Continuity results for parametric nonlinear singular Dirichlet problems

Abstract: In this paper we study from a qualitative point of view the nonlinear singular Dirichlet problem depending on a parameter $\lambda > 0$ that was considered in [32]. Denoting by $S_\lambda$ the set of positive solutions of the problem corresponding to the parameter $\lambda$, we establish the following essential properties of $S_\lambda$:

(i) there exists a smallest element $u_\lambda^*$ in $S_\lambda$, and the mapping $\lambda \mapsto u_\lambda^*$ is (strictly) increasing and left continuous;
(ii) the set-valued mapping $\lambda \mapsto S_\lambda$ is sequentially continuous.

Keywords: Parametric singular elliptic equation, $p$-Laplacian, smallest solution, sequential continuity, monotonicity
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1 Introduction

Elliptic equations with singular terms represent a class of hot-point problems because they are mathematically significant and appear in applications to chemical catalysts processes, non-Newtonian fluids, and in models for the temperature of electrical conductors (see [3, 9]). An extensive literature is devoted to such problems, especially focusing on their theoretical analysis. For instance, Ghergu-Rădulescu [18] established several existence and nonexistence results for boundary value problems with singular terms and parameters; Gasinski-Papageorgiou [15] studied a nonlinear Dirichlet problem with a singular term, a $(p - 1)$-sublinear term, and a Carathéodory perturbation; Hirano-Sacco-Shioji [21] proved Brezis-Nirenberg type theorems for a singular elliptic problem. Related topics and results can be found in Crandall-Rabinowitz-Tartar [7], Cîrstea-Ghergu-Rădulescu [6], Dupaigne-Ghergu-Rădulescu [10], Gasinski-Papageorgiou [17], Averna-Motreanu-Tornatore [2], Papageorgiou-Winkert [33], Carl [4], Faria-Miyagaki-Motreanu [11], Carl-Costa-Tehrani [5], Liu-Motreanu-Zeng [26] Papageorgiou-Rădulescu-Repovš [30], and the references therein.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a $C^2$-boundary $\partial \Omega$ and let $y \in (0, 1)$ and $1 < p < +\infty$. Recently, Papageorgiou-Vetro-Vetro [32] have considered the following parametric nonlinear singular Dirichlet problem

$$
\begin{cases}
-\Delta_p u(x) = \lambda u(x)^y + f(x, u(x)) & \text{in } \Omega \\
u(x) > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

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where the operator $\Delta_p$ stands for the $p$–Laplace differential operator

$$
\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W^{1,p}_0(\Omega).
$$

The nonlinear function $f$ is assumed to satisfy the following conditions:

$H(f) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.e. $x \in \Omega$, $f(x, 0) = 0$, $f(x, s) \geq 0$ for all $s \geq 0$, and

1. for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that $|f(x, s)| \leq a_\rho(x)$ for a.e. $x \in \Omega$ and for all $|s| \leq \rho$;
2. there exists an integer $m \geq 2$ such that

$$
\lim_{s \to +\infty} \frac{f(x, s)}{s^{p-1}} = \hat{\lambda}_m \quad \text{uniformly for a.e. } x \in \Omega,
$$

where $\hat{\lambda}_m$ is the $m$-th eigenvalue of $(-\Delta_p, W^{1,p}_0(\Omega))$, and denoting

$$
F(x, t) = \int_0^s f(x, t) \, dt,
$$

then

$$
pF(x, s) - f(x, s)s \to +\infty \quad \text{as } s \to +\infty, \quad \text{uniformly for a.e. } x \in \Omega;
$$

3. for some $r > p$, there exists $c_0 \geq 0$ such that

$$
0 \leq \liminf_{s \to 0^+} \frac{f(x, s)}{s^{r-1}} \leq \limsup_{s \to 0^+} \frac{f(x, s)}{s^{r-1}} \leq c_0 \quad \text{uniformly for a.e. } x \in \Omega;
$$

4. for every $\rho > 0$, there exists $\tilde{\rho}_\rho > 0$ such that for a.e. $x \in \Omega$ the function

$$
s \mapsto f(x, s) + \tilde{\rho}_\rho s^{p-1}
$$

is nondecreasing on $[0, \rho]$.

The following bifurcation type result is proved in [32, Theorem 2].

**Theorem 1.** If hypotheses $H(f)$ hold, then there exists a critical parameter value $\lambda^* > 0$ such that

(a) for all $\lambda \in (0, \lambda^*)$ problem (1) has at least two positive solutions $u_0, u_1 \in \text{int}(C^1_0(\bar{\Omega}^+))$;

(b) for $\lambda = \lambda^*$ problem (1) has at least one positive solution $u^* \in \text{int}(C^1_0(\bar{\Omega}^+))$;

(c) for all $\lambda > \lambda^*$ problem (1) has no positive solutions.

In what follows, we denote

$$
\mathcal{L} := \{ \lambda > 0 : \text{problem (1) admits a (positive) solution} \} = (0, \lambda^*],
$$

$$
S_\lambda = \{ u \in W^{1,p}_0(\Omega) : u \text{ is a (positive) solution of problem (1)} \}
$$

for $\lambda \in \mathcal{L}$. In this respect, Theorem 1 asserts that the above hypotheses, in conjunction with the nonlinear regularity theory (see Liebermann [24, 25]) and the nonlinear strong maximum principle (see Pucci-Serrin [34]), ensure that there holds

$$
S_\lambda \subset \text{int}(C^1_0(\bar{\Omega}^+)).
$$

Also, we introduce the set-valued mapping $\Lambda : (0, \lambda^*) \to 2^{C^1_0(\bar{\Omega})}$ by

$$
\Lambda(\lambda) = S_\lambda \quad \text{for all } \lambda \in (0, \lambda^*].
$$

The following open questions need to be answered:
1. Is there a smallest positive solution to problem (1) for each $\lambda \in (0, \lambda^*)$?
2. If for each $\lambda \in (0, \lambda^*)$ problem (1) has a smallest positive solution $u^*_\lambda$, then the function $\Gamma : (0, \lambda^*) \to C^1_0(\Omega)$ with $\Gamma(\lambda) = u^*_\lambda$ is it monotone?
3. If for each $\lambda \in (0, \lambda^*)$ problem (1) has a smallest positive solution $u^*_\lambda$, then is the function $\Gamma$ continuous?
4. Is the solution mapping $\Lambda$ upper semicontinuous?
5. Is the solution mapping $\Lambda$ lower semicontinuous?

In this paper we answer in the affirmative the above open questions.

**Theorem 2.** Assume that hypotheses $H(f)$ hold. Then there hold:

(i) the set-valued mapping $\Lambda : L \to 2^{C^1_0(\Omega)}$ is sequentially continuous;
(ii) for each $\lambda \in L$, problem (1) has a smallest positive solution $u^*_\lambda \in \text{int}(C^1_0(\Omega)_+)$, and the map $\Gamma$ from $L$ to $C^1_0(\Omega)$ given by $\Gamma(\lambda) = u^*_\lambda$ is
   (a) (strictly) increasing, that is, if $0 < \mu < \lambda \leq \lambda^*$, then
       $$u^*_\lambda - u^*_\mu \in \text{int}(C^1_0(\Omega)_+);$$
   (b) left continuous.

The rest of the paper is organized as follows. In Section 2 we set forth the preliminary material needed in the sequel. In Section 3 we prove our main results formulated as Theorem 2.

## 2 Preliminaries

In this section we gather the preliminary material that will be used to prove the main result in the paper. For more details we refer to [8, 13, 16, 19, 22, 28, 29, 35].

Let $1 < p < \infty$ and $p'$ be its Hölder conjugate defined by $\frac{1}{p} + \frac{1}{p'} = 1$. In what follows, the Lebesgue space $L^p(\Omega)$ is endowed with the standard norm

$$\|u\|_p = \left( \int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}} \text{ for all } u \in L^p(\Omega).$$

The Sobolev space $W_0^{1,p}(\Omega)$ is equipped with the usual norm

$$\|u\| = \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}} \text{ for all } u \in W_0^{1,p}(\Omega).$$

In addition, we shall use the Banach space

$$C^1_0(\Omega) = \{ u \in C^1(\Omega) : u = 0 \text{ on } \partial \Omega \}.$$  

Its cone of nonnegative functions

$$C^1_0(\Omega)_+ = \{ u \in C^1_0(\Omega) : u \geq 0 \text{ in } \Omega \}$$

has a nonempty interior given by

$$\text{int}(C^1_0(\Omega)_+) = \left\{ u \in C^1_0(\Omega) : u > 0 \text{ in } \Omega \text{ with } \frac{\partial u}{\partial n}|_{\partial \Omega} < 0 \right\},$$
where $\frac{\partial u}{\partial n}$ is the normal derivative of $u$ and $n(\cdot)$ is the outward unit normal to the boundary $\partial \Omega$.

Hereafter by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for $(W^{1,p}(\Omega)^*, W^{1,p}(\Omega))$. Also, we define the nonlinear operator $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ by

$$
\langle A(u), v \rangle = \int_{\Omega} |\nabla u(x)|^{p-2} (\nabla u(x), \nabla v(x))_{\mathbb{R}^n} \, dx \text{ for all } u, v \in W^{1,p}(\Omega). \quad (2)
$$

The following statement is a special case of more general results (see Gasiński-Papageorgiou [14], Motreanu-Motreanu-Papageorgiou [29]).

**Proposition 3.** The map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ introduced in (2) is continuous, bounded (that is, it maps bounded sets to bounded sets), monotone (hence maximal monotone) and of type $(S_+)$, i.e., if $u_n \to u$ in $W^{1,p}(\Omega)$ and

$$
\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0,
$$

then $u_n \to u$ in $W^{1,p}(\Omega)$.

For the sake of clarity we recall the following notion regarding order.

**Definition 4.** Let $(P, \leq)$ be a partially ordered set. A subset $E \subset P$ is called downward directed if for each pair $u, v \in E$ there exists $w \in E$ such that $w \leq u$ and $w \leq v$.

For any $u, v \in W^{1,p}_0(\Omega)$ with $u(x) \leq v(x)$ for a.e. $x \in \Omega$, we set the ordered interval 

$$
[u, v] := \{ w \in W^{1,p}_0(\Omega) : u(x) \leq w(x) \leq v(x) \text{ for a.e. } x \in \Omega \}.
$$

For $s \in \mathbb{R}$, we denote $s^+ = \max\{s, 0\}$. It is clear that if $u \in W^{1,p}_0(\Omega)$ then it holds

$$
|u|^+ \in W^{1,p}_0(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.
$$

We recall a few things regarding upper and lower semicontinuous set-valued mappings.

**Definition 5.** Let $X$ and $Y$ be topological spaces. A set-valued mapping $F : X \to 2^Y$ is called

(i) **upper semicontinuous** (u.s.c., for short) at $x \in X$ if for every open set $O \subset Y$ with $F(x) \subset O$ there exists a neighborhood $N(x)$ of $x$ such that

$$
F(N(x)) := \bigcup_{y \in N(x)} F(y) \subset O;
$$

if this holds for every $x \in X, F$ is called upper semicontinuous;

(ii) **lower semicontinuous** (l.s.c., for short) at $x \in X$ if for every open set $O \subset Y$ with $F(x) \cap O \neq \emptyset$ there exists a neighborhood $N(x)$ of $x$ such that

$$
F(y) \cap O \neq \emptyset \text{ for all } y \in N(x);
$$

if this holds for every $x \in X, F$ is called lower semicontinuous;

(iii) **continuous** at $x \in X$ if $F$ is both upper semicontinuous and lower semicontinuous at $x \in X$ if this holds for every $x \in X, F$ is called continuous.

The propositions below provide criteria of upper and lower semicontinuity.

**Proposition 6.** The following properties are equivalent:

(i) $F : X \to 2^Y$ is u.s.c.;
for every closed subset $C \subset Y$, the set

$$ F^{-}(C) := \{ x \in X \mid F(x) \cap C \neq \emptyset \} $$

is closed in $X$.

**Proposition 7.** The following properties are equivalent:

(a) $F : X \to 2^{Y}$ is l.s.c.;

(b) if $u \in X$, $(u_\lambda)_{\lambda \in J} \subset X$ is a net such that $u_\lambda \to u$, and $u^* \in F(u)$, then for each $\lambda \in J$ there is $u^*_\lambda \in F(u_\lambda)$ with $u^*_\lambda \to u^*$ in $Y$.

**3 Proof of the main result**

In this section we prove Theorem 2. We start with the fact that, for each $\lambda \in \mathcal{L}$, problem (1) has a smallest solution. To this end, we will use the similar technique employed in [12, Lemma 4.1] to show that the solution set $S_\lambda$ is downward directed (see Definition 4).

**Lemma 8.** For each $\lambda \in \mathcal{L} = (0, \lambda^*)$, the solution set $S_\lambda$ of problem (1) is downward directed, i.e., if $u_1, u_2 \in S_\lambda$, then there exists $u \in S_\lambda$ such that

$$ u \leq u_1 \quad \text{and} \quad u \leq u_2. $$

**Proof.** Fix $\lambda \in (0, \lambda^*)$ and $u_1, u_2 \in S_\lambda$. Corresponding to any $\varepsilon > 0$ we introduce the truncation $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$ as follows

$$ \eta_\varepsilon(t) = \begin{cases} 
0 & \text{if } t \leq 0 \\
\frac{\varepsilon}{\varepsilon - t} & \text{if } 0 < t < \varepsilon \\
1 & \text{otherwise}, 
\end{cases} $$

which is Lipschitz continuous. It results from Marcus-Mizel [27] that

$$ \eta_\varepsilon(u_2 - u_1) \in W^{1,p}_0(\Omega) $$

and

$$ \nabla(\eta_\varepsilon(u_2 - u_1)) = \eta'_\varepsilon(u_2 - u_1) \nabla(u_2 - u_1). $$

Then for any function $v \in C_0^\infty(\Omega)$ with $v(x) \geq 0$ for a.e. $x \in \Omega$, we have

$$ \eta_\varepsilon(u_2 - u_1)v \in W^{1,p}_0(\Omega) $$

and

$$ \nabla(\eta_\varepsilon(u_2 - u_1)v) = v\nabla(\eta_\varepsilon(u_2 - u_1)) + \eta_\varepsilon(u_2 - u_1)\nabla v. $$

Since $u_1, u_2 \in S_\lambda$, there hold

$$ \int_{\Omega} |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla \varphi(x))_{\mathbb{R}^N} \, dx = \int_{\Omega} f(x, u_1(x))\varphi(x) \, dx \quad \text{and} \quad \int_{\Omega} f(x, u_1(x))\varphi(x) \, dx \quad \text{for all } \varphi \in W^{1,p}_0(\Omega), \ i = 1, 2.
Inserting \( \varphi = \eta \varepsilon (u_2 - u_1) v \) for \( i = 1 \) and \( \varphi = (1 - \eta \varepsilon (u_2 - u_1)) v \) for \( i = 2 \), and summing the resulting inequalities yield

\[
\int_{\Omega} |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla (\eta \varepsilon (u_2 - u_1) v)(x))_{\mathbb{R}^n} \, dx \\
+ \int_{\Omega} |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla ((1 - \eta \varepsilon (u_2 - u_1)) v)(x))_{\mathbb{R}^n} \, dx \\
= \int_{\Omega} [\lambda u_1(x)^{-\gamma} + f(x, u_1(x))] (\eta \varepsilon (u_2 - u_1) v)(x) \, dx \\
+ \int_{\Omega} [\lambda u_2(x)^{-\gamma} + f(x, u_2(x))] (1 - \eta \varepsilon (u_2 - u_1)) v)(x) \, dx.
\]

We note that

\[
\int_{\Omega} |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla (\eta \varepsilon (u_2 - u_1) v)(x))_{\mathbb{R}^n} \, dx \\
= \frac{1}{\varepsilon} \int_{\{0 < u_2 - u_1 < \varepsilon\}} |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla (u_2 - u_1)(x))_{\mathbb{R}^n} v(x) \, dx \\
+ \int_{\Omega} |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla v(x))_{\mathbb{R}^n} \eta \varepsilon (u_2(x) - u_1(x)) \, dx
\]

and

\[
\int_{\Omega} |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla ((1 - \eta \varepsilon (u_2 - u_1)) v)(x))_{\mathbb{R}^n} \, dx \\
= -\frac{1}{\varepsilon} \int_{\{0 < u_2 - u_1 < \varepsilon\}} |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla (u_2 - u_1)(x))_{\mathbb{R}^n} v(x) \, dx \\
+ \int_{\Omega} |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla v(x))_{\mathbb{R}^n} (1 - \eta \varepsilon (u_2(x) - u_1(x))) \, dx.
\]

Altogether, we obtain

\[
\int_{\Omega} |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla v(x))_{\mathbb{R}^n} \eta \varepsilon (u_2(x) - u_1(x)) \, dx \\
+ \int_{\Omega} |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla v(x))_{\mathbb{R}^n} (1 - \eta \varepsilon (u_2(x) - u_1(x))) \, dx \\
\geq \int_{\Omega} [\lambda u_1(x)^{-\gamma} + f(x, u_1(x))] (\eta \varepsilon (u_2 - u_1) v)(x) \, dx \\
+ \int_{\Omega} [\lambda u_2(x)^{-\gamma} + f(x, u_2(x))] (1 - \eta \varepsilon (u_2 - u_1)) v)(x) \, dx.
\]

Now we pass to the limit as \( \varepsilon \to 0^+ \). Using Lebesgue’s Dominated Convergence Theorem and the fact that

\[ \eta \varepsilon (u_2 - u_1)(x) \to \chi_{\{u_1 = u_2\}}(x) \text{ for a.e. } x \in \Omega \text{ as } \varepsilon \to 0^+, \]

we obtain...
we find
\[
\int |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla v(x))_{\mathbb{R}^N} \, dx \\
+ \int |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla v(x))_{\mathbb{R}^N} \, dx \\
\geq \int \left[ \lambda u_1(x)^\gamma + f(x, u_1(x)) \right] v(x) \, dx \\
+ \int \left[ \lambda u_2(x)^\gamma + f(x, u_2(x)) \right] v(x) \, dx.
\]
(3)

Here the notation \( \chi_D \) stands for the characteristic function of a set \( D \), that is,
\[
\chi_D(t) = \begin{cases} 
1 & \text{if } t \in D \\
0 & \text{otherwise.}
\end{cases}
\]

The gradient of \( u := \min \{ u_1, u_2 \} \in W_0^{1,p}(\Omega) \) is equal to
\[
\nabla u(x) = \begin{cases} 
\nabla u_1(x) & \text{for a.e. } x \in \{ u_1 < u_2 \} \\
\nabla u_2(x) & \text{for a.e. } x \in \{ u_1 \geq u_2 \}.
\end{cases}
\]

Consequently, we can express (3) in the form
\[
\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N} \, dx \geq \int_\Omega \left[ \lambda u(x)^\gamma + f(x, u(x)) \right] v(x) \, dx
\]
(4)

for all \( v \in C_0^\infty(\Omega) \) with \( v(x) \geq 0 \) for a.e. \( x \in \Omega \). Actually, the density of \( C_0^\infty(\Omega) \), in \( W_0^{1,p}(\Omega)_+ \), ensures that (4) is valid for all \( v \in W_0^{1,p}(\Omega)_+ \).

Let \( \tilde{u}_A \) be the unique solution of the purely singular elliptic problem
\[
\begin{cases} 
-\Delta_p u(x) = \lambda u(x)^\gamma & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Proposition 5 of Papageorgiou-Smyrlis [31] guarantees that \( \tilde{u}_A \in \text{int}(C_0^1(\overline{\Omega})_+) \). We claim that
\[
\tilde{u}_A \leq u \text{ for all } u \in S_1.
\]
(5)

For every \( u \in S_1 \), there holds
\[
\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N} \, dx = \int_\Omega \left[ \lambda u(x)^\gamma + f(x, u(x)) \right] v(x) \, dx
\]
(6)
whenever \( v \in W^{1,p}_0(\Omega) \). Inserting \( v = (\tilde{u}_A - u)^+ \in W^{1,p}_0(\Omega) \) in (6) and using the fact that \( f(x, u(x)) \geq 0 \), we derive

\[
\int_{\Omega} |\nabla u(x)|^{p-2}(\nabla u(x), \nabla (\tilde{u}_A - u)^+)(x)\,dx
= \int_{\Omega} [\lambda u(x)^\gamma + f(x, u(x))](\tilde{u}_A - u)^+(x)\,dx
\geq \int_{\Omega} \lambda u(x)^\gamma(\tilde{u}_A - u)^+(x)\,dx
\geq \int_{\Omega} \lambda \tilde{u}_A(x)^\gamma(\tilde{u}_A - u)^+(x)\,dx
= \int_{\Omega} |\nabla \tilde{u}_A|^{p-2}(\nabla \tilde{u}_A(x), \nabla (\tilde{u}_A - u)^+)(x)\,dx.
\]

Then the monotonicity of \( -\Delta_p \) leads to (5).

Since \( u_1, u_2 \in S_A \) and \( u := \min\{u_1, u_2\} \in W^{1,p}_0(\Omega) \), we conclude that \( u \geq \tilde{u}_A \). Corresponding to the truncation

\[
\tilde{g}(x, s) = \begin{cases} 
\lambda \tilde{u}_A(x)^\gamma + f(x, \tilde{u}_A(x)) & \text{if } s < \tilde{u}_A(x) \\
\lambda s^\gamma + f(x, s) & \text{if } \tilde{u}_A(x) \leq s \leq u(x) \\
\lambda u(x)^\gamma + f(x, u(x)) & \text{if } u(x) < s,
\end{cases}
\tag{7}
\]

we consider the intermediate Dirichlet problem

\[
\begin{cases} 
-\Delta_p w(x) = \tilde{g}(x, w(x)) & \text{in } \Omega \\
w > 0 & \text{in } \Omega \\
w(x) = 0 & \text{on } \partial \Omega.
\end{cases}
\tag{8}
\]

By [32, Proposition 7] there exists \( \bar{u} \in W^{1,p}_0(\Omega) \) such that

\[
\langle A(\bar{u}), h \rangle = \int_{\Omega} \tilde{g}(x, \bar{u}(x))h(x)\,dx
\]

for all \( h \in W^{1,p}_0(\Omega) \). Inserting \( h = (\bar{u} - u)^+ \), through (4) and (7), we infer that

\[
\langle A(\bar{u}), (\bar{u} - u)^+ \rangle = \int_{\Omega} [\lambda \bar{u}(x)^\gamma + f(x, \bar{u}(x))](\bar{u} - u)^+(x)\,dx
\leq \langle A(u), (u - u)^+ \rangle.
\]

It turns out that \( \bar{u} \leq u \). Through the same argument, we also imply \( u \geq \tilde{u}_A \). So by virtue of (7) and (8) we arrive at \( \bar{u} \in S_A \) and \( u \leq \min\{u_1, u_2\} \).

We are in a position to prove that problem (1) admits a smallest solution for every \( \lambda \in \mathcal{L} \).

**Lemma 9.** If hypotheses \( H(f) \) hold and \( \lambda \in \mathcal{L} = (0, \lambda^*) \), then problem (1) has a smallest (positive) solution \( u_A^* \in S_A \), that is,

\[ u_A^* \leq u \text{ for all } u \in S_A. \]
**Proof.** Fix $\lambda \in (0, \lambda^*)$. Invoking Hu-Papageorgiou [22, Lemma 3.10], we can find a decreasing sequence $\{u_n\} \subset S_\lambda$ such that

$$\inf S_\lambda = \inf_n u_n.$$  

On the basis of (5) we note that

$$\tilde{u}_n \leq u_n \text{ for all } n. \tag{9}$$

Next we verify that the sequence $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega)$. Arguing by contradiction, suppose that a relabeled subsequence of $\{u_n\}$ satisfies $\|u_n\| \to \infty$. Set $y_n = \frac{u_n}{\|u_n\|}$. This ensures

$$y_n \to y \text{ weakly in } W^{1,p}_0(\Omega) \text{ and } y_n \to y \text{ strongly in } L^p(\Omega) \text{ with } y \geq 0. \tag{10}$$

From (6) and $\{u_n\} \subset S_\lambda$ we have

$$\langle A(y_n), v \rangle = \int_\Omega |\nabla y_n(x)|^{p-2}(\nabla y_n(x), \nabla v(x))_{\mathbb{R}^N} \, dx$$

$$= \frac{1}{\lambda} \int_\Omega \frac{u_n(x)^{p-1}}{\|u_n\|^{p-1}} \left( f(x, u_n(x)) \right) \nu \, dx \tag{11}$$

for all $v \in W^{1,p}_0(\Omega)$. On the other hand, hypotheses $H(f)(i)$ and (ii) entail

$$0 \leq f(x, s) \leq c_1(1 + |s|^{p-1}) \text{ for a.e. } x \in \Omega \text{ and all } s \geq 0, \tag{12}$$

with some $c_1 > 0$. By (10) and (12) we see that the sequence

$$\left\{ \frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \right\} \text{ is bounded in } L^{p'}(\Omega).$$

Due to hypothesis $H(f)(ii)$ and Aizicovici-Papageorgiou-Staicu [1, Proposition 16], we find that

$$\left\{ \frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \right\} \to \widetilde{\lambda}_m y^{p-1} \text{ weakly in } L^{p'}(\Omega).$$

Then inserting $v = y_n - y$ in (11) and using (9) lead to

$$\lim_{n \to \infty} \langle A(y_n), y_n - y \rangle = 0.$$  

We can apply Proposition 3 to obtain $y_n \to y$ in $W^{1,p}_0(\Omega)$. Letting $n \to \infty$ in (11) gives

$$\langle A(y), v \rangle = \tilde{\lambda}_m \int_\Omega y^{p-1} v \, dx \text{ for all } v \in W^{1,p}_0(\Omega),$$

so $y$ is a nontrivial nonnegative solution of the eigenvalue problem

$$\begin{cases} -\Delta_p y(x) = \tilde{\lambda}_m y(x)^{p-1} & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

Consequently, $y$ must be nodal because $m \geq 2$ and $y \neq 0$, which contradicts that $y \geq 0$ in $\Omega$. This contradiction proves that the sequence $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega)$.

Along a relabeled subsequence, we may assume that

$$u_n \to u^*_\lambda \text{ weakly in } W^{1,p}_0(\Omega) \text{ and } u_n \to u^*_\lambda \text{ in } L^p(\Omega), \tag{13}$$

for some $u^*_\lambda \in W^{1,p}_0(\Omega)$. In addition, we may suppose that

$$u_n(x)^{\cdot} \to u^*_\lambda(x)^{\cdot} \text{ for a.e. } x \in \Omega. \tag{14}$$
From \( \tilde{u}_\lambda \in \text{int}(C^1_0(\Omega)_+) \) and (5), through the Lemma in Lazer-McKenna [23], we obtain
\[
0 \leq u_n^\gamma \leq \tilde{u}_\lambda^\gamma \in L^{p'}(\Omega).
\]
(15)

On account of (13)-(15) we have
\[
u_n^\gamma \to (u_\lambda^\star)^\gamma \ \text{weakly in} \ L^{p'}(\Omega)
\]
(see also Gasiński-Papageorgiou [16, p. 38]).

(i) It follows from [32, Proposition 5] that there exists a solution
\[
\text{Proof.}
\]
Lemma 10.
\[
\text{It holds}
\]
fulfills:
\[
\text{On account of (13)-(15) we have}
\]
\[
\text{The desired conclusion is the direct consequence of the inequality}
\]
\[
\text{which completes the proof.}
\]

(ii) Let \( \{\lambda_n\} \subset (0, \lambda^\star) \) and \( \lambda \in (0, \lambda^\star) \) satisfy \( \lambda_n \uparrow \lambda \). Denote for simplicity \( u_n = u_{\lambda_n}^\star = \Gamma(\lambda_n) \in S_{\lambda_n} \subset \text{int}(C^1_0(\Omega)_+) \). It holds
\[
\langle A(u_n), v \rangle = \int_{\Omega} \left[ \lambda_n u_n(x)^{-\gamma} + f(x, u_n(x)) \right] v(x) \, dx
\]
(17)
for all \( v \in W^{1,p}_0(\Omega) \). By assertion (i) we know that
\[
0 \leq u_1 \leq u_n \leq u_\lambda^\star
\]
(18)
Choosing \( v = u_n \) in (17) and proceeding as in the proof of Lemma 9, we verify that the sequence \( \{u_n\} \) is bounded in \( W^{1,p}_0(\Omega) \). Given \( r > N \), it is true that \( (u_\lambda^\star)^r \in \text{int}(C^1_0(\Omega)_+) \), so there is a constant \( c_2 > 0 \) such that
\[
\tilde{u}_1 \leq c_2 (u_\lambda^\star)^r = c_2 u_\lambda^r,
\]
or

\[ u_1^\pm \geq c_2^{\pm} u_1^{\mp}. \]

We can make use of the Lemma in Lazer-McKenna [23] for having

\[ 0 \leq u_n^{\mp} \leq u_1^{\mp} \in L'(\Omega) \quad \text{for all } n. \]

Moreover, hypothesis $H(f)(i)$ and (18) render that

the sequence \{f(\cdot, u_n(\cdot))\} is bounded in $L'(\Omega)$.

Therefore, utilizing Guedda-Véron [20, Proposition 1.3] we obtain the uniform bound

\[ \|u_n\|_{L^\infty(\Omega)} \leq c_3 \quad \text{for all } n, \quad (19) \]

with some $c_3 > 0$. Besides, the linear elliptic problem

\[
\begin{cases}
-\Delta v(x) = g_{\lambda_n}(x) & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $g_{\lambda_n}(\cdot) = \lambda_n u_n(\cdot)^{\gamma} + f(\cdot, u_n(\cdot)) \in L'(\Omega)$, has a unique solution $v_{\lambda_n} \in W^{1,r}_0(\Omega)$ (see, e.g., [19, Theorem 9.15]). Owning to $r > N$, the Sobolev embedding theorem provides

\[ v_{\lambda_n} \in C^{1,\alpha}_0(\overline{\Omega}), \]

with $\alpha = 1 - \frac{N}{r}$. For $w_n := \nabla v_{\lambda_n}$, we have $w_n \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ and

\[
\begin{cases}
-\text{div} \left( |\nabla u_n(x)|^{p-2} \nabla u_n(x) - w_n(x) \right) = 0 & \text{in } \Omega \\
u_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]

This allows us to apply the nonlinear regularity up to the boundary in Liebermann [24, 25] finding that $u_n \in C^{1,\beta}_0(\overline{\Omega})$ with some $\beta \in (0, 1)$ for all $n$. Here the uniform estimate in (19) is essential. The compactness of the embedding of $C^{1,\beta}_0(\overline{\Omega})$ in $C^{1}_0(\overline{\Omega})$ and the monotonicity of the sequence \{u_n\} guarantee

\[ u_n \to \overline{u}_\lambda \quad \text{in } C^{1}_0(\overline{\Omega}) \]

for some $\overline{u}_\lambda \in C^{1}_0(\overline{\Omega})$.

We claim that $\overline{u}_\lambda = u^{\star}_\lambda$. Arguing by contradiction, suppose that there exists $x^{\star} \in \Omega$ satisfying

\[ \overline{u}_\lambda(x^{\star}) < u^{\star}_\lambda(x^{\star}). \]

The known monotonicity property of \{u_n\} entails

\[ u^{\star}_\lambda(x^{\star}) < u_n(x^{\star}) = u^{\star}_{\lambda_n}(x^{\star}) \quad \text{for all } n, \]

which contradicts assertion (i). It results that $\overline{u}_\lambda = u^{\star}_\lambda = \Gamma(\lambda)$, thereby

\[ \Gamma(\lambda_n) = u_n \to \overline{u}_\lambda = \Gamma(\lambda) \quad \text{as } n \to \infty, \]

completing the proof. \hfill \Box

Next we turn to the semicontinuity properties of the set-valued mapping $\Lambda$. \hfill \Box

**Lemma 11.** Assume that hypotheses $H(f)$ hold. Then the set-valued mapping $\Lambda : \mathcal{L} \to 2^{C^{1}_0(\overline{\Omega})}$ is sequentially upper semicontinuous.
By Lemma 10 (i) we know that
\[ A^-(D) := \{ \lambda \in \mathbb{R} : A(\lambda) \cap D \neq \emptyset \} \]
is closed in \( \mathbb{R} \). Let \( \{ \lambda_n \} \subset A^-(D) \) verify \( \lambda_n \to \lambda \) as \( n \to \infty \). So,
\[ A(\lambda_n) \cap D \neq \emptyset, \]
hence there exists a sequence \( \{ u_n \} \subset \text{int}(C^1_0(\overline{D})) \) satisfying
\[ u_n \in A(\lambda_n) \cap D \quad \text{for all } n \in \mathbb{N}, \]
in particular
\[ \int_\Omega |\nabla u_n(x)|^{p-2} (\nabla u_n(x), \nabla v(x))_{\mathbb{R}^N} \, dx = \int_\Omega [\lambda_n u_n(x)^{-\gamma} + f(x, u_n(x))] \gamma v(x) \, dx \tag{20} \]
for all \( v \in W^{1,p}_0(\Omega) \). As in the proof of Lemma 9, we can show that the sequence \( \{ u_n \} \) is bounded in \( W^{1,p}_0(\Omega) \). Therefore we may assume that
\[ u_n \to u \quad \text{weakly in } W^{1,p}_0(\Omega) \quad \text{and } u_n \to u \quad \text{in } L^p(\Omega). \tag{21} \]
for some \( u \in W^{1,p}_0(\Omega) \). Furthermore, the sequences \( \{ f(\cdot, u_n(\cdot)) \} \) and \( \{ u_n^\gamma \} \) are bounded in \( L^p(\Omega) \) as already demonstrated in the proofs of Lemmas 9 and 10. In (20), we choose \( v = u_n - u \in W^{1,p}_0(\Omega) \) and then pass to the limit as \( n \to \infty \). By means of (21) we are led to
\[ \lim_{n \to \infty} \langle A(u_n), u_n - u \rangle = 0. \]
Since \( A \) is of type \( (S_+) \), we can conclude
\[ u_n \to u \quad \text{in } W^{1,p}_0(\Omega). \tag{22} \]
On account of (20), the strong convergence in (22) and Sobolev embedding theorem imply
\[ \int_\Omega |\nabla u(x)|^{p-2} (\nabla u(x), \nabla v(x))_{\mathbb{R}^N} \, dx = \int_\Omega [\lambda u(x)^{-\gamma} + f(x, u(s))] v(x) \, dx \]
for all \( v \in W^{1,p}_0(\Omega) \). This reads as \( u \in S_A = A(\lambda) \).

It remains to check that \( u \in D \). Fix \( \Lambda \in \mathcal{L} \) such that
\[ \Lambda < \lambda_n \leq \Lambda^* \quad \text{for all } n. \]
By Lemma 10 (i) we know that
\[ u_n^* < u_n^\Lambda < u_n \quad \text{for all } n. \]
The same argument as in the proof of Lemma 10 confirms that, for \( r > N \) fixed, the function \( x \mapsto \lambda_n u_n(x)^{-\gamma} + f(x, u_n(x)) \) is bounded in \( L^r(\Omega) \). Let \( g_{\Lambda_n}(x) = \lambda_n u_n(x)^{-\gamma} + f(x, u_n(x)) \) \( (23) \)
in \( L^r(\Omega) \) and consider the linear Dirichlet problem
\[ \begin{cases}
-\Delta v(x) = g_{\Lambda_n}(x) & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega.
\end{cases} \tag{23} \]
The standard existence and regularity theory (see, e.g., Gilbarg-Trudinger [19, Theorem 9.15]) ensure that problem (23) has a unique solution
\[ v_{\Lambda_n} \in W^{2,r}(\Omega) \subset C^{1,a}_0(\overline{\Omega}) \text{ with } \|v_{\Lambda_n}\|_{C^{1,a}_0(\overline{\Omega})} \leq c, \]
with a constant $c_3 > 0$ and $a = 1 - \frac{N}{q}$. Denote $w_n(x) = \nabla \nu_{\lambda_n}(x)$ for all $x \in \Omega$. It holds $w_n \in C^{0,\alpha}(\overline{\Omega})$ thanks to $\nu_{\lambda_n} \in C^{1,\alpha}_0(\overline{\Omega})$. Notice that

$$
\begin{align*}
\begin{cases}
-\text{div} \left( |\nabla u_n(x)|^{p-2} \nabla u_n(x) - w_n(x) \right) = 0 & \text{in } \Omega \\
u_n = 0 & \text{on } \partial \Omega.
\end{cases}
\end{align*}
$$

The nonlinear regularity up to the boundary in Lieberman [24, 25] reveals that $u_n \in C^{1,\beta}_0(\overline{\Omega})$ for all $n \in \mathbb{N}$ with some $\beta \in (0, 1)$. The compactness of the embedding of $C^{1,\beta}_0(\overline{\Omega})$ in $C^{1}_0(\overline{\Omega})$ and (22) yield the strong convergence $u_n \rightharpoonup u$ in $C^1_0(\overline{\Omega})$. Recalling that $D$ is closed in $C^1_0(\overline{\Omega})$ it results that $u \in A(\lambda) \cap D$, i.e., $\lambda \in A^-(D)$.

**Lemma 12.** Suppose that hypotheses $H(f)$ hold. Then the set-valued mapping $\Lambda : \mathcal{L} \to 2^{C^1_0(\overline{\Omega})}$ is sequentially lower semicontinuous.

**Proof.** In order to refer to Proposition 7, let $\{\lambda_n\} \subset \mathcal{L}$ satisfy $\lambda_n \rightharpoonup \lambda \neq 0$ as $n \to \infty$ and let $w \in S_\lambda \subset \text{int}(C^{1,\beta}_0(\overline{\Omega}))$. For each $n \in \mathbb{N}$, we formulate the Dirichlet problem

$$
\begin{align*}
\begin{cases}
-\Delta_p u(x) = \lambda_n w(x)^\gamma + f(x, w(x)) & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{align*}
$$

In view of $w \geq \bar{u}_\lambda \in \text{int}(C^{1,\beta}_0(\overline{\Omega}))$ (see (5)) and

$$
\begin{align*}
\begin{cases}
\lambda_n w(x)^\gamma + f(x, w(x)) \geq 0 & \text{for all } x \in \Omega \\
\lambda_n w(x)^\gamma + f(x, w(x)) \leq 0,
\end{cases}
\end{align*}
$$

it is obvious that problem (24) has a unique solution $u_n^0 \in \text{int}(C^{1,\beta}_0(\overline{\Omega}))$. Relying on the growth condition for $f$ (see hypotheses $H(f)(i)$ and (ii)), through the same argument as in the proof of Lemma 9 we show that the sequence $\{u_n^0\}$ is bounded in $W^{1,p}_0(\Omega)$. Then Proposition 1.3 of Guedda-Véron [20] implies the uniform boundedness

$$
\begin{align*}
u_n^0 \in L^\infty(\Omega) \text{ and } \|u_n^0\|_{L^1(\Omega)} \leq c_5 \text{ for all } n \in \mathbb{N},
\end{align*}
$$

with a constant $c_5 > 0$. As in the proof of Lemma 11, we set $g_{\lambda_n}(x) = \lambda_n w(x)^\gamma + f(x, w(x))$ and consider the Dirichlet problem (23) to obtain that $\{u_n^0\}$ is contained in $C^{1,\beta}_0(\overline{\Omega})$ for some $\beta \in (0, 1)$. Due to the compactness of the embedding of $C^{1,\beta}_0(\overline{\Omega})$ in $C^{1}_0(\overline{\Omega})$, we may assume

$$
\begin{align*}
u_n^0 \rightharpoonup u & \text{ in } C^1_0(\overline{\Omega}) \text{ as } n \to \infty,
\end{align*}
$$

with some $u \in C^1_0(\overline{\Omega})$. Then (24) yields

$$
\begin{align*}
\begin{cases}
-\Delta_p u(x) = \lambda w(x)^\gamma + f(x, w(x)) & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{align*}
$$

Thanks to $w \in A(\lambda)$, a simple comparison justifies $u = w$. Since every convergent subsequence of $\{u_n\}$ converges to the same limit $w$, it is true that

$$
\lim_{n \to \infty} u_n^0 = w.
$$
Next, for each \( n \in \mathbb{N} \), we consider the Dirichlet problem
\[
\begin{aligned}
-\Delta_p u(x) &= \lambda_n u_0(x)^{-p} + f(x, u_0(x)) \quad \text{in } \Omega \\
u &> 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
\]
Carrying on the same reasoning, we can show that this problem has a unique solution \( u^1_n \) belonging to \( \text{int}(C^1_0(\Omega)) \) and that
\[
\lim_{n \to \infty} u^1_n = w.
\]
Continuing the process, we generate a sequence \( \{u^k_n\}_{n,k \geq 1} \) such that
\[
\begin{aligned}
-\Delta_p u^k_n(x) &= \lambda_n u^{k-1}_n(x)^{-p} + f(x, u^{k-1}_n(x)) \quad \text{in } \Omega \\
u^k_n &> 0 \quad \text{in } \Omega \\
u^k_n &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]
and
\[
\lim_{n \to \infty} u^k_n = w \quad \text{for all } k \in \mathbb{N}.
\] (25)

Fix \( n \geq 1 \). As before, based on the nonlinear regularity [24, 25], we notice that the sequence \( \{u^k_n\}_{k \geq 1} \) is relatively compact in \( C^1_0(\Omega) \), so we may suppose
\[
u^k_n \to \nu_n \quad \text{in } C^1_0(\Omega) \quad \text{as } k \to \infty,
\]
for some \( \nu_n \in C^1_0(\Omega) \). Then it appears that
\[
\begin{aligned}
-\Delta_p \nu_n(x) &= \lambda_n \nu_0(x)^{-p} + f(x, \nu_0(x)) \quad \text{in } \Omega \\
\nu_n &> 0 \quad \text{in } \Omega \\
\nu_n &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]
which means that \( \nu_n \in \Lambda(\lambda_n) \).

The convergence in (25) and the double limit lemma (see, e.g., [13, Proposition A.2.35]) result in
\[
u_n \to w \quad \text{in } C^1_0(\Omega) \quad \text{as } n \to \infty.
\]
By Proposition 7 we conclude that \( \Lambda \) is lower semicontinuous. \( \square \)

**Proof of Theorem 2.** (i) It suffices to apply Lemmas 11 and 12.

(ii) The stated conclusion is a direct consequence of Lemmas 9 and 10. \( \square \)

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