Dongsheng Kang*, Mengru Liu, and Liangshun Xu

Critical elliptic systems involving multiple strongly–coupled Hardy–type terms

https://doi.org/10.1515/anona-2020-0029
Received November 25, 2018; accepted March 29, 2019.

Abstract: In this paper, we study the radially–symmetric and strictly–decreasing solutions to a system of critical elliptic equations in $\mathbb{R}^N$, which involves multiple critical nonlinearities and strongly–coupled Hardy–type terms. By the ODEs analysis methods, the asymptotic behaviors at the origin and infinity of solutions are proved. It is found that the singularities of $u$ and $v$ in the solution $(u, v)$ are at the same level. Finally, an explicit form of least energy solutions is found under certain assumptions, which has all of the mentioned properties for the radial decreasing solutions.

Keywords: critical elliptic system; radial decreasing solution; asymptotic property; Hardy term

MSC: 35J47, 35J50

1 Introduction

In this paper, we study the following critical elliptic system involving multiple coupled Hardy–type terms:

$$
\begin{cases}
-\Delta u - \frac{\mu_1 u + \lambda v}{|x|^2} = u^{2^*-1} + \frac{\eta \alpha}{2} u^{\alpha-1} v^\beta, & x \in \mathbb{R}^N \setminus \{0\}, \\
-\Delta v - \frac{\mu_2 u + \lambda u}{|x|^2} = v^{2^*-1} + \frac{\eta \beta}{2} u^\alpha v^{\beta-1}, & x \in \mathbb{R}^N \setminus \{0\}, \\
(u, v) \in (D^{1,2}(\mathbb{R}^N))^2, & u, v > 0, x \in \mathbb{R}^N \setminus \{0\},
\end{cases}
$$

(1.1)

where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $(\int_{\mathbb{R}^N} |\nabla \cdot|^2 dx)^{1/2}$ and the parameters satisfy the assumption:

$$(\Omega_1) \ N \geq 5, \ 1 < \alpha, \beta < 2, \ \alpha + \beta = 2^*, \ \eta > 0, \ \bar{\mu} > \mu_1 \geq \mu_2 > 0, \ \lambda > 0, \ \lambda^2 < \min\{\mu_1 \mu_2, (\bar{\mu} - \mu_1)(\bar{\mu} - \mu_2)\}.$$ 

$2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent and $\bar{\mu} := (\frac{N-2}{2})^2$ is the best constant in the Hardy inequality ([1]):

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx \leq \frac{1}{\bar{\mu}} \int_{\mathbb{R}^N} |\nabla u|^2 dx, \ \forall u \in D^{1,2}(\mathbb{R}^N).$$

Under the assumption $(\Omega_1)$, the matrix $E := \begin{pmatrix} \mu_1 & \lambda \\ \lambda & \mu_2 \end{pmatrix}$ is positive definite and for all $(u, v) \in (D^{1,2}(\mathbb{R}^N))^2$

there holds that $0 < y_1 < y_2 < \bar{\mu}$, and furthermore,

$$y_1(u^2 + v^2) \leq \mu_1 u^2 + 2 \lambda uv + \mu_2 v^2 \leq y_2(u^2 + v^2).$$
where \( y_1, y_2 \), are eigenvalues of the matrix \( E \). According to the Hardy, Sobolev and Young inequalities, the following best constants are well defined ([2–5]):

\[
S(\mu) := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left( |\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx}{\left( \int_{\mathbb{R}^N} |u|^2 \right)^{1/2}}, \quad \mu < \bar{\mu},
\]

\[
S(\mu_1, \mu_2, \lambda) := \inf_{(u, v) \in \mathcal{D}} \frac{\int_{\mathbb{R}^N} \left( |\nabla u|^2 + |\nabla v|^2 \right) - \mu_1 u^2 + 2\lambda uv + \mu_2 v^2 \left( \frac{|x|^2}{|\mu|^2} \right) dx}{\left( \int_{\mathbb{R}^N} (|u|^2 + |v|^2) \right)^{1/2}},
\]

where \( \mathcal{D} := (D^{1,2}(\mathbb{R}^N))^2 \setminus \{(0, 0)\} \). Critical elliptic equations and systems involving the Hardy inequality have been studied by many authors (e.g. [2, 5–10] for the problems of single equation, [3], [4], [11–14] for related elliptic systems). In particular, Terracini proved that \( S(\mu) \) has the unique positive extremals ([5]):

\[
V_\mu(\varepsilon) := \varepsilon \frac{2^N}{\sqrt{\overline{\mu}}} U_\mu(\varepsilon^{-1} x), \quad \forall \varepsilon > 0, \quad \mu \in [0, \bar{\mu}),
\]

(1.3)

where \( U_\mu(x) := \left( \frac{2N(\bar{\mu} - \mu)}{\sqrt{\bar{\mu}}} \right)^{\frac{2\varepsilon}{\varepsilon - 1}} \left( \frac{|x|^{\frac{2\varepsilon - 2}{\varepsilon - 2}}}{\sqrt{\frac{2\varepsilon - 2}{\varepsilon - 2}}} + |x|^{\frac{\varepsilon}{\varepsilon - 2}} \right)^{-\frac{1}{\sqrt{\bar{\mu}}}} \),

by which many elliptic problems involving the Hardy inequality have been studied. Furthermore, elliptic systems corresponding to the case \( \lambda = 0 \) in (1.1) have been studied (e.g. [11] and [3] for the existence of solutions, [4], [13] and [14] for the asymptotic properties of solutions, [15] and [16] for the case without Hardy terms).

In this paper, we study the asymptotic behaviors at the origin and infinity of the radially–symmetric and strictly–decreasing solution \((u, v)\) to (1.1) by the ODEs analysis methods. The existence of this kind of solutions to (1.1) can be proved by the arguments similar to those of [3]. Elliptic regularity argument shows that the solution \((u, v)\) to (1.1) satisfies \( u, v \in C^2(\mathbb{R}^N \setminus \{0\}) \).

The following constants and functions are all well defined under \((\mathcal{H}_1)\):

\[
\mathcal{A} = \mathcal{A}(\mu_1, \mu_2, \lambda) := \frac{2\lambda}{(\mu_1 - \mu_2)^2 + 4\lambda^2 + (\mu_1 - \mu_2)},
\]

\[
l(\mu_1, \mu_2, \lambda) := \sqrt{\overline{\mu} - \frac{\mu_1 + \mu_2 + \sqrt{(\mu_1 - \mu_2)^2 + 4\lambda^2}}{2}},
\]

\[
a(\lambda) := \sqrt{\mu} - l(\mu_1, \mu_2, \lambda), \quad b(\lambda) := \sqrt{\mu} + l(\mu_1, \mu_2, \lambda),
\]

\[
f(\tau) := \frac{\eta \beta}{2} + \tau^a - \tau^{2-\beta} - \frac{\eta a}{2} \tau^2, \quad \tau \geq 0,
\]

\[
g(\tau) := \lambda \tau^2 + (\mu_1 - \mu_2) \tau - \lambda, \quad \tau \geq 0.
\]

(1.4)

(1.5)

Under the assumption \((\mathcal{H}_1)\), since \( f(0) = \frac{\eta \beta}{2} > 0 \), \( f(\infty) = -\infty \), for all \( \eta \in (0, \infty) \), then there exists naturally the smallest positive zero \( \lambda_0 \) of \( f \) (e.g. [16]). Furthermore, \( f(\tau) > 0 \) in \([0, \lambda_0)\) and a direct calculation shows that \( g(\mathcal{A}) = 0 \) and \( g(\tau) < 0 \) for all \( \tau \in [0, \mathcal{A}) \). To study the explicit form minimizers to \( S(\mu_1, \mu_2, \lambda) \), we also define

\[
F(\tau) := \frac{1 + \tau^2}{(1 + \eta \tau^\beta + \tau^2)^{\frac{2}{\beta}}}, \quad \tau \geq 0,
\]

\[
F(\tau_{\min}) := \min_{\tau \geq 0} F(\tau),
\]

(1.6)
\[
\begin{align*}
\mu^* &:= \frac{\mu_1 + 2\lambda \tau_{\text{min}} + \mu_2 (\tau_{\text{min}})^2}{1 + (\tau_{\text{min}})^2}, \\
\mu^* &:= \frac{\mu_1 + \mu_2 + \sqrt{(\mu_1 - \mu_2)^2 + 4\lambda^2}}{2}, \\
S^* &:= \left(1 + \frac{\eta \alpha}{r^2} \right)^{\frac{1}{2}},
\end{align*}
\]
where \( \tau_{\text{min}} \geq 0 \) is a minimal point of \( F(r) \). Direct calculation shows that
\[
2\lambda \leq \mathcal{A} \lambda + \frac{\lambda}{\mathcal{A}}, \quad \mu_1 + \mathcal{A} \lambda = \mu_2 + \frac{\lambda}{\mathcal{A}} = \mu^*, \quad \mu^* \in (0, \bar{\mu}).
\] (1.8)
Note that \( \mathcal{A}_0, \tau_{\text{min}} \) and \( \mu^* \) depend on \( \eta \) but are independent of \( \lambda \). However, \( \mathcal{A} \) and \( \mu^* \) depend on \( \lambda \) but are independent of \( \eta \).

The main results of this paper are summarized in the following theorems and they are new to the best of our knowledge. It can be checked that the intervals for the parameters are not empty.

**Theorem 1.1.** Suppose that \((\mathcal{C}_1)\) holds and \((u, v)\) is a radially–symmetric and strictly–decreasing solution to the problem (1.1). Set \( r = |x|, \ x \in \mathbb{R}^N \setminus \{0\} \).

(i) Then
\[
\lim_{r \to 0^+} \frac{v(r)}{u(r)} = \mathcal{A}, \quad \lim_{r \to 0^+} \frac{u'(r)}{u(r)} = \lim_{r \to 0^+} \frac{v'(r)}{v(r)} = -a(l),
\]
\[
\lim_{r \to +\infty} \frac{v(r)}{u(r)} = \mathcal{A}, \quad \lim_{r \to +\infty} \frac{u'(r)}{u(r)} = \lim_{r \to +\infty} \frac{v'(r)}{v(r)} = -b(l).
\]

(ii) Assume furthermore that \( \mathcal{A} < \mathcal{A}_0 \), then
\[
\inf_{r > 0} \frac{v(r)}{u(r)} = \mathcal{A}, \quad \inf_{r > 0} \frac{u'(r)}{u(r)} = -b(l), \quad \sup_{r > 0} \frac{u'(r)}{u(r)} = -a(l),
\]
and none of above infimums and supremum can be achieved.

Let \((u, v)\) be a radial decreasing solution to (1.1) and set \( r = |x| = e^t, \ x \in \mathbb{R}^N \setminus \{0\}, \ t \in \mathbb{R} \). By \((\mathcal{C}_1), (1.4)\) and above Theorem 1.1, we have that \( \mathcal{A} \mathcal{A} < \bar{\mu} - \mu_1 \) and the following constant and function are well defined:
\[
T_0 := \inf \left\{ T > 0 \left| \bar{\mu} - \mu_1 - \lambda \frac{\nu(e^t)}{u(e^t)} > 0, \ \forall \ t \in (-\infty, -T) \cup (T, \infty) \right. \right\},
\]
\[
\omega(t) := \sqrt{\bar{\mu} - \mu_1 - \lambda \frac{\nu(e^t)}{u(e^t)}}, \quad t \in (-\infty, -T_0) \cup (T_0, \infty).
\]

**Theorem 1.2.** Suppose that \((\mathcal{C}_1)\) holds with \( \mathcal{A} < \mathcal{A}_0 \) and \((u, v)\) is a radially–symmetric and strictly–decreasing solution to the problem (1.1). Set \( r = |x| = e^t, \ x \in \mathbb{R}^N \setminus \{0\} \). Then there exist the constants \( C_1, C_2 > 0 \), such that
\[
\lim_{t \to +\infty} e^{\nu(e^t)} u(e^t) = C_1, \quad \lim_{t \to -\infty} e^{(\nu(e^t)+1) t} u(e^t) = C_2.
\]
\[
\lim_{t \to +\infty} e^{\nu(e^t)} v(e^t) = C_1 \mathcal{A}, \quad \lim_{t \to -\infty} e^{(\nu(e^t)+1) t} v(e^t) = C_2 \mathcal{A}.
\]
In the following theorem, we find the explicit radially–symmetric and strictly–decreasing minimizers to \( S(\mu_1, \mu_2, \lambda) \) under certain assumptions, among which there exists an explicit form of least energy solutions to (1.1) satisfying all of the properties mentioned in Theorems 1.1 and 1.2.
Theorem 1.3. Suppose that (H1) holds and \( V^e_{\mu} \) is the minimizer of \( S(\mu) \) defined as in (1.3). Assume furthermore that \( \tau_{\text{min}} = A \). Then

\[
\mu_\varepsilon = \mu^*, \quad S(\mu_1, \mu_2, \lambda) = F(\Lambda)S(\mu_\varepsilon) = F(A)S(\mu^*),
\]

and \( S(\mu_1, \mu_2, \lambda) \) has the radially–symmetric and strictly–decreasing minimizers of the form \( \{ C(V^e_{\mu_1}, AV^e_{\mu_2}), C, \varepsilon > 0 \} \), among which the problem (1.1) has the explicit form of least energy solutions

\[
\{(s^*, V^e_{\mu_1}(x), s^*AV^e_{\mu_2}(x)), \varepsilon > 0\}.
\]

Corollary 1.4. Suppose that (H1) holds with \( \beta < \alpha \), \( \eta \geq \frac{N}{N-2} \). Then there exist \( \mu_1, \mu_2, \lambda \in (0, \mu) \) such that \( \tau_{\text{min}} = A \), \( S(\mu_1, \mu_2, \lambda) = F(A)S(\mu_\varepsilon) \), and thus \( S(\mu_1, \mu_2, \lambda) \) has the minimizers of the form \( \{ C(V^e_{\mu_1}(x), AV^e_{\mu_2}(x)), C, \varepsilon > 0 \} \).

Remark 1.5. Suppose that (H1) holds. Then the existence of radial decreasing ground state solutions to (1.1) can be proved by the arguments similar to those of [3], where the case \( \lambda = 0 \) was studied and by which \( S(\mu_1, \mu_2, \lambda) \) is achieved. Taking in (1.1) \( \mu_1 > \mu_2, \eta = 2, \alpha = \beta = \frac{\varepsilon_0}{T} \), then a direct calculation shows that \( 0 < A < 1, A_0 = 1 \), and thus \( A < A_0 \) holds naturally.

Remark 1.6. Suppose that (H1) holds with \( A < A_0 \). Taking \( T = \max \{|t_1|, |t_2|\} \) and by Lemma 3.3 of this paper, the radial decreasing solution \((u, v)\) to (1.1) satisfies

\[
\lim_{t \to \pm \infty} \omega(t) = l(\mu_1, \mu_2, \lambda) > 0, \quad \omega(t) < l(\mu_1, \mu_2, \lambda), \forall t \in \mathbb{R},
\]

\[
\lim_{t \to \pm \infty} \frac{\sqrt{\mu}t + \int_{T_0}^{t} \omega(s)ds}{b(l)t} = 1, \quad \lim_{t \to \pm \infty} \frac{\sqrt{\mu}t + \int_{t}^{-T_0} \omega(s)ds}{a(l)t} = 1,
\]

which together with Theorem 1.2 reveals that \( u \) and \( v \) have the singularities of same level at the origin and infinity. When \( \lambda = 0 \) and the other parameters satisfy (H1), then there exists a critical surface \( \Pi \) for the positive solution \((u, v)\) to (1.1) \((\text{e.g.} \ [14]):\)

\[
\Pi : \quad \alpha \sqrt{\mu} - \mu_1 = (2 - \beta) \sqrt{\mu} - \mu_2,
\]

such that the singularities of \( v \) are different above and below \( \Pi \), which implies that the strongly–coupled critical terms in (1.1) plays a key role for the asymptotic properties of \( u, v \). However, when (H1) holds with \( \lambda > 0 \), then the asymptotic properties of \( u, v \) depend only on the Hardy terms in (1.1). Hence, the methods and conclusions of this paper have crucial differences with those of [14].

Remark 1.7. Suppose that (H1) holds. Then Theorem 1.3 shows that the minimizers of \( S(\mu_1, \mu_2, \lambda) \) has an explicit relation with those of \( S(\mu^\varepsilon) \) if \( \tau_{\text{min}} = A \). In our following work, we will study singularities of solutions to (1.1) when \( \tau_{\text{min}} \neq A \). Then combining

with the conclusions of this paper, the asymptotic properties of the radial decreasing minimizer \((U, V)\) to \( S(\mu_1, \mu_2, \lambda) \) will be clear and further studies on (1.1) and related problems can be done, even without the explicit forms of \((U, V)\).

This paper is organized as follows. Some preliminary results are established in Section 2, Theorems 1.1 and 1.2 are proved in Section 3, and Theorem 1.3 is verified in Section 4. For convenience, we always denote positive constants as \( C \) and omit \( dx \) in integrals if no confusion is caused.
2 Preliminary results

Assume that \((\mathcal{C}_1)\) holds. Let \(r = |x|\) and \((u(r), v(r))\) be a radially–symmetric and strictly–decreasing solution to (1.1). Then from (1.1) it follows that

\[
\begin{aligned}
    (r^{N–1} u'(r))' &= -r^{N–1} \left( \mu_1 \frac{u}{r^2} + \lambda \frac{v}{r^2} + u^{2'–1} + \frac{\eta \alpha}{2} u^{a–1} v^\beta \right), \\
    (r^{N–1} v'(r))' &= -r^{N–1} \left( \mu_2 \frac{v}{r^2} + \mu_1 \frac{v}{r^2} + v^{2'–1} + \frac{\eta \beta}{2} u^{a} v^{\beta–1} \right).
\end{aligned}
\]

(2.1)

which implies that \(r^{N–1} u'(r)\) and \(r^{N–1} v'(r)\) are strictly decreasing in \((0, +\infty)\). Since \(u'(r), v'(r) \leq 0\), then we obtain that \(u'(r), v'(r) < 0\) in \((0, +\infty)\). Set

\[
\begin{aligned}
    t &= \ln r, \quad r \in (0, +\infty), \quad t \in \mathbb{R}, \\
    y_1(t) &= r^{\sqrt{\eta}} u(r), \quad z_1(t) = r^{\sqrt{\mu}} u'(r), \\
    y_2(t) &= r^{\sqrt{\eta}} v(r), \quad z_2(t) = r^{\sqrt{\mu}} v'(r).
\end{aligned}
\]

(2.2)

Since \(u(r), v(r) \in C^2(0, +\infty)\), \(y_1(t), y_2(t) \in C^2(\mathbb{R})\), by (2.1) and (2.2) we have that

\[
\begin{aligned}
    y_1' &= \sqrt{\bar{\mu}} y_1 + z_1, \\
    z_1' &= -\sqrt{\bar{\mu}} z_1 - \mu_1 y_1 - \lambda y_2 - \left( y_1^{z_1–1} + \frac{\eta \alpha}{2} y_1^{a–1} y_2^\beta \right), \\
    y_2' &= \sqrt{\bar{\eta}} y_2 + z_2, \\
    z_2' &= -\sqrt{\bar{\eta}} z_2 - \lambda y_1 - \mu_2 y_2 - \left( y_2^{z_2–1} + \frac{\eta \beta}{2} y_1^a y_2^{\beta–1} \right).
\end{aligned}
\]

(2.3)

which implies that \(y_1\) and \(y_2\) satisfy the following ODEs system:

\[
\begin{aligned}
    y_1'' &= (\bar{\mu} - \mu_1) y_1 - \lambda y_2 - y_1^{z_1–1} - \frac{\eta \alpha}{2} y_1^{a–1} y_2^\beta, \\
    y_2'' &= (\bar{\eta} - \mu_2) y_2 - \lambda y_1 - y_2^{z_2–1} - \frac{\eta \beta}{2} y_1^a y_2^{\beta–1}.
\end{aligned}
\]

(2.4)

The complete integral of (2.4) is given by

\[
V(y_1, y_2, y'_1, y'_2) := \frac{1}{2} \left( |y'_1|^2 + |y'_2|^2 \right) - \frac{1}{2} (\bar{\mu} - \mu_1) y_1^2 - \frac{1}{2} (\bar{\eta} - \mu_2) y_2^2 + \lambda y_1 y_2 + \frac{1}{2} \left( y_1^{z_1} + y_2^{z_2} + \eta y_1^a y_2^\beta \right).
\]

Set \(V(t) = V(y_1(t), y_2(t), y'_1(t), y'_2(t))\). From (2.4) it follows that \(V'(t) = 0\). Then \(V(t)\) is a constant and we set

\[
V(t) \equiv K_0, \quad \forall t \in \mathbb{R}.
\]

Lemma 2.1. Suppose that \((\mathcal{C}_1)\) holds. Then \(K_0 = 0\). Furthermore,

\[
\lim_{t \to \pm \infty} y_i(t) = \lim_{t \to \pm \infty} y'_i(t) = \lim_{t \to \pm \infty} y''_i(t) = 0, \quad i = 1, 2.
\]

(2.5)

Proof. The arguments are similar to those of Lemmas 2.1 and 2.2 in [14], where the case \(\lambda = 0\) of (1.1) was studied. The details are omitted for simplicity. \(\square\)
3 Asymptotic properties

Assume that (H_{1}) holds and the functions \( y_1(t), y_2(t) \), are defined as in (2.2). For all \( l \in \mathbb{R} \), \( i = 1, 2 \), a direct calculation shows that

\[
e^{it} y_i(t) = y_i(0) e^{\int_0^t \left( \frac{y_i'(s)}{y_i(s)} \right) ds}, \quad t \in (0, +\infty),
\]

(3.1)

Furthermore,

\[
e^{-it} y_i(t) = y_i(0) e^{\int_0^t \left( \frac{-y_i'(s)}{y_i(s)} \right) ds}, \quad t \in (-\infty, 0).
\]

(3.2)

For any \( \varepsilon > 0 \), from (3.2) it follows that

\[
g_i(\sqrt{\mu_i - \mu_i} + \varepsilon) = +\infty, \quad g_i(0) = -\infty.
\]

Therefore the following constants are well defined:

\[
l_i^* := \sup \{ l \mid g_i(l) < +\infty \}, \quad l_i^{**} := \inf \{ l \mid g_i(l) > -\infty \}.
\]

Since \( \varepsilon > 0 \) is arbitrary, from (3.3) it follows that

\[
0 \leq l_i^* \leq l_i^{**} \leq \sqrt{\mu_i - \mu_i}, \quad i = 1, 2.
\]

(3.4)

Lemma 3.1. Suppose that (H_{1}) holds. Then \( l_i^* = l_i^{**} \geq 0, \quad i = 1, 2 \).

Proof. Assume that \( l_i^{**} < l_i^* \). Then there exist \( a, b \in \mathbb{R} \) such that \( l_i^* \leq b < a \leq l_i^{**} \) and \( -\infty < g_i(b) < g_i(a) < +\infty \). Furthermore,

\[
g_i(a) - g_i(b) = \int_0^b (a - b)ds = +\infty,
\]

a contradiction. From (3.4) it follows that \( l_i^* = l_i^{**} \geq 0 \).

According to Lemma 3.1, the following constants are well defined:

\[
l_i := \sup \{ l \mid g_i(l) < +\infty \} = \inf \{ l \mid g_i(l) > -\infty \}, \quad i = 1, 2.
\]

Similarly, we define the functions

\[
g_{i+2}(l) := \int_0^l \left( l - \frac{y_i'(s)}{y_i(s)} \right) ds, \quad i = 1, 2.
\]

Arguing as above, the following constants are well defined

\[
l_i := \sup \{ l \mid g_i(l) < +\infty \} = \inf \{ l \mid g_i(l) > -\infty \}, \quad i = 3, 4.
\]
Furthermore, \[ 0 \leq l_1, l_3 \leq \sqrt{\bar{\mu} - \mu_1}, \quad 0 \leq l_2, l_4 \leq \sqrt{\bar{\mu} - \mu_2}. \]

Therefore, we can assume that \( l \geq 0 \) when studying \( l_i, \ 1 \leq i \leq 4 \).

**Lemma 3.2.** Suppose that \((\mathcal{H}_1)\) holds. Then \( l_1 = l_2, \ l_3 = l_4. \)

**Proof.** For all \( \varepsilon > 0 \), from the definitions of \( l_i \) it follows that
\[
\int_0^t \left( l_i \pm \varepsilon + \frac{y_i'(s)}{y_i(s)} \right) ds \to \pm \infty \text{ as } t \to +\infty, \quad \varepsilon > 0, \quad i = 1, 2,
\]
which together with (3.1) implies that
\[
e^{(l_i - \varepsilon)t} y_i(t) = y_i(0) e^{\int_0^t (l_i - \varepsilon + \frac{y_i'(s)}{y_i(s)}) ds} \to 0 \text{ as } t \to +\infty, \quad i = 1, 2,
\]
\[
e^{(l_i + \varepsilon)t} y_i(t) = y_i(0) e^{\int_0^t (l_i + \varepsilon + \frac{y_i'(s)}{y_i(s)}) ds} \to +\infty \text{ as } t \to +\infty, \quad i = 1, 2.
\]

Consequently,
\[
0 < e^{(l_i - l - 2\varepsilon)t} y_i(t) < e^{(l_i - l + 2\varepsilon)t} y_i(t) \text{ as } t \to +\infty, \quad \varepsilon > 0,
\]
which implies that
\[
e^{(l_i - l - 2\varepsilon)t} y_i(t) < \frac{y_2}{y_1} < e^{(l_i - l + 2\varepsilon)t} y_i(t) \text{ as } t \to +\infty, \quad \varepsilon > 0,
\]
If \( l_1 > l_2 \), then by taking \( 2\varepsilon < l_1 - l_2 \) we have that \( \lim_{t \to +\infty} \frac{y_2}{y_1} = +\infty \). From (2.4) and the L'Hospital rule it follows that
\[
\lim_{t \to +\infty} \frac{|y_1'|^2}{y_1^2} = \lim_{t \to +\infty} \frac{y_1''}{y_1} = \lim_{t \to +\infty} \left( \bar{\mu} - \mu_1 - \lambda \frac{\nu y_1}{y_2} - \frac{\eta \alpha}{2} y_1 \bar{\mu} y_2^2 \right) = -\infty,
\]
a contradiction. If \( l_1 < l_2 \), then \( \lim_{t \to +\infty} \frac{y_1}{y_2} = +\infty \) and therefore
\[
\lim_{t \to +\infty} \frac{|y_2'|^2}{y_2^2} = \lim_{t \to +\infty} \frac{y_2''}{y_2} = \lim_{t \to +\infty} \left( \bar{\mu} - \mu_2 - \lambda \frac{\nu y_2}{y_2} - \frac{\eta \beta}{2} y_2 \bar{\mu} y_2^2 \right) = -\infty,
\]
a contradiction.

Therefore \( l_1 = l_2 \) must hold and similarly \( l_3 = l_4 \) also holds. \( \square \)

**Lemma 3.3.** Suppose that \((\mathcal{H}_1)\) holds. Let \( A \) and \( l(\mu_1, \mu_2, \lambda) \) be defined as in (1.4). Then
\[
\lim_{t \to +\infty} \frac{y_1(t)}{y_1(t)} = \lim_{t \to -\infty} \frac{y_1(t)}{y_1(t)} = A,
\]
\[
\lim_{t \to +\infty} \frac{y_1'(t)}{y_1(t)} = -\lim_{t \to -\infty} \frac{y_1'(t)}{y_1(t)} = -l(\mu_1, \mu_2, \lambda), \quad i = 1, 2.
\]

**Proof.** From (3.5) and Lemma 3.2 it follows that
\[
\lim_{t \to +\infty} y_i^{\alpha - \frac{3}{2}} y_2^{\beta} = 0, \quad \lim_{t \to -\infty} y_i^{\alpha} y_2^{\beta - 2} = 0.
\]

(i) We claim that either \( \frac{y_2(t)}{y_1(t)} \) or \( \frac{y_1(t)}{y_2(t)} \) is bounded in \((0, +\infty)\).

In fact, if neither \( \frac{y_2(t)}{y_1(t)} \) nor \( \frac{y_1(t)}{y_2(t)} \) is bounded in \((0, +\infty)\), then the continuity implies that \( \frac{y_2(t)}{y_1(t)} \) must have sequences of local minimum points \( \{ \sigma_n \} \subset (0, +\infty) \) such that
\[
\lim_{n \to +\infty} \sigma_n = +\infty, \quad \lim_{n \to +\infty} \frac{y_2(\sigma_n)}{y_1(\sigma_n)} = 0, \quad \left( \frac{y_2(t)}{y_1(t)} \right)' \bigg|_{t = \sigma_n} = 0, \quad \left( \frac{y_2(t)}{y_1(t)} \right)'' \bigg|_{t = \sigma_n} \geq 0.
\]
Then for all \( n \in \mathbb{N} \), a direct calculation shows that

\[
\frac{y_2'(\sigma_n)}{y_1'(\sigma_n)} = \frac{y_2(\sigma_n)}{y_1(\sigma_n)} \cdot \left( \frac{y_2'(t)}{y_1'(t)} - \frac{y_2'(t)}{y_1'(t)} \right)_{t=\sigma_n} \geq 0,
\]

which together with (2.4) imply that

\[
\left( \hat{\mu} - \mu_2 - \lambda \frac{\nu_1}{\tau_2} - \nu_1^{\nu'_2} - \frac{\eta \beta}{2} y_2^{\nu_2} \right)_{t=\sigma_n} \geq \left( \hat{\mu} - \mu_1 - \lambda \frac{\nu_2}{\tau_1} - \nu_1^{\nu'_2} - \frac{\eta \beta}{2} y_2^{\nu_2} \right)_{t=\sigma_n}.
\]

Since

\[
\lim_{n \to \infty} y_1(\sigma_n) = \lim_{n \to \infty} y_2(\sigma_n) = 0, \quad \lim_{n \to \infty} \frac{y_1(\sigma_n)}{y_2(\sigma_n)} = +\infty,
\]

by (3.6) and (3.7) we get a contradiction, which implies that either \( \frac{y_2(t)}{y_1(t)} \) or \( \frac{y_1(t)}{y_2(t)} \) is bounded in \((0, +\infty)\).

(ii) We claim that if \( \lim_{t \to +\infty} \frac{y_2(t)}{y_1(t)} \) is bounded, then there exists the limit \( \lim_{t \to +\infty} \frac{y_2(t)}{y_1(t)} = A \).

In fact, if \( \lim_{t \to +\infty} \frac{y_2(t)}{y_1(t)} \) doesn't exist, then the continuity implies that \( \frac{y_2(t)}{y_1(t)} \) must have sequences of local minimum points \( \{ \sigma_n \} \subset (0, +\infty) \) and of local maximum points \( \{ \tau_n \} \subset (0, +\infty) \), such that

\[
\lim_{n \to \infty} \sigma_n = \infty, \quad \left( \frac{y_2(t)}{y_1(t)} \right)'_{t=\sigma_n} = 0, \quad \left( \frac{y_2(t)}{y_1(t)} \right)''_{t=\sigma_n} \geq 0,
\]

\[
\lim_{n \to \infty} \tau_n = \infty, \quad \left( \frac{y_2(t)}{y_1(t)} \right)'_{t=\tau_n} = 0, \quad \left( \frac{y_2(t)}{y_1(t)} \right)''_{t=\tau_n} \leq 0.
\]

Then for all \( n \in \mathbb{N} \), a direct calculation shows that

\[
\frac{y_2'(\sigma_n)}{y_1'(\sigma_n)} = y_2(\sigma_n) \cdot \left( \frac{y_2'(\sigma_n)}{y_1'(\sigma_n)} \right) = y_2(\sigma_n), \quad \frac{y_2'(\tau_n)}{y_1'(\tau_n)} = y_2(\tau_n),
\]

\[
\left( \frac{y''_2(t)}{y_1(t)} - \frac{y_2''(t)}{y_1(t)} \right)_{t=\sigma_n} \geq 0, \quad \left( \frac{y''_2(t)}{y_1(t)} - \frac{y_2''(t)}{y_1(t)} \right)_{t=\tau_n} \leq 0.
\]

For any local minimum points \( \{ \sigma_n \} \subset (0, +\infty) \) of \( \frac{y_2(t)}{y_1(t)} \), since \( \sigma_n \) is a local maximum point of \( \frac{y_1(t)}{y_2(t)} \), from (3.6) and (3.7) it follows that

\[
\mu_1 - \mu_2 + o(1) \geq \lambda \left( \frac{y_1(\sigma_n)}{y_2(\sigma_n)} - \frac{y_2(\sigma_n)}{y_1(\sigma_n)} \right), \quad \text{as } n \to \infty,
\]

which implies that

\[
\mu_1 - \mu_2 + o(1) \geq \lambda \left( \frac{y_1(\tau_n)}{y_2(\tau_n)} - \frac{y_2(\tau_n)}{y_1(\tau_n)} \right), \quad \text{as } t \to +\infty.
\]

Similarly, for any local maximum points \( \{ \tau_n \} \subset (0, +\infty) \) of \( \frac{y_2(t)}{y_1(t)} \), since \( \tau_n \) is a local minimum point of \( \frac{y_1(t)}{y_2(t)} \) such that

\[
\mu_1 - \mu_2 + o(1) \leq \lambda \left( \frac{y_1(\tau_n)}{y_2(\tau_n)} - \frac{y_2(\tau_n)}{y_1(\tau_n)} \right), \quad \text{as } n \to \infty,
\]

and therefore

\[
\mu_1 - \mu_2 + o(1) \leq \lambda \left( \frac{y_1(t)}{y_2(t)} - \frac{y_2(t)}{y_1(t)} \right), \quad \text{as } t \to +\infty.
\]

Consequently,

\[
\lim_{t \to +\infty} \left( \frac{y_1(t)}{y_2(t)} - \frac{y_2(t)}{y_1(t)} \right) = \frac{\mu_1 - \mu_2}{\lambda},
\]

which implies that

\[
\lim_{t \to +\infty} \left( \frac{y_1(t)}{y_2(t)} + \frac{y_2(t)}{y_1(t)} \right) = \sqrt{\left( \frac{\mu_1 - \mu_2}{\lambda} \right)^2 + 4}.
\]
From (3.8) and (3.9) it follows that \( \lim_{t \to +\infty} \frac{y_2(t)}{y_1(t)} \) exists, a contradiction.

Therefore, there must exist the limit \( B := \lim_{t \to +\infty} \frac{y_2(t)}{y_1(t)} \). Since \( l_1 = l_2 \), from (2.4) and the L'Hospital rule it follows that

\[
B = \lim_{t \to +\infty} \frac{y_2(t)}{y_1(t)} = \lim_{t \to +\infty} \frac{y_2'(t)}{y_1'(t)} = \lim_{t \to +\infty} \frac{\frac{y_2''(t)}{y_1''(t)}(\frac{\mu - \mu_2)B - \lambda}{\mu - \mu_1 - \lambda B}}{
\]

which implies that

\[
B = \lim_{t \to +\infty} \frac{y_2(t)}{y_1(t)} = \mathcal{A} = \frac{2\lambda}{\sqrt{(\mu_1 - \mu_2)^2 + 4\lambda^2} + (\mu_1 - \mu_2)} > 0.
\]

(3.10)

(iii) Similarly, if \( y_1(t), y_2(t) \) is bounded, then there exists the limit \( \lim_{t \to +\infty} \frac{y_1(t)}{y_2(t)} = \frac{1}{\mathcal{A}} \).

From (i)–(iii) it follows that (3.10) holds under (\( \mathcal{H}_1 \)). Arguing as above, under (\( \mathcal{H}_1 \)) we also have that

\[
\lim_{t \to +\infty} \frac{y_2(t)}{y_1(t)} = \mathcal{A}.
\]

(3.11)

Let \( l(\mu_1, \mu_2, \lambda) \) be defined as in (1.4). Then a direct calculation shows that

\[
l(\mu_1, \mu_2, \lambda) = \sqrt{\overline{\mu} - \overline{\mu}_1} - \frac{\lambda}{\mathcal{A}} = \sqrt{\overline{\mu} - \overline{\mu}_2} - \frac{\lambda}{\mathcal{A}}.
\]

(3.12)

From (2.4), (3.6), (3.11), (3.12) and the L'Hospital rule it follows that

\[
\lim_{t \to +\infty} \frac{|y_1'(t)|}{y_1(t)} = \lim_{t \to +\infty} \frac{y_1''(t)}{y_1'(t)} = \lim_{t \to +\infty} \frac{\overline{\mu} - \mu - \lambda y_2}{y_1'} = l(\mu_1, \mu_2, \lambda),
\]

\[
\lim_{t \to +\infty} \frac{|y_2'(t)|}{y_2(t)} = \lim_{t \to +\infty} \frac{y_2''(t)}{y_2'(t)} = \lim_{t \to +\infty} \frac{\overline{\mu} - \mu - \lambda y_2}{y_2'} = l(\mu_1, \mu_2, \lambda).
\]

(3.13)

By (3.13) we deduce that \( y_1''(t), y_2''(t) > 0 \) as \( |t| \) large enough, which together with (2.5) implies that \( y_1', y_2' > 0 \) as \( t \to +\infty \) and \( y_1', y_2' > 0 \) as \( t \to -\infty \). Consequently,

\[
\lim_{t \to +\infty} \frac{y_1'(t)}{y_1(t)} = -\lim_{t \to +\infty} \frac{y_2'(t)}{y_2(t)} = -l(\mu_1, \mu_2, \lambda), \quad i = 1, 2.
\]

(3.14)

The proof is complete.

\[\square\]

**Lemma 3.4.** Suppose that (\( \mathcal{H}_1 \)) holds and \( \mathcal{A} < \Lambda_0 \). Then

\[
\inf_{t \in \mathbb{R}} \frac{y_2(t)}{y_1(t)} = \mathcal{A}, \quad \sup_{t \in \mathbb{R}} \frac{y_1'(t)}{y_1(t)} = l(\mu_1, \mu_2, \lambda), \quad \inf_{t \in \mathbb{R}} \frac{y_1'(t)}{y_1(t)} = -l(\mu_1, \mu_2, \lambda).
\]

**Proof.** If \( \inf_{t \in \mathbb{R}} \frac{y_2(t)}{y_1(t)} \) can't be achieved at any finite point, then from Lemma 3.3 it follows that \( \inf_{t \in \mathbb{R}} \frac{y_2(t)}{y_1(t)} = \mathcal{A} \). If the infimum is achieved at a finite point \( t_1 \in \mathbb{R} \), then

\[
0 < \inf_{t \in \mathbb{R}} \frac{y_2(t)}{y_1(t)} = \frac{y_2(t_1)}{y_1(t_1)} \leq \mathcal{A},
\]

\[
\left. \left( \frac{y_2}{y_1} \right) ' \right|_{t=t_1} = \left[ \frac{y_2}{y_1} \left( \frac{y_2''}{y_2} - \frac{y_1''}{y_1} \right) \right]_{t=t_1} = 0,
\]

\[
\left. \left( \frac{y_2}{y_1} \right) '' \right|_{t=t_1} = \left[ \frac{y_2}{y_1} \left( \frac{y_2''}{y_2} - \frac{y_1''}{y_1} \right) \right]_{t=t_1} \geq 0.
\]

(3.15)
Let the functions $f(\tau)$ and $g(\tau)$ be defined as in (1.5) and $A_0$ be the smallest positive zero of $f(\tau)$. Then $f(\tau) > 0$ for all $\tau \in [0, A_0)$, $g(A) = 0$ and $g(\tau) < 0$ for all $\tau \in [0, A)$. Since $A < A_0$, from (2.4) and (3.15) it follows that

$$0 > g\left(\frac{y_2(t_3)}{y_1(t_3)}\right) > f\left(\frac{y_2(t_3)}{y_1(t_3)}\right) \left(\frac{y_2(t_3)}{y_1(t_3)}\right)^{\beta - 1} y_1(t_3)^{\gamma - 2} > 0,$$

a contradiction, which together with (3.10) implies that $\inf_{t \in \mathbb{R}} \frac{y_2(t)}{y_1(t)}$ can't be achieved at any finite point and furthermore,

$$\lim_{t \to -\infty} \frac{y_2(t)}{y_1(t)} = \inf_{t \in \mathbb{R}} \frac{y_2(t)}{y_1(t)} = \Lambda,$$

(3.16)

Consider the supremum of $\left|\frac{y_1'(t)}{y_1(t)}\right|$ in $\mathbb{R}$. If the supremum can't be achieved at any finite point, then (3.16) implies that $\sup_{t \in \mathbb{R}} \frac{y_1'(t)}{y_1(t)} = l(\mu_1, \mu_2, \lambda)$. If the supremum is achieved at finite point $t_4 \in \mathbb{R}$, then

$$\left|\frac{\left|\frac{y_1'(t_4)}{y_1(t_4)}\right|}{\left|\frac{y_1'(t_4)}{y_1(t_4)}\right|_{t=t_4}} = \sup_{t \in \mathbb{R}} \frac{y_1'(t)}{y_1(t)} \geq l(\mu_1, \mu_2, \lambda).$$

Furthermore,

$$\left|\frac{\left|\frac{y_1'(t_4)}{y_1(t_4)}\right|}{\left|\frac{y_1'(t_4)}{y_1(t_4)}\right|_{t=t_4}} = \left(\frac{y_1'(t_4)}{y_1(t_4)}\right)^2\right|_{t=t_4} = 0.$$

(3.17)

Since $\inf_{t \in \mathbb{R}} \frac{y_2(t)}{y_1(t)}$ can't be achieved at any finite point, from (2.4), (3.16) and (3.17) it follows that

$$\left|\frac{\left|\frac{y_1'(t_4)}{y_1(t_4)}\right|}{\left|\frac{y_1'(t_4)}{y_1(t_4)}\right|_{t=t_4}} = \left\{\left(\frac{\mu - \mu_1}{\Lambda y_1} - y_1^{\gamma - 2} \cdot \frac{\eta A}{2} y_1^{\alpha - 2} y_2^2\right)\right\}_{t=t_4} \leq \left(\frac{\mu - \mu_1}{\Lambda y_1} - y_1^{\gamma - 2} \cdot \frac{\eta A}{2} y_1^{\alpha - 2} y_2^2\right),$$

$$\leq \frac{\mu - \mu_1}{\Lambda y_1} - y_1^{\gamma - 2} \cdot \frac{\eta A}{2} y_1^{\alpha - 2} y_2^2,$$

a contradiction, which implies that $\sup_{t \in \mathbb{R}} \frac{y_1'(t)}{y_1(t)}$ can't be achieved at any finite points. Then from (3.14) and (3.16) it follows that

$$\lim_{t \to -\infty} \frac{y_1'(t)}{y_1(t)} = \inf_{t \in \mathbb{R}} \frac{y_1'(t)}{y_1(t)} = -\sqrt{\mu - \mu_1 - \Lambda} = -l(\mu_1, \mu_2, \lambda),$$

$$\lim_{t \to -\infty} \frac{y_1'(t)}{y_1(t)} = \sup_{t \in \mathbb{R}} \frac{y_1'(t)}{y_1(t)} = \sqrt{\mu - \mu_1 - \Lambda} = l(\mu_1, \mu_2, \lambda).$$

(3.18)

The proof is complete.

Lemma 3.5. Suppose that (3.1) holds and $\Lambda < A_0$. Then $l_i = l(\mu_1, \mu_2, \lambda)$, $1 \leq i \leq 4$.

Proof. For any $\epsilon > 0$, by the definition of $l_1$ we have that

$$\int_0^{\infty} \left(\left(\frac{l_1}{y_1(t)} + \frac{\eta A}{2} y_1^{\alpha - 2} y_2^2\right) + \frac{\eta A}{2} y_1^{\alpha - 2} y_2^2\right)\frac{y_1'(t)}{y_1(t)} ds$$

$$= \int_0^{\infty} \left(\frac{l_1}{y_1(t)} + \frac{\eta A}{2} y_1^{\alpha - 2} y_2^2\right)\frac{y_1'(t)}{y_1(t)} ds = +\infty.$$

From Lemma 3.4 it follows that

$$l_1 + \epsilon - l(\mu_1, \mu_2, \lambda) > 0, \quad \forall \epsilon > 0,$$
that is, \( l_1 - l(\mu_1, \mu_2, \lambda) \geq 0 \). For any \( \varepsilon > 0 \), from Lemma 3.4 it follows that
\[
g_1(l(\mu_1, \mu_2, \lambda) + \varepsilon) = +\infty.
\]
Then \( l_1 \leq l(\mu_1, \mu_2, \lambda) \). Therefore, \( l_1 = l(\mu_1, \mu_2, \lambda) \), which together with Lemma 3.2 implies that \( l_1 = l_2 = l(\mu_1, \mu_2, \lambda) \).

Under the assumption \( (3\_1) \), arguing similarly as above we also have that
\[
\lim_{t \to +\infty} \frac{Y_1(t)}{Y_2(t)} = \lim_{t \to -\infty} \frac{Y_1(t)}{Y_2(t)} = l(\mu_1, \mu_2, \lambda), \quad l_3 = l_4 = l(\mu_1, \mu_2, \lambda).
\]

The proof is complete. \( \square \)

**Lemma 3.6.** Suppose that \( (3\_1) \) holds with \( \mathcal{A} < A_0 \) and let \( t_1, t_2 \), be defined as in Theorem 1.1. Then there exist the positive constants \( C_1, C_2 > 0 \), such that
\[
\lim_{t \to +\infty} e^{\int_{t_1}^{t_2} \sqrt{\mu - \mu_1 - \lambda \frac{Y_2(t)}{Y_1(t)}} \, ds} Y_1(t) = C_1, \quad \lim_{t \to -\infty} e^{\int_{t_2}^{t_1} \sqrt{\mu - \mu_1 - \lambda \frac{Y_2(t)}{Y_1(t)}} \, ds} Y_1(t) = C_2,
\]
\[
\lim_{t \to +\infty} e^{\int_{t_1}^{t_2} \sqrt{\mu - \mu_1 - \lambda \frac{Y_2(t)}{Y_1(t)}} \, ds} Y_2(t) = C_1 \mathcal{A}, \quad \lim_{t \to -\infty} e^{\int_{t_2}^{t_1} \sqrt{\mu - \mu_1 - \lambda \frac{Y_2(t)}{Y_1(t)}} \, ds} Y_2(t) = C_2 \mathcal{A}.
\]

**Proof.** We only prove the first equality. The second one can be verified similarly and the last two can be concluded by the first two equalities and (3.10).

Set \( H_1(s) := \frac{Y_1'(s)}{Y_1(s)} \). From (2.4) and Lemma 3.4 it follows that
\[
l(\mu_1, \mu_2, \lambda) \geq H_1(s) > 0, \quad \forall s \in \mathbb{R}.
\]
Note that \( \mathcal{A} < A_0 \) and
\[
\lim_{t \to +\infty} \left( \hat{\mu} - \mu_1 - \lambda \frac{Y_2(t)}{Y_1(t)} \right) = l(\mu_1, \mu_2, \lambda) > 0.
\]
Arguing as in the proof of Lemma 3.4, we have that
\[
\hat{\mu} - \mu_1 - \lambda \frac{Y_2(t)}{Y_1(t)} > 0, \quad \forall t \in (T_0, +\infty).
\]

According to (2.4) and by direct calculation we have that
\[
H_1'(s) = \mu - \mu_1 - \lambda \frac{Y_2(s)}{Y_1(s)} - H_1^2(s) - y_1^{2-2} - \frac{\eta a_1}{T} y_1^{-2} y_2^\beta,
\]
which implies that
\[
\sqrt{\mu - \mu_1 - \lambda \frac{Y_2(s)}{Y_1(s)}} + H_1(s) = \frac{H_1'(s) + y_1^{2-2} + \frac{\eta a_1}{T} y_1^{-2} y_2^\beta}{\sqrt{\mu - \mu_1 - \lambda \frac{Y_2(s)}{Y_1(s)}} - H_1(s)}, \quad \forall t > T_0.
\]

Define
\[
I_1 := \int_{T_0}^{+\infty} \frac{H_1'(s) + y_1^{2-2} + \frac{\eta a_1}{T} y_1^{-2} y_2^\beta}{\sqrt{\mu - \mu_1 - \lambda \frac{Y_2(s)}{Y_1(s)}} - H_1(s)} \, ds,
\]
\[
I_2 := \int_{T_0}^{+\infty} \frac{y_1^{2-2} + \frac{\eta a_1}{T} y_1^{-2} y_2^\beta}{\sqrt{\mu - \mu_1 - \lambda \frac{Y_2(s)}{Y_1(s)}} - H_1(s)} \, ds.
\]
We claim that the integral \( I := \int_{T_0}^{+\infty} \left( \sqrt{\tilde{\mu} - \mu_1 - \lambda \frac{y_1(s)}{y_1(s)} + H_1(s)} \right) ds \) converges.

In fact, from (3.18) and (3.19) it follows that

\[
\lim_{t \to +\infty} \int_{T}^{+\infty} \frac{H'_1(s)}{\sqrt{\tilde{\mu} - \mu_1 - \lambda \frac{y_1(s)}{y_1(s)} - H_1(s)}} ds = l(\mu_1, \mu_2, \lambda), \quad \lim_{t \to +\infty} H_1(s) = -l(\mu_1, \mu_2, \lambda). \tag{3.21}
\]

Since \( A < A_0 \), arguing as in the proof of Lemma 3.4 we have that \( \frac{y_1(s)}{y_1(s)} \) and \( H_1(s) \) are strictly decreasing as \( s \to +\infty \). Taking \( \tilde{T} > T_0 \) large enough we have that \( H'_1(s) < 0 \) for all \( s > \tilde{T} \) and

\[
\begin{align*}
\lim_{t \to +\infty} \int_{T}^{+\infty} \frac{H'_1(s)}{\sqrt{\tilde{\mu} - \mu_1 - \lambda \frac{y_1(s)}{y_1(s)} - H_1(s)}} ds &= \int_{T}^{+\infty} \frac{H'_1(s)}{l(\mu_1, \mu_2, \lambda) - H_1(s)} \frac{l(\mu_1, \mu_2, \lambda) - H_1(s)}{\sqrt{\tilde{\mu} - \mu_1 - \lambda \frac{y_1(s)}{y_1(s)} - H_1(s)}} ds \\
&\leq C \int_{T}^{+\infty} \frac{H'_1(s)}{l(\mu_1, \mu_2, \lambda) - H_1(s)} ds \\
&\leq C \ln \frac{2l(\mu_1, \mu_2, \lambda)}{l(\mu_1, \mu_2, \lambda) - H_1(\tilde{T})} < +\infty.
\end{align*}
\]

Then the integral \( I_1 \) converges. Furthermore, (3.21) implies that there exists \( R > 0 \) such that

\[
\sqrt{\tilde{\mu} - \mu_1 - \lambda \frac{y_1(s)}{y_1(s)} - H_1(s)} > l(\mu_1, \mu_2, \lambda), \quad \forall s > R.
\]

Therefore,

\[
0 < I_2 := \int_{R}^{+\infty} \frac{2 \sqrt{y_1^2 - y_1^2 + \eta a y_1^2 - y_2^2}}{\sqrt{\tilde{\mu} - \mu_1 - \lambda \frac{y_1(s)}{y_1(s)} - H_1(s)}} ds
\]

\[
< \frac{1}{l(\mu_1, \mu_2, \lambda)} \int_{R}^{+\infty} e^{-2s}((2s - 2)l(\mu_1, \mu_2, \lambda) - (2s + 2)\epsilon) ds.
\]

By taking \( \epsilon \to 0^+ \) we have that

\[-(2s - 2)l(\mu_1, \mu_2, \lambda) - (2s + 2)\epsilon < 0.\]

Then the integral \( \int_{T_0}^{+\infty} (2 \sqrt{y_1^2 - y_1^2 + \eta a y_1^2 - y_2^2}) ds \) converges, which together with (3.23) implies that the integral \( I_2 \) converges.

By (3.22) and (3.23) we have that the integral \( I = I_1 + I_2 \) converges. Furthermore,

\[
\lim_{t \to +\infty} \int_{T_0}^{t} e^{\int_{T_0}^{t} \sqrt{\tilde{\mu} - \mu_1 - \lambda \frac{y_1(s)}{y_1(s)}} ds} y_1(t)
\]

\[
= y_1(T_0) \lim_{t \to +\infty} \int_{T_0}^{t} e^{\int_{T_0}^{t} \sqrt{\tilde{\mu} - \mu_1 - \lambda \frac{y_1(s)}{y_1(s)}} + H_1(s) ds}
\]

\[
= y_1(T_0) e^{\int_{T_0}^{T_0} \left( \sqrt{\tilde{\mu} - \mu_1 - \lambda \frac{y_1(s)}{y_1(s)}} + H_1(s) \right) ds} = C_1,
\]

where \( C_1 := y_1(T_0) e^{\int_{T_0}^{T_0} \left( \sqrt{\tilde{\mu} - \mu_1 - \lambda \frac{y_1(s)}{y_1(s)}} + H_1(s) \right) ds} > 0. \]

\( \square \)
4 Explicit form solutions

In this section, we study the explicit form of radially–symmetric and strictly–decreasing minimizers to \( S(\mu_1, \mu_2, \lambda) \), among which there exists an explicit form of least energy solutions to (1.1), satisfying all of the properties in Theorems 1.1 and 1.2. For convenience we set \( k(\tau) := -f(\tau), \tau > 0 \), where \( f(\tau) \) is defined as in (1.5).

Proof of Theorem 1.3. Suppose that (H_1) holds with \( \tau_{\min} = \lambda \). We first investigate the functions \( F(\tau) \) and \( k(\tau) \). A direct calculation shows that

\[
F'(\tau) = \frac{2\tau^{\beta-1}k(\tau)}{(1 + \eta \tau^\beta + \tau^2)^{\frac{\beta}{2}+1}}, \quad \tau > 0.
\]  

Note that

\[
limit_{\tau \to 0^+} F(\tau) = \frac{2}{1 + \eta \tau^\beta + \tau^2}, \quad \lim_{\tau \to +\infty} F(\tau) = 1,
\]

\[k(\tau) < 0 \text{ as } \tau \to 0^+, \quad k(\tau) > 0 \text{ as } \tau \to +\infty.\]

Then \( \min_{\tau \geq 0} F(\tau) \) must be achieved at finite \( \tau_{\min} > 0 \) and from (4.1) it follows that

\[
F'(\tau_{\min}) = 0, \quad k(\tau_{\min}) = 0, \quad 0 < F(\tau_{\min}) < 1.
\]

For all \( w \in D^{1,2}(\mathbb{R}^N) \setminus \{0\} \), testing the second Rayleigh quotient in (1.2) by \( (w, \tau_{\min} w) \), we have that

\[
S(\mu_1, \mu_2, \lambda) \leq F(\tau_{\min})S(\mu_\ast),
\]

which together with (1.2) implies that

\[
S(\mu_1, \mu_2, \lambda) \leq F(\tau_{\min})S(\mu_\ast). \tag{4.2}
\]

Let \( \{ (u_n, v_n) \} \subset \mathcal{D} \) be a minimizing sequence of \( S(\mu_1, \mu_2, \lambda) \) and define \( z_n = s_n v_n \), where

\[
s_n = \left( \left( \int_{\mathbb{R}^N} |v_n|^2 \right)^{-1} \int_{\mathbb{R}^N} |u_n|^2 \right)^{\frac{1}{2}},
\]

which implies that

\[
\int_{\mathbb{R}^N} |z_n|^2 = \int_{\mathbb{R}^N} |u_n|^2. \tag{4.3}
\]

By the Young inequality and (4.3) we have that

\[
\int_{\mathbb{R}^N} |u_n|^\beta |z_n|^\beta \leq \frac{\alpha}{2} \int_{\mathbb{R}^N} |u_n|^2 + \frac{\beta}{2} \int_{\mathbb{R}^N} |z_n|^2 = \int_{\mathbb{R}^N} |u_n|^2 = \int_{\mathbb{R}^N} |z_n|^2. \tag{4.4}
\]
Then from (1.2), (1.6), (1.8) and (4.4) it follows that

\[
\int_{\mathbb{R}^N} \left( |\nabla u_n|^2 + |\nabla v_n|^2 - \frac{\mu_1 u_n^2 + 2\lambda u_n v_n + \mu_2 v^2}{|x|^2} \right) \geq \int_{\mathbb{R}^N} \left( |u_n|^2 + \eta |u_n|^\beta |v_n|^\beta + |v_n|^2 \right)^{\frac{2}{\beta}}
\]

\[
\geq \int_{\mathbb{R}^N} \left( |\nabla u_n|^2 + |\nabla v_n|^2 -(\mu_1 + \lambda A)\frac{u_n^2}{|x|^2} - (\mu_2 + \lambda A)\frac{v^2}{|x|^2} \right)
\]

\[
\left( (1 + \eta s_n^{-\beta} + s_n^{-2(\beta)}) \int_{\mathbb{R}^N} |u_n|^2 \right)^{\frac{2}{\beta}}
\]

\[
\geq \frac{\int_{\mathbb{R}^N} \left( |\nabla u_n|^2 - \mu^* \frac{u_n^2}{|x|^2} \right)}{(1 + \eta s_n^{-\beta} + s_n^{-2(\beta)}) \int_{\mathbb{R}^N} |u_n|^2}
\]

\[
\geq \frac{1}{s_n^2} \int_{\mathbb{R}^N} \left( |\nabla z_n|^2 - \mu^* \frac{z_n^2}{|x|^2} \right)
\]

\[
+ \frac{1}{s_n^2} \int_{\mathbb{R}^N} \left( |\nabla z_n|^2 - \mu^* \frac{z_n^2}{|x|^2} \right)
\]

\[
\geq F(s_n^{-1})S(\mu^*)
\]

\[
\geq F(\tau_{\text{min}})S(\mu^*).
\]

Taking \( n \to \infty \) we have that

\[
S(\mu_1, \mu_2, \lambda) \geq F(\tau_{\text{min}})S(\mu^*). \quad (4.5)
\]

A direct calculation shows that

\[
\tau_{\text{min}} = A \iff \mu^* = \frac{\lambda}{\mu}. \quad (4.6)
\]

Then from (4.2), (4.5) and (4.6) it follows that

\[
S(\mu_1, \mu_2, \lambda) = F(\lambda)S(\mu^*) = F(\lambda)S(\mu^*),
\]

which implies that \( S(\mu_1, \mu_2, \lambda) \) has the minimizers of the form:

\[
\{ C(V_{\mu^*}^\varepsilon(x), \lambda V_{\mu^*}^\varepsilon(x)), \quad C, \varepsilon > 0 \}.
\]

Since \( k(\tau_{\text{min}}) = k(A) = 0 \), a direct calculation shows that the problem (1.1) has the explicit form of least energy solutions

\[
\{(s^*V_{\mu^*}^\varepsilon(x), \ s^* \lambda V_{\mu^*}^\varepsilon(x)), \quad \varepsilon > 0 \},
\]

that is, \((u_\varepsilon, v_\varepsilon) := (s^*V_{\mu^*}^\varepsilon(x), \ s^* \lambda V_{\mu^*}^\varepsilon(x))\) is a solution to (1.1) such that

\[
\int_{\mathbb{R}^N} \left( |\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 - \frac{\mu_1 u_\varepsilon^2 + 2\lambda u_\varepsilon v_\varepsilon + \mu_2 v_\varepsilon^2}{|x|^2} \right)
\]

\[
= \int_{\mathbb{R}^N} \left( u_\varepsilon^2 + v_\varepsilon^2 + \eta u_\varepsilon^\alpha v_\varepsilon^\beta \right)
\]
where \( s^* \) is defined as in (1.7). By (3.18) we have that \( l(\mu_1, \mu_2, \lambda) = \sqrt{\mu_1 - \lambda \lambda} \) and therefore the solution \((s^* V_{\mu^*}^e(x), s^* A V_{\mu^*}^e(x))\) satisfies all of the properties mentioned in Theorems 1.1 and 1.2 with \( T_0 = 0 \).

The proof is complete. \( \square \)

**Proof of Corollary 1.4.** Since \( \tau_{\min} \) depends on \( \alpha, \beta, \eta, \) and \( A \) depends on \( \mu_1, \mu_2, \lambda, \) then \( \tau_{\min} \) is independent of \( A \). Obviously, by (4.1) a sufficient condition to ensure \( \tau_{\min} = \lambda \) is

\[
k(\lambda) = 0, \quad k(\tau) < 0 \text{ in } (0, \lambda), \quad k(\tau) > 0 \text{ in } (\lambda, +\infty).
\]

In the following discussion, we only consider the case \( k(\tau) > 0 \) in \([1, +\infty)\).

Suppose that \( 1 < \beta < \alpha < 2, \eta \geq \frac{N}{N-2} \). Since

\[
k'(\tau) = \tau^{1-\beta} \left( 2 - \beta + \frac{2\eta \alpha}{2^*} \tau - \alpha \tau^{2^*-2} \right),
\]

then we have that

\[
k(1) = \frac{\eta}{2}(\alpha - \beta) > 0,
\]

\[
2 - \beta > 0, \quad \beta > 2^* - 2, \quad \frac{2\eta \alpha}{2^*>2} \geq \alpha,
\]

\[
k'(\tau) > 0, \quad k(\tau) > k(1) > 0, \quad \forall \tau \in (1, +\infty).
\]

Since \( k(\tau) < 0 \) as \( \tau \to 0^+ \), from (4.1) and (4.7) it follows that \( \min_{\tau \in (0, 1)} F(\tau) \) must be achieved at finite \( \tau_{\min} \in (0, 1) \).

Noting that \( 0 < \lambda \leq 1, \ A \to 0 \) as \( \lambda \to 0 \) and \( A = 1 \) as \( \mu_1 = \mu_2, \) and \( A(\mu_1, \mu_2, \lambda) \) is a continuous function, there must exist certain \( \mu_1, \mu_2, \lambda \in (0, \bar{\lambda}) \), such that \( A(\mu_1, \mu_2, \lambda) = \tau_{\min} \). Then the desired result follows directly from Theorem 1.3. \( \square \)

**Acknowledgement**

The authors acknowledge the anonymous referee for carefully reading this paper and making many important comments.

This work is supported by the Fundamental Research Funds for the Central Universities of China, South–Central University for Nationalities (No. CZT18008).

**References**


