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Bifurcations of nontrivial solutions of a cubic Helmholtz system

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Abstract: This paper presents local and global bifurcation results for radially symmetric solutions of the cubic Helmholtz system

$$\begin{cases} -\Delta u - \mu u = \left(u^2 + b\, v^2\right) u & \text{on } \mathbb{R}^3, \\ -\Delta v - \nu v = \left(v^2 + b\, u^2\right) v & \text{on } \mathbb{R}^3. \end{cases}$$

It is shown that every point along any given branch of radial semitrivial solutions $(u_0, 0, b)$ or diagonal solutions (u_b, u_b, b) (for $\mu = \nu$) is a bifurcation point. Our analysis is based on a detailed investigation of the oscillatory behavior and the decay of solutions at infinity.

Keywords: Nonlinear Helmholtz sytem, bifurcation

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1 Introduction and main results

Systems of two coupled nonlinear Helmholtz equations arise, for instance, in models of nonlinear optics. In this paper, we analyze the physically relevant and technically easiest case of a Kerr-type nonlinearity in $N = 3$ space dimensions, that is, we study the system

$$\begin{cases} -\Delta u - \mu u = \left(u^2 + b\, v^2\right) u & \text{on } \mathbb{R}^3, \\ -\Delta v - \nu v = \left(v^2 + b\, u^2\right) v & \text{on } \mathbb{R}^3 \end{cases} \tag{H}$$

for given $\mu, \nu > 0$ and a constant coupling parameter $b \in \mathbb{R}$. We are mostly interested in existence results for fully nontrivial radially symmetric solutions of this system that we will obtain using bifurcation theory. Such an approach is new in the context of nonlinear Helmholtz equations or systems. In order to describe the methods used in related works we briefly discuss the available results for scalar nonlinear Helmholtz equations of the form

$$-\Delta u - \lambda u = Q(x)|u|^{p-2}u \qquad \text{on } \mathbb{R}^N, \quad \lambda > 0. \tag{1}$$

Here, the main difficulty is that solutions typically oscillate and do not belong to $H^1(\mathbb{R}^N)$. In the past years, Evéquoz and Weth have developed several methods allowing to find nontrivial solutions of (1) under certain conditions on Q and p, some of which we wish to mention. In [1, 2], they discuss the case of compactly supported Q and $2 < p < 2^\star := \frac{2N}{N-2}$. The idea in [1] is to solve an exterior problem where the nonlinearity vanishes and knowledge about the far-field expansion of solutions is available. The remaining problem on a bounded domain can be solved using variational techniques. The approach in [2] uses Leray-Schauder continuation

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with respect to the parameter λ in order to find solutions of (1). Existence of solutions under the assumption that $Q \in L^\infty(\mathbb{R}^N)$ decays as $|x| \to \infty$ or is periodic is proved in [3] using a dual variational approach, which yields (dual) ground state solutions and, in the case of decaying Q, infinitely many bound states. The technique relies on the Limiting Absorption Principle of Gutiérrez, see Theorem 6 in [4], which leads to the additional constraint $\frac{2(N+1)}{N-1} < p < 2^*$. Furthermore, assuming that Q is radial, the existence of a continuum of radially symmetric solutions of (1) has been shown by Montefusco, Pellacci and the first author in [5], generalizing earlier results in [1]. Their results rely on ODE techniques and only require $p > 2$ and a monotonicity assumption on Q.

To our knowledge, the only available result on nonlinear Helmholtz systems like (H) has been provided by the authors in [6] where, using the methods developed in [3], the existence of a nontrivial dual ground state solution is proved for the system

$$\begin{cases} -\Delta u - \mu u = a(x)\left(|u|^{\frac{p}{2}} + b(x)|v|^{\frac{p}{2}}\right)|u|^{\frac{p}{2}-2}u & \text{on } \mathbb{R}^N, \\ -\Delta v - \nu v \;= a(x)\left(|v|^{\frac{p}{2}} + b(x)|u|^{\frac{p}{2}}\right)|v|^{\frac{p}{2}-2}v & \text{on } \mathbb{R}^N, \\ u, v \in L^p(\mathbb{R}^N) \end{cases}$$

for $N \geq 2$, $\mathbb{Z}^N$-periodic coefficients $a, b \in L^\infty(\mathbb{R}^N)$ with $a(x) \geq a_0 > 0$, $0 \leq b(x) \leq p-1$ and $\frac{2(N+1)}{N-1} < p < 2^*$. Under additional easily verifiable assumptions the ground state can be shown to be fully nontrivial, i.e., both components are nontrivial. Assuming constant coefficients and working on spaces of radially symmetric functions, this variational existence result for dual ground states extends to the case $p = 4$, $N = 3$ which we dicuss in the present paper. In contrast to [6] we construct fully nontrivial radial solutions for arbitrarily large and small $b \in \mathbb{R}$ that, however, need not be dual ground states.

Motivated by the decay properties of radial solutions of nonlinear Helmholtz equations in [5], e.g. Theorem 1.2 (iii), we look for solutions of (H) in the Banach space X_1 where, for $q \geq 1$,

$$X_q := \left\{ w \in C_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{R}) \mid \|w\|_{X_q} < \infty \right\} \quad \text{with} \quad \|w\|_{X_q} := \sup_{x \in \mathbb{R}^3} (1 + |x|^2)^{\frac{q}{2}} |w(x)|. \tag{2}$$

Working on these spaces, we will be able to derive compactness properties which are crucial when proving our bifurcation results. Throughout, we discuss classical, radially symmetric solutions $u, v \in X_1 \cap C^2(\mathbb{R}^3)$ of the system (H) and related equations. Let us remark here only briefly that, using elliptic regularity, all weak solutions $u, v \in L^4_{\mathrm{rad}}(\mathbb{R}^3)$ are actually smooth and, thanks to Proposition 6 in the next section, belong to $X_1 \cap C^2(\mathbb{R}^3)$.

We study bifurcation of solutions (u, v, b) of the nonlinear Helmholtz system (H) from a branch of semitrivial solutions of the form

$$\mathcal{T}_{u_0} := \{(u_0, 0, b) \mid b \in \mathbb{R}\} \subseteq X_1 \times X_1 \times \mathbb{R}$$

in the Banach space $X_1 \times X_1 \times \mathbb{R}$. Here $u_0 : \mathbb{R}^3 \to \mathbb{R}$ denotes any of the uncountably many nontrivial radial solutions of the scalar Helmholtz equation

$$-\Delta u_0 - \mu u_0 = u_0^3 \qquad \text{on } \mathbb{R}^3, \tag{h}$$

which all belong to the space X_1, see [5]. In contrast to the Schrödinger case, we will demonstrate that every point in $\mathcal{T}_{u_0}$ is a bifurcation point for fully nontrivial solutions of (H). Our strategy is to use bifurcation from simple eigenvalues with b acting as a bifurcation parameter. The existence of isolated and algebraically simple eigenvalues will be ensured by assuming radial symmetry and by imposing suitable conditions on the asymptotic behavior of the solutions u, v. For $\tau, \omega \in [0, \pi)$, we define $\mathcal{S} \subseteq X_1 \times X_1 \times \mathbb{R} \setminus \mathcal{T}_{u_0}$ as the set of all solutions $(u, v, b) \in X_1 \times X_1 \times \mathbb{R} \setminus \mathcal{T}_{u_0}$ of (H) satisfying the asymptotic conditions

$$\begin{aligned} u(x) - u_0(x) &= c_1 \frac{\sin(|x|\sqrt{\mu} + \tau)}{|x|} + O\left(\frac{1}{|x|^2}\right) \\ v(x) &= c_2 \frac{\sin(|x|\sqrt{\nu} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \end{aligned} \qquad \text{as } |x| \to \infty \tag{A}$$

for some $c_1, c_2 \in \mathbb{R}$. Propositions 4 and 6 below show that it is natural to assume such an asymptotic behavior for solutions of (H). For notational convenience, we do not denote the dependence of $\mathcal{S}$ and of the asymptotic conditions (A) on the choice $\tau, \omega \in [0, \pi)$. As we will show in Proposition 6, there exists a unique $\tau_0 = \tau_0(u_0) \in [0, \pi)$ such that, for $\tau \in [0, \pi)$,

$$\begin{cases} -\Delta w - \mu w = 3u_0^2(x)\, w & \text{on } \mathbb{R}^3, \\ w(x) = \frac{\sin(|x|\sqrt{\mu}+\tau)}{|x|} + O\left(\frac{1}{|x|^2}\right) & \text{as } |x| \to \infty \end{cases} \quad \text{has no radial solution for } \tau \neq \tau_0. \tag{N}$$

With that, we obtain the following

Theorem 1. *Let $\mu, \nu > 0$, fix any $u_0 \in X_1 \setminus \{0\}$ solving the nonlinear Helmholtz equation* (h) *and choose $\tau \in [0, \pi) \setminus \{\tau_0\}$ according to* (N). *Then, for every $\omega \in [0, \pi)$, there exists a strictly increasing sequence $(b_k(\omega))_{k \in \mathbb{Z}}$ such that $(u_0, 0, b_k(\omega)) \in \overline{\mathcal{S}}$ where $\mathcal{S}$ denotes the set of all solutions $(u, v, b) \in X_1 \times X_1 \times \mathbb{R} \setminus \mathcal{T}_{u_0}$ of* (H) *satisfying* (A). *Moreover,*

(i) *the connected component $\mathcal{C}_k$ of $(u_0, 0, b_k(\omega))$ in $\overline{\mathcal{S}}$ is unbounded in $X_1 \times X_1 \times \mathbb{R}$; and*
(ii) *each bifurcation point $(u_0, 0, b_k(\omega))$ has a neighborhood where $\mathcal{C}_k$ is a smooth curve in $X_1 \times X_1 \times \mathbb{R}$ which, except for the bifurcation point, consists of fully nontrivial solutions.*

The main tools in proving this statement are the Crandall-Rabinowitz Bifurcation Theorem, which will be used to show the local statement (ii) of Theorem 1, and Rabinowitz' Global Bifurcation Theorem, which will provide (i). For a reference, see [7], Theorem 1.7 and [8], Theorem 1.3. We add some remarks the proof of which will also be given in Section 3.

Remark 2. *(a) We will also see that fully nontrivial solutions of* (H) *satisfying the asymptotic condition* (A) *bifurcate from some point $(u_0, 0, b) \in \mathcal{T}_{u_0}$ if and only if $b = b_k(\omega)$ for some $k \in \mathbb{Z}$. Moreover, the proof will show that the values $b_k(\omega)$ do not depend on the choice of τ.*
(b) The map $\mathbb{R} \to \mathbb{R}$, $k\pi + \omega \mapsto b_k(\omega)$ where $0 \le \omega < \pi$, $k \in \mathbb{Z}$ is strictly increasing and onto with $b_k(\omega) \to \pm\infty$ as $k \to \pm\infty$. In particular, every point $(u_0, 0, b) \in \mathcal{T}_{u_0}$, $b \in \mathbb{R}$, is a bifurcation point for fully nontrivial radial solutions of (H), *which is in contrast to Schrödinger systems where bifurcation points are isolated, cf. [9], Satz 2.1.6.*
(c) Close to the respective bifurcation point $(u_0, 0, b_k(\omega)) \in \mathcal{T}_{u_0}$, each continuum $\mathcal{C}_k$ is characterized by a phase parameter $\omega_v(v^2 + b\, u^2) = \omega + k\pi$ derived from the asymptotic behavior of v (see (10)*). It seems that, in the Helmholtz case of oscillating solutions, the integer k takes the role of the nodal characterizations in the Schrödinger case, cf. Satz 2.1.6 in [9]. That phase parameter is constant on connected subsets of the continuum until it possibly runs into another family of semitrivial solutions $\mathcal{T}_{u_1}$ with $u_1 \neq u_0$; unfortunately we cannot provide criteria deciding whether or not this happens. For this reason we cannot claim that $\mathcal{C}_k$ contains an unbounded sequence of fully nontrivial solutions.*
(d) For $\delta \neq 0$, let us assume that $u_\delta \in C^2(\mathbb{R}^3) \cap X_1$ solves $-\Delta u_\delta - \mu u_\delta = u_\delta^3$ on $\mathbb{R}^3$ with $u_\delta(0) = u_0(0) + \delta$, see Theorem 1.2 in [5]. Then $w := \frac{\mathrm{d}}{\mathrm{d}\delta}\big|_{\delta=0} u_\delta$ satisfies $-\Delta w - \mu w = 3u_0^2\, w$ on $\mathbb{R}^3$, $w(0) = 1$. We define $\tau_0 \in [0, \pi)$ as the constant appearing in the asymptotic expansion of w,

$$w(x) = c\, \frac{\sin(|x|\sqrt{\mu} + \tau_0)}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty$$

for some unique $c \neq 0$ and $\tau_0 \in [0, \pi)$, see Proposition 6. With that in mind, the condition $\tau \neq \tau_0$ is a nondegeneracy condition which by means of (N) *ensures that the simplicity requirements of the above-mentioned bifurcation theorems are satisfied.*

Our results are inspired by known bifurcation results for the nonlinear Schrödinger system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + b\, uv^2 & \text{on } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + b\, vu^2 & \text{on } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u > 0, v > 0 \end{cases} \tag{3}$$

where one assumes $\lambda_1, \lambda_2 > 0$ in contrast to (H). We focus on bifurcation results by Bartsch, Wang and Wei in [10] and Bartsch, Dancer and Wang in [11] and refer to the respective introductory sections for a general overview of methods and results for (3). In Theorem 1.1 of [10] the authors show that a continuum consisting of positive radially symmetric solutions $(u, v, \lambda_1, \lambda_2, \mu_1, \mu_2, b)$ of (3) with topological dimension at least 5 bifurcates from a two-dimensional set of semipositive solutions $(u, v) = (u_{\lambda_1,\mu_1}, 0)$ parametrized by $\lambda_1, \mu_1 > 0$. The existence of countably many bifurcation points giving rise to sign-changing radially symmetric solutions was proved by the first author in his dissertation thesis (Satz 2.1.6 of [9]).

In Theorem 1 above, we analyze the corresponding case of bifurcation from a semitrivial family $\mathcal{T}_{u_0}$ in the Helmholtz case. In contrast to the Schrödinger case, our result shows bifurcation at every point in the topology of $X_1 \times X_1 \times \mathbb{R}$, see Remark 2 (b). Looking more closely, we find the same structure of discrete bifurcation points as in the Schrödinger case when fixing parameters τ, ω prescribing the oscillatory behavior of solutions as $|x| \to \infty$ as in the condition (A). In the Schrödinger case, the bifurcating solutions are characterized by their nodal structure; in the Helmholtz case, we use instead a condition on the "asymptotic phase" of the solution (disguised as an integral), which at least close to the j-th bifurcation point takes the value $\omega + j\pi$ as described in Remark 2 (c).

Similar observations can be made for bifurcation from families of diagonal solutions of the Schrödinger system (3) in the special case $N = 2, 3$ and $\lambda_1 = \lambda_2 > 0$ and $\mu_1, \mu_2 > 0$; in order to keep the presentation short, we assume in addition $\mu_1 = \mu_2 = 1$. Bartsch, Dancer and Wang proved in [11] the existence of countably many mutually disjoint global continua of solutions bifurcating from some diagonal solution family of the form

$$\{(u_b, u_b, b) : b > -1\} \subset H^1_{\mathrm{rad}}(\mathbb{R}^N) \times H^1_{\mathrm{rad}}(\mathbb{R}^N) \times \mathbb{R}$$

with a concentration of bifurcation points as $b \searrow -1$. Here $u_b := (1 + b)^{-1/2} u_0$ where $u_0 \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ is a nondegenerate solution of $-\Delta u + u = u^3$. Moreover, having introduced a suitable labeling of the continua, the authors showed that the k-th continuum consists of solutions where the radial profile of $u - v$ has exactly $k - 1$ nodes, cf. Theorem 2.3 in [11].

We provide a counterpart for the Helmholtz system (H) in our second result, Theorem 3, using the same functional analytical setup as in Theorem 1. Here we assume $\nu = \mu$. For nonzero u_0 solving (h), we can then introduce the diagonal solution family

$$\mathfrak{T}_{u_0} := \{(u_b, u_b, b) \mid b > -1\} \subseteq X_1 \times X_1 \times \mathbb{R} \qquad \text{with } u_b := (1 + b)^{-1/2}\, u_0.$$

Given $\tau, \omega \in [0, \pi)$, we denote by $\mathfrak{S}$ the set of all solutions $(u, v, b) \in X_1 \times X_1 \times \mathbb{R} \setminus \mathfrak{T}_{u_0}$ of the nonlinear Helmholtz system (H) with

$$\left.\begin{aligned} u(x) + v(x) &= 2u_b(x) + c_1 \frac{\sin(|x|\sqrt{\mu} + \tau)}{|x|} + O\left(\frac{1}{|x|^2}\right) \\ u(x) - v(x) &= c_2 \frac{\sin(|x|\sqrt{\mu} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \end{aligned}\right\} \quad \text{as } |x| \to \infty \tag{A^{diag}}$$

for some $c_1, c_2 \in \mathbb{R}$. Our existence result for fully nontrivial solutions of (H) bifurcating from $\mathfrak{T}_{u_0}$ with asymptotics (A^{diag}) reads as follows.

Theorem 3. *Let $\nu = \mu > 0$, fix any $u_0 \in X_1 \setminus \{0\}$ solving the nonlinear Helmholtz equation* (h) *and choose $\tau \in [0, \pi) \setminus \{\tau_0\}$ according to* (N). *Then, for every $\omega \in [0, \pi)$, there exists a sequence $(\mathfrak{b}_k(\omega))_{k\in\mathbb{N}}$ such that $(u_{\mathfrak{b}_k(\omega)}, u_{\mathfrak{b}_k(\omega)}, \mathfrak{b}_k(\omega)) \in \overline{\mathfrak{S}}$ where $\mathfrak{S}$ denotes the set of all solutions $(u, v, b) \in X_1 \times X_1 \times \mathbb{R} \setminus \mathfrak{T}_{u_0}$ of* (H) *satisfying (A^{diag}). Moreover,*

(i) the connected component $\mathfrak{C}_k$ of $(u_{\mathfrak{b}_k(\omega)}, u_{\mathfrak{b}_k(\omega)}, \mathfrak{b}_k(\omega))$ in $\overline{\mathfrak{S}}$ is unbounded in $X_1 \times X_1 \times \mathbb{R}$; and

(ii) each bifurcation point $(u_{\mathfrak{b}_k(\omega)}, u_{\mathfrak{b}_k(\omega)}, \mathfrak{b}_k(\omega))$ has a neighborhood where the set $\mathfrak{C}_k$ contains a smooth curve in $X_1 \times X_1 \times \mathbb{R}$ which, except for the bifurcation point, consists of fully nontrivial, non-diagonal solutions.

Again, similar statements as in Remark 2 can be proved. In particular, one can check that every point on $\mathfrak{T}_{u_0}$ is a bifurcating point by a suitable choice of ω.

We point out that our methods in Theorems 1 and 3 also apply for nontrivial radial solutions of

$$-\Delta u_0 - \mu u_0 = -u_0^3 \quad \text{on } \mathbb{R}^3$$

and corresponding modifications in the system (H). Such solutions u_0 exist in the Helmholtz case (but not in the Schrödinger case) and belong to the space X_1, see Theorem 1.2 in [5].

Let us give a short outline of this paper. In Section 2, we introduce the concepts and technical results we use in the proof of Theorems 1 and 3, which are presented in Section 3 and Section 4. In the final section, we provide the proofs of the auxiliary results of Section 2.

2 Properties of the scalar problem

The main challenge in proving Theorem 1 is the thorough analysis of the linearized problem provided in this chapter. Throughout, we fix $\lambda > 0$ and discuss the linear Helmholtz equation

$$-\Delta w - \lambda w = f \qquad \text{on } \mathbb{R}^3 \tag{4}$$

for some $f \in X_3$, where X_3 is defined in (2). We will frequently identify radially symmetric functions $x \mapsto w(x)$ with their profiles; in particular, we denote by $w' := \partial_r w$, $w'' = \partial_r^2 w$ the radial derivatives. The results we establish in this section will demonstrate how to rewrite the system (H) in a way suitable for Bifurcation Theory.

2.1 Representation Formulas

First, we discuss a representation formula for solutions of the linear Helmholtz equation (4). The results resemble a more general Representation Theorem by Agmon, Theorem 4.3 in [12]. We introduce the fundamental solutions

$$\Psi_\lambda, \tilde{\Psi}_\lambda : \mathbb{R}^3 \to \mathbb{R}, \quad \Psi_\lambda(x) := \frac{\cos(\sqrt{\lambda}|x|)}{4\pi|x|} \quad \text{and} \quad \tilde{\Psi}_\lambda(x) := \frac{\sin(\sqrt{\lambda}|x|)}{4\pi|x|} \quad (x \neq 0) \tag{5}$$

of the equation $-\Delta w - \lambda w = 0$ on $\mathbb{R}^3$. We observe that $\tilde{\Psi}_\lambda$ is, up to multiplication with a constant, its unique global classical solution. We will frequently require knowledge of the mapping properties of convolutions with Ψ_λ resp. $\tilde{\Psi}_\lambda$. Various results of such type have been found by Evéquoz and Weth in [3] and further publications, assuming $f \in L^{p'}(\mathbb{R}^N)$, $w \in L^p(\mathbb{R}^N)$ for suitable $p, p' \in (1, \infty)$. In the spaces X_3 resp. X_1, which satisfy the continuous embeddings

$$X_1 \hookrightarrow L^p_{\mathrm{rad}}(\mathbb{R}^3) \text{ for } 3 < p \le \infty, \qquad X_3 \hookrightarrow L^q_{\mathrm{rad}}(\mathbb{R}^3) \text{ for } 1 < q \le \infty, \tag{6}$$

we prove the following statements.

Proposition 4. *For constants $\alpha, \tilde{\alpha} \in \mathbb{R}$, we let $\mathcal{R}_\lambda f := (\alpha\Psi_\lambda + \tilde{\alpha}\tilde{\Psi}_\lambda) * f$. Then,*

(a) the linear map $L^{\frac{4}{3}}_{\mathrm{rad}}(\mathbb{R}^3) \to L^4_{\mathrm{rad}}(\mathbb{R}^3)$, $f \mapsto \mathcal{R}_\lambda f$ is well-defined and continuous;
(b) the linear map $X_3 \to X_1$, $f \mapsto \mathcal{R}_\lambda f$ is well-defined, continuous and compact;
(c) for $f \in X_3$, we have $w := \mathcal{R}_\lambda f \in X_1 \cap C^2(\mathbb{R}^3)$ with $-\Delta w - \lambda w = \alpha \cdot f$ on $\mathbb{R}^3$; and
(d) for $f \in X_3$, the profile of $w := \mathcal{R}_\lambda f$ and its radial derivative satisfy as $r \to \infty$

$$\begin{aligned} w(r) &= \sqrt{\frac{\pi}{2}}\,\hat{f}(\sqrt{\lambda})\,\frac{\alpha\cos(r\sqrt{\lambda}) + \tilde{\alpha}\sin(r\sqrt{\lambda})}{r} + O\left(\frac{1}{r^2}\right), \\ w'(r) &= \sqrt{\frac{\pi}{2}}\,\hat{f}(\sqrt{\lambda})\,\frac{-\alpha\sqrt{\lambda}\sin(r\sqrt{\lambda}) + \tilde{\alpha}\sqrt{\lambda}\cos(r\sqrt{\lambda})}{r} + O\left(\frac{1}{r^2}\right) \end{aligned} \tag{7}$$

where $\hat{f}(\sqrt{\lambda}) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(r) \frac{\sin(r\sqrt{\lambda})}{r\sqrt{\lambda}}\, r^2 \,\mathrm{d}r$. *Further,* $\tilde{\Psi}_\lambda \star f = 4\pi\sqrt{\frac{\pi}{2}}\, \hat{f}(\sqrt{\lambda}) \cdot \tilde{\Psi}_\lambda$.

As a consequence of Proposition 4, we state the representation formulae we require later to construct the functional analytic setting in the proof of Theorem 1. For $\omega \in (0, \pi)$, we define

$$\mathcal{R}_\lambda^\omega : X_3 \to X_1, \quad f \mapsto \Psi_\lambda \star f + \cot(\omega)\, \tilde{\Psi}_\lambda \star f \tag{8}$$

which provide solutions of the Helmholtz equation (4) the asymptotic behavior of which is described by the phase parameter ω as follows.

Corollary 5. *Let $\omega \in (0, \pi)$ and $f \in X_3$. Then, for $w \in X_1$, we have $w = \mathcal{R}_\lambda^\omega f$ if and only if w is a C^2 solution of $-\Delta w - \lambda w = f$ on $\mathbb{R}^3$ with, for some $c \in \mathbb{R}$,*

$$w(x) = c \cdot \frac{\sin(|x|\sqrt{\lambda} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \qquad \text{as } |x| \to \infty.$$

We point out that the operator $\mathcal{R}_\lambda^\omega$ is not well-defined for $\omega = 0$ due to the pole of the cotangent. We will comment on suitable modifications during the proofs of Theorems 1 and 3.

2.2 The Asymptotic Phase

Frequently, equations of interest will take the form (4) with $f = g \cdot w$ for some $g \in X_2$, see (2). We can then use ODE methods, more specifically the Prüfer transformation, to discuss the corresponding initial value problem for the profiles,

$$-w'' - \frac{2}{r} w' - \lambda w = g(r)\, w \qquad \text{on } (0, \infty) \qquad \text{with } w(0) = 1,\ w'(0) = 0. \tag{9}$$

Proposition 6. *Assume $g \in X_2$. Then the initial value problem (9) has a unique (global) solution $w : [0, \infty) \to \mathbb{R}$ which satisfies*

$$w(r) = \varrho_\lambda(g)\, \frac{\sin(r\sqrt{\lambda} + \omega_\lambda(g))}{r} + O\left(\frac{1}{r^2}\right),$$
$$w'(r) = \varrho_\lambda(g)\sqrt{\lambda}\, \frac{\cos(r\sqrt{\lambda} + \omega_\lambda(g))}{r} + O\left(\frac{1}{r^2}\right)$$

as $r \to \infty$ for some $\varrho_\lambda(g) > 0$ and $\omega_\lambda(g) \in \mathbb{R}$. Here, the value of $\omega_\lambda(g)$ is given by

$$\begin{aligned} &\omega_\lambda(g) = \frac{1}{\sqrt{\lambda}} \int_0^\infty g(r) \sin^2(\phi(r)\sqrt{\lambda})\, \mathrm{d}r \\ &\textit{where } \phi : [0, \infty) \to \mathbb{R} \textit{ solves } \begin{cases} \phi' = 1 + \frac{1}{\lambda} g(r) \sin^2(\phi\sqrt{\lambda}), \\ \phi(0) = 0. \end{cases} \end{aligned} \tag{10}$$

In particular, given $u_0 \in X_1 \setminus \{0\}$ solving (h), Proposition 6 with $g := 3u_0^2 \in X_2$ shows that the nondegeneracy condition (N) holds with $\tau_0 \in [0, \pi)$ such that $\omega_\mu(3\, u_0^2) \in \tau_0 + \pi\mathbb{Z}$.

Comparing Proposition 6 with Corollary 5, we observe that Proposition 6 guarantees $\varrho_\lambda(g) > 0$, that is, the solution has a nonvanishing term of leading order as $r \to \infty$. The asymptotic conditions imposed in Corollary 5 with $f = g \cdot w$ now take the form $\omega_\lambda(g) \in \omega + \pi\mathbb{Z}$. Such boundary conditions at infinity will provide operators with spectral properties suitable for building the functional analytic framework in which to prove Theorem 1. As a first auxiliary result, we prove the following continuity property.

Proposition 7. *The asymptotic phase is continuous as a map $\omega_\lambda : X_2 \to \mathbb{R}, g \mapsto \omega_\lambda(g)$.*

When studying eigenvalue problems of a linearization of (H), we need to know the dependence of the asymptotic phase $\omega_\lambda(b\,u_0^2)$ on the parameter $b \in \mathbb{R}$.

Proposition 8. *Let $u_0 \in X_1 \cap C^2(\mathbb{R}^3)$ be some nonzero solution of* (h). *Then the map $\mathbb{R} \to \mathbb{R},\ b \mapsto \omega_\lambda(b\,u_0^2)$ is continuous, strictly increasing and onto with $\omega_\lambda(0) = 0$.*

2.3 The spectrum of the linearization

In the proof of Theorem 1, we will rewrite the nonlinear Helmholtz system (H) in the form

$$u = \mathcal{R}_\mu^\tau(u(u^2 + b\,v^2)), \qquad v = \mathcal{R}_\nu^\omega(v(v^2 + b\,u^2)), \qquad u, v \in X_1$$

for some $\tau, \omega \in (0, \pi)$, which additionally imposes a certain asymptotic behavior on the solutions, see Corollary 5. In order to analyze the linearized problem, we fix some nontrivial $u_0 \in X_1 \cap C^2(\mathbb{R}^3)$ with $-\Delta u_0 - \mu u_0 = u_0^3$ on $\mathbb{R}^3$ and study the spectra of the linear operators

$$\mathbf{R}_\lambda^\omega : X_1 \to X_1, \qquad w \mapsto \mathcal{R}_\lambda^\omega(u_0^2\,w) = \big(\Psi_\lambda + \cot(\omega)\,\tilde{\Psi}_\lambda\big) * [u_0^2\,w], \tag{11}$$

which are compact thanks to Proposition 4 (b).

Proposition 9. *Let $\omega \in (0, \pi)$, $\lambda > 0$ and u_0 as before. Then the spectrum of $\mathbf{R}_\lambda^\omega$ is*

$$\sigma(\mathbf{R}_\lambda^\omega) = \{0\} \cup \sigma_{\mathrm{p}}(\mathbf{R}_\lambda^\omega), \qquad \sigma_{\mathrm{p}}(\mathbf{R}_\lambda^\omega) = \left\{ \frac{1}{b_k(\omega, \lambda, u_0^2)} \;\middle|\; k \in \mathbb{Z} \right\}$$

where, for $k \in \mathbb{Z}$, $b = b_k(\omega, \lambda, u_0^2) \in \mathbb{R}$ is the unique solution of $\omega_\lambda(b\,u_0^2) = \omega + k\pi$, see Proposition 8. Moreover, all eigenvalues are algebraically simple, and the sequence $(b_k(\omega, \lambda, u_0^2))_{k\in\mathbb{Z}}$ is strictly increasing and unbounded below and above.

This excludes the case $\omega = 0$, even though the values $b_k(0, \lambda, u_0^2) \in \mathbb{R}$, $k \in \mathbb{Z}$, can be defined accordingly. Indeed, the first step of the proof of Proposition 9 above shows for all $\omega \in [0, \pi)$:

Remark 10. *Fix $\omega \in [0, \pi)$. Then the problem*

$$-\Delta w - \lambda w = b u_0^2\, w \quad \text{on } \mathbb{R}^3, \qquad w(x) = \frac{\sin(|x|\sqrt{\lambda} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty$$

has a nontrivial radial solution $w \in X_1 \cap C^2(\mathbb{R}^3)$ if and only if $b \in \{b_k(\omega, \lambda, u_0^2) \mid k \in \mathbb{Z}\}$.

3 Proof of Theorem 1

We first discuss the case $\omega, \tau \in (0, \pi)$, $\tau \neq \tau_0$. Afterwards, we sketch the modifications required if $\omega = 0$ or $\tau = 0$.

The case $\omega \in (0, \pi)$ and $\tau \in (0, \pi) \setminus \{\tau_0\}$.

Step 1: *The Setting.*

We define the map

$$F : \quad X_1 \times X_1 \times \mathbb{R} \to X_1 \times X_1,$$

$$F(w,v,b) := \begin{pmatrix} w - \mathcal{R}_\mu^\tau(w^3 + 3u_0w^2 + 3u_0^2w + b\,(u_0+w)v^2) \\ v - \mathcal{R}_\nu^\omega(v^3 + bv(u_0+w)^2) \end{pmatrix}$$

with the convolution operators $\mathcal{R}_\mu^\tau, \mathcal{R}_\nu^\omega : X_3 \to X_1$ from (8). Observe that F is well-defined since $u, v, w \in X_1$ implies $uvw \in X_3$. Recalling Corollary 5 and (h), we have

$$F(w,v,b) = 0 \quad \Leftrightarrow \quad (u,v,b) := (u_0 + w, v, b) \text{ satisfies (H) with asymptotics } (A).$$

So we aim to find nontrivial zeros of F. Second, we observe that F has a trivial solution family, that is $F(0,0,b) = 0$ holds for every $b \in \mathbb{R}$. Third, $F(\cdot, b)$ is a compact perturbation of the identity on $X_1 \times X_1$ since the operators $\mathcal{R}_\mu^\tau, \mathcal{R}_\nu^\omega : X_3 \to X_1$ are compact thanks to Proposition 4 (b). Moreover, F is twice continuously Fréchet differentiable; we have for $\varphi, \psi \in X_1$ and $b \in \mathbb{R}$, denoting by D the Fréchet derivative w.r.t. the w and v components,

$$DF(0,0,b)[(\varphi,\psi)] = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} 3\,\mathcal{R}_\mu^\tau(u_0^2\,\varphi) \\ b\,\mathcal{R}_\nu^\omega(u_0^2\,\psi) \end{pmatrix} = \begin{pmatrix} \varphi - 3\,\mathbf{R}_\mu^\tau\varphi \\ \psi - b\,\mathbf{R}_\nu^\omega\psi \end{pmatrix} \tag{12}$$

with compact linear operators $\mathbf{R}_\mu^\tau, \mathbf{R}_\nu^\omega : X_1 \to X_1$ as in (11). We deduce that, due to (N) and $\tau \neq \tau_0$, $DF(0,0,b)[\varphi,\psi] = 0$ implies $\varphi = 0$. So nontrivial elements of $\ker DF(0,0,b)$ are of the form $(0,\psi)$ where ψ satisfies $\psi = b\,\mathbf{R}_\nu^\omega\psi$. By Proposition 9, a nontrivial solution exists if and only if $b = b_k(\omega, \nu, u_0^2)$, i.e. $\omega_\nu(b\,u_0^2) = k\pi + \omega$ for some $k \in \mathbb{Z}$, and that the associated eigenspaces are one-dimensional. We abbreviate $b_k(\omega) := b_k(\omega,\nu,u_0^2)$ and write

$$\ker DF(0,0,b_k(\omega)) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ \psi_k \end{pmatrix}\right\}$$

for some $\psi_k \in X_1 \setminus \{0\}$. Thus $b \in \{b_k(\omega) \mid k \in \mathbb{Z}\}$ is a necessary condition for bifurcation of solutions of $F(w,v,b) = 0$ from $(0,0,b)$. We show in the following that it is also sufficient.

Step 2: *Local Bifurcation.*

We apply the Crandall-Rabinowitz Theorem at the point $(0,0,b_k(\omega))$. As $F(\cdot,b)$ is a compact perturbation of the identity on $X_1 \times X_1$, the Riesz-Schauder Theorem implies that $DF(0,0,b_k(\omega))$ is a Fredholm operator of index zero with one-dimensional kernel spanned by $(0,\psi_k)$, see above. To verify the transversality condition, we first compute

$$\partial_b DF(0,0,b_k(\omega))[(0,\psi_k)] \overset{(12)}{=} -\begin{pmatrix} 0 \\ \mathbf{R}_\nu^\omega\psi_k \end{pmatrix} = -\frac{1}{b_k(\omega)}\begin{pmatrix} 0 \\ \psi_k \end{pmatrix}.$$

Then, assuming there is $v \in X_1$ with $v - b_k(\omega)\,\mathbf{R}_\nu^\omega v = \psi_k$, we conclude

$$v \in \ker(I - b_k(\omega)\,\mathbf{R}_\nu^\omega)^2 \setminus \ker(I - b_k(\omega)\,\mathbf{R}_\nu^\omega),$$

which contradicts the algebraic simplicity of the eigenvalue $b_k(\omega)^{-1}$ of $\mathbf{R}_\nu^\omega$ proved in Proposition 9. Thus $\partial_b DF(0,0,b_k(\omega))[(0,\psi_k)] \notin \operatorname{ran} DF(0,0,b_k(\omega))$, and the Crandall-Rabinowitz Theorem provides the smooth curve of solutions of $F(w,v,b) = 0$ as in (ii). Further, possibly shrinking the neighborhood where the local result holds, we may w.l.o.g. assume fully nontrivial solutions $(u_0 + w, v)$ of (H) since the direction of bifurcation is given by $(0,\psi_k)$.

Step 3: *Global Bifurcation.*

We have already seen that $F(\cdot,b)$, $b \in \mathbb{R}$, is a compact perturbation of the identity on $X_1 \times X_1$. Thus the application of Rabinowitz' Global Bifurcation Theorem only requires to verify that the index of $F(\cdot,b)$ in $(0,0)$ changes sign at each value $b = b_k(\omega)$, $k \in \mathbb{Z}$. By the identity (12), for $b \notin \{b_k(\omega) \mid k \in \mathbb{Z}\}$,

$$\begin{aligned} \operatorname{ind}_{X_1\times X_1}\big(F(\cdot,b),(0,0)\big) &= \operatorname{ind}_{X_1\times X_1}\big(DF(0,0,b),(0,0)\big) \\ &\overset{(12)}{=} \operatorname{ind}_{X_1}\big(I - 3\,\mathbf{R}_\mu^\tau, 0\big)\cdot\operatorname{ind}_{X_1}\big(I - b\,\mathbf{R}_\nu^\omega, 0\big), \end{aligned}$$

and hence $\mathrm{ind}_{X_1\times X_1}\big(F(\cdot\,,b),(0,0)\big)$ changes sign at $b=b_k(\omega)$ if and only if so does $\mathrm{ind}_{X_1}\big(I-b\,\mathbf{R}_\nu^\omega,0\big)$. The latter change of index occurs since $b_k(\omega)$ is an isolated eigenvalue of algebraic multiplicity 1 of $\mathbf{R}_\nu^\omega$, see Proposition 9.

The Global Bifurcation Theorem by Rabinowitz asserts that $(u_0,0,b_k(\omega))\in\overline{\mathcal{S}}$ and that the associated connected component $\mathcal{C}_k$ of $\overline{\mathcal{S}}$ is unbounded or returns to $\mathcal{T}_{u_0}$ at some point $(u_0,0,b^\star)$. We prove that, in any case, the component is unbounded.

The asymptotic phase satisfies $\omega_\nu(b_k(\omega)u_0^2)=\omega+k\pi$ by definition of $b_k(\omega)$, see Step 1, and $\omega_\nu(v^2+bu^2)\in\omega+\pi\mathbb{Z}$ for all $(u,v,b)\in\mathcal{C}_k$ with $v\neq 0$. This is due to (A) and Proposition 6. So if all elements $(u,v,b)\in\mathcal{C}_k\setminus\mathcal{T}_{u_0}$ satisfy $v\neq 0$, then as a consequence of the continuity of ω_ν (see Proposition 7) and of the fact that $\mathcal{C}_k$ is connected, we infer that $\omega_\nu(v^2+bu^2)=\omega+k\pi$ for all $(u,v,b)\in\mathcal{C}_k$. Let us now assume that $\mathcal{C}_k$ returns to the trivial family in some point $(u_0,0,b^\star)\in\mathcal{T}_{u_0}$, $b^\star\neq b_k(\omega)$. Then $\omega_\nu(b^\star u_0^2)\neq\omega+k\pi$ by strict monotonicity (see Proposition 8), hence $(u,v,b)\mapsto\omega_\nu(v^2+bu^2)$ is not constant on $\mathcal{C}_k$. Thus, there exists a semitrivial element $(u_1,0,b_1)\in\mathcal{C}_k\setminus\mathcal{T}_{u_0}$, $u_1\neq u_0$. Since $\mathcal{C}_k$ is maximal connected, it contains the unbounded semitrivial family $\mathcal{T}_{u_1}=\{(u_1,0,b)\mid b\in\mathbb{R}\}$.

The case $\omega=0$ and $\tau\in(0,\pi)\setminus\{\tau_0\}$.

Step 1: *The Setting.*

We recall that, in case $\omega=0$, the map F resp. $\mathcal{R}_\nu^\omega$ is not well-defined due to the pole of the cotangent. To write down a suitable replacement, we use the Hahn-Banach Theorem to define functionals $\alpha^{(\nu)},\beta^{(\nu)}\in X_1'$ with the following property: For $w\in X_1$ with

$$w(x)=\alpha_w\frac{\sin(|x|\sqrt{\nu})}{4\pi|x|}+\beta_w\frac{\cos(|x|\sqrt{\nu})}{4\pi|x|}+O\left(\frac{1}{|x|^2}\right)\qquad\text{as }|x|\to\infty \tag{13}$$

for some $\alpha_w,\beta_w\in\mathbb{R}$, we have $\alpha^{(\nu)}(w):=\alpha_w$ and $\beta^{(\nu)}(w):=\beta_w$. We then define for $\sigma=\pm1$

$$\begin{aligned}G_\sigma:\quad & X_1\times X_1\times\mathbb{R}\to X_1\times X_1,\\ G_\sigma(w,v,b):=&\begin{pmatrix} w-\mathcal{R}_\mu^\tau(w^3+3u_0w^2+3u_0^2w+b\,(u_0+w)v^2)\\ v-\Psi_\nu\star[v\,(v^2+b\,(w+u_0)^2)]-(\alpha^{(\nu)}(v)+\sigma\beta^{(\nu)}(v))\cdot\tilde{\Psi}_\nu\end{pmatrix}.\end{aligned}$$

Using Proposition 4, we see that $G_\sigma(w,v,b)=0$ if and only if (u_0+w,v,b) solves the nonlinear Helmholtz system (H) with asymptotics (A), $\omega=0$. Indeed, by (13), one sees that $G_\sigma(w,v,b)=0$ implies $\beta^{(\nu)}(v)=0$. Further, recalling the property (N), $(\varphi,\psi)\in\ker DG_\sigma(0,0,b)$ if and only if

$$\varphi\equiv 0,\quad -\Delta\psi-\nu\psi=b\,u_0^2\,\psi,\quad \psi(x)\overset{\beta^{(\nu)}(\psi)=0}{=}c\,\frac{\sin(|x|\sqrt{\nu})}{|x|}+O\left(\frac{1}{|x|^2}\right)$$

for some $c\neq 0$. As before, Propositions 6 and 8 allow to conclude that solutions of (H), (A) for $\omega=0$ can bifurcate from $(u_0,0,b)\in\mathcal{T}_{u_0}$ only if $b=b_k(0)$ for some $k\in\mathbb{Z}$, and that there exist $\psi_k\in X_1\setminus\{0\}$ with

$$\ker DG_\sigma(0,0,b_k(0))=\operatorname{span}\left\{\begin{pmatrix}0\\ \psi_k\end{pmatrix}\right\}.$$

Step 2: *Local Bifurcation.*

The proof of transversality as required in the Crandall-Rabinowitz Theorem has to be adapted. Assuming for contradiction that there are $\varphi,\psi\in X_1$ with $DG_\sigma(0,0,b_k(0))[(\varphi,\psi)]=\partial_b DG_\sigma(0,0,b_k(0))[(0,\psi_k)]$, a short calculation gives $\varphi=0$ (due to (N)) and

$$\psi=b_k(0)\,\Psi_\nu\star[u_0^2\,\psi]+(\alpha^{(\nu)}(\psi)+\sigma\beta^{(\nu)}(\psi))\cdot\tilde{\Psi}_\nu-\Psi_\nu\star[u_0^2\,\psi_k].$$

Applying the functional $\alpha^{(\nu)}$ to this identity, we infer $\beta^{(\nu)}(\psi) = 0$. Moreover, Proposition 4 (c) gives $-\psi'' - \frac{2}{r}\psi' - \nu\psi = b_k(0)u_0^2\,\psi - u_0^2\,\psi_k$. Further, by Step 1, $\beta^{(\nu)}(\psi_k) = 0$ as well as $-\psi_k'' - \frac{2}{r}\psi_k' - \nu\psi_k = b_k(0)u_0^2\,\psi_k$. Using these differential equations, one finds

$$(r^2(\psi_k\psi' - \psi\psi_k'))' = r^2u_0^2(r)\psi_k^2 \qquad \text{for } r > 0.$$

Integrating by parts and exploiting the asymptotic behavior of ψ resp. ψ_k and their derivatives, see Proposition 4 (d) and equation (13), this finally implies

$$\int_0^R r^2u_0^2(r)\psi_k^2(r)\,\mathrm{d}r = R^2\left(\psi_k(R)\psi'(R) - \psi(R)\psi_k'(R)\right) = O\left(\frac{1}{R}\right),$$

which is a contradiction as $R \to \infty$; hence transversality holds.

Step 3: *Global Bifurcation.*

We apply Rabinowitz' Global Bifurcation Theorem from [13], Theorem II.3.3, which as above yields unbounded connected components $\mathcal{C}_k \subseteq \overline{\mathcal{S}}$ once we show that the index

$$\mathrm{ind}_{X_1}\left(I - K_b, 0\right) \qquad \text{where } K_b := b\,\Psi_\nu \star [u_0^2\,\cdot\,] + (\alpha^{(\nu)} + \sigma\beta^{(\nu)})\cdot\tilde{\Psi}_\nu$$

changes sign at $b = b_k(0)$, $k \in \mathbb{Z}$. More precisely, we analyze bifurcation at $b_k(0) \geq 0$ using the map G_+ and at $b_k(0) < 0$ using G_-. In the following, we present the main ideas how to verify that 1 is an algebraically simple eigenvalue of $K_{b_k(0)}$ and that the corresponding perturbed eigenvalue $\lambda_b \approx 1$ of K_b for $b \approx b_k(0)$ crosses 1 as b crosses $b_k(0)$. For the existence, algebraic simplicity and continuous dependence of λ_b on b we refer to Kielhöfer's book [13], p. 203.

Algebraic Simplicity.

We adapt the proof of Proposition 9 to the case $\omega = 0$. Assuming $\ker(I - K_{b_k(0)}) = \mathrm{span}\{w\}$ and $v \in \ker(I - K_{b_k(0)})^2 \setminus \ker(I - K_{b_k(0)})$, we have w.l.o.g.

$$w = K_{b_k(0)}w \qquad \text{and} \qquad v = K_{b_k(0)}(v + w). \tag{14}$$

Then, Proposition 4 (c) implies that the profiles satisfy

$$-w'' - \frac{2}{r}w' - \nu w = b_k(0)u_0^2\,w, \quad -v'' - \frac{2}{r}v' - \nu v = b_k(0)u_0^2\,(v + w) \qquad \text{on } \mathbb{R}^3. \tag{15}$$

We let $q(r) := r^2\left(w(r)v'(r) - v(r)w'(r)\right)$ for $r \geq 0$. Using (15), we find

$$q'(r) = -r^2b_k(0)u_0^2(r)w^2(r) \quad (r > 0), \qquad q(0) = 0$$

hence q is nondecreasing if $b_k(0) \leq 0$ and nonincreasing if $b_k(0) \geq 0$. On the other hand, applying $\alpha^{(\nu)}$ to (14), we infer $\beta^{(\nu)}(w) = 0$ and $\beta^{(\nu)}(v) = -\sigma\alpha^{(\nu)}(w)$. Then the asymptotic expansions of v, w due to equation (14) and Proposition 4 (d) imply as $r \to \infty$

$$q(r) = \sigma\cdot\frac{\alpha^{(\nu)}(w)^2}{(4\pi)^2}\sqrt{\nu} + O\left(\frac{1}{r}\right).$$

Since $\alpha^{(\nu)}(w) \neq 0$ by Proposition 6, and since we choose $\sigma = +1$ for $b_k(0) \geq 0$ and $\sigma = -1$ for $b_k(0) < 0$, this contradicts the monotonicity of q. Hence $\ker(I - K_{b_k(0)}) = \ker(I - K_{b_k(0)})^2$.

Perturbation of the eigenvalue.

Throughout the following lines, we consider a perturbed value $b \approx b_k(0), b \neq b_k(0)$ and the corresponding eigenpair with $K_b w_b = \lambda_b w_b$. The latter implies

$$(\lambda_b - 1)\alpha^{(v)}(w_b) = \sigma\beta^{(v)}(w_b) \tag{16}$$

and hence $\lambda_b \neq 1$ due to $\beta^{(v)}(w_b) \neq 0$, see Proposition 8. We recall that

$$\omega_v(b_k(0)u_0^2) \in \pi\mathbb{Z} \qquad \text{and} \qquad \frac{\alpha^{(v)}(w_b)}{\beta^{(v)}(w_b)} = \cot(\omega_v(b\lambda_b^{-1}u_0^2)) \quad (b \neq b_k(0), b \approx b_k(0)) \tag{17}$$

where the second identity can be deduced comparing the expansions in equation (13) and in Corollary 5 resp. Proposition 6.

We now discuss the values $b_k(0) \geq 0$, i.e. $\sigma = +1$. In case $b > b_k(0)$ we show that $\lambda_b > 1$. Assuming $\lambda_b < 1$, we infer from (16) that $\operatorname{sgn} \alpha^{(v)}(w_b) \neq \operatorname{sgn} \beta^{(v)}(w_b)$ and thus $\omega_v(b\lambda_b^{-1}u_0^2) \in \left(-\frac{\pi}{2}, 0\right) + \pi\mathbb{Z}$ due to (17). But since $b\lambda_b^{-1} > b_k(0)$, the monotonicity stated in Proposition 8 implies $\omega_v(b\lambda_b^{-1}u_0^2) \in \omega_v(b_k(0)u_0^2) + \left(0, \frac{\pi}{2}\right) \subseteq \left(0, \frac{\pi}{2}\right) + \pi\mathbb{Z}$, a contradiction. In the same way, for $b < b_k(0)$, we can show that $\lambda_b < 1$. Following the same strategy, we see for $b_k(0) < 0$, i.e. $\sigma = -1$, that $b > b_k(0)$ implies $\lambda_b < 1$ and $b < b_k(0)$ implies $\lambda_b > 1$.

We have thus proved that, as b crosses $b_k(0)$, the perturbed eigenvalue λ_b crosses $\lambda_{b_k(0)} = 1$ and hence the sign of the Leray-Schauder index $\operatorname{ind}_{X_1\times X_1}\left(G_\sigma(\cdot\,, b), (0,0)\right)$ changes at $b = b_k(0)$ for all $k \in \mathbb{Z}$ and for $\sigma \in \{\pm 1\}$ chosen as above.

The case $\tau = 0$.

This is covered by redefining the first components of F resp. G_σ,

$$(w, v, b) \mapsto w - \Psi_\mu \star [w^3 + 3u_0w^2 + 3u_0^2w + b\,(u_0 + w)v^2] - \left[\alpha^{(\mu)}(w) + \beta^{(\mu)}(w)\right] \cdot \tilde{\Psi}_\mu$$

instead of $(w, v, b) \mapsto w - \mathcal{R}_\mu^\tau(w^3 + 3u_0w^2 + 3u_0^2w + b\,(u_0 + w)v^2)$ similar to the modification of the second component in the case $\omega = 0$. The argumentation works as before, which is why we omit the details. □

Proof of Remark 2

(a) The Steps 1 of the proof above in fact show that solutions of (H), (A) bifurcate from $(u_0, 0, b) \in \mathcal{T}_{u_0}$ only if $b = b_k(\omega)$ for $k \in \mathbb{Z}$; Steps 2 show that this is also sufficient. Moreover, since $b_k(\omega)$ is the unique solution of $\omega_v(b\,u_0^2) = \omega + k\pi$, its value does not change when choosing another asymptotic parameter τ in (A).

(b) By Proposition 8, the map $q : \mathbb{R} \to \mathbb{R},\ q(b) := \omega_v(b\,u_0^2)$ is strictly increasing and onto. Having chosen $b_k(\omega) = q^{-1}(\omega + k\pi)$ for $\omega \in [0, \pi), k \in \mathbb{Z}$, we infer strict monotonicity and surjectivity of the map $\mathbb{R} \to \mathbb{R},\ \omega + k\pi \mapsto b_k(\omega)$.

(c) In Steps 2 we have seen that in a neighborhood of the bifurcation point $(u_0, 0, b_k(\omega))$, the continuum $\mathcal{C}_k$ contains only fully nontrivial solutions apart from $(u_0, 0, b_k(\omega))$ itself. In Step 3, we infer for all $(u, v, b) \in \mathcal{C}_k$ from this neighborhood that the asymptotic phase of v satisfies $\omega_v(v^2 + bu^2) = \omega + k\pi$. More generally, $\omega_v(v^2 + bu^2) = \omega + k\pi$ holds on every connected subset of $\mathcal{C}_k$ containing $(u_0, 0, b_k(\omega))$ but no other semitrivial solution with $v = 0$.

(d) By Proposition 6, the (formally derived) initial value problem has a unique radial solution with $c = \varrho_\mu(3u_0^2) \neq 0$ and $\tau_0 = \omega_\mu(3u_0^2) \in [0, \pi)$. □

4 Proof of Theorem 3

We now prove the occurence of bifurcations from the diagonal solution family $\mathfrak{T}_{u_0}$. To this end we first rewrite the system (H) in an equivalent but more convenient way. Looking for solutions $(u, v, b) \in X_1 \times X_1 \times \mathbb{R} \setminus \mathfrak{T}_{u_0}$, we introduce the functions $w_1, w_2 \in X_1$ via

$$u =: u_b + w_1 - w_2, \qquad v =: u_b + w_1 + w_2.$$

A few computations then yield that bifurcation at the point (u_b, u_b, b) occurs if and only if we have bifurcation from the trivial solution of the nonlinear Helmholtz system

$$\begin{cases} -\Delta w_1 - \mu w_1 = (1+b)\big((w_1+u_b)^3 - u_b^3\big) + (3-b)(w_1+u_b)w_2^2 & \text{on } \mathbb{R}^3, \\ -\Delta w_2 - \mu w_2 = (1+b)w_2^3 + (3-b)(w_1+u_b)^2 w_2 & \text{on } \mathbb{R}^3, \end{cases} \tag{18}$$

and the asymptotic conditions (A^{diag}) are equivalent to

$$w_1(x) = c_1 \frac{\sin(|x|\sqrt{\mu}+\tau)}{|x|} + O\left(\frac{1}{|x|^2}\right), \quad w_2(x) = c_2 \frac{\sin(|x|\sqrt{\mu}+\omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \tag{19}$$

as $|x| \to \infty$ for some $c_1, c_2 \in \mathbb{R}$. As in the proof of Theorem 1, the functional analytical setting in the special cases $\omega = 0$ or $\tau = 0$ is different from the general one since a substitute for the operators $\mathcal{R}_\mu^\tau, \mathcal{R}_\mu^\omega$ has to be found, see the definition of G_σ in the proof of Theorem 1. In order to keep the presentation short we only discuss the case $\tau, \omega \in (0, \pi)$ and refer to the proof of Theorem 1 for the modifications in the remaining cases. So we introduce the map $F : X_1 \times X_1 \times (-1, \infty) \to X_1 \times X_1$ via

$$F(w_1, w_2, b) := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \begin{pmatrix} \mathcal{R}_\mu^\tau\Big((1+b)\big((w_1+u_b)^3 - u_b^3\big) + (3-b)(w_1+u_b)w_2^2\Big) \\ \mathcal{R}_\mu^\omega\Big((1+b)w_2^3 + (3-b)(w_1+u_b)^2 w_2\Big) \end{pmatrix}.$$

Then $F(0, 0, b) = 0$ for $b > -1$, $F(\,\cdot\,, b)$ is a compact perturbation of the identity on $X_1 \times X_1$ and it remains to find bifurcation points for $F(w_1, w_2, b) = 0$. First we identify candidates for bifurcation points, i.e. those $b \in (-1, \infty)$ where $\ker DF(0, 0, b)$ is nontrivial. Using

$$DF(0,0,b)[(\phi_1,\phi_2)] = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} - \begin{pmatrix} \mathcal{R}_\mu^\tau(3(1+b)u_b^2\phi_1) \\ \mathcal{R}_\mu^\omega((3-b)u_b^2\phi_2) \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} - \begin{pmatrix} 3\mathbf{R}_\mu^\tau \phi_1 \\ \frac{3-b}{1+b}\cdot \mathbf{R}_\mu^\omega \phi_2 \end{pmatrix},$$

we get that nontrivial kernels occur exactly if $\frac{3-b}{1+b} = b_k(\omega)$ for some $k \in \mathbb{Z}$, cf. Steps 1 in the previous proof. For the analogous result in the Schrödinger case, see Lemma 3.1 [11]. So

$$\ker DF(0,0,b) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ \psi_k \end{pmatrix}\right\} \quad \text{provided } b = \frac{3 - b_k(\omega)}{1 + b_k(\omega)} > -1$$

for some $\psi_k \in X_1 \setminus \{0\}$. Using the algebraic simplicity of ψ_k proved in Proposition 9 we infer exactly as in the proof of Theorem 1 that the transversality condition holds and that the Leray-Schauder index changes at the bifurcation point. So, choosing $\mathfrak{b}_k(\omega) := \frac{3-b_k(\omega)}{1+b_k(\omega)}$ for all $k \in \mathbb{Z}$ with $b_k(\omega) > -1$, the Crandall-Rabinowitz Theorem and Rabinowitz' Global Bifurcation Theorem yield statements (ii) and (i) of the Theorem, respectively. We remark that, to be consistent with the labeling in the Theorem, we might have to shift the index in such way that $b_0(\omega) \le -1 < b_1(\omega)$.

Unboundedness of the components can also be deduced as before. Indeed, assuming that $\mathfrak{C}_k$ is bounded, it returns to $\mathfrak{T}_{u_0}$ at some point $(u_{b^\star}, u_{b^\star}, b^\star) \neq (u_{\mathfrak{b}_k(\omega)}, u_{\mathfrak{b}_k(\omega)}, \mathfrak{b}_k(\omega))$ by Rabinowitz' Theorem. We then infer that the phase $\omega_\mu((1+b)w_2^2 + (3-b)(w_1+u_b)^2)$ cannot be constant along $\mathfrak{C}_k$. Due to Proposition 6 applied to w_2 in (18), this requires the existence of some element $(u, v, b) \in \mathfrak{C}_k$ with $w_2 = \frac{1}{2}(v-u) = 0$, and hence the associated unbounded diagonal family belongs to $\mathfrak{C}_k$. □

5 Proofs of the Results in Section 2

Before proving Proposition 4, we state two auxiliary results. The first one provides a formula for the Fourier transform of radially symmetric functions, see e.g. [14], p. 430.

Lemma 11. *For $f \in X_3$ and $x \in \mathbb{R}^3 \setminus \{0\}$, we have*

$$\hat{f}(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(r)\, \frac{\sin(|x|r)}{|x|r}\, r^2 \,\mathrm{d}r.$$

We denote by $\Phi_\lambda(x) := \frac{e^{i\sqrt{\lambda}|x|}}{4\pi|x|} \overset{(5)}{=} \Psi_\lambda(x) + i\,\tilde{\Psi}_\lambda(x)$ for $\lambda > 0, x \neq 0$ a (complex) fundamental solution of the Helmholtz equation $-\Delta\phi - \lambda\phi = 0$ on $\mathbb{R}^3$. A short calculation using spherical coordinates provides the following pointwise formula for convolutions with kernel Φ_λ.

Lemma 12. *For $f \in X_3$ and $x \in \mathbb{R}^3 \setminus \{0\}$, we have*

$$\begin{aligned}(\Phi_\lambda * f)(x) &= \frac{e^{i\sqrt{\lambda}|x|}}{|x|} \cdot \int_0^{|x|} \frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r} \cdot f(r)\, r^2 \,\mathrm{d}r + \frac{\sin(\sqrt{\lambda}|x|)}{|x|} \cdot \int_{|x|}^\infty \frac{e^{i\sqrt{\lambda}r}}{\sqrt{\lambda}r} \cdot f(r)\, r^2 \,\mathrm{d}r \\ &= \sqrt{\frac{\pi}{2}}\, \hat{f}(\sqrt{\lambda}) \cdot \frac{e^{i\sqrt{\lambda}|x|}}{|x|} + \int_{|x|}^\infty f(r) \cdot \frac{e^{i\sqrt{\lambda}r}\sin(\sqrt{\lambda}|x|) - e^{i\sqrt{\lambda}|x|}\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r|x|}\, r^2 \mathrm{d}r.\end{aligned}$$

5.1 Proof of Proposition 4

We now prove, one by one, the assertions of Proposition 4 for convolutions with Φ_λ in place of $\alpha\Psi_\lambda + \tilde{\alpha}\tilde{\Psi}_\lambda$. The latter (real-valued) case can be deduced from the former via $\Phi_\lambda = \Psi_\lambda + i\,\tilde{\Psi}_\lambda$. Unless stated otherwise, we extend norms defined on spaces of real-valued functions to complex-valued functions $g : \mathbb{R}^3 \to \mathbb{C}$ by considering the respective norm of $|g| : \mathbb{R}^3 \to \mathbb{R}$. (a) is a consequence of Theorem 2.1 in [3]. The solution properties stated in (c) can be verified by direct computation.

Step 1: *Proof of (b), first part. Continuity.*

Due to the continuous embedding $X_3 \subseteq L^{\frac{4}{3}}_{\mathrm{rad}}(\mathbb{R}^3)$, see (6), the convolution is well-defined for $f \in X_3$. Using Young's convolution inequality, we get

$$\begin{aligned}|(\Phi_\lambda * f)(x)| &\le \left\|\left(\mathbb{1}_{B_1(0)}\Phi_\lambda\right) * f\right\|_{L^\infty(\mathbb{R}^3)} + \left\|\left(\mathbb{1}_{\mathbb{R}^3\setminus B_1(0)}\Phi_\lambda\right) * f\right\|_{L^\infty(\mathbb{R}^3)} \\ &\le \left\|\mathbb{1}_{B_1(0)}\Phi_\lambda\right\|_{L^1(\mathbb{R}^3)} \|f\|_{L^\infty(\mathbb{R}^3)} + \left\|\mathbb{1}_{\mathbb{R}^3\setminus B_1(0)}\Phi_\lambda\right\|_{L^4(\mathbb{R}^3)} \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \\ &\le \|f\|_{L^\infty(\mathbb{R}^3)} \int_{B_1(0)} \frac{\mathrm{d}y}{4\pi|y|} + \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \left(\int_{\mathbb{R}^3\setminus B_1(0)} \frac{\mathrm{d}y}{(4\pi|y|)^4} \right)^{\frac{1}{4}} \\ &\overset{(6)}{\le} C_1 \cdot \|f\|_{X_3}\end{aligned}$$

for some $C_1 \ge 0$. Next, by means of Lemma 12, we estimate for $x \in \mathbb{R}^3 \setminus \{0\}$

$$\begin{aligned}||x| \cdot (\Phi_\lambda * f)(x)| &= \left| e^{i\sqrt{\lambda}|x|} \cdot \int_0^{|x|} \frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r} \cdot f(r)\, r^2 \,\mathrm{d}r + \sin(\sqrt{\lambda}|x|) \cdot \int_{|x|}^\infty \frac{e^{i\sqrt{\lambda}r}}{\sqrt{\lambda}r} \cdot f(r)\, r^2 \,\mathrm{d}r \right| \\ &\le \int_0^{|x|} \frac{1}{\sqrt{\lambda}r} \cdot \frac{\|f\|_{X_3}}{(1+r^2)^{\frac{3}{2}}}\, r^2 \,\mathrm{d}r + \int_{|x|}^\infty \frac{1}{\sqrt{\lambda}r} \cdot \frac{\|f\|_{X_3}}{(1+r^2)^{\frac{3}{2}}}\, r^2 \,\mathrm{d}r\end{aligned}$$

$$= C_2 \cdot \|f\|_{X_3}$$

with some $C_2 \geq 0$. Combining both estimates, we have $\|\Phi_\lambda * f\|_{X_1} \leq [C_1 + C_2] \cdot \|f\|_{X_3}$.

Step 2: *Proof of (d). Asymptotics of w and w'.*

Given $f \in X_3$, we let $w := \Phi_\lambda * f$. Then for $r = |x| > 0$, Lemma 12 implies

$$\begin{aligned} \left| w(r) - \sqrt{\frac{\pi}{2}}\hat{f}(\sqrt{\lambda})\frac{e^{i\sqrt{\lambda}r}}{r} \right| &= \left| \int_r^\infty f(s)\frac{e^{i\sqrt{\lambda}s}\sin(\sqrt{\lambda}r) - e^{i\sqrt{\lambda}r}\sin(\sqrt{\lambda}s)}{\sqrt{\lambda}sr} s^2 ds \right| \\ &\leq \int_r^\infty \frac{\|f\|_{X_3}}{(1+s^2)^{\frac{3}{2}}} \cdot \frac{2}{\sqrt{\lambda}r} s ds \\ &= \|f\|_{X_3} \cdot \frac{2}{\sqrt{\lambda}r^2}. \end{aligned} \tag{20}$$

To understand the asymptotic behavior of the radial derivative w', a short calculation shows that the auxiliary function $\delta(r) := r \cdot w(r) - \sqrt{\frac{\pi}{2}}\hat{f}(\sqrt{\lambda}) \cdot e^{i\sqrt{\lambda}r}$ satisfies

$$\delta(r) = O\left(\frac{1}{r}\right), \quad \delta''(r) = -\lambda\delta(r) - r \cdot f(r) = O\left(\frac{1}{r}\right) \quad \text{as } r \to \infty.$$

Then for $r > 0$, we find $\tau_r \in (0, 1)$ with $\delta(r + 1) = \delta(r) + \delta'(r) + \frac{1}{2}\delta''(r + \tau_r)$, whence also $\delta'(r) = O\left(\frac{1}{r}\right)$. This shows the asserted properties of w' since

$$r \cdot w'(r) = i\sqrt{\lambda} \cdot \sqrt{\frac{\pi}{2}}\hat{f}(\sqrt{\lambda}) \cdot e^{i\sqrt{\lambda}r} - w(r) + O\left(\frac{1}{r}\right) \quad \text{as } r \to \infty.$$

As a consequence of equation (20), we derive the formula stated for $\tilde{\Psi}_\lambda * f$. Due to (c), $\tilde{\Psi}_\lambda * f$ is a radial solution of the homogeneous Helmholtz equation $-\Delta w - \lambda w = 0$ on $\mathbb{R}^3$ and hence a scalar multiple of $\tilde{\Psi}_\lambda$ itself. The asymptotics in (20) justify the asserted constant.

Step 3: *Proof of (b), second part. Compactness.*

We consider a bounded sequence $(f_n)_n$ in the space X_3 and aim to prove convergence of a subsequence of $(u_n)_n$, $u_n := \Phi_\lambda * f_n$, in the space X_1. First, due to the continuous embeddings into reflexive L^p spaces stated in (6), we can pass to a subsequence with

$$f_{n_k} \rightharpoonup f \text{ weakly in } L^4(\mathbb{R}^3) \cap L^{\frac{4}{3}}(\mathbb{R}^3), \quad u_{n_k} \rightharpoonup u \text{ weakly in } L^4(\mathbb{R}^3)$$

for some $f \in L^4_{\text{rad}}(\mathbb{R}^3) \cap L^{\frac{4}{3}}_{\text{rad}}(\mathbb{R}^3)$, $u \in L^4_{\text{rad}}(\mathbb{R}^3)$. Then the regularity properties in Proposition A.1 in [3] and the Rellich-Kondrachov Embedding Theorem 6.3 in [15] allow to extract a subsequence with $u_{n_k} \to u$ strongly in $C^1_{\text{loc}}(\mathbb{R}^3)$, in particular for any $R > 0$

$$\left\| \mathbb{1}_{B_R(0)} \cdot u_{n_k} - \mathbb{1}_{B_R(0)} \cdot u_{n_l} \right\|_{X_1} \to 0 \quad \text{as } k, l \to \infty. \tag{21}$$

On the other hand, on the unbounded set $\mathbb{R}^3 \backslash B_R(0)$, convergence in X_1 follows essentially from the asymptotic expansion in equation (20) where w.l.o.g. $\hat{f}_{n_k}(\sqrt{\lambda}) \to \hat{f}(\sqrt{\lambda})$ as $k \to \infty$. Then,

$$\begin{aligned} &\left\| \mathbb{1}_{\mathbb{R}^3\backslash B_R(0)} \cdot u_{n_k} - \mathbb{1}_{\mathbb{R}^3\backslash B_R(0)} \cdot u_{n_l} \right\|_{X_1} \\ &\overset{(20)}{\leq} \sup_{|x|\geq R} \left[\sqrt{\frac{\pi}{2}} |\hat{f}_{n_k}(\sqrt{\lambda}) - \hat{f}_{n_l}(\sqrt{\lambda})| \frac{(1+|x|^2)^{\frac{1}{2}}}{|x|} + \frac{2(1+|x|^2)^{\frac{1}{2}}}{\sqrt{\lambda}|x|^2} \cdot \|f_{n_k} - f_{n_l}\|_{X_3} \right] \\ &\leq \sqrt{\pi} |\hat{f}_{n_k}(\sqrt{\lambda}) - \hat{f}_{n_l}(\sqrt{\lambda})| + \frac{1}{R} \cdot \frac{4\sqrt{2}}{\sqrt{\lambda}} \sup_{n\in\mathbb{N}} \|f_n\|_{X_3}. \end{aligned} \tag{22}$$

Thus given $\varepsilon > 0$, we can choose $R(\varepsilon) > 0$ large enough and $k(\varepsilon) \in \mathbb{N}$ such that (21) and (22) imply $\|u_{n_k} - u_{n_l}\|_{X_1} < \varepsilon$ for all $k, l \geq k(\varepsilon)$. Hence $(u_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence in X_1. □

5.2 Proof of Proposition 6

Let $g \in X_2$. Then the profile $w : [0,\infty) \to \mathbb{R}$ is a (global) solution of the initial value problem (9) if and only if $y : [0,\infty) \to \mathbb{R}$, $y(r) = r \cdot w(r)$ solves

$$\begin{cases} -y'' - \lambda y = g(r) \cdot y & \text{on } (0,\infty), \\ y(0) = 0,\, y'(0) = 1. \end{cases} \tag{23}$$

Moreover, $w \in X_1$ if y is bounded. Global existence and uniqueness of such $y \in C^2([0,\infty))$ are consequences of the Picard-Lindelöf Theorem and of Gronwall's Lemma since $g \in L^1([0,\infty))$. We apply the Prüfer transformation. Since $y \not\equiv 0$, uniqueness implies that $y(r)^2 + y'(r)^2 > 0$ for all $r \geq 0$. We thus parametrize using polar coordinates in the phase space

$$y(r) = \varrho(r) \cdot \sin(\phi(r)\sqrt{\lambda}), \qquad y'(r) = \varrho(r) \cdot \sqrt{\lambda}\cos(\phi(r)\sqrt{\lambda}) \qquad (r \geq 0) \tag{24}$$

with functions $\varrho : [0,\infty) \to (0,\infty)$ and $\phi : [0,\infty) \to \mathbb{R}$. A short calculation shows that we thus obtain a solution of (23) if and only if ϱ and ϕ satisfy the first-order system

$$\begin{cases} (\log \varrho)' = -\frac{g(r)}{2\sqrt{\lambda}} \sin(2\phi\sqrt{\lambda}) & \text{on } (0,\infty), \\ \phi' = 1 + \frac{g(r)}{\lambda} \sin^2(\phi\sqrt{\lambda}) & \text{on } (0,\infty), \\ \varrho(0) = \frac{1}{\sqrt{\lambda}}, \quad \phi(0) = 0. \end{cases} \tag{25}$$

Equivalently, for $r \geq 0$,

$$\begin{aligned} \varrho(r) &= \frac{1}{\sqrt{\lambda}} \exp\left(-\int_0^r \frac{g(t)}{2\sqrt{\lambda}} \sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t\right), \\ \phi(r) &= r + \int_0^r \frac{g(t)}{\lambda} \sin^2(\phi(t)\sqrt{\lambda})\,\mathrm{d}t. \end{aligned} \tag{26}$$

We will frequently refer to the estimate

$$\forall\, r \geq 0 \qquad \frac{1}{\sqrt{\lambda}} \exp\left(\left|\int_0^r \frac{g(t)}{2\sqrt{\lambda}} \sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t\right|\right) \leq \frac{1}{\sqrt{\lambda}} \exp\left(\frac{\pi}{4\sqrt{\lambda}}\, \|g\|_{X_2}\right) =: C_g. \tag{27}$$

Indeed, (26) and (27) immediately yield boundedness of y and moreover convergence of the integrals in

$$\omega_\lambda(g) := \int_0^\infty \frac{g(t)}{\sqrt{\lambda}} \sin^2(\phi(t)\sqrt{\lambda})\,\mathrm{d}t \;\text{ and }\; \varrho_\lambda(g) := \frac{1}{\sqrt{\lambda}} \cdot \exp\left(-\int_0^\infty \frac{g(t)}{2\sqrt{\lambda}} \sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t\right).$$

Thus $\varrho_\lambda(g) > 0$, and we verify the asserted asymptotic behavior of y as $r \to \infty$:

$$\begin{aligned} &\left|y(r) - \varrho_\lambda(g)\sin(r\sqrt{\lambda} + \omega_\lambda(g))\right| \\ &\overset{(24)}{=} \left|\varrho(r)\sin(\phi(r)\sqrt{\lambda}) - \varrho_\lambda(g)\sin(r\sqrt{\lambda} + \omega_\lambda(g))\right| \\ &\leq \left|[\varrho(r) - \varrho_\lambda(g)]\,\sin(\phi(r)\sqrt{\lambda})\right| + \left|\varrho_\lambda(g)\left[\sin(\phi(r)\sqrt{\lambda}) - \sin(r\sqrt{\lambda} + \omega_\lambda(g))\right]\right| \\ &\overset{(27)}{\leq} C_g \cdot \left(\left|\frac{\varrho(r)}{\varrho_\lambda(g)} - 1\right| + \left|\sin(\phi(r)\sqrt{\lambda}) - \sin(r\sqrt{\lambda} + \omega_\lambda(g))\right|\right) \end{aligned}$$

where we estimate both terms as follows, using $|g(r)| \leq \|g\|_{X_2} \cdot (1 + r^2)^{-1}$ for $r > 0$,

$$\left|\frac{\varrho(r)}{\varrho_\lambda(g)} - 1\right| + \left|\sin(\phi(r)\sqrt{\lambda}) - \sin(r\sqrt{\lambda} + \omega_\lambda(g))\right|$$

$$\overset{(26)}{\le} \left| \exp\left(\int_r^\infty \frac{g(t)}{2\sqrt{\lambda}} \sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t \right) - 1 \right| + \left| \phi(r)\sqrt{\lambda} - r\sqrt{\lambda} - \omega_\lambda(g) \right|$$
$$\overset{(26),(27)}{\le} C_g \cdot \left| \int_r^\infty \frac{g(t)}{2\sqrt{\lambda}} \sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t \right| + \left| \int_r^\infty \frac{g(t)}{\sqrt{\lambda}} \sin^2(\phi(t)\sqrt{\lambda})\,\mathrm{d}t \right|$$
$$\le \left(\frac{C_g}{2} + 1 \right) \cdot \frac{\|g\|_{X_2}}{\sqrt{\lambda}} \cdot \frac{1}{r}.$$

Thus $y(r) - \varrho_\lambda(g)\sin(r\sqrt{\lambda} + \omega_\lambda(g)) = O\left(\frac{1}{r}\right)$, and similarly $y'(r) - \varrho_\lambda(g)\sqrt{\lambda}\cos(r\sqrt{\lambda} + \omega_\lambda(g)) = O\left(\frac{1}{r}\right)$ as $r \to \infty$. Since $y(r) = r \cdot w(r)$, all assertions are proved. □

5.3 Proof of Proposition 7

We consider g_n, g_0 in X_2 with $g_n \to g_0$ in X_2 and aim to show that $\omega_\lambda(g_n) \to \omega_\lambda(g_0)$. By $\phi_n \in C^1((0,\infty)) \cap C([0,\infty))$ we denote the unique solution of

$$\phi_n' = 1 + \frac{g_n(r)}{\lambda} \sin^2(\phi_n\sqrt{\lambda}), \qquad \phi_n(0) = 0.$$

Then we have pointwise convergence, $\phi_n(r) \to \phi_0(r)$ for all $r \ge 0$. Indeed, let us fix any $R > 0$ and estimate for $0 \le r \le R$ and $n \in \mathbb{N}$

$$\left|\phi_n(r) - \phi_0(r)\right| = \left| \int_0^r \frac{g_n(t)}{\lambda} \sin^2(\phi_n(t)\sqrt{\lambda}) - \frac{g_0(t)}{\lambda} \sin^2(\phi_0(t)\sqrt{\lambda})\,\mathrm{d}t \right|$$
$$\le \frac{1}{\lambda} \int_0^r \left|g_n(t) - g_0(t)\right|\,\mathrm{d}t + \frac{1}{\lambda} \int_0^r \left|g_0(t)\right| \cdot \left|\sin^2(\phi_n(t)\sqrt{\lambda}) - \sin^2(\phi_0(t)\sqrt{\lambda})\right|\,\mathrm{d}t$$
$$\le \frac{1}{\lambda} \int_0^\infty \|g_n - g_0\|_{X_2} \frac{\mathrm{d}t}{1+t^2} + \frac{2\,\|g_0\|_\infty}{\sqrt{\lambda}} \int_0^r \left|\phi_n(t) - \phi_0(t)\right|\,\mathrm{d}t$$
$$\le \frac{\pi}{2\lambda} \|g_n - g_0\|_{X_2} + \frac{2\,\|g_0\|_\infty}{\sqrt{\lambda}} \int_0^r \left|\phi_n(t) - \phi_0(t)\right|\,\mathrm{d}t.$$

Thus, by Gronwall's Lemma, we have for $0 \le r \le R$

$$\left|\phi_n(r) - \phi_0(r)\right| \le \frac{\pi}{2\lambda} \|g_n - g_0\|_{X_2} \cdot \mathrm{e}^{\frac{2\|g_0\|_\infty}{\sqrt{\lambda}} r} \le \frac{\pi}{2\lambda} \|g_n - g_0\|_{X_2} \cdot \mathrm{e}^{\frac{2\|g_0\|_\infty}{\sqrt{\lambda}} R}.$$

Since $g_n \to g_0$ in X_2, we conclude $\phi_n \to \phi_0$ locally uniformly on $[0,\infty)$, in particular pointwise. Now we can deduce the convergence of the asymptotic phase,

$$\omega_\lambda(g_n) = \frac{1}{\sqrt{\lambda}} \int_0^\infty g_n(r)\sin^2(\phi_n(r)\sqrt{\lambda})\,\mathrm{d}r \to \frac{1}{\sqrt{\lambda}} \int_0^\infty g_0(r)\sin^2(\phi_0(r)\sqrt{\lambda})\,\mathrm{d}r = \omega_\lambda(g_0),$$

which follows by dominated convergence since $\sup_{n\in\mathbb{N}} \|g_n\|_{X_2} < \infty$. □

5.4 Proof of Proposition 8

Let us first recall that, given the assumptions of Proposition 8, equation (10) implies for $b \in \mathbb{R}$

$$\omega_\lambda(b\,u_0^2) = \frac{b}{\sqrt{\lambda}} \int_0^\infty u_0^2(r)\sin^2(\phi_b(r)\sqrt{\lambda})\,\mathrm{d}r$$

where ϕ_b satisfies $\phi_b' = 1 + \frac{b}{\lambda}u_0^2(r)\sin^2(\phi_b\sqrt{\lambda})$ on $(0,\infty)$, $\phi_b(0) = 0$. We immediately see that $\omega_\lambda(0) = 0$ and $\operatorname{sgn}\omega_\lambda(b\,u_0^2) = \operatorname{sgn}(b)$ for all $b \in \mathbb{R}\setminus\{0\}$. Further, continuity of $b \mapsto \omega_\lambda(b\,u_0^2)$ is a consequence of Proposition 7. The assertions are proved once we show that $b \mapsto \omega_\lambda(b\,u_0^2)$ is strictly increasing with $\omega_\lambda(b\,u_0^2) \to \pm\infty$ as $b \to \pm\infty$.

Step 1: *Strict monotonicity.* We let $b_1 < b_2$, define

$$\chi(r) := \begin{cases} \frac{\sin^2(\phi_{b_2}(r)\sqrt{\lambda})-\sin^2(\phi_{b_1}(r)\sqrt{\lambda})}{\phi_{b_2}(r)\sqrt{\lambda}-\phi_{b_1}(r)\sqrt{\lambda}} & \text{if } \phi_{b_2}(r) \neq \phi_{b_1}(r), \\ 2\sin(\phi_{b_1}(r)\sqrt{\lambda})\cos(\phi_{b_1}(r)\sqrt{\lambda}) & \text{else} \end{cases}$$

and observe that χ is bounded with $0 \le |\chi(r)| \le 2$ and continuous. $\psi := \phi_{b_2} - \phi_{b_1}$ satisfies

$$\psi' = \frac{b_2 - b_1}{\lambda}u_0^2(r)\sin^2(\phi_{b_2}(r)\sqrt{\lambda}) + \frac{b_1}{\sqrt{\lambda}}u_0^2(r)\chi(r)\psi, \qquad \psi(0) = 0.$$

The unique solution is given by the Variation of Constants formula. We have

$$\omega_\lambda(b_2\,u_0^2) - \omega_\lambda(b_1\,u_0^2) = \sqrt{\lambda}\lim_{r\to\infty}\psi(r) = \int_0^\infty \frac{b_2 - b_1}{\sqrt{\lambda}}u_0^2(\varrho)\sin^2(\phi_{b_2}(\varrho)\sqrt{\lambda})\mathrm{e}^{\int_\varrho^\infty \frac{b_1}{\sqrt{\lambda}}u_0^2(t)\chi(t)\,\mathrm{d}t}\,\mathrm{d}\varrho > 0$$

since the integrand is nonnegative and not identically zero.

Step 2: *Asymptotic behavior as $b \to \infty$.*

By the uniqueness statement of the Picard-Lindelöf Theorem, $u_0 \not\equiv 0$ requires $u_0(0) \neq 0$. We can thus choose $r_0 > 0$ with

$$\frac{1}{2}u_0^2(0) < u_0^2(r) < \frac{3}{2}u_0^2(0) \qquad \text{for all } r \in [0, r_0]. \tag{28}$$

To keep notation short, we let in this paragraph $\xi := \frac{1}{2\lambda}u_0^2(0)$. We have for $b > 0$

$$\phi_b' = 1 + \frac{b}{\lambda}u_0^2(r)\sin^2(\phi_b\sqrt{\lambda}) \overset{(28)}{\geq} 1 + b\cdot\xi\sin^2(\phi_b\sqrt{\lambda}) \quad \text{on } [0, r_0], \qquad \phi_b(0) = 0.$$

We now study the modified initial value problem

$$\psi_b' = 1 + b\cdot\xi\sin^2(\psi_b\sqrt{\lambda}) \quad \text{on } [0, r_0], \qquad \psi_b(0) = 0.$$

For $0 \le r \le r_0$ with $r \notin \frac{\pi}{2} + \pi\mathbb{Z}$, its unique solution is given by the expression

$$\psi_b(r) = \frac{1}{\sqrt{\lambda}}\left[n\pi + \arctan\left(\frac{\tan\left(r\sqrt{\lambda}\sqrt{1+b\cdot\xi}\right)}{\sqrt{1+b\cdot\xi}}\right)\right] \quad \text{for } \left|\sqrt{1+b\cdot\xi}\sqrt{\lambda}\,r - n\pi\right| < \frac{\pi}{2}$$

where $n \in \mathbb{N}_0$. We deduce immediately $\psi_b(r_0) \to \infty$ as $b \to \infty$. Since by construction $\phi_b \ge \psi_b$ on $[0, r_0]$, this implies $\phi_b(r_0) \to \infty$ as $b \to \infty$. Substituting $t := \sqrt{\lambda}\phi_b(r)$, we estimate

$$\begin{aligned} \omega_\lambda(b\,u_0^2) &= \frac{b}{\sqrt{\lambda}}\int_0^\infty u_0^2(r)\sin^2(\phi_b(r)\sqrt{\lambda})\,\mathrm{d}r \\ &\overset{(28)}{\geq} b\cdot\xi\sqrt{\lambda}\int_0^{r_0}\sin^2(\phi_b(r)\sqrt{\lambda})\,\mathrm{d}r \\ &= b\cdot\xi\int_0^{\sqrt{\lambda}\phi_b(r_0)}\sin^2(t)\,\frac{\mathrm{d}t}{\phi_b'(\phi_b^{-1}(\lambda^{-\frac{1}{2}}t))} \end{aligned}$$

$$\overset{(28)}{\geq} \frac{b\cdot\xi}{1+3b\cdot\xi} \int_0^{\sqrt{\lambda}\phi_b(r_0)} \sin^2(t)\,\mathrm{d}t$$

and hence $\phi_b(r_0) \to \infty$ implies $\omega_\lambda(b\,u_0^2) \to \infty$ as $b \to \infty$.

Step 3: *Asymptotic behavior as $b \to -\infty$.*

For $b < -1$, we introduce

$$r_b := \max\left\{ r > 0 \;\middle|\; \phi_b(r)\sqrt{\lambda} = \arcsin(|b|^{-\frac{1}{4}}) \right\},$$

which is well-defined due to $1 - \frac{|b|}{\lambda}\frac{\|u_0\|_{X_1}^2}{1+r^2} \leq \phi_b' \leq 1$ and $\phi_b(0) = 0$. In particular, we have

$$\phi_b(r_b)\sqrt{\lambda} = \arcsin(|b|^{-\frac{1}{4}}) \quad \text{and} \quad \phi_b(r)\sqrt{\lambda} > \arcsin(|b|^{-\frac{1}{4}}) \text{ for all } r > r_b. \tag{29}$$

We prove below that $r_b \to \infty$ as $b \to -\infty$. Then for $r \geq r_b$, equation (29) and $\phi_b' \leq 1$ imply

$$\phi_b(r)\sqrt{\lambda} \leq \phi_b(r_b)\sqrt{\lambda} + (r - r_b)\sqrt{\lambda} = r\sqrt{\lambda} + \left(\arcsin(|b|^{-\frac{1}{4}}) - r_b\sqrt{\lambda}\right).$$

Then the asymptotic phase satisfies

$$\omega_\lambda(b\,u_0^2) = \sqrt{\lambda}\cdot \lim_{r\to\infty}(\phi_b(r) - r) \leq \arcsin(|b|^{-\frac{1}{4}}) - r_b\sqrt{\lambda} \to -\infty \quad \text{as } b \to -\infty.$$

It remains to prove that $r_b \to \infty$ as $b \to -\infty$. We assume for contradiction that we find a subsequence $(b_k)_{k\in\mathbb{N}}$ and $\tilde{r} > 0$ with $b_k \searrow -\infty$, $r_{b_k} \to \tilde{r}$ as $k \to \infty$. Then, since $\phi_{b_k}' \leq 1$ and due to equation (29), we have for sufficiently large $k \in \mathbb{N}$

$$\arcsin(|b_k|^{-\frac{1}{4}}) \leq \phi_{b_k}(r)\sqrt{\lambda} \leq \arcsin(|b_k|^{-\frac{1}{4}}) + 1 < \frac{\pi}{2} \qquad \text{for } r_{b_k} \leq r \leq r_{b_k} + \frac{1}{\sqrt{\lambda}}. \tag{30}$$

We conclude $\sin(\phi_{b_k}(r)\sqrt{\lambda}) \geq |b_k|^{-\frac{1}{4}}$ and hence as $k \to \infty$

$$\begin{aligned}
\phi_{b_k}\left(r_{b_k} + \frac{1}{\sqrt{\lambda}}\right) &= \phi_{b_k}(r_{b_k}) + \int_0^{\frac{1}{\sqrt{\lambda}}} \phi_{b_k}'(r_{b_k} + t)\,\mathrm{d}t \\
&\overset{(29)}{=} \frac{1}{\sqrt{\lambda}}\arcsin(|b_k|^{-\frac{1}{4}}) + \int_0^{\frac{1}{\sqrt{\lambda}}} \left[1 - \frac{|b_k|}{\lambda}u_0^2(r_{b_k}+t)\sin^2(\phi_{b_k}(r_{b_k}+t)\sqrt{\lambda})\right]\,\mathrm{d}t \\
&\overset{(30)}{\leq} \frac{2}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} - \frac{\sqrt{|b_k|}}{\lambda}\cdot \int_0^{\frac{1}{\sqrt{\lambda}}} u_0^2(r_{b_k}+t)\,\mathrm{d}t \\
&\to -\infty
\end{aligned}$$

since $u_0^2 > 0$ almost everywhere and $r_{b_k} \to \tilde{r}$. On the other hand, for every $k \in \mathbb{N}$, the differential equation $\phi' = 1 + \frac{b_k}{\lambda}u_0^2(r)\sin^2(\phi\sqrt{\lambda})$ states that $\phi_{b_k}(r) = 0$ implies $\phi_{b_k}'(r) = 1$. Thus ϕ_{b_k} cannot attain negative values, which contradicts the limit calculated before. □

5.5 Proof of Proposition 9

For $\omega \in (0, \pi)$ and $\lambda > 0$, we compute the spectrum of

$$\mathbf{R}_\lambda^\omega : X_1 \to X_1, \quad w \mapsto \mathcal{R}_\lambda^\omega[u_0^2\,w] = \left(\Psi_\lambda + \cot(\omega)\tilde{\Psi}_\lambda\right) \star [u_0^2\,w].$$

Compactness of $\mathbf{R}^\omega_\lambda$ is a consequence of Proposition 4 (b). Then immediately $\sigma(\mathbf{R}^\omega_\lambda) = \{0\} \cup \sigma_p(\mathbf{R}^\omega_\lambda)$ with discrete eigenvalues of finite multiplicity.

Step 1: *Eigenvalues.*

We find the eigenfunctions of $\mathbf{R}^\omega_\lambda$, that is, we look for such $\eta \in \mathbb{R}, \eta \neq 0$ and nontrivial $w \in X_1$ that $\mathbf{R}^\omega_\lambda w = \eta \cdot w$. Corollary 5 implies that this is equivalent to $\eta \in \mathbb{R}, \eta \neq 0$ and nontrivial $w \in X_1 \cap C^2(\mathbb{R}^3)$,

$$\begin{aligned} -\Delta w - \lambda w &= \frac{1}{\eta} \cdot u_0^2(x)\, w \ \text{ on } \mathbb{R}^3 \\ \text{with } w(x) &= c\, \frac{\sin(|x|\sqrt{\lambda} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \ \text{as } |x| \to \infty \end{aligned}$$

for some $c \in \mathbb{R}$. By Proposition 6, such an eigenfunction exists if and only if $\omega_\lambda\left(\frac{1}{\eta}\, u_0^2\right) = \omega + k\pi$ for some $k \in \mathbb{Z}$; in this case, $c \neq 0$ and every eigenspace is one-dimensional since the radially symmetric solution w is unique up to multiplication by a constant. Since we have seen in Proposition 8 that $\mathbb{R} \to \mathbb{R},\ b \mapsto \omega_\lambda(b\, u_0^2)$ is strictly increasing and onto, we can define $b_k(\omega, \lambda, u_0^2)$ via $\omega_\lambda(b_k(\omega, \lambda, u_0^2)\, u_0^2) = \omega + k\pi$ for all $k \in \mathbb{Z}$, and conclude

$$\sigma_p(\mathbf{R}^\omega_\lambda) = \left\{ \frac{1}{b_k(\omega, \lambda, u_0^2)} \,\middle|\, k \in \mathbb{Z} \right\}.$$

Step 2: *Simplicity.*

It remains to show that the eigenvalues are algebraically simple. We consider an eigenvalue $\eta := \frac{1}{b_k(\omega,\lambda,u_0^2)}$ of $\mathbf{R}^\omega_\lambda$ with eigenspace $\ker\left(\mathbf{R}^\omega_\lambda - \eta I_{X_1}\right) = \operatorname{span}\{w\}$. We have to prove that

$$\ker\left(\mathbf{R}^\omega_\lambda - \eta I_{X_1}\right)^2 = \ker\left(\mathbf{R}^\omega_\lambda - \eta I_{X_1}\right).$$

So let now $v \in \ker\left(\mathbf{R}^\omega_\lambda - \eta I_{X_1}\right)^2$. We assume for contradiction that $v \notin \ker\left(\mathbf{R}^\omega_\lambda - \eta I_{X_1}\right)$. By assumption on v, we have $\mathbf{R}^\omega_\lambda v - \eta v \in \ker\left(\mathbf{R}^\omega_\lambda - \eta I_{X_1}\right) \setminus \{0\}$, and since $\eta \neq 0$ we may assume w.l.o.g. $\mathbf{R}^\omega_\lambda v - \eta v = -\eta w = -\mathbf{R}^\omega_\lambda w$. Then by Proposition 4 $v, w \in C^2(\mathbb{R}^3)$ as well as

$$-w'' - \frac{2}{r} w' - \lambda w = \frac{1}{\eta}\, u_0^2(r) \cdot w, \quad -v'' - \frac{2}{r} v' - \lambda v = \frac{1}{\eta}\, u_0^2(r) \cdot (v + w) \qquad \text{on } (0, \infty). \tag{31}$$

Furthermore, Proposition 4 (d) implies

$$\begin{aligned} w(r) &= c_1 \frac{\sin(r\sqrt{\lambda} + \omega)}{r} + O\left(\frac{1}{r^2}\right), \quad w'(r) = c_1\sqrt{\lambda}\, \frac{\cos(r\sqrt{\lambda} + \omega)}{r} + O\left(\frac{1}{r^2}\right), \\ v(r) &= c_2 \frac{\sin(r\sqrt{\lambda} + \omega)}{r} + O\left(\frac{1}{r^2}\right), \quad v'(r) = c_2\sqrt{\lambda}\, \frac{\cos(r\sqrt{\lambda} + \omega)}{r} + O\left(\frac{1}{r^2}\right) \end{aligned} \tag{32}$$

for some $c_1, c_2 \in \mathbb{R}$. Let us define $q(r) = r^2(w'(r)v(r) - v'(r)w(r))$ for $r \geq 0$. Then, using (31), we find $q'(r) = \frac{1}{\eta}\, r^2 u_0^2(r) \cdot w^2(r)$ for $r \geq 0$. Hence q is monotone on $[0, \infty)$ with $q(0) = 0$. On the other hand, the asymptotic expansions in (32) imply $q(r) = O\left(\frac{1}{r}\right)$ as $r \to \infty$. We conclude $q(r) = 0$ for $r \geq 0$. Since all zeros of w are simple, one can deduce that $v(r) = c \cdot w(r)$ for all $r \geq 0$ and some $c \in \mathbb{R}$, and thus $v \in \ker\left(\mathbf{R}^\omega_\lambda - \eta I_{X_1}\right)$, a contradiction. □

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