

Using this and (4.5), we then have that for $z \in B(0, r\varepsilon^\theta \lambda_l^{-1})$,

$$\begin{aligned} \left(\frac{\lambda_j^2 + |m_{lj}(0)|^2}{\lambda_j^2 + |m_{lj}(\lambda_l z)|^2} \right)^{\frac{n-2}{2}} &= \left(\frac{\lambda_j^2 + |m_{lj}(0)|^2}{\lambda_j^2 + (|m_{lj}(0)| + O(\lambda_l z))^2} \right)^{\frac{n-2}{2}} \\ &= \left(\frac{\varepsilon^{2\alpha} d_j^2 + \varepsilon^{2\beta} |\tau_l - \tau_j|^2 + o(\varepsilon^{2\beta})}{\varepsilon^{2\alpha} d_j^2 + (\varepsilon^\beta |\tau_l - \tau_j| + o(\varepsilon^\beta) + O(\varepsilon^\theta))^2} \right)^{\frac{n-2}{2}} \rightarrow 1 \end{aligned}$$

as $\varepsilon \rightarrow 0$. Moreover, since m_{lj} converges to the identity map and $|g(\exp_{\xi_l}(\lambda_l z))| \rightarrow 1$ pointwise as $\varepsilon \rightarrow 0$, by the dominated convergence theorem, we get

$$\int_{B(0, \varepsilon^\theta \lambda_l^{-1})} [\chi_r(\lambda_l z) U(z)]^{2^*-1+\varepsilon} \chi_r(m_{lj}(\lambda_l z)) \left(\frac{\lambda_j^2 + |m_{lj}(0)|^2}{\lambda_j^2 + |m_{lj}(\lambda_l z)|^2} \right)^{\frac{n-2}{2}} \sqrt{|g(\exp_{\xi_l}(\lambda_l z))|} dz \rightarrow \int_{\mathbb{R}^n} U(z)^{2^*-1} dz$$

as $\varepsilon \rightarrow 0$. Thus, we derive

$$L_1 = \varepsilon^{(n-2)(\alpha-\beta)} \frac{d_l^{\frac{n-2}{2}} d_j^{\frac{n-2}{2}}}{|\tau_l - \tau_j|^{n-2}} \alpha_n \int_{\mathbb{R}^n} U(z)^{2^*-1} dz + o(\varepsilon^{(n-2)(\alpha-\beta)}). \quad (4.7)$$

On the other hand,

$$\begin{aligned} |L_2| &= \int_{B(0, r) \setminus B(0, \varepsilon^\theta)} \left[\chi_r(y) \lambda_l^{\frac{2-n}{2}} U\left(\frac{y}{\lambda_l}\right) \right]^{2^*-1+\varepsilon} \chi_r(y) \lambda_j^{\frac{2-n}{2}} U\left(\frac{\exp_{\xi_j}^{-1}(\exp_{\xi_l}(y))}{\lambda_j}\right) \sqrt{|g(\exp_{\xi_l}(y))|} dy \\ &\leq C \int_{B(0, r) \setminus B(0, \varepsilon^\theta)} \left[\lambda_l^{\frac{2-n}{2}} U\left(\frac{y}{\lambda_l}\right) \right]^{2^*-1+\varepsilon} \lambda_j^{\frac{2-n}{2}} U\left(\frac{\exp_{\xi_j}^{-1}(\exp_{\xi_l}(y))}{\lambda_j}\right) dy \end{aligned}$$

$$\begin{aligned}
&\leq C \int_{B(0,r) \setminus B(0,\varepsilon^\theta)} \frac{\lambda_l^{\frac{n+2}{2} + \frac{n-2}{2}\varepsilon}}{(\lambda_l^2 + |y|^2)^{\frac{n+2}{2} + \frac{n-2}{2}\varepsilon}} \frac{\lambda_j^{\frac{n-2}{2}}}{(\lambda_j^2 + |m_{lj}(y)|^2)^{\frac{n-2}{2}}} dy \\
&\leq C \lambda_l^{\frac{n+2}{2} + \frac{n-2}{2}\varepsilon} \lambda_j^{\frac{n-2}{2}} \int_{B(0,r) \setminus B(0,\varepsilon^\theta)} \frac{1}{|y|^{n+2+(n-2)\varepsilon}} \frac{1}{|m_{lj}(y)|^{n-2}} dy.
\end{aligned}$$

Since $\beta < \theta < \alpha$, we have

$$\begin{aligned}
|m_{lj}(y)| &= |\exp_{\xi_j}^{-1}(\exp_{\xi_l}(y))| = |m_{lj}(0)| + O(|y|) \\
&= \varepsilon^\beta |\tau_l - \tau_j| + o(\varepsilon^\beta) + O(|y|) \\
&\geq \varepsilon^\beta |\tau_l - \tau_j| + o(\varepsilon^\beta) + O(\varepsilon^\theta) \quad (\text{since } y \in B(0, r) \setminus B(0, \varepsilon^\theta)) \\
&\geq \varepsilon^\beta |\tau_l - \tau_j| + o(\varepsilon^\beta) \quad (\text{since } \beta < \theta) \\
&\geq \varepsilon^\beta R^{-1} + o(\varepsilon^\beta) \quad (\text{since } |\tau_l - \tau_j| > R^{-1}).
\end{aligned}$$

Thus, we find

$$\begin{aligned}
|L_2| &\leq C \varepsilon^{n\alpha - (n-2)\beta} \int_{B(0,r) \setminus B(0,\varepsilon^\theta)} \frac{1}{|y|^{n+2+(n-2)\varepsilon}} dy \\
&\leq C \varepsilon^{n\alpha - (n-2)\beta - 2\theta} \int_{\mathbb{R}^n \setminus B(0,1)} \frac{1}{|z|^{n+2+(n-2)\varepsilon}} dz \leq C \varepsilon^{(n-2)(\alpha-\beta) + 2(\alpha-\theta)} = o(\varepsilon^{(n-2)(\alpha-\beta)}). \quad (4.8)
\end{aligned}$$

By (4.4), (4.7) and (4.8), we get (4.3). $\square$

Lemma 4.5. *We have*

$$\sum_{l \neq j} \int_M [\nabla_g W_{\lambda_j, \xi_j} \nabla_g W_{\lambda_l, \xi_l} + a(x) W_{\lambda_j, \xi_j} W_{\lambda_l, \xi_l} - W_{\lambda_l, \xi_l}^{2^*-1+\varepsilon} W_{\lambda_j, \xi_j}] d\mu_g = O(\varepsilon^{(n-2)(\alpha-\beta) + 2\beta}) = o(\varepsilon^{(n-2)(\alpha-\beta)}) \quad (4.9)$$

as $\varepsilon \rightarrow 0$, C^0 -uniformly with respect to $\bar{\tau}$ in $(\mathbb{R}^n)^k$ and to $\bar{d}$ in compact subsets of $(\mathbb{R}^+)^k$.

Proof. Using the standard properties of the exponential map, for $x \in B(0, r)$,

$$-\Delta_g u = -\Delta u - (g^{ij} - \delta^{ij}) \partial_{ij}^2 u + g^{ij} \Gamma_{ij}^k \partial_k u,$$

with

$$g^{ij} = \delta^{ij} - \frac{1}{3} R_{iabj} x_a x_b + O(|x|^3) \quad \text{and} \quad g^{ij} \Gamma_{ij}^k = \partial_l \Gamma_{ii}^k x_l + O(|x|^2).$$

Thus,

$$|\Delta_g W_{\lambda_l, \xi_l} + a(x) W_{\lambda_l, \xi_l} - W_{\lambda_l, \xi_l}^{2^*-1+\varepsilon}| (\exp_{\xi_0}(x)) \leq C \lambda_l^{\frac{2-n}{2}} U\left(\frac{\exp_{\xi_l}^{-1}(x)}{\lambda_l}\right).$$

For $l \neq j$, we then have

$$\begin{aligned}
&\int_M [\nabla_g W_{\lambda_j, \xi_j} \nabla_g W_{\lambda_l, \xi_l} + a(x) W_{\lambda_j, \xi_j} W_{\lambda_l, \xi_l} - W_{\lambda_l, \xi_l}^{2^*-1+\varepsilon} W_{\lambda_j, \xi_j}] d\mu_g \\
&= \int_M [\Delta_g W_{\lambda_l, \xi_l} + a(x) W_{\lambda_l, \xi_l} - W_{\lambda_l, \xi_l}^{2^*-1+\varepsilon}] W_{\lambda_j, \xi_j} d\mu_g \\
&\leq C \int_{B_g(0,r)} \lambda_l^{\frac{2-n}{2}} U\left(\frac{\exp_{\xi_l}^{-1}(x)}{\lambda_l}\right) \lambda_j^{\frac{2-n}{2}} U\left(\frac{\exp_{\xi_j}^{-1}(x)}{\lambda_j}\right) d\mu_g \\
&\leq C \int_{B_g(0,r)} \frac{\lambda_l^{\frac{n-2}{2}}}{(\lambda_l^2 + |\exp_{\xi_l}^{-1}(x)|^2)^{\frac{n-2}{2}}} \frac{\lambda_j^{\frac{n-2}{2}}}{(\lambda_j^2 + |\exp_{\xi_j}^{-1}(x)|^2)^{\frac{n-2}{2}}} d\mu_g \\
&\leq C \int_{B(0,r)} \frac{\lambda_l^{\frac{n-2}{2}}}{(\lambda_l^2 + |y|^2)^{\frac{n-2}{2}}} \frac{\lambda_j^{\frac{n-2}{2}}}{(\lambda_j^2 + |m_{lj}(y)|^2)^{\frac{n-2}{2}}} dy. \quad (4.10)
\end{aligned}$$

We divide the domain $B(0, r)$ into three disjoint sets, i.e., we set

$$\begin{aligned}\Omega_1 &:= \left\{ y : y \in B(0, r), |m_{lj}(y)| \leq \frac{\varepsilon^\beta |\tau_l - \tau_j|}{2} \right\}, \\ \Omega_2 &:= B(0, 10R\varepsilon^\beta) \setminus \Omega_1, \\ \Omega_3 &:= B(0, r) \setminus B(0, 10R\varepsilon^\beta),\end{aligned}$$

where R is given in (2.4). From (4.6), for $y \in \Omega_1$, we have

$$|y| \geq \frac{9}{10}(|m_{lj}(0)| - |m_{lj}(y)|) \geq \frac{9}{10} \left(\frac{9}{10} - \frac{1}{2} \right) \varepsilon^\beta |\tau_l - \tau_j| \geq \frac{2}{10R} \varepsilon^\beta.$$

Then

$$\begin{aligned}\int_{\Omega_1} \frac{\lambda_l^{\frac{n-2}{2}}}{(\lambda_l^2 + |y|^2)^{\frac{n-2}{2}}} \frac{\lambda_j^{\frac{n-2}{2}}}{(\lambda_j^2 + |m_{lj}(y)|^2)^{\frac{n-2}{2}}} dy &\leq C\varepsilon^{(n-2)\alpha} \int_{\Omega_1} \frac{1}{|y|^{n-2}} \frac{1}{|m_{lj}(y)|^{n-2}} dy \\ &\leq C\varepsilon^{(n-2)(\alpha-\beta)} \int_{\{|m_{lj}(y)| \leq \varepsilon^\beta |\tau_l - \tau_j|/2\}} \frac{1}{|m_{lj}(y)|^{n-2}} dy.\end{aligned}$$

Take the change of variable $m_{lj}(y) \mapsto z$, and use the facts that $|\tau_l - \tau_j| \leq |\tau_l| + |\tau_j| \leq 2R$, the map m_{lj} is injective in its domain $B(0, i_M)$ and $\min_{B(0, r)} |Dm_{lj}(z)| \geq \frac{1}{2}$ for $\varepsilon > 0$ and $r > 0$ small enough. Thus, we find

$$\begin{aligned}\int_{\Omega_1} \frac{\lambda_l^{\frac{n-2}{2}}}{(\lambda_l^2 + |y|^2)^{\frac{n-2}{2}}} \frac{\lambda_j^{\frac{n-2}{2}}}{(\lambda_j^2 + |m_{lj}(y)|^2)^{\frac{n-2}{2}}} dy &\leq C \frac{\varepsilon^{(n-2)(\alpha-\beta)}}{\min_{B(0, r)} |Dm_{lj}(z)|} \int_{\{|z| \leq \varepsilon^\beta |\tau_l - \tau_j|/2\}} \frac{1}{|z|^{n-2}} dz \\ &\leq C\varepsilon^{(n-2)(\alpha-\beta)+2\beta} = o(\varepsilon^{(n-2)(\alpha-\beta)}).\end{aligned}\quad (4.11)$$

If $y \in \Omega_2$, we have that $|y| \leq 10R\varepsilon^\beta$ and $|m_{lj}(y)| > \frac{\varepsilon^\beta |\tau_l - \tau_j|}{2} \geq \frac{\varepsilon^\beta}{2R}$. Then

$$\begin{aligned}\int_{\Omega_2} \frac{\lambda_l^{\frac{n-2}{2}}}{(\lambda_l^2 + |y|^2)^{\frac{n-2}{2}}} \frac{\lambda_j^{\frac{n-2}{2}}}{(\lambda_j^2 + |m_{lj}(y)|^2)^{\frac{n-2}{2}}} dy &\leq C\varepsilon^{(n-2)\alpha} \int_{\Omega_2} \frac{1}{|y|^{n-2}} \frac{1}{|m_{lj}(y)|^{n-2}} dy \\ &\leq C\varepsilon^{(n-2)(\alpha-\beta)} \int_{\{|y| \leq 10R\varepsilon^\beta\}} \frac{1}{|y|^{n-2}} dy \\ &\leq C\varepsilon^{(n-2)(\alpha-\beta)+2\beta} = o(\varepsilon^{(n-2)(\alpha-\beta)}).\end{aligned}\quad (4.12)$$

If $y \in \Omega_3$, we have that $|y| < r$ and $|y| \geq 10R\varepsilon^\beta$. From (4.5), (4.6) and $|\tau_l| \leq R$, $|\tau_j| \leq R$, for $y \in \Omega_3$, we get

$$|m_{lj}(y)| \geq |m_{lj}(y) - m_{lj}(0)| - |m_{lj}(0)| = (|y| + o(1)) - \varepsilon^\beta (|\tau_l - \tau_j| + o(1)) \geq \frac{9}{10}|y| - 2R\varepsilon^\beta \geq \frac{7}{10}|y|.$$

Thus,

$$\begin{aligned}\int_{\Omega_3} \frac{\lambda_l^{\frac{n-2}{2}}}{(\lambda_l^2 + |y|^2)^{\frac{n-2}{2}}} \frac{\lambda_j^{\frac{n-2}{2}}}{(\lambda_j^2 + |m_{lj}(y)|^2)^{\frac{n-2}{2}}} dy &\leq C\varepsilon^{(n-2)\alpha} \int_{\Omega_3} \frac{1}{|y|^{n-2}} \frac{1}{|m_{lj}(y)|^{n-2}} dy \\ &\leq C\varepsilon^{(n-2)\alpha} \int_{\{10R\varepsilon^\beta \leq |y| \leq r\}} \frac{1}{|y|^{2(n-2)}} dy \\ &\leq C\varepsilon^{(n-2)\alpha} \varepsilon^{-(n-4)\beta} = C\varepsilon^{(n-2)(\alpha-\beta)+2\beta} = o(\varepsilon^{(n-2)(\alpha-\beta)}).\end{aligned}\quad (4.13)$$

Therefore, (4.9) follows from (4.10)–(4.13). $\square$

Lemma 4.6. *We have*

$$\int_M \left[\left(\sum_{j=1}^k W_{\lambda_j, \xi_j} \right)^{2^* + \varepsilon} - \sum_{j=1}^k W_{\lambda_j, \xi_j}^{2^* + \varepsilon} - (2^* + \varepsilon) \sum_{j \neq l} W_{\lambda_l, \xi_l}^{2^* - 1 + \varepsilon} W_{\lambda_j, \xi_j} \right] d\mu_g = o(\varepsilon^{(n-2)(\alpha-\beta)}) \quad (4.14)$$

as $\varepsilon \rightarrow 0$, C^0 -uniformly with respect to $\bar{\tau}$ in $(\mathbb{R}^n)^k$ and to $\bar{d}$ in compact subsets of $(\mathbb{R}^+)^k$.

Proof. Let $B_h := B_g(\xi_h, \frac{\varepsilon^\beta}{4R})$, where $B_g(\xi_h, \frac{\varepsilon^\beta}{4R})$ denotes the geodesic ball in (M, g) with center at ξ_h and $R > 0$ is given in (2.4). We write

$$\begin{aligned} & \int_M \left[\left(\sum_{j=1}^k W_{\lambda_j, \xi_j} \right)^{2^* + \varepsilon} - \sum_{j=1}^k W_{\lambda_j, \xi_j}^{2^* + \varepsilon} - (2^* + \varepsilon) \sum_{j \neq l} W_{\lambda_l, \xi_l}^{2^* - 1 + \varepsilon} W_{\lambda_j, \xi_j} \right] d\mu_g \\ &= \int_{M \setminus \bigcup_{h=1}^k B_h} \left[\left(\sum_{j=1}^k W_{\lambda_j, \xi_j} \right)^{2^* + \varepsilon} - \sum_{j=1}^k W_{\lambda_j, \xi_j}^{2^* + \varepsilon} - (2^* + \varepsilon) \sum_{j \neq l} W_{\lambda_l, \xi_l}^{2^* - 1 + \varepsilon} W_{\lambda_j, \xi_j} \right] d\mu_g \\ & \quad + \sum_{h=1}^k \int_{B_h} \left[\left(\sum_{j=1}^k W_{\lambda_j, \xi_j} \right)^{2^* + \varepsilon} - \sum_{j=1}^k W_{\lambda_j, \xi_j}^{2^* + \varepsilon} - (2^* + \varepsilon) \sum_{j \neq l} W_{\lambda_l, \xi_l}^{2^* - 1 + \varepsilon} W_{\lambda_j, \xi_j} \right] d\mu_g \\ &:= I + \sum_{h=1}^k I_h. \end{aligned} \quad (4.15)$$

We first have

$$\begin{aligned} |I| &\leq C \sum_{j \neq l} \int_{M \setminus \bigcup_{h=1}^k B_h} W_{\lambda_l, \xi_l}^{2^* - 2 + \varepsilon} W_{\lambda_j, \xi_j}^2 d\mu_g \\ &\leq C \sum_{j \neq l} \int_{M \setminus (B_l \cup B_j)} W_{\lambda_l, \xi_l}^{2^* - 2 + \varepsilon} W_{\lambda_j, \xi_j}^2 d\mu_g \\ &\leq C \sum_{j \neq l} \int_{B_g(0, r) \setminus (B_g(\xi_l, \frac{\varepsilon^\beta}{4R}) \cup B_g(\xi_j, \frac{\varepsilon^\beta}{4R}))} \frac{\lambda_l^{2 + \frac{n-2}{2}\varepsilon}}{(\lambda_l^2 + |\exp_{\xi_l}^{-1}(x)|^2)^{2 + \frac{n-2}{2}\varepsilon}} \frac{\lambda_j^{n-2}}{(\lambda_j^2 + |\exp_{\xi_j}^{-1}(x)|^2)^{n-2}} d\mu_g \\ &\leq C \sum_{j \neq l} \int_{B_g(0, r) \setminus (B_g(\xi_l, \frac{\varepsilon^\beta}{4R}) \cup B_g(\xi_j, \frac{\varepsilon^\beta}{4R}))} \frac{\lambda_l^{2 + \frac{n-2}{2}\varepsilon}}{|\exp_{\xi_l}^{-1}(x)|^{4 + n - 2\varepsilon}} \frac{\lambda_j^{n-2}}{|\exp_{\xi_j}^{-1}(x)|^{2(n-2)}} d\mu_g \\ &\leq C \sum_{j \neq l} \int_{B(0, r) \setminus (B(0, \frac{\varepsilon^\beta}{4R}) \cup \exp_{\xi_l}^{-1}(B_g(\xi_j, \frac{\varepsilon^\beta}{4R})))} \frac{\lambda_l^{2 + \frac{n-2}{2}\varepsilon}}{|y|^{4 + (n-2)\varepsilon}} \frac{\lambda_j^{n-2}}{|m_{lj}(y)|^{2(n-2)}} dy \quad (y = \exp_{\xi_l}^{-1}(x)) \\ &\leq C \varepsilon^{n\alpha} \varepsilon^{-4\beta} \sum_{j \neq l} \int_{B(0, r) \setminus (B(0, \frac{\varepsilon^\beta}{4R}) \cup \exp_{\xi_l}^{-1}(B_g(\xi_j, \frac{\varepsilon^\beta}{4R})))} \frac{1}{|m_{lj}(y)|^{2(n-2)}} dy \\ &\leq C \varepsilon^{n\alpha} \varepsilon^{-4\beta} \sum_{j \neq l} \int_{|m_{lj}(y)| \geq \frac{\varepsilon^\beta}{4R}} \frac{1}{|m_{lj}(y)|^{2(n-2)}} dy \\ &\leq C \varepsilon^{n\alpha} \varepsilon^{-4\beta} \sum_{j \neq l} \int_{|z| \geq \frac{\varepsilon^\beta}{4R}} \frac{1}{|z|^{2(n-2)}} dy \\ &\leq C \varepsilon^{n(\alpha - \beta)}. \end{aligned} \quad (4.16)$$

Moreover,

$$\begin{aligned} |I_h| &= \left| \int_{B_h} \left[\left(W_{\lambda_h, \xi_h} + \sum_{j \neq h} W_{\lambda_j, \xi_j} \right)^{2^* + \varepsilon} - W_{\lambda_h, \xi_h}^{2^* + \varepsilon} - \sum_{j \neq h} W_{\lambda_j, \xi_j}^{2^* + \varepsilon} - (2^* + \varepsilon) \sum_{j \neq l} W_{\lambda_l, \xi_l}^{2^* - 1 + \varepsilon} W_{\lambda_j, \xi_j} \right] d\mu_g \right| \\ &\leq C \sum_{j \neq h} \left[\int_{B_h} W_{\lambda_h, \xi_h}^{2^* - 2 + \varepsilon} W_{\lambda_j, \xi_j}^2 d\mu_g + \int_{B_h} W_{\lambda_j, \xi_j}^{2^* + \varepsilon} d\mu_g + \sum_{j \neq l} \int_{B_h} W_{\lambda_l, \xi_l}^{2^* - 1 + \varepsilon} W_{\lambda_j, \xi_j} d\mu_g \right]. \end{aligned} \quad (4.17)$$

Since

$$\text{dist}_g(B_h, \xi_j) \geq d_g(\xi_j, \xi_h) - \frac{\varepsilon^\beta}{4R} \geq \frac{\varepsilon^\beta}{4R} \quad \text{for } j \neq h,$$

we have

$$\begin{aligned}
 \int_{B_h} W_{\lambda_h, \xi_h}^{2^*-2+\varepsilon} W_{\lambda_j, \xi_j}^2 d\mu_g &\leq C \int_{B_h} \frac{\lambda_h^{2+\frac{n-2}{2}\varepsilon}}{(\lambda_h^2 + |\exp_{\xi_h}^{-1}(x)|^2)^{2+\frac{n-2}{2}\varepsilon}} \frac{\lambda_j^{n-2}}{(\lambda_j^2 + |\exp_{\xi_j}^{-1}(x)|^2)^{n-2}} d\mu_g \\
 &\leq C \int_{B_h} \frac{\lambda_h^{2+\frac{n-2}{2}\varepsilon}}{|\exp_{\xi_h}^{-1}(x)|^{4+n-2\varepsilon}} \frac{\lambda_j^{n-2}}{|\exp_{\xi_j}^{-1}(x)|^{2(n-2)}} d\mu_g \\
 &\leq C \int_{B(0, \frac{\varepsilon\beta}{4R})} \frac{\lambda_h^{2+\frac{n-2}{2}\varepsilon}}{|y|^{4+(n-2)\varepsilon}} \frac{\lambda_j^{n-2}}{|m_{lj}(y)|^{2(n-2)}} dy \quad (y = \exp_{\xi_l}^{-1}(x)) \\
 &\leq C\varepsilon^{n\alpha} \varepsilon^{-(2(n-2)\beta)} \int_{B(0, \frac{\varepsilon\beta}{4R})} \frac{1}{|y|^{4+(n-2)\varepsilon}} dy \\
 &\leq C\varepsilon^{n(\alpha-\beta)}.
 \end{aligned}$$

For $j \neq h$,

$$\int_{B_h} W_{\lambda_j, \xi_j}^{2^*+\varepsilon} d\mu_g \leq C \int_{B_h} \frac{\lambda_j^{n+\frac{n-2}{2}\varepsilon}}{(\lambda_j^2 + |\exp_{\xi_j}^{-1}(x)|^2)^{n+\frac{n-2}{2}\varepsilon}} d\mu_g \leq C\varepsilon^{n\alpha} \int_{B(o, r) \setminus B(0, \frac{\varepsilon\beta}{4R})} \frac{1}{|y|^{2n+(n-2)\varepsilon}} dy \leq C\varepsilon^{n(\alpha-\beta)}.$$

For the third term in (4.17), if $l \neq h$, using Hölder's inequality and the fact that the estimate for the second integral is bounded by $C\varepsilon^{n(\alpha-\beta)}$, then we can get that the third term is bounded by $C\varepsilon^{n(\alpha-\beta)}$. Moreover, for $j \neq h$ and $l = h$, we have

$$\begin{aligned}
 \int_{B_h} W_{\lambda_l, \xi_l}^{2^*-1+\varepsilon} W_{\lambda_j, \xi_j} d\mu_g &= \int_{B_h} W_{\lambda_j, \xi_j}^{2^*-1+\varepsilon} W_{\lambda_h, \xi_h} d\mu_g \\
 &\leq C \int_{B_h} \frac{\lambda_j^{\frac{n+2}{2}+\frac{n-2}{2}\varepsilon}}{(\lambda_j^2 + |\exp_{\xi_j}^{-1}(x)|^2)^{\frac{n+2}{2}+\frac{n-2}{2}\varepsilon}} \frac{\lambda_h^{\frac{n-2}{2}}}{(\lambda_h^2 + |\exp_{\xi_h}^{-1}(x)|^2)^{\frac{n-2}{2}}} d\mu_g \\
 &\leq C\varepsilon^{n\alpha} \int_{B_h} \frac{1}{|\exp_{\xi_j}^{-1}(x)|^{n+2+(n-2)\varepsilon}} \frac{1}{|\exp_{\xi_h}^{-1}(x)|^{n-2}} d\mu_g \\
 &\leq C\varepsilon^{n\alpha} \varepsilon^{-(n+2)\beta} \int_{B_h} \frac{1}{|\exp_{\xi_j}^{-1}(x)|^{n-2}} d\mu_g \\
 &\leq C\varepsilon^{n\alpha} \varepsilon^{-(n+2)\beta} \int_{B(0, \frac{\varepsilon\beta}{4R})} \frac{1}{|y|^{(n-2)}} dy \\
 &\leq C\varepsilon^{n(\alpha-\beta)},
 \end{aligned}$$

where we used the fact that

$$|\exp_{\xi_j}^{-1}(x)| = \text{dist}_g(x, \xi_j) \geq \text{dist}_g(B_h, \xi_j) \geq d_g(\xi_j, \xi_h) - \frac{\varepsilon\beta}{4R} \geq \frac{\varepsilon\beta}{4R}$$

for $j \neq h$ and $x \in B_h$. Thus, we have $|I_h| \leq C\varepsilon^{n(\alpha-\beta)}$. Therefore, (4.14) follows from (4.15), (4.16) and the last inequality. $\square$

Proof of Proposition 3.2 (ii). From (4.1) and Lemmas 4.2–4.6, we get that if $n \geq 5$ and $(\bar{d}, \bar{\tau}) \in \mathcal{O}_{\eta, R}^k$ satisfies (2.4), then

$$\begin{aligned}
 \tilde{J}_\varepsilon(\bar{d}, \bar{\tau}) &= kc_0 + \varepsilon k(c_1 + c_2 \ln \varepsilon) + \sum_{j=1}^k [c_3 \ln(d_j) + c_4 d_j^2 \varphi(\xi_j)] \varepsilon \\
 &\quad - c_5 \varepsilon^{(n-2)(\alpha-\beta)} \sum_{j < l} \frac{d_l^{\frac{n-2}{2}} d_j^{\frac{n-2}{2}}}{|\tau_l - \tau_j|^{n-2}} + o(\varepsilon^{(n-2)(\alpha-\beta)})
 \end{aligned} \tag{4.18}$$

as $\varepsilon \rightarrow 0$, C^0 -uniformly with respect to $\bar{\tau}$ in $(\mathbb{R}^n)^k$ and to $\bar{d}$ in compact subsets of $(\mathbb{R}^+)^k$, where

$$\begin{aligned} c_0 &= \frac{K_n^{-n}}{n}, & c_1 &= c_0 C_n, & c_2 &= c_0 \frac{(n-2)^2}{8}, \\ c_3 &= \frac{(n-2)^2}{4} c_0, & c_4 &= \frac{2(n-1)}{(n-2)(n-4)} c_0, & c_5 &= C_n \int_{\mathbb{R}^n} U(z)^{2^*-1} dz, \end{aligned}$$

with C_n being a positive constant, which is defined in (4.2), and φ given by (1.5). By Taylor's theorem and the assumptions $\nabla \varphi(\xi_0) = 0$, we have

$$\varphi(\xi_j) = \varphi(\xi_0) + \varepsilon^{2\beta} D^2 \varphi(\xi_0)[\tau_j, \tau_j] + o(\varepsilon^{2\beta}). \quad (4.19)$$

Therefore, (3.5) follows from (4.18) and (4.19). $\square$

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