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A class of semipositone $p$-Laplacian problems with a critical growth reaction term

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Abstract: We prove the existence of ground state positive solutions for a class of semipositone $p$-Laplacian problems with a critical growth reaction term. The proofs are established by obtaining crucial uniform $C^{1,\alpha}$ a priori estimates and by concentration compactness arguments. Our results are new even in the semilinear case $p = 2$.

Keywords: critical semipositone $p$-Laplacian problems, ground state positive solutions, concentration compactness, uniform $C^{1,\alpha}$ a priori estimates

MSC: Primary 35B33, Secondary 35J92, 35B09, 35B45

1 Introduction

Consider the $p$-superlinear semipositone $p$-Laplacian problem

$$
-\Delta_p u = u^{q-1} - \mu \quad \text{in } \Omega,
$$

$$
\begin{cases}
-\Delta_p u = u^{q-1} - \mu & \text{in } \Omega, \\
\quad u > 0 & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

(1.1)

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $1 < p < N$, $p < q < p^*$, $\mu > 0$ is a parameter, and $p^* = Np/(N-p)$ is the critical Sobolev exponent. The scaling $u \mapsto \mu^{1/(q-1)} u$ transforms the first equation in (1.1) into

$$
-\Delta_p u = \mu^{(q-p)/(q-1)} \left( u^{q-1} - 1 \right),
$$

so in the subcritical case $q < p^*$, it follows from the results in Castro et al.[1] and Chhetri et al.[2] that this problem has a weak positive solution for sufficiently small $\mu > 0$ when $p > 1$ (see also Unsurangie [3], Allegretto et al.[4], Ambrosetti et al.[5], and Caldwell et al.[6] for the case when $p = 2$). On the other hand, in the critical case $q = p^*$, it follows from a standard argument involving the Pohozaev identity for the $p$-Laplacian (see Guedda and Véron [7, Theorem 1.1]) that problem (1.1) has no solution for any $\mu > 0$ when $\Omega$ is star-shaped. The purpose of the present paper is to show that this situation can be reversed by the addition of lower-order terms, as was observed in the positone case by Brézis and Nirenberg in the celebrated paper [8]. However, this extension to the semipositone case is not straightforward as $u = 0$ is no longer a subsolution, making it much harder to find a positive solution as was pointed out in Lions [9]. The positive solutions that we obtain here are ground states, i.e., they minimize the energy among all positive solutions.

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We study the Brézis-Nirenberg type critical semipositone \( p \)-Laplacian problem

\[
\begin{aligned}
-\Delta_p u &= \lambda u^{p-1} + u^{p^*-1} - \mu \quad \text{in } \Omega \\
 u &> 0 \quad \text{in } \Omega \\
 u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

(1.2)

where \( \lambda, \mu > 0 \) are parameters. Let \( W_{0}^{1,p}(\Omega) \) be the usual Sobolev space with the norm given by

\[
||u||^p = \int_{\Omega} |\nabla u|^p \, dx.
\]

For a given \( \lambda > 0 \), the energy of a weak solution \( u \in W_{0}^{1,p}(\Omega) \) of problem (1.2) is given by

\[
I_{\mu}(u) = \int_{\Omega} \left( \frac{|\nabla u|^p}{p} - \frac{\lambda u^p}{p} - \frac{u^{p^*}}{p^*} + \mu u \right) \, dx,
\]

and clearly all weak solutions lie on the set

\[
N_{\mu} = \left\{ u \in W_{0}^{1,p}(\Omega) : u > 0 \text{ in } \Omega \text{ and } \int_{\Omega} |\nabla u|^p \, dx = \int_{\Omega} \left( \lambda u^p + u^{p^*} - \mu u \right) \, dx \right\}.
\]

We will refer to a weak solution that minimizes \( I_{\mu} \) on \( N_{\mu} \) as a ground state. Let

\[
\lambda_1 = \inf_{u \in W_{0}^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} |u|^p \, dx}
\]

(1.3)

be the first Dirichlet eigenvalue of the \( p \)-Laplacian, which is positive. We will prove the following existence theorem.

**Theorem 1.1.** If \( N \geq p^2 \) and \( \lambda \in (0, \lambda_1) \), then there exists \( \mu^* > 0 \) such that for all \( \mu \in (0, \mu^*) \), problem (1.2) has a ground state solution \( u_{\mu} \in C^{1,\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \).

The scaling \( u \mapsto \mu^{1/(p^*-p)} u \) transforms the first equation in the critical semipositone \( p \)-Laplacian problem

\[
\begin{aligned}
-\Delta_p u &= \lambda u^{p-1} + \mu \left( u^{p^*-1} - 1 \right) \quad \text{in } \Omega \\
 u &> 0 \quad \text{in } \Omega \\
 u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

(1.4)

into

\[
-\Delta_p u = \lambda u^{p-1} + u^{p^*-1} - \mu^{(p^*-1)/(p^*-p)},
\]

so as an immediate corollary we have the following existence theorem for problem (1.4).

**Theorem 1.2.** If \( N \geq p^2 \) and \( \lambda \in (0, \lambda_1) \), then there exists \( \mu^* > 0 \) such that for all \( \mu \in (0, \mu^*) \), problem (1.4) has a ground state solution \( u_{\mu} \in C^{1,\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \).

We would like to emphasize that Theorems 1.1 and 1.2 are new even in the semilinear case \( p = 2 \).

The outline of the proof of Theorem 1.1 is as follows. We consider the modified problem

\[
\begin{aligned}
-\Delta_p u &= \lambda u^{p-1} + u^{p^*-1} - \mu f(u) \quad \text{in } \Omega \\
 u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

(1.5)
where \( u_+(x) = \max \{ u(x), 0 \} \) and
\[
 f(t) = \begin{cases} 
 1, & t \geq 0 \\
 1 - |t|^{p-1}, & -1 < t < 0 \\
 0, & t \leq -1.
\end{cases}
\]

Weak solutions of this problem coincide with critical points of the \( C^1 \)-functional
\[
 I_\mu(u) = \int_\Omega \left( \frac{\nabla u}{p} - \frac{\lambda u}{p} - \frac{u^\mu}{p'} \right) \, dx + \mu \left[ \int_{\{u>0\}} u \, dx - \frac{|u|^{p-1} u}{p} \right] \, dx - \left( 1 - \frac{1}{p} \right) \int \left( |u| - 1 \right) \, dx, \quad u \in W^{1,p}_0(\Omega),
\]
where \(|\cdot|\) denotes the Lebesgue measure in \( \mathbb{R}^N \). Recall that \( I_\mu \) satisfies the Palais-Smale compactness condition at the level \( c \in \mathbb{R} \), or the (PS)\( c \) condition for short, if every sequence \( (u_j) \subset W^{1,p}_0(\Omega) \) such that \( I_\mu(u_j) \to c \) and \( I'_\mu(u_j) \to 0 \), called a (PS)\( c \) sequence for \( I_\mu \), has a convergent subsequence. As we will see in Lemma 2.1 in the next section, it follows from concentration compactness arguments that \( I_\mu \) satisfies the (PS)\( c \) condition for all
\[
 c < \frac{1}{N} S^{N/p} - \left( 1 - \frac{1}{p} \right) \mu |\Omega|,
\]
where \( S \) is the best Sobolev constant (see (2.1)). First we will construct a mountain pass level below this threshold for compactness for all sufficiently small \( \mu > 0 \). This part of the proof is more or less standard. The novelty of the paper lies in the fact that the solution \( u_\mu \) of the modified problem (1.5) thus obtained is positive, and hence also a solution of our original problem (1.2), if \( \mu \) is further restricted. Note that this does not follow from the strong maximum principle as usual since \( -\mu f(0) < 0 \). This is precisely the main difficulty in finding positive solutions of semipositone problems (see Lions [9]). We will prove that for every sequence \( \mu_j \to 0 \), a subsequence of \( u_{\mu_j} \) is positive in \( \Omega \). The idea is to show that a subsequence of \( u_{\mu_j} \) converges in \( C_0^1(\overline{\Omega}) \) to a solution of the limit problem
\[
 \begin{cases} 
 -\Delta_p u = \lambda u^{p-1} + u^{p-1} & \text{in } \Omega \\
 u > 0 & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
This requires a uniform \( C^{1,\alpha}(\overline{\Omega}) \) estimate of \( u_{\mu_j} \) for some \( \alpha \in (0, 1) \). We will obtain such an estimate by showing that \( u_{\mu_j} \) is uniformly bounded in \( W^{1,p}_0(\Omega) \) and uniformly equi-integrable in \( L^p(\Omega) \), and applying a result of de Figueiredo et al.[10]. The proof of uniform equi-integrability in \( L^p(\Omega) \) involves a second (nonstandard) application of the concentration compactness principle. Finally, we use the mountain pass characterization of our solution to show that it is indeed a ground state.

**Remark 1.3.** Establishing the existence of solutions to the critical semipositone problem
\[
 \begin{cases} 
 -\Delta_p u = \mu \left( u^{p-1} + u^{p-1} - 1 \right) & \text{in } \Omega \\
 u > 0 & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega
\end{cases}
\]
for small \( \mu \) remains open.
2 Preliminaries

Let

\[
S = \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\left( \int_\Omega |u|^{p'} \, dx \right)^{p/p'}}
\]

be the best constant in the Sobolev inequality, which is independent of \(\Omega\). The proof of Theorem 1.1 will make use of the following compactness result.

**Lemma 2.1.** For any fixed \(\lambda, \mu > 0\), \(I_\mu\) satisfies the (PS)_c condition for all

\[
c < \frac{1}{N} S^{N/p} - \left( 1 - \frac{1}{p} \right) \mu |\Omega|.
\]

**Proof.** Let \((u_j)\) be a (PS)_c sequence. First we show that \((u_j)\) is bounded. We have

\[
I_\mu(u_j) = \int_\Omega \left( \frac{|\nabla u_j|^p}{p} - \frac{\lambda u_j^{p-1}}{p} - \frac{u_j^{p'}}{p'} \right) \, dx + \mu \left[ \int_{\{u_j > 0\}} u_j \, dx \right]
\]

and

\[
I_\mu'(u_j) v = \int_\Omega \left( |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v - \lambda u_j^{p-1} v - u_j^{p-1} \frac{v}{u_j} \right) \, dx + \mu \left[ \int_{\{u_j > 0\}} v \, dx \right]
\]

Taking \(v = u_j\) in (2.4), dividing by \(p\), and subtracting from (2.3) gives

\[
\frac{1}{N} \int_\Omega u_j^p \, dx \leq c + \left( 1 - \frac{1}{p} \right) \mu |\Omega| + o(1) (\|u_j\| + 1),
\]

and it follows from this, (2.3), and the H"older inequality that \((u_j)\) is bounded in \(W^{1,p}_0(\Omega)\).

Since \((u_j)\) is bounded, so is \((u_{j+})\), a renamed subsequence of which then converges to some \(v \geq 0\) weakly in \(W^{1,p}_0(\Omega)\), strongly in \(L^q(\Omega)\) for all \(q \in [1, p^*)\) and a.e. in \(\Omega\), and

\[
|\nabla u_{j+}|^p \, dx \overset{w^*}{\to} \kappa, \quad u_{j+}^p \, dx \overset{w^*}{\to} v
\]

in the sense of measures, where \(\kappa\) and \(v\) are bounded nonnegative measures on \(\overline{\Omega}\) (see, e.g., Folland [11]). By the concentration compactness principle of Lions [12, 13], then there exist an at most countable index set \(I\) and points \(x_i \in \overline{\Omega}, \ i \in I\) such that

\[
\kappa \geq |\nabla v|^p \, dx + \sum_{i \in I} \chi_i \delta_{x_i}, \quad v = v^p \, dx + \sum_{i \in I} v_1 \delta_{x_i},
\]

where \(\chi_i \geq 0\) and \(\sum_{i \in I} \chi_i = 1\).
Let \( \kappa_i, v_i > 0 \) and \( v_i^{1/p^*} \leq \kappa_i/S \). We claim that \( I = \emptyset \). Suppose by contradiction that there exists \( i \in I \). Let \( \varphi : \mathbb{R}^N \to [0, 1] \) be a smooth function such that \( \varphi(x) = 1 \) for \( |x| \leq 1 \) and \( \varphi(x) = 0 \) for \( |x| \geq 2 \). Then set
\[
\varphi_{i, \rho}(x) = \varphi \left( \frac{x - x_i}{\rho} \right), \quad x \in \mathbb{R}^N
\]
for \( i \in I \) and \( \rho > 0 \), and note that \( \varphi_{i, \rho} : \mathbb{R}^N \to [0, 1] \) is a smooth function such that \( \varphi_{i, \rho}(x) = 1 \) for \( |x - x_i| \leq \rho \) and \( \varphi_{i, \rho}(x) = 0 \) for \( |x - x_i| \geq 2\rho \). The sequence \( (\varphi_{i, \rho} u_j) \) is bounded in \( W_0^{1, p}(\Omega) \) and hence taking \( v = \varphi_{i, \rho} u_j \) in (2.4) gives
\[
\int_{\Omega} (\varphi_{i, \rho} |\nabla u_j|^{p} + u_j |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla \varphi_{i, \rho} - \lambda \varphi_{i, \rho} u_j^{p} - \mu \varphi_{i, \rho} u_j^{p^*} + \mu \varphi_{i, \rho} u_j) \, dx = 0(1). \tag{2.8}
\]
By (2.6),
\[
\int_{\Omega} \varphi_{i, \rho} |\nabla u_j|^{p} \, dx \to \int_{\Omega} \varphi_{i, \rho} \, dx,
\]
\[
\int_{\Omega} \varphi_{i, \rho} u_j^{p^*} \, dx \to \int_{\Omega} \varphi_{i, \rho} \, dx.
\]
Denoting by \( C \) a generic positive constant independent of \( j \) and \( \rho \),
\[
\left| \int_{\Omega} (u_j |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla \varphi_{i, \rho} - \lambda \varphi_{i, \rho} u_j^{p} + \mu \varphi_{i, \rho} u_j) \, dx \right| \leq C \left[ \left( \frac{1}{\rho} + \mu \right) I_j^{1/p} + I_j \right],
\]
where
\[
I_j := \int_{\Omega \cap B_{\rho}(x_i)} u_j^{p^*} \, dx \to \int_{\Omega \cap B_{\rho}(x_i)} v^{p^*} \, dx \leq C \rho^p \left( \int_{\Omega \cap B_{\rho}(x_i)} v^{p^*} \, dx \right)^{p/p^*}.
\]
So passing to the limit in (2.8) gives
\[
\int_{\Omega} \varphi_{i, \rho} \, dx - \int_{\Omega} \varphi_{i, \rho} v \, dx \leq C \left[ (1 + \mu p) \left( \int_{\Omega \cap B_{\rho}(x_i)} v^{p^*} \, dx \right)^{1/p^*} + \int_{\Omega \cap B_{\rho}(x_i)} v^{p} \, dx \right].
\]
Letting \( \rho \to 0 \) and using (2.7) now gives \( \kappa_i \leq v_i \), which together with \( v_i > 0 \) and \( v_i^{p/p^*} \leq \kappa_i/S \) then gives \( v_i \geq S^{N/p} \). On the other hand, passing to the limit in (2.5) and using (2.6) and (2.7) gives
\[
v_i \leq N \left[ c + \left( 1 - \frac{1}{p} \right) \mu |\Omega| \right] < S^{N/p}
\]
by (2.2), a contradiction. Hence \( I = \emptyset \) and
\[
\int_{\Omega} u_j^{p^*} \, dx \to \int_{\Omega} v^{p^*} \, dx. \tag{2.9}
\]
Passing to a further subsequence, \( u_j \) converges to some \( u \) weakly in \( W_0^{1, p}(\Omega) \), strongly in \( L^q(\Omega) \) for all \( q \in [1, p^*] \), and a.e.\( \) in \( \Omega \). Since
\[
|u_j^{p^*} - 1 (u_j - u)| \leq u_j^{p^*} + u_j^{p^* - 1} |u| \leq \left( 2 - \frac{1}{p^*} \right) u_j^{p^*} + \frac{1}{p^*} |u|^{p^*}
\]
by Young's inequality,
\[
\int_{\Omega} u_j^{p^* - 1} (u_j - u) \, dx \to 0
\]
by (2.9) and the dominated convergence theorem. Then taking \( v = u_j - u \) in (2.4) gives
\[
\int_{\Omega} |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla (u_j - u) \, dx \to 0,
\]
so \( u_j \to u \) in \( W_0^{1, p}(\Omega) \) for a renamed subsequence (see, e.g., Perera et al. [14, Proposition 1.3]).
The infimum in (2.1) is attained by the family of functions

\[ u_\varepsilon(x) = \frac{C_{N,p} (N-p)^p}{(x + |x| (p-1)(N-p))}, \quad \varepsilon > 0 \]

when \( \Omega = \mathbb{R}^N \), where the constant \( C_{N,p} > 0 \) is chosen so that

\[ \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p \, dx = \int_{\mathbb{R}^N} u_\varepsilon^p \, dx = S^{N/p}. \]

Without loss of generality, we may assume that \( 0 \in \Omega \). Let \( r > 0 \) be so small that \( B_{2r}(0) \subset \Omega \), take a function \( \psi \in C^\infty_0(B_{2r}(0), [0,1]) \) such that \( \psi = 1 \) on \( B_r(0) \), and set

\[ \tilde{u}_\varepsilon(x) = \psi(x) u_\varepsilon(x), \quad v_\varepsilon(x) = \frac{\tilde{u}_\varepsilon(x)}{\left( \int_\Omega \tilde{u}_\varepsilon^p \, dx \right)^{1/p}}, \]

so that \( \int_\Omega v_\varepsilon^p \, dx = 1 \). Then we have the well-known estimates

\[ \int_\Omega |\nabla v_\varepsilon|^p \, dx \leq S + C \varepsilon^{(N-p)/p}, \quad (2.10) \]

\[ \int_\Omega v_\varepsilon^p \, dx \geq \begin{cases} \frac{1}{C} \varepsilon^{p-1}, & N > p^2 \\ \frac{1}{C} \varepsilon^{p-1} |\log \varepsilon|, & N = p^2, \end{cases} \quad (2.11) \]

where \( C = C(N, p) > 0 \) is a constant (see, e.g., Drábek and Huang [15]).

3 Proof of Theorem 1.1

First we show that \( I_\mu \) has a uniformly positive mountain pass level below the threshold for compactness given in Lemma 2.1 for all sufficiently small \( \mu > 0 \). Let \( v_\varepsilon \) be as in the last section.

**Lemma 3.1.** There exist \( \mu_0, \rho, c_0 > 0, R > \rho, \) and \( \beta < \frac{1}{N} S^{N/p} \) such that the following hold for all \( \mu \in (0, \mu_0) \):

(i) \( \|u\| = \rho \Rightarrow I_\mu(u) \geq c_0 \),

(ii) \( I_\mu(tv_\varepsilon) \leq 0 \) for all \( t \geq R \) and \( \varepsilon \in (0,1] \),

(iii) denoting by \( \Gamma = \left\{ y \in C([0,1], W^{1,p}_0(\Omega)) : y(0) = 0, y(1) = R v_\varepsilon \right\} \) the class of paths joining the origin to \( R v_\varepsilon \),

\[ c_0 \leq c_\mu := \inf_{y \in \Gamma} \max_{u \in y([0,1])} I_\mu(u) \leq \beta - \left( 1 - \frac{1}{p} \right) \mu |\Omega| \]

for all sufficiently small \( \varepsilon > 0 \),

(iv) \( I_\mu \) has a critical point \( u_\mu \) at the level \( c_\mu \).

**Proof.** By (1.3) and (2.1),

\[ I_\mu(u) \geq \frac{1}{p} \left( 1 - \frac{1}{\lambda} \right) \|u\|^p - S^{p'/p} \|u\|^{p'} - \left( 1 - \frac{1}{p} \right) \mu |\Omega|, \]

and (i) follows from this for sufficiently small \( \rho, c_0, \mu > 0 \) since \( \lambda < \lambda_1 \).
Since \( \nu_\varepsilon \geq 0 \),
\[
I_\mu(t \nu_\varepsilon) = \frac{t^p}{p} \int_\Omega \left( |\nabla \nu_\varepsilon|^{p} - \lambda \nu_\varepsilon^{p} \right) dx - \frac{t^{p^*}}{p^*} + \mu t \int_\Omega \nu_\varepsilon \, dx
\]
for \( t \geq 0 \). By the Hölder’s and Young’s inequalities,
\[
\mu t \int_\Omega \nu_\varepsilon \, dx \leq \mu |\Omega|^{1-1/p} \left( \int_\Omega \nu_\varepsilon^{p} \, dx \right)^{1/p} \leq C_\lambda \mu^{p/(p-1)} + \frac{\lambda \mu^{p}}{2p^*} \int_\Omega \nu_\varepsilon^{p} \, dx,
\]
where
\[
C_\lambda = \left( 1 - \frac{1}{p} \right) \left( \frac{2}{\lambda} \right)^{1/(p-1)} |\Omega|,
\]
so
\[
I_\mu(t \nu_\varepsilon) \leq \frac{t^p}{p} \int_\Omega \left( |\nabla \nu_\varepsilon|^{p} - \lambda \nu_\varepsilon^{p} \right) dx - \frac{t^{p^*}}{p^*} + C_\lambda \mu^{p/(p-1)}.
\]

Then by (2.10) and for \( \varepsilon, \mu \in (0, 1) \),
\[
I_\mu(t \nu_\varepsilon) \leq (S + C) \frac{t^p}{p} - \frac{t^{p^*}}{p^*} + C_\lambda,
\]
from which (ii) follows for sufficiently large \( R > \rho \).

The first inequality in (3.1) is immediate from (i) since \( R > \rho \). Maximizing the right-hand side of (3.2) over \( t \geq 0 \) gives
\[
c_\mu \leq \frac{1}{N} \left[ \int_\Omega \left( |\nabla \nu_\varepsilon|^{p} - \lambda \nu_\varepsilon^{p} \right) dx \right]^{N/p} + C_\lambda \mu^{p/(p-1)},
\]
and (2.10) and (2.11) imply that the integral on the right-hand side is strictly less than \( S \) for all sufficiently small \( \varepsilon > 0 \) since \( N \geq p^2 \) and \( \lambda > 0 \), so the second inequality in (3.1) holds for sufficiently small \( \mu > 0 \).

Finally, (iv) follows from (i)–(iii), Lemma 2.1, and the mountain pass lemma (see Ambrosetti and Rabinowitz [16]).

Next we show that \( u_\mu \) is uniformly bounded in \( W^{1,p}_0(\Omega) \) and uniformly equi-integrable in \( L^{p^*}(\Omega) \), and hence also uniformly bounded in \( C^{1,\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \) by de Figueiredo et al. [10, Proposition 3.7], for all sufficiently small \( \mu \in (0, \mu_0) \).

**Lemma 3.2.** There exists \( \mu^* \in (0, \mu_0) \) such that the following hold for all \( \mu \in (0, \mu^*) \):

(i) \( u_\mu \) is uniformly bounded in \( W^{1,p}_0(\Omega) \),

(ii) \( \int_{\Omega} |u_\mu|^{p^*} \, dx \to 0 \) as \( |E| \to 0 \), uniformly in \( \mu \),

(iii) \( u_\mu \) is uniformly bounded in \( C^{1,\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \).

**Proof.** We have
\[
I_\mu(u_\mu) = \int_\Omega \left( \frac{|\nabla u_\mu|^{p}}{p} - \frac{\lambda u_\mu^{p}}{p} - \frac{u_\mu^{p^*}}{p^*} \right) \, dx + \mu \int_{\{u_\mu \leq 0\}} u_\mu \, dx
\]
\[
+ \int_{\{-1 < u_\mu < 0\}} \left( u_\mu - \frac{|u_\mu|^{p-1} u_\mu}{p} \right) \, dx - \left( 1 - \frac{1}{p} \right) \int_{\{u_\mu \leq -1\}} = c_\mu
\]
and

\[ I_\mu'(u_\mu) v = \int_\Omega \left( |\nabla u_\mu|^{p-2} \nabla u_\mu \cdot \nabla v - \lambda u_\mu^{p-1} v - u_\mu^{p-1} v \right) dx + \mu \left( \int_{\{u_\mu > 0\}} v \right) dx + \int_{\{-1 < u_\mu < 0\}} \left( 1 - |u_\mu|^{p-1} \right) v dx = 0 \quad \forall v \in W^{1,p}_0(\Omega). \quad (3.4) \]

Taking \( v = u_\mu \) in (3.4), dividing by \( p \) and subtracting from (3.3) gives

\[ \frac{1}{N} \int_\Omega u_\mu^p dx \leq c_\mu + \left( 1 - \frac{1}{p} \right) \mu |\Omega| \leq \beta \quad (3.5) \]

by (3.1), and (i) follows from this, (3.4) with \( v = u_\mu \), and the Hölder inequality.

If (ii) does not hold, then there exist sequences \( \mu_j \to 0 \) and \( (E_j) \) with \( |E_j| \to 0 \) such that

\[ \lim_{E_j} \int_{E_j} u_\mu^1 dx > 0. \quad (3.6) \]

Since \( (u_\mu) \) is bounded by (i), so is \( (u_\mu^+) \), a renamed subsequence of which then converges to some \( v \geq 0 \) weakly in \( W^{1,p}_0(\Omega) \), strongly in \( L^q(\Omega) \) for all \( q \in [1, p^*) \) and a.e. in \( \Omega \), and

\[ |\nabla u_\mu|^p dx \xrightarrow{w^*} \kappa, \quad u_\mu^{p^*} dx \xrightarrow{w^*} v \quad (3.7) \]

in the sense of measures, where \( \kappa \) and \( v \) are bounded nonnegative measures on \( \Omega \). By Lions [12, 13], then there exist an at most countable index set \( I \) and points \( x_i \in \Omega \), \( i \in I \) such that

\[ \kappa \geq |\nabla v|^p dx + \sum_{i \in I} \kappa_i \delta_{x_i}, \quad v = v_i^p dx + \sum_{i \in I} v_i \delta_{x_i}, \quad (3.8) \]

where \( \kappa_i, v_i > 0 \) and \( v_i^{p^*/p} \leq \kappa_i / S \). Suppose \( I \) is nonempty, say, \( i \in I \). An argument similar to that in the proof of Lemma 2.1 shows that \( \kappa_i \leq v_i \), so \( v_i \geq S^{N/p} \). On the other hand, passing to the limit in (3.5) with \( \mu = \mu_j \) and using (3.7) and (3.8) gives \( v_i \leq N \beta < S^{N/p} \), a contradiction. Hence \( I = \emptyset \) and

\[ \int_\Omega u_\mu^{p^*} dx \to \int_\Omega v^{p^*} dx. \]

As in the proof of Lemma 2.1, a further subsequence of \( (u_\mu) \) then converges to some \( u \) in \( W^{1,p}_0(\Omega) \), and hence also in \( L^{p^*}(\Omega) \), and a.e. in \( \Omega \). Then

\[ \int_{E_j} |u_\mu|^p dx \leq \int_\Omega |u_\mu|^p dx + \int_{E_j} |u|^p dx \to 0, \]

contradicting (3.6).

Finally, (iii) follows from (i), (ii), and de Figueiredo et al.[10, Proposition 3.7].

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** We claim that \( u_\mu \) is positive in \( \Omega \), and hence a weak solution of problem (1.2), for all sufficiently small \( \mu \in (0, \mu_*) \). It suffices to show that for every sequence \( \mu_j \to 0 \), a subsequence of \( u_\mu \) is
positive in $\Omega$. By Lemma 3.2 (iii), a renamed subsequence of $u_{\mu_j}$ converges to some $u$ in $C^1_0(\overline{\Omega})$. We have

$$I_{\mu_j}(u_{\mu_j}) = \int_{\Omega} \left( \frac{\|\nabla u_{\mu_j}\|^p}{p} - \frac{\lambda u_{\mu_j}^p}{p} - \frac{u_{\mu_j}^p}{p^*} \right) dx + \mu_j \left[ \int_{\{u_{\mu_j} \geq 0\}} u_{\mu_j} dx \right.$$

$$+ \int_{\{-1 < u_{\mu_j} < 0\}} \left( u_{\mu_j} - \frac{|u_{\mu_j}|^{p-1} u_{\mu_j}}{p} \right) dx - \left(1 - \frac{1}{p}\right) |\{u_{\mu_j} \leq -1\}| = c_{\mu_j} \geq c_0$$

by (3.1) and

$$I'_{\mu_j}(u_{\mu_j}) v = \int_{\Omega} \left( \frac{\|\nabla u_{\mu_j}\|^p}{p} - \frac{\lambda u_{\mu_j}^p}{p} - \frac{u_{\mu_j}^p}{p^*} \right) \nabla u_{\mu_j} \cdot \nabla v + \lambda u_{\mu_j}^{p-1} v - u_{\mu_j}^{p-1} v \right) dx + \mu_j \left[ \int_{\{u_{\mu_j} \geq 0\}} v dx \right.$$

$$+ \int_{\{-1 < u_{\mu_j} < 0\}} \left(1 - |u_{\mu_j}|^{p-1}\right) v dx = 0 \quad \forall v \in W^1_0(\Omega),$$

and passing to the limits gives

$$\int_{\Omega} \left( \frac{\|\nabla u\|^p}{p} - \frac{\lambda u^p}{p} - \frac{u^p}{p^*} \right) dx \geq c_0$$

and

$$\int_{\Omega} \left( \frac{\|\nabla u\|^p}{p} - \frac{\lambda u^p}{p} - \frac{u^p}{p^*} \right) dx = 0 \quad \forall v \in W^1_0(\Omega),$$

so $u$ is a nontrivial weak solution of the problem

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} + u^{p-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega. \end{cases}$$

Then $u > 0$ in $\Omega$ and its interior normal derivative $\partial u/\partial \nu > 0$ on $\partial \Omega$ by the strong maximum principle and the Hopf lemma for the $p$-Laplacian (see Vázquez [17]). Since $u_{\mu_j} \to u$ in $C^1_0(\overline{\Omega})$, then $u_{\mu_j} > 0$ in $\Omega$ for all sufficiently large $j$.

It remains to show that $u_{\mu_j}$ minimizes $I_{\mu}$ on $N_{\mu_j}$ when it is positive. For each $w \in N_{\mu_j}$, we will construct a path $y_w \in \Gamma$ such that

$$\max_{u \in y_w([0,1])} I_{\mu}(u) = I_{\mu}(w).$$

Since

$$I_{\mu}(u_{\mu_j}) = c_{\mu_j} \leq \max_{u \in y_w([0,1])} I_{\mu}(u)$$

by the definition of $c_{\mu_j}$, the desired conclusion will then follow. First we note that the function

$$g(t) = I_{\mu}(t w) = \frac{t^p}{p} \int_{\Omega} \left( \|\nabla w\|^p - \lambda w^p \right) dx - \frac{t^{p^*}}{p^*} \int_{\Omega} w^{p^*} dx + \mu t \int_{\Omega} w dx, \quad t \geq 0$$

has a unique maximum at $t = 1$. Indeed,

$$g'(t) = t^{p-1} \int_{\Omega} \left( \|\nabla w\|^p - \lambda w^p \right) dx - t^{p^*-1} \int_{\Omega} w^{p^*} dx + \mu \int_{\Omega} w dx$$

$$= \left( t^{p-1} - t^{p^*-1} \right) \int_{\Omega} \left( \|\nabla w\|^p - \lambda w^p \right) dx + \left(1 - t^{p^*-1} \right) \mu \int_{\Omega} w dx$$
since \( w \in N_\mu \), and the last two integrals are positive since \( \lambda < \lambda_1 \) and \( w > 0 \), so \( g'(t) > 0 \) for \( 0 \leq t < 1 \), \( g'(1) = 0 \), and \( g'(t) < 0 \) for \( t > 1 \). Hence
\[
\max_{t \geq 0} I_\mu(tw) = I_\mu(w) > 0
\]
since \( g(0) = 0 \). In view of Lemma 3.1 (ii), now it suffices to observe that there exists \( \tilde{R} > \max \{1, R\} \) such that
\[
I_\mu(\tilde{R}u) = \frac{\tilde{R}^p}{p} \int_\Omega (|\nabla u|^p - \lambda u^p) \, dx - \frac{\tilde{R}^{p^*}}{p^*} \int_\Omega u^{p^*} \, dx + \mu \tilde{R} \int_\Omega u \, dx \leq 0
\]
for all \( u \) on the line segment joining \( w \) to \( v_\varepsilon \) since all norms on a finite dimensional space are equivalent. \( \square \)

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References


