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Sign-changing multi-bump solutions for the Chern-Simons-Schrödinger equations in $\mathbb{R}^2$

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Abstract: In this paper we consider the nonlinear Chern-Simons-Schrödinger equations with general nonlinearity

$$-\Delta u + \lambda V(|x|)u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds \right) u = f(u), \quad x \in \mathbb{R}^2,$$

where $\lambda > 0$, V is an external potential and

$$h(s) = \frac{1}{2} \int_0^s r u^2(r) dr = \frac{1}{4\pi} \int_{B_s} u^2(x) dx$$

is the so-called Chern-Simons term. Assuming that the external potential V is nonnegative continuous function with a potential well $\Omega := \text{int} V^{-1}(0)$ consisting of $k + 1$ disjoint components $\Omega_0, \Omega_1, \Omega_2, \dots, \Omega_k$, and the nonlinearity f has a general subcritical growth condition, we are able to establish the existence of sign-changing multi-bump solutions by using variational methods. Moreover, the concentration behavior of solutions as $\lambda \rightarrow +\infty$ are also considered.

Keywords: Chern-Simons-Schrödinger equations; sign-changing solution; potential well; concentration behavior

MSC: 35J20; 58E50

1 Introduction and main results

In this paper we are interested in the following nonlinear Schrödinger system with the gauge field

$$\begin{cases} iD_0\phi + (D_1D_1 + D_2D_2)\phi + g(\phi) = 0, \\ \partial_0A_1 - \partial_1A_0 = -\text{Im}(\bar{\phi}D_2\phi), \\ \partial_0A_2 - \partial_2A_0 = \text{Im}(\bar{\phi}D_1\phi), \\ \partial_1A_2 - \partial_2A_1 = -\frac{1}{2}|\phi|^2, \end{cases} \quad (1.1)$$

where i denotes the imaginary unit, $\partial_0 = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$ for $(t, x_1, x_2) \in \mathbb{R}^{1+2}$, $\phi : \mathbb{R}^{1+2} \rightarrow \mathbb{C}$ is the complex scalar field, $A_\kappa : \mathbb{R}^{1+2} \rightarrow \mathbb{R}$ is the gauge field and $D_\kappa = \partial_\kappa + iA_\kappa$ is the covariant derivative for $\kappa = 0, 1, 2$. This model (1.1) was first proposed and studied in [22–24], and is sometimes called the Chern-Simons-Schrödinger equations. The two-dimensional Chern-Simons-Schrödinger equations is a nonrelativistic quantum model describing the dynamics of a large number of particles in the plane, which interact both

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directly and via a self-generated electromagnetic field. Moreover, it describes an external uniform magnetic field which is of great phenomenological interest for applications of Chern-Simons theory to the quantum Hall effect.

As usual in Chern-Simons theory, system (1.1) is invariant under gauge transformation

$$\phi \rightarrow \phi e^{i\chi}, \quad A_\kappa \rightarrow A_\kappa - \partial_\kappa \chi$$

for any arbitrary C^∞ function χ . The existence of standing wave solutions for system (1.1) with power type nonlinearity, that is, $g(u) = \lambda|u|^{p-1}u$ ($p > 1$ and $\lambda > 0$), has been investigated recently by a number of authors. For example, see [6, 7, 20, 25, 32] and the references therein. The standing wave solutions of system (1.1) have the following form

$$\begin{aligned} \phi(t, x) &= u(|x|)e^{i\omega t}, \quad A_0(t, x) = k(|x|), \\ A_1(t, x) &= \frac{x_2}{|x|^2}h(|x|), \quad A_2(t, x) = -\frac{x_1}{|x|^2}h(|x|), \end{aligned} \quad (1.2)$$

where $\omega > 0$ is a given frequency, u, k, h are real valued functions depending only on $|x|$. Note that the ansatz (1.2) satisfies the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$. Inserting the ansatz (1.2) into the system (1.1), Byeon et al.[6] got the following nonlocal semilinear elliptic equation

$$-\Delta u + \omega u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds \right) u = \lambda|u|^{p-2}u \text{ in } \mathbb{R}^2, \quad (1.3)$$

where

$$h(s) = h_u(s) = \frac{1}{2} \int_0^s r u^2(r) dr = \frac{1}{4\pi} \int_{B_s} u^2(x) dx.$$

Mathematically, equation (1.3) is not a pointwise identity as the appearance of the Chern-Simons term

$$\left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds \right) u.$$

Hence problem (1.3) is called a nonlocal problem and is quite different from the usual semi-linear Schrödinger equation. From the variational point of view, the nonlocal term causes some mathematical difficulties that make the study of problem (1.3) more interesting.

Following [6], equation (1.3) possesses a variational structure, that is, the standing wave solutions are obtained as critical points of the energy functional associated to (1.3) defined by

$$\mathcal{L}(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + \omega u^2) dx + \frac{1}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right)^2 dx - \frac{\lambda}{p} \int_{\mathbb{R}^2} |u|^p dx, \quad (1.4)$$

$u \in H_r^1(\mathbb{R}^2)$, where $H_r^1(\mathbb{R}^2) := \{u \in H^1(\mathbb{R}^2) \mid u(x) = u(|x|)\}$. In [6, 10, 20, 29, 30, 33, 39, 40, 45], the critical points of $\mathcal{L}$ are found by using variational methods. It is shown that the value $p = 4$ is critical for this problem. Indeed, for $p > 4$, it is known that the energy functional is unbounded from below and satisfies a mountain-pass geometry. In a certain sense, in this case the local nonlinearity dominates the nonlocal term. However the existence of a solution is not so direct, since for $p \in (4, 6)$ the (PS)-condition is not known to hold. In the spirit of [34], this problem is bypassed in [6] by using a constrained minimization taking into account the Nehari-Pohozaev manifold.

A special case is $p = 4$: in this case, solutions have been explicitly found in [6, 7] as optimizers of a certain inequality. An alternative approach would be to pass to a self-dual equation, which leads to a Liouville equation that can be solved explicitly. For more information on the self-dual equations, see [14, 24]. For the

case $p \geq 4$, [29] and [30] proved the existence, multiplicity, quantitative property and asymptotic behavior of normalized solutions with prescribed L^2 -norm.

The situation is different if $p \in (2, 4)$, solutions are found in [6] as minimizers on a L^2 -sphere. Later, the results has been extended by Pomponio and Ruiz [32] by investigating the geometry of the energy functional under the different range of frequency ω . Moreover, Pomponio and Ruiz in [33] also studied the bounded domain case for $p \in (2, 4)$. By using singular perturbation arguments based on a Lyapunov-Schmidt reduction, they obtained some results on boundary concentration of solutions. Wan and Tan [40] studied the existence and multiplicity of standing waves for asymptotically linear nonlinearity case, and see [44] for the sublinear case. Cunha et al.[10] obtained a multiplicity result when the nonlinearity satisfies the general hypotheses introduced by Berestycki and Lions [8]. For more results about the initial value problem, well-posedness, existence and blow-up, scattering and uniqueness results for some nonlocal problems, we refer readers to [5, 11, 12, 19, 21, 26, 27, 42, 43] and references therein.

When $p > 6$, by using the symmetric mountain pass theorem, Huh [20] obtained the existence of infinitely many radially symmetric solutions for equation (1.3). Recently, this result has been extended to more general nonlinearity model by Zhang et al.[45]. Besides, Deng et al.[16] and Li et al.[31] investigated the existence and asymptotic behavior of radial sign-changing solutions by using constraint minimization method and quantitative deformation lemma for equation (1.3). Liu et al.[28] obtained a multiplicity result of sign-changing solutions via a novel perturbation approach and the method of invariant sets of descending flow.

Very recently, for the general 6-superlinear nonlinearity case, Tang et al.[39] considered the following nonlocal Schrödinger equation with the gauge field and deepening potential well

$$-\Delta u + \lambda V(|x|)u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds \right) u = f(u) \text{ in } \mathbb{R}^2, \quad (1.5)$$

where the potential V is a continuous function satisfies

(V_1) $V(|x|) \in C(\mathbb{R}^2)$ and $V(|x|) \geq 0$ on $\mathbb{R}^2$;

(V_2) there exists a constant $b > 0$ such that the set $V_b := \{x \in \mathbb{R}^2 | V(|x|) < b\}$ is nonempty and has finite measure;

(V_3) there is bounded symmetric domain Ω such that $\Omega = \text{int} V^{-1}(0)$ with smooth boundary $\partial\Omega$ and $\bar{\Omega} = V^{-1}(0)$.

Under some suitable conditions on the nonlinearity f , the second and third authors proved the existence and multiplicity of solutions (possibly positive, negative or sign-changing) by using mountain pass theorem. Moreover, the concentration behavior of these solutions on the set Ω as $\lambda \rightarrow +\infty$ are also studied.

Involving the Chern-Simons-Schrödinger equations with potential wells, there is only the work [39] so far. As described above, the shape of solutions obtained in [39] may be single-bump. However, nothing is known for the existence of multi-bump type solutions. Motivated by the above facts, we intend in the present paper to study the existence of sign-changing multi-bump solutions for equation (1.5) with deepening potential well. To the best of our knowledge, it seems that such a problem was not considered in literature before. In order to state our statements, for the potential V we need to assume that the following conditions besides (V_2) and (V_3),

(V_1) $V \in C^1(\mathbb{R}^2)$ and $V(|x|) \geq 0$ on $\mathbb{R}^2$;

(V_4) there are $k + 1$ disjoint open bounded components $\Omega_0, \Omega_1, \Omega_2, \dots, \Omega_k$ ($k \geq 2$) such that $\Omega = \text{int} V^{-1}(0) = \bigcup_{i=0}^k \Omega_i$ and $\text{dist}(\Omega_i, \Omega_j) > 0$ for $i \neq j$, $i, j = 0, 1, 2, \dots, k$, where $\Omega_0 = \{x \in \mathbb{R}^2, r_0 = 0 \leq |x| \leq r'_0\}$, $\Omega_i = \{x \in \mathbb{R}^2, r_i \leq |x| \leq r'_i\}$.

Moreover, we suppose that the nonlinearity f satisfies

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$, and there exist constants $C > 0$ and $q_0 \in (4, +\infty)$ such that

$$|f(t)| \leq C(|t| + |t|^{q_0-1});$$

(f_2) $f(t) = o(t)$ as $t \rightarrow 0$;

(f_3) $\lim_{t \rightarrow \infty} \frac{F(t)}{t^6} = +\infty$ where $F(t) = \int_0^t f(s)ds$;

(f_4) the function $\frac{f(t)}{t^5}$ is increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$.

Before stating our results we first need to introduce some notations. Throughout this paper, we define

$$H_\lambda := \left\{ u \in H_r^1(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} \lambda V(|x|)u^2 dx < +\infty \right\}$$

with the norm

$$\|u\|_\lambda^2 = \int_{\mathbb{R}^2} |\nabla u|^2 + \lambda V(|x|)u^2 dx.$$

Clearly, the embedding $H_\lambda \hookrightarrow H_r^1(\mathbb{R}^2)$ is continuous due to (V_1) and (V_2) . We will give the proof later. Define the energy functional $\mathcal{J}_\lambda : H_\lambda \rightarrow \mathbb{R}$ by

$$\mathcal{J}_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + \lambda V(|x|)u^2) dx + \frac{1}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right)^2 dx - \int_{\mathbb{R}^2} F(u) dx, \quad (1.6)$$

Then, our hypotheses imply that the functional $\mathcal{J}_\lambda \in C^1(H_\lambda, \mathbb{R})$, and for any $u, \varphi \in H_\lambda$, we have

$$\begin{aligned} \langle \mathcal{J}'_\lambda(u), \varphi \rangle &= \int_{\mathbb{R}^2} \nabla u \nabla \varphi + \lambda V(|x|)u\varphi dx + \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u\varphi dx \\ &\quad + \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right) \left(\int_0^{|x|} s u(s)\varphi(s) ds \right) dx - \int_{\mathbb{R}^2} f(u)\varphi dx. \end{aligned} \quad (1.7)$$

Obviously, critical points of $\mathcal{J}_\lambda$ are the weak solutions for equation (1.5). Furthermore, if $u \in H_\lambda$ is a solution of (1.5) and $u^\pm \neq 0$, then u is a sign-changing solution of (1.5), where

$$u^+(x) = \max\{u(x), 0\} \text{ and } u^-(x) = \min\{u(x), 0\}.$$

Our main result can be stated as follows.

Theorem 1.1. Suppose that (V_1) – (V_4) and (f_1) – (f_4) hold. Then, for any non-empty subset $T \subset \{0, 1, 2, 3, \dots, k\}$ with

$$T = T_1 \cup T_2 \cup T_3 \text{ and } T_i \cap T_j = \emptyset \text{ for } i \neq j, \ i, j = 1, 2, 3. \quad (1.8)$$

There exists a constant $\Lambda_T > 0$ such that for $\lambda > \Lambda_T$, equation (1.5) has a sign-changing multi-bump solution u_λ , which possesses the following property: for any sequence $\{\lambda_n\}$ with $\lambda_n \rightarrow +\infty$ as $n \rightarrow \infty$, there is a subsequences $\{u_{\lambda_{n_i}}\}$ converges strongly to u in $H_r^1(\mathbb{R}^2)$, where $u \in H_r^1(\mathbb{R}^2)$ is a nontrivial solution of the equation

$$\begin{cases} -\Delta u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{r_k} \frac{h(s)}{s} u^2(s) ds \right) u = f(u), & x \in \Omega_T = \cup_{i \in T} \Omega_i, \\ u = 0 & x \in \mathbb{R}^2 \setminus \Omega_T. \end{cases} \quad (1.9)$$

Moreover, $u|_{\Omega_i}$ is positive for $i \in T_1$, $u|_{\Omega_i}$ is negative for $i \in T_2$, and $u|_{\Omega_i}$ changes sign exactly once for $i \in T_3$.

The motivation of the present paper arises from the study of the local Schrödinger equations with deepening potential well

$$-\Delta u + (\lambda V(x) + a(x))u = f(x, u) \text{ in } \mathbb{R}^N. \quad (1.10)$$

We remark that conditions (V_1) – (V_3) have been first introduced by Bartsch, Pankov and Wang [9] in studying the Schrödinger equation (1.10). They obtained some results on the existence of multiple solutions, and also showed that the solutions concentrated at the bottom of the potential well as $\lambda \rightarrow +\infty$. The existence and

characterization of the solutions for equation (1.10) were considered in [1, 2, 15, 35] under conditions (V_1) – (V_4) . For example, Ding and Tanaka [15] first constructed the existence of multi-bump positive solutions u_λ for (1.10), and they also proved that, up to a subsequence, u_λ converges strongly in $H^1(\mathbb{R}^N)$ to a function u , which satisfies $u = 0$ outside Ω_T and $u|_{\Omega_i}$ is the positive least energy solution to the equation

$$-\Delta u + a(x)u = f(x, u), \quad u \in H_0^1(\Omega_i). \quad (1.11)$$

Inspired by [15], Alves [1] and Sato and Tanaka [35] investigated the sign-changing multi-bump solutions to equation (1.10) independently. Later, Alves and Pereira [2] obtained a similar result for the critical growth case. We must point out that the equation (1.11) (called limit equation of (1.10)) plays an important role in the study of multi-bump solutions for equation (1.10). Because the positive, negative and sign-changing solutions of (1.11) are used as building bricks to construct the multi-bump solutions of (1.10) by using of gluing techniques. Recently, there are some works focused on study of multi-bump solutions and ground states solutions for other nonlocal problems. For instance, see [18, 36, 37] for Schrödinger-Kirchhoff equation, [3, 13, 38] for Schrödinger-Poisson system, and [4] for Choquard equation and so on.

From the commentaries above, it is quite natural to ask if the results in [1, 35] still hold for the Chern-Simons-Schrödinger equations. Unfortunately, we can not draw a similar conclusion in a straight way. Since problem (1.5) is a nonlocal one as the appearance of the Chern-Simons term, then the solutions of problem (1.5) need their global information. Thus, we cannot use the same arguments explored in [1, 35] to solve the corresponding limit equation (1.9) separately on each Ω_i . This result in the effective methods used [1, 35] for local Schrödinger equations cannot be applied to nonlocal problem (1.5) directly, there arises a technical problem one should overcome. As we all know, in order to obtain the existence of sign-changing multi-bump solutions for problem (1.5), we need to use the existence and some properties of the least energy sign-changing solutions of limit equation (1.9). However, it is not easy to prove the existence and sign properties of the least energy solution for limit equation (1.9). Hence, the first step is to consider the limit equation (1.9) and to look for the existence of least energy sign-changing solution that is nonzero on each component Ω_i , $i \in T$.

Our result on the limit equation (1.9) can be stated as follows.

Theorem 1.2. *Suppose that (f_1) – (f_4) hold, then, for any non-empty subset T , equation (1.9) has a nontrivial solution u with $u|_{\Omega_i}$ is positive for $i \in T_1$, $u|_{\Omega_i}$ is negative for $i \in T_2$, and $u|_{\Omega_i}$ changes sign exactly once for $i \in T_3$. Moreover, u is the least energy solution among all solutions with those sign properties.*

To prove our results, some arguments are in order. First, to obtain the least energy sign-changing solution solutions with prescribed sign properties for limit equation (1.9), we first prove the set $\mathcal{M}_T$ (see (2.1)) is nonempty and then we seek minimizers of the energy functional on $\mathcal{M}_T$. Observe that $\mathcal{M}_T$ is not a C^1 -manifold, we will take advantage of constraint minimization method and quantitative deformation lemma to obtain the existence of minimizers of the energy functional. Second, we will use the penalization technique explored by del Pino and Felmer in [17] to cut off the nonlinearity f , then, to control the order of growth of nonlinearity f outside the potential well Ω_T . In such a way, we build a modification of the energy functional associated to (1.5) and give some energy relations of problems (1.5) and (1.9) which play a key role in getting the critical point of (1.5) (see Lemma 4.3). Moreover, in order to show a critical point associated to the modified functional is indeed a solution to the original problem, we also need give a delicate L^∞ -estimation for the solutions of the modified problem. Finally, we study a special minimax value of the modified functional, which is crucial for proving Theorem 1.1. Furthermore, via a rather precise analysis of deformation flow to the modified functional, we prove the existence of sign-changing multi-bump solutions for (1.5).

The paper is organized as follows. In Section 2, we give some preliminary lemmas and the proof of Theorem 1.2. In Section 3, we define a penalization problem and modified functional, and give a L^∞ -estimation for the solutions of the modified problem. In Section 4, we study a special minimax value of the modified functional. At last, we give the proof of Theorem 1.1 in Section 5.

Throughout the sequel, we denote the usual Lebesgue space with norms $\|u\|_p = \left(\int_{\mathbb{R}^2} |u|^p dx\right)^{\frac{1}{p}}$ by $L^p(\mathbb{R}^2)$, where $1 \leq p < \infty$, and C denotes different positive constant in different place.

2 Mixed type sign-changing solutions to limit problem

In this section, we study the existence of solutions for the limited equation (1.9) with prescribed sign properties. Firstly, we restrict the nonlinearity $\tilde{f}(x, t) = 0$ if

$$(i) \ t < 0 \text{ and } x \in \Omega_i \text{ for } i \in T_1; \text{ or } (ii) \ t > 0 \text{ and } x \in \Omega_i \text{ for } i \in T_2.$$

Denoted

$$H_T = \left\{ u \in H_r^1(\mathbb{R}^2) \mid u = 0 \text{ in } x \in \mathbb{R}^2 \setminus \Omega_T \right\}$$

with the norm

$$\|u\|_T^2 = \int_{\Omega_T} |\nabla u|^2 dx.$$

Now we define the energy functional J_T corresponding to limit problem (1.9) on H_T

$$J_T(u) := \frac{1}{2} \int_{\Omega_T} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_T} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right)^2 dx - \int_{\Omega_T} \tilde{F}(x, u) dx, \quad u \in H_T,$$

and the set

$$\begin{aligned} \mathcal{M}_T = \{ & u \in H_T \mid \langle J'_T(u), u_i \rangle = 0 \text{ for } i \in T, u_i^+ \neq 0 \text{ for } i \in T_1, \\ & u_i^- \neq 0 \text{ for } i \in T_2 \text{ and } \langle J'_T(u), u_i^+ \rangle = 0, u_i^+ \neq 0 \text{ for } i \in T_3 \}, \end{aligned} \quad (2.1)$$

where $u_i := u|_{\Omega_i}$ and $T = T_1 \cup T_2 \cup T_3 \subset \{0, 1, 2, 3, \dots, k\}$ satisfies (1.8), and $\tilde{F}(x, u) = \int_0^u \tilde{f}(x, t) dt$. Let

$$m = \inf_{u \in \mathcal{M}_T} J_T(u).$$

If m is attained by $u_0 \in \mathcal{M}_T$ and $J'_T(u_0) = 0$, then u_0 be called a the least energy sign-changing solution of limit problem (1.9).

Without loss of generality, we consider the case $T_1 = \{1\}$, $T_2 = \{2\}$ and $T_3 = \{3\}$ for simplify. In this case, $\Omega_T = \cup_{i=1}^3 \Omega_i$ with $\text{dist}(\Omega_i, \Omega_j) > 0$ for $i \neq j$, $i, j = 1, 2, 3$. To simplify the notations, we use $\Omega, \mathcal{M}, H$ to denote the sets $\Omega_T, \mathcal{M}_T, H_T$ respectively. Moreover, we define the functional on H as follows

$$J(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right)^2 dx - \int_{\Omega} \tilde{F}(x, u) dx, \quad u \in H. \quad (2.2)$$

The functional J is well-defined and belongs to $C^1(H, \mathbb{R})$. Moreover, for any $u, \varphi \in H$, we have

$$\begin{aligned} \langle J'(u), \varphi \rangle = & \int_{\Omega} \nabla u \nabla \varphi dx + \int_{\Omega} \frac{h^2(|x|)}{|x|^2} u \varphi dx + \int_{\Omega} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right) \left(\int_0^{|x|} s u(s) \varphi(s) ds \right) dx \\ & - \int_{\Omega} \tilde{f}(x, u) \varphi dx. \end{aligned} \quad (2.3)$$

Clearly, critical points of J are the weak solutions for limit problem (1.9).

We use constraint minimizer on $\mathcal{M}$ to seek a critical point of J with nonzero component. We first check that the set $\mathcal{M}$ is nonempty in H .

Lemma 2.1. Assume that (f_1) – (f_4) hold. For $u \in H$ with $u_1^+ \neq 0$, $u_2^- \neq 0$ and $u_3^+ \neq 0$, then there exists a unique 4-tuple $(s_1, s_2, s_3, s_4) \in (\mathbb{R}_+)^4$ such that

$$s_1 u_1 + s_2 u_2 + s_3 u_3^+ + s_4 u_3^- \in \mathcal{M}.$$

Proof. For $u \in H$ with $u_1^+ \neq 0$, $u_2^- \neq 0$ and $u_3^+ \neq 0$, we define

$$F(u) = (F_1(u), F_2(u), F_3(u), F_4(u)), \quad (2.4)$$

where

$$F_i(u) = \begin{cases} J'(u)u_i, & i = 1, 2, \\ J'(u)(u_3^+), & i = 3, \\ J'(u)(u_3^-), & i = 4. \end{cases}$$

Obviously, $t_1 u_1 + t_2 u_2 + t_3 u_3^+ + t_4 u_3^- \in \mathcal{M}$ if and only if

$$F(t_1 u_1 + t_2 u_2 + t_3 u_3^+ + t_4 u_3^-) = 0. \quad (2.5)$$

Next, we obtain the desired results by proving two claims.

Claim 1. For $u \in H$ with $u_1 \neq 0$, $u_2 \neq 0$ and $u_3^\pm \neq 0$, there exists at least one solution for (2.5).

Firstly, there exists a unique $\bar{t}_1 > 0$ such that

$$F_1(\bar{t}_1 u_1 + t_2 u_2 + t_3 u_3^+ + t_4 u_3^-) = 0. \quad (2.6)$$

for fixed $(t_2, t_3, t_4) \in (\mathbb{R}_+)^3$. In fact, we define

$$\begin{aligned} k(t) := & t^2 \int_{\Omega} |\nabla u_1|^2 dx + \int_{\Omega} \left(\int_0^{|x|} \frac{s}{2} (t^2 u_1^2(s) + t_2^2 u_2^2(s) + t_3^2 u_3^{+2}(s) + t_4^2 u_4^{+2}(s)) ds \right)^2 \frac{t^2 u_1^2}{|x|^2} dx \\ & + \int_{\Omega} \frac{t^2 u_1^2 + t_2^2 u_2^2 + t_3^2 u_3^{+2} + t_4^2 u_4^{+2}}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} (t^2 u_1^2(s) + t_2^2 u_2^2(s) + t_3^2 u_3^{+2}(s) + t_4^2 u_4^{+2}(s)) ds \right) \\ & \left(\int_0^{|x|} s t^2 u_1^2(s) ds \right) dx - \int_{\Omega} \tilde{f}(x, tu_1) tu_1 dx. \end{aligned}$$

By (f_2) , (f_3) and (f_4) , it is easy to see that $k(t) > 0$ for $t > 0$ small enough and $k(t) < 0$ for $t > 0$ large enough. Hence, there exists $\bar{t}_1 > 0$ such that $k(\bar{t}_1) = 0$. Moreover, by (f_4) we can deduce that $\bar{t}_1$ is unique. Thus, we can get a function $\eta_1 : (\mathbb{R}_+)^3 \rightarrow (0, +\infty)$ defined by

$$\eta_1(t_2, t_3, t_4) = \bar{t}_1,$$

such that $F_1(\bar{t}_1 u_1 + t_2 u_2 + t_3 u_3^+ + t_4 u_3^-) = 0$.

By the same arguments as above, we can define function $\eta_i : (\mathbb{R}_+)^3 \rightarrow (0, +\infty)$, $i = 2, 3, 4$, given by

$$\eta_2(t_1, t_3, t_4) = \bar{t}_2, \quad \eta_3(t_1, t_2, t_4) = \bar{t}_3, \quad \eta_4(t_1, t_2, t_3) = \bar{t}_4,$$

satisfying $F_2(t_1 u_1 + \bar{t}_2 u_2 + t_3 u_3^+ + t_4 u_3^-) = 0$, $F_3(t_1 u_1 + t_2 u_2 + \bar{t}_3 u_3^+ + t_4 u_3^-) = 0$, and $F_4(t_1 u_1 + t_2 u_2 + t_3 u_3^+ + \bar{t}_4 u_3^-) = 0$.

Moreover, the functions $\eta_i : (\mathbb{R}_+)^3 \rightarrow (0, +\infty)$, $i = 1, 2, 3, 4$, have the following three properties:

- (i) For any $(a_1, a_2, a_3, a_4) \in (\mathbb{R}_+)^4$, we have $\eta_i(a_i | a_1, a_2, a_3, a_4) > 0$;
- (ii) η_i are continuous on $[0, +\infty)^3$;
- (iii) If $a_{\max}^i$ large enough, then

$$\bar{a}_i = \eta_i(a_i | a_1, a_2, a_3, a_4) < a_{\max}^i := \max\{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_4\}$$

where $(a_i | a_1, a_2, a_3, a_4) := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_4)$.

By the property (iii), there exists $M_1 > 0$ such that $\eta_i(a_i | a_1, a_2, a_3, a_4) \leq a_{\max}^i$ for $a_{\max}^i > M_1$. From the property (i), we get

$$M_2 := \max_i \{\max \eta_i(a_i | a_1, a_2, a_3, a_4) : a_{\max}^i \leq M_1, i = 1, 2, 3, 4\} > 0.$$

Thus, $M_0 := \max\{M_1, M_2\} > 0$. For any $(a_1, a_2, a_3, a_4) \in [0, M_0]^4$, it follows from (iii) that $\eta_i(a_i | a_1, a_2, a_3, a_4) \leq M_0$.

Hence, we can define $L : [0, M_0]^4 \rightarrow [0, M_0]^4$ by

$$L(a_1, a_2, a_3, a_4) := (\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4),$$

where $\bar{a}_i = \eta_i(a_i | a_1, a_2, a_3, a_4)$ satisfying $F_1(\bar{a}_1 u_1 + a_2 u_2 + a_3 u_3^+ + a_4 u_3^-) = 0$, $F_2(a_1 u_1 + \bar{a}_2 u_2 + a_3 u_3^+ + a_4 u_3^-) = 0$, $F_3(a_1 u_1 + a_2 u_2 + \bar{a}_3 u_3^+ + a_4 u_3^-) = 0$, $F_4(a_1 u_1 + a_2 u_2 + a_3 u_3^+ + \bar{a}_4 u_3^-) = 0$.

Obviously, $L(a_1, a_2, a_3, a_4)$ is continuous on $[0, M_0]^4$. Now, by applying the Brouwer Fixed Point Theorem, there exists $(s_1, s_2, s_3, s_4) \in [0, M_0]^4$ such that

$$L(s_1, s_2, s_3, s_4) = (s_1, s_2, s_3, s_4).$$

Thus, (s_1, s_2, s_3, s_4) is a solution of (2.5).

Claim 2. (s_1, s_2, s_3, s_4) obtained by Claim 1 is the unique solution of (2.5). To show Claim 2, the proof will be carried out in two cases.

Case 1. $u \in \mathcal{M}$. In this case, the 4-tuple $(1, 1, 1, 1)$ is a solution of (2.5). We prove that $(1, 1, 1, 1)$ is the unique solution of (2.5) in $(\mathbb{R}^+)^4$. Indeed, suppose (a_1, a_2, a_3, a_4) be any other solution, then

$$F_i(a_1 u_1 + a_2 u_2 + a_3 u_3^+ + a_4 u_3^-) = 0, \quad i = 1, 2, 3, 4.$$

Without loss of generality, we suppose $a_1 = \max\{a_1, a_2, a_3, a_4\}$. That is

$$\begin{aligned} & a_1^2 \int_{\Omega} |\nabla u_1|^2 dx + \int_{\Omega} \left(\int_0^{|x|} \frac{s}{2} (a_1^2 u_1^2(s) + a_2^2 u_2^2(s) + a_3^2 u_3^{+2}(s) + a_4^2 u_4^{+2}(s)) ds \right)^2 \frac{a_1^2 u_1^2}{|x|^2} dx \\ & + \int_{\Omega} \frac{a_1^2 u_1^2 + a_2^2 u_2^2 + a_3^2 u_3^{+2} + a_4^2 u_4^{+2}}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} (a_1^2 u_1^2(s) + a_2^2 u_2^2(s) + a_3^2 u_3^{+2}(s) + a_4^2 u_4^{+2}(s)) ds \right) \\ & \left(\int_0^{|x|} s a_1^2 u_1^2(s) ds \right) dx = \int_{\Omega} \tilde{f}(x, a_1 u_1) a_1 u_1 dx. \end{aligned}$$

Since $u \in \mathcal{M}$, we have

$$\begin{aligned} & \int_{\Omega} |\nabla u_1|^2 dx + \int_{\Omega} \left(\int_0^{|x|} \frac{s}{2} (u_1^2(s) + u_2^2(s) + u_3^{+2}(s) + u_4^{+2}(s)) ds \right)^2 \frac{u_1^2}{|x|^2} dx \\ & + \int_{\Omega} \frac{u_1^2 + u_2^2 + u_3^{+2} + u_4^{+2}}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} (u_1^2(s) + u_2^2(s) + u_3^{+2}(s) + u_4^{+2}(s)) ds \right) \\ & \left(\int_0^{|x|} s u_1^2(s) ds \right) dx = \int_{\Omega} \tilde{f}(x, u_1) u_1 dx. \end{aligned}$$

So we obtain

$$\left(\frac{1}{a_1^4} - 1 \right) \int_{\Omega} |\nabla u_1|^2 dx \geq \int_{\Omega} \left(\frac{\tilde{f}(x, a_1 u_1)}{(a_1 u_1)^5} - \frac{\tilde{f}(x, u_1)}{(u_1)^5} \right) u_1^6 dx.$$

By (f_1) – (f_4) , it imply that $a_1 = \max\{a_1, a_2, a_3, a_4\} \leq 1$. By the same arguments, we can easily conclude that $\min\{a_1, a_2, a_3, a_4\} \geq 1$. Consequently, the 4-tuple $(1, 1, 1, 1)$ is the unique solution of (2.5).

Case 2. $u \notin \mathcal{M}$. If $(u_1, u_2, u_3, u_4) \notin \mathcal{M}$, then by Claim 1, we know that (2.5) has a solution (s_1, s_2, s_3, s_4) . Assume that (s'_1, s'_2, s'_3, s'_4) also be a solution. Then we have

$$\frac{s'_1}{s_1} s_1 u_1 + \frac{s'_2}{s_2} s_2 u_2 + \frac{s'_3}{s_3} s_3 u_3^+ + \frac{s'_4}{s_4} s_4 u_3^- \in \mathcal{M}.$$

Since $s_1 u_1 + s_2 u_2 + s_3 u_3^+ + s_4 u_3^- \in \mathcal{M}$, by Case 1 we get

$$\frac{s'_1}{s_1} = \frac{s'_2}{s_2} = \frac{s'_3}{s_3} = \frac{s'_4}{s_4} = 1.$$

Thus, (s_1, s_2, s_3, s_4) is the unique solution of (2.5) in $(\mathbb{R}_+)^4$.

Lemma 2.2. Assume that (f_1) – (f_4) hold, then the unique 4-tuple $(t_1, t_2, t_3, t_4) \in (\mathbb{R}_+)^4$ obtained by Lemma 2.1 is the unique maximum point of the function $\varphi : (\mathbb{R}_+)^4 \rightarrow \mathbb{R}$ defined by

$$\varphi(s_1, s_2, s_3, s_4) = J(s_1 u_1 + s_2 u_2 + s_3 u_3^+ + s_4 u_3^-).$$

Proof. From the proof of Lemma 2.1, we know that (t_1, t_2, t_3, t_4) is the unique critical point of φ in $(\mathbb{R}_+)^4$. By the assumption (f_3) , we deduce that $\varphi(t_1, t_2, t_3, t_4) \rightarrow -\infty$ as $|(t_1, t_2, t_3, t_4)| \rightarrow \infty$, so it is sufficient to check that a maximum point cannot be achieved on the boundary of $(\mathbb{R}_+)^4$. Choose $(s_1, s_2, s_3, s_4) \in \partial(\mathbb{R}_+)^4$, without loss of generality, we may assume that $s_1 = 0$. But, it is obviously that

$$\begin{aligned} l(s) &= \varphi(s, s_2, s_3, s_4) = J(su_1 + s_2 u_2 + s_3 u_3^+ + s_4 u_3^-) \\ &= \frac{1}{2} \int_{\Omega} |\nabla(su_1 + s_2 u_2 + s_3 u_3^+ + s_4 u_3^-)|^2 dx \\ &\quad + \frac{1}{2} \int_{\Omega} \frac{(su_1 + s_2 u_2 + s_3 u_3^+ + s_4 u_3^-)^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} (su_1(s) + s_2 u_2(s) + s_3 u_3^+(s) + s_4 u_3^-(s))^2 ds \right)^2 dx \\ &\quad - \int_{\Omega} \tilde{F}(x, su_1 + s_2 u_2 + s_3 u_3^+ + s_4 u_3^-) dx \end{aligned}$$

is an increasing function with respect to s if $s > 0$ is small enough, $(0, s_2, s_3, s_4)$ is impossible to be a maximum point of φ .

Lemma 2.3. Assume that (f_1) – (f_4) hold, let $u \in H$ with $u_1^+ \neq 0$, $u_2^- \neq 0$ and $u_3^{\pm} \neq 0$ such that, $F_i(u) \leq 0$ for $i = 1, 2, 3, 4$, where F_i are given by (2.4). Then the unique 4-tuples $(t_1, t_2, t_3, t_4) \in (\mathbb{R}_+)^4$ obtained by Lemma 2.1 satisfied $(t_1, t_2, t_3, t_4) \in (0, 1]^4$.

Proof. Without loss of generality, we suppose that $t_1 = \max\{t_1, t_2, t_3, t_4\}$. Since $t_1 u_1 + t_2 u_2 + t_3 u_3^+ + t_4 u_3^- \in \mathcal{M}$, we get

$$\begin{aligned} &t_1^2 \int_{\Omega} |\nabla u_1|^2 dx + \int_{\Omega} \left(\int_0^{|x|} \frac{s}{2} (t_1^2 u_1^2(s) + t_2^2 u_2^2(s) + t_3^2 u_3^{+2}(s) + t_4^2 u_4^{+2}(s)) ds \right)^2 \frac{t_1^2 u_1^2}{|x|^2} dx \\ &\quad + \int_{\Omega} \frac{t_1^2 u_1^2 + t_2^2 u_2^2 + t_3^2 u_3^{+2} + t_4^2 u_4^{+2}}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} (t_1^2 u_1^2(s) + t_2^2 u_2^2(s) + t_3^2 u_3^{+2}(s) + t_4^2 u_4^{+2}(s)) ds \right) \\ &\quad \left(\int_0^{|x|} s t_1^2 u_1^2(s) ds \right) dx = \int_{\Omega} \tilde{f}(x, t_1 u_1) t_1 u_1 dx. \end{aligned} \quad (2.7)$$

At the same time, using $F_i(u) \leq 0$, one has

$$\begin{aligned} &\int_{\Omega} |\nabla u_1|^2 dx + \int_{\Omega} \left(\int_0^{|x|} \frac{s}{2} (u_1^2(s) + u_2^2(s) + u_3^{+2}(s) + u_4^{+2}(s)) ds \right)^2 \frac{u_1^2}{|x|^2} dx \\ &\quad + \int_{\Omega} \frac{u_1^2 + u_2^2 + u_3^{+2} + u_4^{+2}}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} (u_1^2(s) + u_2^2(s) + u_3^{+2}(s) + u_4^{+2}(s)) ds \right) \\ &\quad \left(\int_0^{|x|} s u_1^2(s) ds \right) dx \leq \int_{\Omega} \tilde{f}(x, u_1) u_1 dx. \end{aligned} \quad (2.8)$$

Therefore, (2.7) and (2.8) imply that

$$\left(\frac{1}{t_1^4} - 1\right) \int_{\Omega} |\nabla u_1|^2 dx \geq \int_{\Omega} \left(\frac{\tilde{f}(x, t_1 u_1)}{t_1^5 u_1^5} - \frac{\tilde{f}(x, u_1)}{u_1^5} \right) u_1^6 dx.$$

By (f_4) , we obtain $t_1 \leq 1$. Thus we complete the proof.

Next, we consider the following constrained minimization problem $m := \inf_{u \in \mathcal{M}} J(u)$.

Lemma 2.4. Assume that (f_1) – (f_4) hold, then $m > 0$ and m can be achieved by a function $v \in \mathcal{M}$.

Proof. Let $u \in \mathcal{M}$, then

$$\int_{\Omega} |\nabla u_i|^2 dx \leq \int_{\Omega} \tilde{f}(x, u_i) u_i dx \quad \text{for } i = 1, 2,$$

and

$$\int_{\Omega} |\nabla u_3^\pm|^2 dx \leq \int_{\Omega} \tilde{f}(x, u_3^\pm) u_3^\pm dx.$$

Thus, by (f_1) , (f_2) and Sobolev embedding theorem, there exists a positive constant C such that

$$\|u_1^+\|, \|u_2^-\|, \|u_3^+\|, \|u_3^-\| \geq C.$$

Therefore,

$$\begin{aligned} J(u) &= J(u) - \frac{1}{6} \sum_{i=1}^3 \langle J'(u), u_i \rangle = \sum_{i=1}^3 \left(\frac{1}{3} \int_{\Omega} |\nabla u_i|^2 dx + \int_{\Omega} \frac{1}{6} \tilde{f}(x, u_i) u_i - \tilde{F}(x, u_i) dx \right) \\ &\geq \frac{1}{3} \sum_{i=1}^3 \int_{\Omega} |\nabla u_i|^2 dx \geq C, \end{aligned}$$

this implies that $m > 0$. Suppose that $\{v_n\} \subset \mathcal{M}$ satisfying

$$\lim_{n \rightarrow +\infty} J(v_n) = m,$$

we can easily to get that

$$0 < C_1 \leq \|v_{n,i}\| \leq C_2 \quad \text{for } i = 1, 2 \text{ and } C_1 \leq \|v_{n,3}^\pm\| \leq C_2.$$

Passing to a subsequences, one has

$$\begin{aligned} v_n &\rightharpoonup v \quad \text{weakly in } H, \\ v_n &\rightarrow v \quad \text{strongly in } L^p(\Omega) \text{ for } 2 \leq p < \infty, \\ v_n &\rightarrow v \quad \text{a.e. in } \Omega. \end{aligned} \tag{2.9}$$

Assumptions (f_1) and (f_2) give

$$\int_{\Omega} \tilde{f}(x, v_3^\pm) v_3^\pm dx = \lim_{n \rightarrow +\infty} \int_{\Omega} \tilde{f}(x, v_{n,3}^\pm) v_{n,3}^\pm dx \geq \liminf_{n \rightarrow +\infty} \|v_{n,3}^\pm\|^2 \geq C_1.$$

Thus, $v_3^\pm \neq 0$. By the same arguments, we conclude that $v_1^+ \neq 0$ and $v_2^- \neq 0$. By Lemma 2.1, there exists a unique 4-tuple $(t_1, t_2, t_3, t_4) \in (\mathbb{R}_+)^4$ such that

$$\bar{v} := t_1 v_1 + t_2 v_2 + t_3 v_3^+ + t_4 v_3^- \in \mathcal{M}.$$

On the other hand, due to $\{v_n\} \subset \mathcal{M}$, we have that

$$F_i(v) \leq 0 \quad \text{for } i = 1, 2, 3.$$

Thus, Lemma 2.3 deduces that

$$(t_1, t_2, t_3, t_4) \in (0, 1]^4. \tag{2.10}$$

Thanks to the function $sf(s) - 6F(s)$ is a non-negative function, increasing on $(0, +\infty)$, decreasing on $(-\infty, 0)$, it follow from (2.10) and Lemma 2.2, we have

$$\begin{aligned}
 6m &\leq 6J(\bar{v}) = 6J(\bar{v}) - \sum_{i=1}^2 \langle J'(\bar{v}), t_i v_i \rangle - \langle J'(\bar{v}), t_3 v_3^+ \rangle - \langle J'(\bar{v}), t_4 v_4^- \rangle \\
 &= \sum_{i=1}^2 2t_i^2 \|v_i\|^2 + 2t_3^2 \|v_3^+\|^2 + 2t_4^2 \|v_3^-\|^2 + \sum_{i=1}^2 \int_{\Omega} (\tilde{f}(x, t_i v_i) t_i v_i - 6\tilde{F}(x, t_i v_i)) dx \\
 &\quad + \int_{\Omega} (\tilde{f}(x, t_3 v_3^+) t_3 v_3^+ - 6\tilde{F}(x, t_3 v_3^+)) dx + \int_{\Omega} (\tilde{f}(x, t_4 v_3^-) t_4 v_3^- - 6\tilde{F}(x, t_4 v_3^-)) dx \\
 &\leq \liminf_{n \rightarrow +\infty} \left\{ \sum_{i=1}^2 2\|v_{n,i}\|^2 + 2\|v_{n,3}^+\|^2 + 2\|v_{n,3}^-\|^2 + \sum_{i=1}^2 \int_{\Omega} (\tilde{f}(x, v_{n,i}) v_{n,i} - 6\tilde{F}(x, v_{n,i})) dx \right. \\
 &\quad \left. + \int_{\Omega} (\tilde{f}(x, v_{n,3}^+) v_{n,3}^+ - 6\tilde{F}(x, v_{n,3}^+)) dx + \int_{\Omega} (\tilde{f}(x, v_{n,3}^-) v_{n,3}^- - 6\tilde{F}(x, v_{n,3}^-)) dx \right\} \\
 &\leq \liminf_{n \rightarrow +\infty} 6J(v_n) = 6m.
 \end{aligned}$$

Thus, $t_1 = t_2 = t_3 = t_4 = 1$ and m is attained by v . Since the restriction on $\tilde{f}(x, u)$, we can get that $v_1 \geq 0$, $v_2 \leq 0$ and $(v_3^*) \neq 0$. So we prove the conclusion.

Proof of Theorem 1.2. We prove indirectly and assume that $\Phi'(v) \neq 0$. Then there exist $\delta > 0$ and $\varrho > 0$ such that

$$\|u - v\| \leq 3\delta \Rightarrow \|\Phi'(u)\|_{H^*} \geq \varrho. \quad (2.11)$$

For convenient, we define the function

$$g(s_1, s_2, s_3, s_4) = s_1 v_1 + s_2 v_2 + s_3 v_3^+ + s_4 v_3^-, \quad (s_1, s_2, s_3, s_4) \in D = \left(\frac{1}{2}, \frac{3}{2}\right)^4.$$

By Lemma 2.2, we have

$$\bar{m} := \max_{(s_1, s_2, s_3, s_4) \in \partial D} J(g(s_1, s_2, s_3, s_4)) < m. \quad (2.12)$$

For $\varepsilon := \min\{(m - \bar{m})/2, \varrho\delta/8\}$ and $S := B(v, \delta)$, then [41] yields a deformation $\eta \in C([0, 1] \times H, H)$ such that

- (i) $\eta(1, u) = u$ if $J(u) < m - 2\varepsilon$ or $J(u) > m + 2\varepsilon$;
- (ii) $\eta(1, J^{m_0+\varepsilon} \cap B(v, \delta)) \subset J^{m-\varepsilon}$;
- (iii) $J(\eta(1, u)) \leq J(u)$, $\forall u \in H$;
- (iv) $\eta(1, u)$ is a homeomorphism of H .

It is clear that

$$\max_{(s_1, s_2, s_3, s_4) \in \bar{D}} J(\eta(1, g(s_1, s_2, s_3, s_4))) < m. \quad (2.13)$$

We claim that $\eta(1, g(D)) \cap \mathcal{M} \neq \emptyset$. In fact, define

$$h(s_1, s_2, s_3, s_4) := \eta(1, g(s_1, s_2, s_3, s_4))$$

and

$$\begin{aligned}
 \Psi_0(s_1, s_2, s_3, s_4) &:= \left(\frac{1}{s_1} F_1 \circ g, \frac{1}{s_2} F_2 \circ g, \frac{1}{s_3} F_3 \circ g, \frac{1}{s_4} F_4 \circ g\right)(s_1, s_2, s_3, s_4), \\
 \Psi_1(s_1, s_2, s_3, s_4) &:= \left(\frac{1}{s_1} F_1 \circ h, \frac{1}{s_2} F_2 \circ h, \frac{1}{s_3} F_3 \circ h, \frac{1}{s_4} F_4 \circ h\right)(s_1, s_2, s_3, s_4),
 \end{aligned}$$

Then the degree theory and Lemma 2.1 yield

$$\deg(\Psi_0, D, 0) = 1.$$

It follows from (i), one has $g = h$ on ∂D . Thus, we obtain

$$\deg(\Psi_1, D, 0) = \deg(\Psi_0, D, 0) = 1$$

Thus, $\Psi_1(\bar{s}_1, \bar{s}_2, \bar{s}_3, \bar{s}_4) = 0$ for some $(\bar{s}_1, \bar{s}_2, \bar{s}_3, \bar{s}_4) \in D$, this implies that $\eta(1, g(D)) \cap \mathcal{M} \neq \emptyset$.

By (2.13) and the definition of m we reach a contradiction. Thus, $J'(v) = 0$. We can get $v \in C^2(\Omega)$ by standard elliptic regularity theory. So $v_1 > 0$, $v_2 < 0$ by the Maximum Principle. At last, we show that v_3 indeed has two nodal domains. Arguing by contradiction, we may assume that

$$v_3 = w_1 + w_2 + w_3, w_1, w_2, w_3 \in H_0^1(\Omega_3),$$

with $w_i \neq 0$, $w_1 \geq 0$, $w_2 \leq 0$ and $\text{suppt}(w_i) \cap \text{suppt}(w_j) = \emptyset$, for $i \neq j$, $i, j = 1, 2, 3$ and

$$\langle J'(v), w_i \rangle = 0, \text{ for } i = 1, 2, 3.$$

Let $u := w_1 + w_2$, we see that $u^+ = w_1$ and $u^- = w_2$. Setting

$$\bar{v} := v_1 + v_2 + u \in H_0^1(\Omega).$$

By Lemma 2.1, there exists a unique 4-tuple $(\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4) \in (\mathbb{R}_+)^4$ such that

$$\bar{t}_1 v_1 + \bar{t}_2 v_2 + \bar{t}_3 u^+ + \bar{t}_4 u^- \in \mathcal{M}.$$

Therefore

$$J(\bar{t}_1 v_1 + \bar{t}_2 v_2 + \bar{t}_3 u^+ + \bar{t}_4 u^-) \geq m. \quad (2.14)$$

Moreover, $J'(v)w_i = 0$ implies that $F_i(\bar{v}) < 0$ for $i = 1, 2, 3, 4$. Using Lemma 2.3, we deduce that $(\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4) \in (0, 1]^4$. At the same time,

$$\begin{aligned} 0 &= \frac{1}{6} \langle J'(v), w_3 \rangle = \frac{1}{6} \int_{\Omega} |\nabla w_3|^2 dx \\ &\quad + \frac{1}{6} \int_{\Omega} \left(\int_0^{|x|} \frac{s}{2} (v_1^2(s) + v_2^2(s) + w_1^2(s) + w_2^2(s) + w_3^2(s)) ds \right)^2 \frac{w_3^2}{|x|^2} dx \\ &\quad + \frac{1}{6} \int_{\Omega} \frac{v_1^2 + v_2^2 + w_1^2 + w_2^2 + w_3^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} (v_1^2(s) + v_2^2(s) + w_1^2(s) \right. \\ &\quad \left. + w_2^2(s) + w_3^2(s)) ds \right) \left(\int_0^{|x|} s w_3^2(s) ds \right) dx - \frac{1}{6} \int_{\Omega} \tilde{f}(x, w_3) w_3 dx \\ &< \frac{1}{2} \int_{\Omega} |\nabla w_3|^2 dx + \frac{1}{6} \int_{\Omega} \left(\int_0^{|x|} \frac{s}{2} (v_1^2(s) + v_2^2(s) + w_1^2(s) + w_2^2(s) + w_3^2(s)) ds \right)^2 \frac{w_3^2}{|x|^2} dx \\ &\quad + \frac{1}{6} \int_{\Omega} \frac{v_1^2 + v_2^2 + w_1^2 + w_2^2 + w_3^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} (v_1^2(s) + v_2^2(s) + w_1^2(s) \right. \\ &\quad \left. + w_2^2(s) + w_3^2(s)) ds \right) \left(\int_0^{|x|} s w_3^2(s) ds \right) dx - \int_{\Omega} \tilde{F}(x, w_3) dx \end{aligned} \quad (2.15)$$

and

$$\begin{aligned}
 & J(\bar{t}_1 v_1 + \bar{t}_2 v_2 + \bar{t}_3 u^+ + \bar{t}_4 u^-) \\
 &= J(\cdot) - \frac{1}{6} \left(\sum_{i=1}^2 \langle J'(\cdot), \bar{t}_i v_i \rangle + \langle J'(\cdot), \bar{t}_3 u_3^+ \rangle + \langle J'(\cdot), \bar{t}_4 u_4^- \rangle \right) \\
 &= \frac{1}{3} \sum_{i=1}^2 \bar{t}_i^2 \|v_i\|^2 + \frac{1}{3} \bar{t}_3^2 \|u_3^+\|^2 + \frac{1}{3} \bar{t}_4^2 \|u_3^-\|^2 + \frac{1}{6} \sum_{i=1}^2 \int_{\Omega} (\tilde{f}(x, t_i v_i) t_i v_i - 6\tilde{F}(x, t_i v_i)) dx \\
 &\quad + \frac{1}{6} \int_{\Omega} (\tilde{f}(x, t_3 u_3^+) t_3 u_3^+ - 6\tilde{F}(x, t_3 u_3^+)) dx + \frac{1}{6} \int_{\Omega} (\tilde{f}(x, t_4 u_3^-) t_4 u_3^- - 6\tilde{F}(x, t_4 u_3^-)) dx \\
 &\leq \sum_{i=1}^2 \left(\frac{1}{3} \|v_i\|^2 + \frac{1}{6} \int_{\Omega} (\tilde{f}(x, v_i) v_i - 6\tilde{F}(x, v_i)) dx \right) + \frac{1}{3} \|u_3^+\|^2 + \frac{1}{3} \|u_3^-\|^2 \\
 &\quad + \frac{1}{6} \int_{\Omega} (\tilde{f}(x, u_3^+) u_3^+ - 6\tilde{F}(x, u_3^+)) dx + \frac{1}{6} \int_{\Omega} (\tilde{f}(x, u_3^-) u_3^- - 6\tilde{F}(x, u_3^-)) dx.
 \end{aligned} \tag{2.16}$$

By using (2.14)-(2.16) and the fact that $u^+ = w_1$, $u^- = w_2$, we have

$$\begin{aligned}
 m &\leq J(\bar{t}_1 v_1 + \bar{t}_2 v_2 + \bar{t}_3 u^+ + \bar{t}_4 u^-) \\
 &< \sum_{i=1}^2 \left(\frac{1}{3} \|v_i\|^2 + \frac{1}{6} \int_{\Omega} (\tilde{f}(x, v_i) v_i - 6\tilde{F}(x, v_i)) dx \right) + \frac{1}{3} \|u^+\|^2 + \frac{1}{3} \|u^-\|^2 \\
 &\quad + \frac{1}{6} \int_{\Omega} (\tilde{f}(x, u^+) u^+ - 6\tilde{F}(x, u^+)) dx + \frac{1}{6} \int_{\Omega} (\tilde{f}(x, u^-) u^- - 6\tilde{F}(x, u^-)) dx \\
 &\quad + \frac{1}{2} \int_{\Omega} |\nabla w_3|^2 dx + \frac{1}{2} \int_{\Omega} \left(\int_0^{|x|} \frac{s}{2} (v_1^2(s) + v_2^2(s) + w_1^2(s) + w_2^2(s) + w_3^2(s)) ds \right)^2 \frac{w_3^2}{|x|^2} dx \\
 &\quad + \frac{1}{2} \int_{\Omega} \frac{v_1^2 + v_2^2 + w_1^2 + w_2^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} (v_1^2(s) + v_2^2(s) + w_1^2(s) \right. \\
 &\quad \left. + w_2^2(s)) ds \right) \left(\int_0^{|x|} s w_3^2(s) ds \right) dx \\
 &\quad + \frac{1}{2} \int_{\Omega} \frac{v_1^2 + v_2^2 + w_1^2 + w_2^2 + w_3^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} w_3^2(s) ds \right)^2 dx - \int_{\Omega} \tilde{F}(x, w_3) dx \\
 &= J(v) = m,
 \end{aligned} \tag{2.17}$$

which is a contradiction. So $w_3 = 0$, and v_3 has indeed two nodal domains.

3 Penalization of the nonlinearity and L^∞ -estimation

In this section, we will modify the functional I_λ by penalizing the nonlinearity $f(u)$. It plays a key role in establishing the relation between m_λ and m (will be defined later). On the other hand, we also give a delicate L^∞ -estimation for the critical points of the modified functional.

Since

$$\Omega = \cup_{i=1}^3 \Omega_i \text{ and } \text{dist}(\Omega_i, \Omega_j) > 0 \text{ for } i \neq j, \ i, j = 1, 2, 3,$$

there exist open sets $\Omega_i^\rho = \{x \in \mathbb{R}^2 : \text{dist}(x, \Omega_i) < \rho\}$ for $i = 1, 2, 3$ with smooth boundary such that $\text{dist}(\Omega_i^\rho, \Omega_j^\rho) > 0$ for $i \neq j, i, j = 1, 2, 3$. Denote $\Omega^\rho := \cup_{i=1}^3 \Omega_i^\rho$. For open set $\Theta \subset \mathbb{R}^2$, we define

$$H_\lambda(\Theta) := \left\{ u \in H_r^1(\Theta) \mid \int_\Theta V(|x|)u^2 dx < +\infty \text{ and } u = 0 \text{ in } \mathbb{R}^2 \setminus \Theta \right\}$$

with norm

$$\|u\|_{\lambda, \Theta}^2 = \int_\Theta |\nabla u|^2 + \lambda V(|x|)u^2 dx.$$

By (V_1) and (V_2) , there exists a positive constant v_0 such that

$$v_0 \int_{\mathbb{R}^2 \setminus \Omega^\rho} u^2 dx \leq \frac{1}{2} \|u\|_{\lambda, \mathbb{R}^2 \setminus \Omega^\rho}^2 \text{ for all } u \in H_\lambda(\mathbb{R}^2 \setminus \Omega^\rho). \quad (3.1)$$

Let $a_0 > 0$ satisfy

$$0 < \max \left\{ \frac{f(a_0)}{a_0}, -\frac{f(-a_0)}{a_0} \right\} \leq v_0,$$

and $\tilde{f}, \tilde{F} : \mathbb{R} \rightarrow \mathbb{R}$ are the functions given by

$$\tilde{f}(s) = \begin{cases} -\frac{f(-a_0)}{a_0}s, & \text{if } s < -a_0, \\ f(s), & \text{if } |s| \leq a_0, \\ \frac{f(a_0)}{a_0}s, & \text{if } s > a_0, \end{cases}$$

and

$$\tilde{F}(s) = \int_0^s \tilde{f}(\tau) d\tau.$$

Using the above notations, we denote

$$g(|x|, s) := \chi_{\Omega^\rho}(|x|)f(s) + (1 - \chi_{\Omega^\rho}(|x|))\tilde{f}(s)$$

and

$$G(|x|, s) := \int_0^s g(|x|, t) dt = \chi_{\Omega^\rho}(|x|)F(s) + (1 - \chi_{\Omega^\rho}(|x|))\tilde{F}(s),$$

where χ_{Ω^ρ} denotes the characteristic function of the set Ω^ρ . We define the functional $\Phi_\lambda : H_\lambda \rightarrow \mathbb{R}$

$$\Phi_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + \lambda V(|x|)u^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right)^2 dx - \int_{\mathbb{R}^2} G(|x|, u) dx \quad (3.2)$$

and the critical points of Φ_λ are weak solutions of

$$-\Delta u + \lambda V(|x|)u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) ds \right) u = g(|x|, u), \quad x \in \mathbb{R}^2. \quad (3.3)$$

The next Proposition is about the asymptotic behavior of the critical points of Φ_λ as $\lambda \rightarrow +\infty$.

Proposition 3.1 Suppose $\lambda_n \rightarrow +\infty$ as $n \rightarrow \infty$ and $\{u_{\lambda_n}\} \subset H_{\lambda_n}$ satisfying

$$\Phi_{\lambda_n}(u_{\lambda_n}) \rightarrow c \text{ and } \|\Phi'_{\lambda_n}(u_{\lambda_n})\|_{H_{\lambda_n}^*} \rightarrow 0.$$

Then, up to a subsequence, there exists $u \in H_r^1(\mathbb{R}^2)$ such that

(i) $\|u_n - u\|_{\lambda_n} \rightarrow 0$, consequently $u_{\lambda_n} \rightarrow u$ in $H_r^1(\mathbb{R}^2)$;

(ii) $u = 0$ in $\mathbb{R}^2 \setminus \Omega$ and u is a solution to equation (1.9);

(iii) $\Phi_{\lambda_n}(u_{\lambda_n}) \rightarrow J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right)^2 dx - \int_{\Omega} F(u) dx$.

Proof. It is easy to know that $\{u_{\lambda_n}\}$ is bounded in $H_{\lambda_n}(\mathbb{R}^2)$ and hence $\{u_{\lambda_n}\}$ is bounded in $H_r^1(\mathbb{R}^2)$. Passing to a subsequences, one has

$$\begin{aligned} u_n &\rightharpoonup u \text{ weakly in } H_r^1(\mathbb{R}^2); \\ u_n &\rightarrow u \text{ strongly in } L^p(\mathbb{R}^N) \text{ for } 2 < p < \infty; \\ u_n &\rightarrow u \text{ a.e. in } \mathbb{R}^2. \end{aligned} \quad (3.4)$$

We prove (ii) firstly. Let $m \in \mathbb{N}^+$, set $S_m = \{x \in \mathbb{R}^2 : V(|x|) \geq \frac{1}{m}\}$, one has

$$\int_{S_m} u_{\lambda_n}^2 dx \leq \frac{2m}{\lambda_n} \int_{S_m} \lambda_n V(|x|) u_{\lambda_n}^2 dx \leq \frac{2mC}{\lambda_n}, \quad (3.5)$$

and hence

$$\int_{S_m} u^2 dx \leq \liminf_{n \rightarrow \infty} \int_{S_m} u_{\lambda_n}^2 dx = 0.$$

It implies that $u \equiv 0$ in S_m . So we prove the $u \equiv 0$ in $\mathbb{R}^2 \setminus \bar{\Omega}$. Now, for any $\varphi \in C_0^\infty(\Omega)$, since $\langle \Phi'_{\lambda_n}(u_n), \varphi \rangle = 0$, we can deduce that u is a solution to equation (1.9).

Next, we prove $u_{\lambda_n} \rightarrow u$ in $H_r^1(\mathbb{R}^2)$. For convenience, let

$$Y(u) = \frac{1}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right)^2 dx$$

and $u_n = u_{\lambda_n}$. By virtue of $\langle \Phi'_{\lambda_n}(u_n), u_n - u \rangle = \langle \Phi'_{\lambda_n}(u), u_n - u \rangle = 0$ when $n \rightarrow +\infty$, we have

$$(u_n, u_n - u)_{\lambda_n} + \langle Y'(u_n), u_n - u \rangle = \int_{\mathbb{R}^2} g(|x|, u_n)(u_n - u) dx, \quad (3.6)$$

$$(u, u_n - u)_{\lambda_n} + \langle Y'(u), u_n - u \rangle = \int_{\mathbb{R}^2} g(|x|, u)(u_n - u) dx. \quad (3.7)$$

There, by Lemma 3.2 in [6], we have

$$\lim_{n \rightarrow \infty} (\langle Y'(u_n), u_n - u \rangle - \langle Y'(u), u_n - u \rangle) = 0. \quad (3.8)$$

Using the standard argument, we can deduce that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} g(|x|, u_n)(u_n - u) - g(|x|, u)(u_n - u) dx = 0.$$

Therefore, by (3.6), (3.7) and (3.8), we get

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{\lambda_n}^2 = 0.$$

On the other hand, the embedding $H_{\lambda} \hookrightarrow H_r^1(\mathbb{R}^2)$ is continuous. Indeed, by (V_1) and (V_2) , we can deduce that

$$V(x) \geq \delta_0 > 0, \quad x \in \mathbb{R}^2 \setminus \Omega^\rho,$$

so it is easy to get

$$\int_{\mathbb{R}^2 \setminus \Omega^\rho} |\nabla u|^2 + u^2 dx \leq C \int_{\mathbb{R}^2} |\nabla u|^2 + \lambda V(|x|) u^2 dx.$$

Thus we only need to show that

$$\int_{\Omega^p} |\nabla u|^2 + u^2 dx \leq C \int_{\mathbb{R}^2} |\nabla u|^2 + \lambda V(|x|) u^2 dx.$$

We choose a cut-off function $\Psi \in C^\infty(\mathbb{R}^2)$ such that $0 \leq \Psi \leq 1$ in $\mathbb{R}^2$, $\Psi(x) = 1$ for each $x \in \Omega^p$ and $\Psi(x) = 0$ for $x \in \mathbb{R}^2 \setminus \Omega^{2p}$ and $|\nabla \Psi| < C$. By Sobolev's embedding inequality,

$$\begin{aligned} \int_{\Omega^p} u^2 dx &\leq \int_{\Omega^{2p}} |u\Psi|^2 dx \leq C \int_{\Omega^{2p}} |\nabla(u\Psi)|^2 dx \\ &\leq 2C \int_{\Omega^{2p}} |\nabla u|^2 dx + 2C \int_{\Omega^{2p} \setminus \Omega^p} u^2 dx \\ &\leq C \int_{\mathbb{R}^2} |\nabla u|^2 + \lambda V(|x|) u^2 dx. \end{aligned}$$

So we get $\int_{\mathbb{R}^2} |\nabla u|^2 + u^2 dx \leq C \int_{\mathbb{R}^2} |\nabla u|^2 + \lambda V(|x|) u^2 dx$. Thus we can get that

$$\|u_n - u\|_{H^1_r(\mathbb{R}^2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Combining with (i), it is easy to prove the (iii).

The next Lemma is important which indicates that the critical points u_λ of Φ_λ with bounded energy are the solutions of the original problem (1.9) if λ large enough.

Lemma 3.2. Fix $M > 0$, for any critical points u_λ of $\Phi_\lambda(u_\lambda) \leq M$. Then there exists $\Lambda_0 > 0$ such that $\lambda \geq \Lambda_0$, one has

$$|u_\lambda(x)| \leq a_0 \text{ for all } x \in \mathbb{R}^2 \setminus \Omega^p. \quad (3.9)$$

Proof. We prove Lemma 3.2 by Moser's iteration. By Proposition 3.1, it is easy to get that

$$\lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}^2 \setminus \Omega} |u_\lambda|^q dx = 0, \quad \forall 2 < q < \infty. \quad (3.10)$$

So, for any small $\eta_0 > 0$ and λ large enough,

$$\int_{\mathbb{R}^2 \setminus \Omega} |u_\lambda|^6 dx \leq 2\eta_0. \quad (3.11)$$

Let ψ be a smooth cut-off function and $\beta > 1$, both of them will be specified later. For $R > 0$, we define

$$u_\lambda^R = \begin{cases} R, & \text{if } u_\lambda > R, \\ u_\lambda, & \text{if } |u_\lambda| \leq R, \\ -R, & \text{if } u_\lambda < -R, \end{cases}$$

and multiply (3.3) by $\psi^2 |u_\lambda^R|^{\beta-1} u_\lambda$, then

$$\begin{aligned} &\int_{\mathbb{R}^2} \nabla(\psi^2 |u_\lambda^R|^{\beta-1} u_\lambda) \nabla u_\lambda dx + \lambda \int_{\mathbb{R}^2} V(|x|) \psi^2 |u_\lambda^R|^{\beta-1} u_\lambda^2 dx \\ &+ \int_{\mathbb{R}^2} \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) ds \right) \psi^2 |u_\lambda^R|^{\beta-1} u_\lambda^2 dx \\ &= \int_{\mathbb{R}^2} g(|x|, u_\lambda) \psi^2 |u_\lambda^R|^{\beta-1} u_\lambda dx. \end{aligned} \quad (3.12)$$

That is

$$\begin{aligned} & \int_{\mathbb{R}^2} \psi^2 |u_\lambda^R|^{\beta-1} |\nabla u_\lambda|^2 dx + (\beta-1) \int_{\mathbb{R}^2} \psi^2 |u_\lambda^R|^{\beta-3} u_\lambda^R u_\lambda \nabla u_\lambda^R \nabla u_\lambda dx \\ & + 2 \int_{\mathbb{R}^2} \psi |u_\lambda^R|^{\beta-1} u_\lambda \nabla \psi \nabla u_\lambda dx + \lambda \int_{\mathbb{R}^2} V(|x|) \psi^2 |u_\lambda^R|^{\beta-1} u_\lambda^2 dx \\ & \leq \int_{\mathbb{R}^2} g(|x|, u_\lambda) \psi^2 |u_\lambda^R|^{\beta-1} u_\lambda dx. \end{aligned} \quad (3.13)$$

On the other hand, by Hölder's inequality and Young's inequality, one has

$$\left| \int_{\mathbb{R}^2} \psi |u_\lambda^R|^{\beta-1} u_\lambda \nabla \psi \nabla u_\lambda dx \right| \leq \frac{1}{4} \int_{\mathbb{R}^2} \psi^2 |u_\lambda^R|^{\beta-1} |\nabla u_\lambda|^2 dx + C \int_{\mathbb{R}^2} |\nabla \psi|^2 |u_\lambda^R|^{\beta-1} u_\lambda^2 dx.$$

Note $|g(|x|, u)u| \leq |u|^2 + C_0|u|^{q_0}$, so the inequality (3.13) leads to

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^2} \psi^2 |u_\lambda^R|^{\beta-1} |\nabla u_\lambda|^2 dx + (\beta-1) \int_{\mathbb{R}^2} \psi^2 |u_\lambda^R|^{\beta-3} u_\lambda^R u_\lambda \nabla u_\lambda^R \nabla u_\lambda dx \\ & \leq 2C \int_{\mathbb{R}^2} |\nabla \psi|^2 |u_\lambda^R|^{\beta-1} u_\lambda^2 dx + \int_{\mathbb{R}^2} \psi^2 u_\lambda^2 |u_\lambda^R|^{\beta-1} dx + C_0 \int_{\mathbb{R}^2} \psi^2 u_\lambda^{q_0} |u_\lambda^R|^{\beta-1} dx. \end{aligned} \quad (3.14)$$

By sobolev imbedding theorem, we have

$$\begin{aligned} & S(p) \left(\int_{\mathbb{R}^2} (\psi |u_\lambda^R|^{\frac{\beta-1}{2}} u_\lambda)^p dx \right)^{\frac{2}{p}} \leq \int_{\mathbb{R}^2} |\nabla (\psi |u_\lambda^R|^{\frac{\beta-1}{2}} u_\lambda)|^2 dx \\ & \leq \frac{(\beta+1)^2}{2} \int_{\mathbb{R}^2} \psi^2 |u_\lambda^R|^{\beta-1} |\nabla u_\lambda|^2 dx + 2 \int_{\mathbb{R}^2} |\nabla \psi|^2 |u_\lambda^R|^{\beta-1} u_\lambda^2 dx, \end{aligned} \quad (3.15)$$

where $S(p)$ is imbedding constant. Using (3.14) and (3.15), one has

$$\begin{aligned} & S(p) \left(\int_{\mathbb{R}^2} (\psi |u_\lambda^R|^{\frac{\beta-1}{2}} u_\lambda)^p dx \right)^{\frac{2}{p}} \leq (2 + 2(\beta+1)^2 C) \int_{\mathbb{R}^2} |\nabla \psi|^2 |u_\lambda^R|^{\beta-1} u_\lambda^2 dx \\ & + C_0(\beta+1)^2 \int_{\mathbb{R}^2} \psi^2 u_\lambda^{q_0} |u_\lambda^R|^{\beta-1} dx + (\beta+1)^2 \int_{\mathbb{R}^2} \psi^2 |u_\lambda^R|^{\beta-1} u_\lambda^2 dx. \end{aligned} \quad (3.16)$$

Now, for $y \in \mathbb{R}^2 \setminus \Omega^\rho$ and fix a r which $0 < r < \frac{\rho}{8}$. Then take the cut-off function ψ by

$$\psi(x) = \begin{cases} 1, & x \in B_{2r}(y), \\ 0, & \mathbb{R}^2 \setminus B_{4r}(y), \end{cases}$$

and $0 \leq \psi \leq 1$, $|\nabla \psi| \leq \frac{C}{r}$. Using Hölder's inequality and (3.10), we have

$$\begin{aligned} & \int_{\mathbb{R}^2} \psi^2 u_\lambda^{q_0} |u_\lambda^R|^{\beta-1} dx \leq \left(\int_{\mathbb{R}^2} (\psi^2 |u_\lambda^R|^{\beta-1} u_\lambda^2)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^2 \setminus \Omega} (u_\lambda^{q_0-2})^{\frac{p}{p-2}} dx \right)^{\frac{p-2}{p}} \\ & \leq \frac{S(p)}{2C_0(\beta+1)^2} \left(\int_{\mathbb{R}^2} (\psi^2 |u_\lambda^R|^{\beta-1} u_\lambda^2)^{\frac{p}{2}} dx \right)^{\frac{2}{p}}. \end{aligned} \quad (3.17)$$

Combing (3.16) and (3.17), we have

$$\begin{aligned} S(p) \left(\int_{\mathbb{R}^2} (\psi |u_\lambda^R|^{\frac{\beta-1}{2}} u_\lambda)^p dx \right)^{\frac{2}{p}} &\leq (4 + 4(\beta + 1)^2 C) \int_{\mathbb{R}^2} |\nabla \psi|^2 |u_\lambda^R|^{\beta-1} u_\lambda^2 dx \\ &+ 2(\beta + 1)^2 \int_{\mathbb{R}^2} \psi^2 |u_\lambda^R|^{\beta-1} u_\lambda^2 dx. \end{aligned} \quad (3.18)$$

Taking limit $R \rightarrow +\infty$ and $\beta = 5$ in (3.18), which implies that for any $y \in \mathbb{R}^2 \setminus \Omega^\rho$,

$$\left(\int_{B_{2r}(y)} |u_\lambda|^{3p} dx \right)^{\frac{2}{p}} \leq C(r, p) \int_{B_{4r}(y)} |u_\lambda|^6 dx. \quad (3.19)$$

Because $p \in (2, +\infty)$, we choose $p = \frac{3}{2}q_0 - 3 > 2$ since $q_0 > 4$, one has

$$\left(\int_{B_{2r}(y)} |u_\lambda|^{\frac{3}{2}q_0-9} dx \right)^{\frac{4}{3q_0-6}} \leq C(r) \int_{B_{4r}(y)} |u_\lambda|^6 dx. \quad (3.20)$$

Now, we use the above estimation combining with Moser's iteration argument to complete the proof. Let $Z_\lambda = |u_\lambda^R|^{\frac{\beta-1}{2}} u_\lambda$, where $\beta > 1$ will choose later, then (3.16) becomes

$$\begin{aligned} S(6) \left(\int_{\mathbb{R}^2} (\psi Z_\lambda)^6 dx \right)^{\frac{1}{3}} &\leq (2 + 2(\beta + 1)^2 C) \int_{\mathbb{R}^2} |\nabla \psi|^2 Z_\lambda^2 dx \\ &+ C_0(\beta + 1)^2 \int_{\mathbb{R}^2} \psi^2 Z_\lambda^2 u_\lambda^{q_0-2} dx + (\beta + 1)^2 \int_{\mathbb{R}^2} \psi^2 Z_\lambda^2 dx, \end{aligned} \quad (3.21)$$

where ψ is a cut-off function supported in $B_{2r}(y)$ with $y \in \mathbb{R}^2 \setminus \Omega^\rho$ and $r \leq \frac{\rho}{4}$. By the Hölder's inequality, we get

$$\int_{\mathbb{R}^2} \psi^2 Z_\lambda^2 u_\lambda^{q_0-2} dx \leq \left(\int_{\mathbb{R}^2} (\psi Z_\lambda)^{\frac{18}{7}} dx \right)^{\frac{7}{9}} \left(\int_{B_{2r}(y)} u_\lambda^{\frac{9}{2}q_0-9} dx \right)^{\frac{2}{9}}.$$

Since $2 < \frac{18}{7} < 6$, thus, for any $\epsilon > 0$,

$$\|\psi Z_\lambda\|_{\frac{18}{7}}^2 \leq \epsilon \|\psi Z_\lambda\|_6^2 + \epsilon^{-\frac{1}{2}} \|\psi Z_\lambda\|_2^2.$$

By (3.21) and above estimate, it deduces that

$$\begin{aligned} S(6) \left(\int_{\mathbb{R}^2} (\psi Z_\lambda)^6 dx \right)^{\frac{1}{3}} &\leq (2 + 2(\beta + 1)^2 C) \int_{\mathbb{R}^2} |\nabla \psi|^2 Z_\lambda^2 dx \\ &+ C_0 C_1(r)(\beta + 1)^2 (\epsilon \|\psi Z_\lambda\|_6^2 + \epsilon^{-\frac{1}{2}} \|\psi Z_\lambda\|_2^2) + (\beta + 1)^2 \int_{\mathbb{R}^2} \psi^2 Z_\lambda^2 dx, \end{aligned} \quad (3.22)$$

where

$$C_1(r) = \left(\int_{B_{2r}(y)} u_\lambda^{\frac{9}{2}q_0-9} dx \right)^{\frac{2}{9}} \leq \left(C(r) \int_{B_{4r}(y)} |u_\lambda|^6 dx \right)^{\frac{q_0-2}{6}} \leq [2\eta_0 C(r)]^{\frac{q_0-2}{6}}.$$

Setting $\epsilon = S(6)[2C_0 C_1(r)(\beta + 1)^2]^{-1}$, we obtain from (3.22) that

$$\left(\int_{\mathbb{R}^2} (\psi Z_\lambda)^6 dx \right)^{\frac{1}{3}} \leq \frac{(4 + 4(\beta + 1)^2 C)}{S} \int_{\mathbb{R}^2} |\nabla \psi|^2 Z_\lambda^2 dx + \frac{C_2}{r^{\frac{q_0-2}{4}}} (\beta + 1)^3 \int_{\mathbb{R}^2} \psi^2 Z_\lambda^2 dx. \quad (3.23)$$

Now, for $r \leq r_2 < r_1 \leq 2r$, we choose ψ such that $\psi \equiv 1$ in $B_{r_2}(y)$, $\psi \equiv 0$ in $\mathbb{R}^2 \setminus B_{r_1}(y)$ and $0 \leq \psi \leq 1$, $|\nabla \psi| \leq \frac{C}{r_1 - r_2}$. Then we obtain that

$$\|Z_\lambda\|_{L^6(B_{r_2}(y))} \leq \frac{C_3}{(r_1 - r_2)} h^{\frac{3}{2}} (r^{\frac{q_0-2}{4}})^{-\frac{1}{2}} \|Z_\lambda\|_{L^2(B_{r_1}(y))}, \quad (3.24)$$

where $h = (1 + \beta)$. Set

$$L(p, r) := \left(\int_{B_r(y)} |u_\lambda|^p dx \right)^{\frac{1}{p}}.$$

When $R \rightarrow +\infty$ in (3.24), then we have

$$L(3h, r_2) \leq \left(\frac{C_3}{(r_1 - r_2)} \right)^{\frac{2}{h}} h^{\frac{3}{h}} (r^{\frac{q_0-2}{4}})^{-\frac{1}{h}} L(h, r_1). \quad (3.25)$$

Let $h = h_m = 6 \cdot 3^m$, $r_m = r(1 + 2^{-m})$ for $m = 0, 1, 2, \dots$, by (3.25), we get

$$\begin{aligned} L(6 \cdot 3^{m+1}, r^{m+1}) &= L(3h_m, r_{m+1}) \leq \left(\frac{C_3}{r_m - r_{m+1}} \right)^{\frac{2}{h_m}} h_m^{\frac{3}{h_m}} (r^{\frac{q_0-2}{4}})^{-\frac{1}{h_m}} L(h_m, r_m) \\ &= \left(\frac{2C_3}{r^{\frac{q_0+6}{8}}} 6^{\frac{3}{2}} \right)^{\frac{1}{3 \cdot 3^m}} \left(2 \cdot 3^{\frac{3}{2}} \right)^{\frac{m}{3 \cdot 3^m}} L(h_m, r_m) \leq \dots \\ &\leq \left(\frac{2C_3}{r^{\frac{q_0+6}{8}}} 6^{\frac{3}{2}} \right)^{\frac{1}{3} \sum_{j=0}^{\infty} 3^{-j}} \left(2 \cdot 3^{\frac{3}{2}} \right)^{\frac{1}{3} \sum_{j=0}^{\infty} j 3^{-j}} L(6, 2r). \end{aligned} \quad (3.26)$$

Let $m \rightarrow \infty$, we have

$$\sup_{x \in B_r(y)} |u(x)| = \lim_{s \rightarrow \infty} L(s, r) \leq C_4(r)(2\eta_0)^{\frac{1}{6}}. \quad (3.27)$$

We can choose Λ_0 , when $\lambda \geq \Lambda_0$, we have $C_4(r)(2\eta_0)^{\frac{1}{6}} \leq a_0$. So we can get

$$\|u_{\lambda_n}\|_{L^\infty(\mathbb{R}^2 \setminus \Omega^\rho)} \leq a_0$$

for $\lambda \geq \Lambda_0$. Therefore we complete the proof.

4 A special minimax value for the modified functional

We investigate a special minimax value for the modified functional Φ_λ , which is used to get a key Lemma. We define a new functional

$$\bar{J}_\lambda(u) = \frac{1}{2} \int_{\Omega^\rho} |\nabla u|^2 + \lambda V(|x|)u^2 dx + \frac{1}{2} \int_{\Omega^\rho} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right)^2 dx - \int_{\Omega^\rho} F(u) dx \quad (4.1)$$

which is well defined and belongs to $C^1(H_\lambda(\Omega^\rho), \mathbb{R})$. We define set

$$\tilde{\mathcal{M}} = \{u \in H_\lambda(\Omega^\rho) \mid \langle \bar{J}'(u), u_i \rangle = 0 \text{ for } i = 1, 2, u_1^+ \neq 0, u_2^- \neq 0 \text{ and } \langle \bar{J}'(u), u_3^+ \rangle = 0, u_3^+ \neq 0\}$$

and

$$\bar{m}_\lambda := \inf_{u \in \tilde{\mathcal{M}}} \bar{J}_\lambda(u).$$

Similar to the proof of Section 2, we deduce that there exists $\bar{v}_\lambda \in H_\lambda(\Omega^\rho)$ such that

$$\bar{J}_\lambda(\bar{v}_\lambda) = \bar{m}_\lambda \text{ and } \bar{J}'_\lambda(\bar{v}_\lambda) = 0.$$

Lemma 4.1. *There holds that*

(i) $0 < \bar{m}_\lambda \leq m$, for all $\lambda > 0$;

(ii) $\bar{m}_\lambda \rightarrow m$, as $\lambda \rightarrow +\infty$.

Proof. The proof of (i) is trivial since $u \in \mathcal{M}$ which also belongs to $\tilde{\mathcal{M}}$ by zero extension.

Now we are going to prove (ii). Let $\{\lambda_n\}$ be a sequence with $\lambda_n \rightarrow +\infty$. For each λ_n , there exists $u_{\lambda_n} \in H_{\lambda_n}(\Omega^\rho)$ with

$$\bar{J}_{\lambda_n}(u_{\lambda_n}) = \bar{m}_{\lambda_n} \text{ and } \bar{J}'_{\lambda_n}(u_{\lambda_n}) = 0. \quad (4.2)$$

Since $\bar{m}_{\lambda_n} \leq m$, we can suppose $\{\bar{J}(u_{\lambda_n})\}$ convergence (up to a subsequence) and $\bar{J}'_{\lambda_n}(u_{\lambda_n}) = 0$. It is easy to know that there exists $u \in H_0^1(\Omega) \cap H_r^1(\Omega) \subset H_\lambda(\Omega^\rho)$ such that

$$u_{\lambda_n} \rightarrow u \text{ in } H_\lambda(\Omega^\rho) \text{ as } n \rightarrow +\infty$$

and $(u|_{\Omega_1})^+, (u|_{\Omega_2})^-, (u|_{\Omega_3})^+ \neq 0$. Moreover,

$$\bar{m}_{\lambda_n} = \bar{J}_{\lambda_n}(u_{\lambda_n}) \rightarrow J(u),$$

and

$$0 = \bar{J}'_{\lambda_n}(u_{\lambda_n}) \rightarrow J'(u).$$

By the definition of m , one has that

$$\lim_{\lambda_n \rightarrow +\infty} \bar{m}_{\lambda_n} = J(u) \geq m.$$

Using conclusion (i), we obtain that $\bar{m}_{\lambda_n} \rightarrow m$ as $n \rightarrow +\infty$.

In Section 2, we have known that there exists $v \in H$, that is

$$v \in \mathcal{M}, J(v) = m, J'(v) = 0. \quad (4.3)$$

and $v_1 = v|_{\Omega_1}$ is positive, $v_2 = v|_{\Omega_2}$ is negative and $v_3 = v|_{\Omega_3}$ changes sign exactly once. At the same time, we can find two positive constants $\tau_2 > \tau_1$ such that

$$\tau_1 \leq \|v_1\|, \|v_2\|, \|v_3^+\|, \|v_3^-\| \leq \tau_2. \quad (4.4)$$

We define $y_0 : [\frac{1}{2}, \frac{3}{2}] \rightarrow H_\lambda$ by

$$y_0(t_1, t_2, t_3, t_4) := t_1 v_1 + t_2 v_2 + t_3 v_3^+ + t_4 v_3^- \quad (4.5)$$

and

$$m_\lambda := \inf_{y \in \Sigma_\lambda} \max_{t \in [\frac{1}{2}, \frac{3}{2}]^4} \Phi_\lambda(y(t)), \quad (4.6)$$

where

$$\Sigma_\lambda := \left\{ y \in C\left([\frac{1}{2}, \frac{3}{2}]^4, H_\lambda\right) : \|y(t)\|_\lambda \leq 6\tau_2 + \tau_1, (y|_{\Omega_1^\rho})^+, (y|_{\Omega_2^\rho})^-, (y|_{\Omega_3^\rho})^\pm \neq 0 \right. \\ \left. \text{and } y = y_0 \text{ on } \partial\left[\frac{1}{2}, \frac{3}{2}\right]^4 \right\}. \quad (4.7)$$

It is easy to check $y_0 \in \Sigma_\lambda$, so $\Sigma_\lambda \neq \emptyset$ and m_λ is well defined.

The next Lemma is trivial by degree theory, so we omit the detail.

Lemma 4.2. For any $y \in \Sigma_\lambda$, there exists an 4-tuple $t^* = (t_1^*, t_2^*, t_3^*, t_4^*) \in D = (\frac{1}{2}, \frac{3}{2})^4$ such that $\langle \bar{J}'_\lambda(y(t^*)|_{\Omega^\rho}), y_1^+(t^*) \rangle = \langle \bar{J}'_\lambda(y(t^*)|_{\Omega^\rho}), y_2^-(t^*) \rangle = 0$ and $\langle \bar{J}'_\lambda(y(t^*)|_{\Omega^\rho}), y_3^\pm(t^*) \rangle = 0$ where $y_i(t) = y(t)|_{\Omega_i^\rho}$ for $i = 1, 2, 3$.

Lemma 4.3. There holds that

(i) $\bar{m}_\lambda \leq m_\lambda \leq m$ for all $\lambda \geq 1$;

(ii) $m_\lambda \rightarrow m$ as $\lambda \rightarrow +\infty$;

(iii) There exists $\epsilon_0 > 0$ such that $\Phi_\lambda(y(t)) < m - \epsilon_0$ for all $\lambda \geq 0$, $y \in \Sigma_\lambda$ and $t = (t_1, t_2, t_3, t_4) \in \partial[\frac{1}{2}, \frac{3}{2}]^4$.

Proof. (i) Since $y_0 \in \Sigma_\lambda$, we have

$$m_\lambda \leq \max_{t \in [\frac{1}{2}, \frac{3}{2}]^4} \Phi_\lambda(y_0(t)) = \max_{t \in [\frac{1}{2}, \frac{3}{2}]^4} J(y_0(t)) = m.$$

Now, fixing $t^* \in (\frac{1}{2}, \frac{3}{2})^4$ given by Lemma 4.2, it implies

$$\bar{m}_\lambda \leq \bar{J}_\lambda(y(t^*))|_{\Omega^\rho}.$$

By the definition of $g(|x|, u)$, we deduce that $|G(|x|, u)| \leq \frac{v_0}{2} u^2$ for $x \in \mathbb{R}^2 \setminus \Omega^\rho$. By (3.1) we can get

$$\Phi_\lambda(y(t^*)) \geq \bar{J}_\lambda(y(t^*))|_{\Omega^\rho}.$$

Therefore,

$$\max_{t \in [\frac{1}{2}, \frac{3}{2}]^4} \Phi_\lambda(y(t)) \geq \bar{J}_\lambda(y(t^*))|_{\Omega^\rho} \geq \bar{m}_\lambda, \text{ for each } y \in \Sigma_\lambda.$$

So $m_\lambda \geq \bar{m}_\lambda$.

(ii) it is obtained by Lemma 4.1 (ii) and Lemma 4.3 (i).

(iii) For $t = (t_1, t_2, t_3, t_4) \in \partial[\frac{1}{2}, \frac{3}{2}]^4$, we have

$$\Phi_\lambda(y(t)) = J(y_0(t)) \text{ for } t = (t_1, t_2, t_3, t_4) \in \partial[\frac{1}{2}, \frac{3}{2}]^4.$$

By Lemma 2.1, it is to get

$$\Phi_\lambda(y(t)) < m - \epsilon_0 \text{ for } t = (t_1, t_2, t_3, t_4) \in \partial[\frac{1}{2}, \frac{3}{2}]^4,$$

where ϵ_0 is a small positive constant.

5 Proof of Theorem 1.1

In this section, we prove our main results. Define

$$S := \{u \in \mathcal{M} \mid J(u) = m\}.$$

Then we need further to study the properties of the set S .

Lemma 5.1. S is compact in H .

Proof. The proof is standard, we omit it immediately.

Lemma 5.2. Let $d > 0$ be a fixed number and let $\{u_n\} \subset S^d$ be a sequence. Then, up to a subsequence, $u_n \rightharpoonup u_0$ in H_λ as $n \rightarrow \infty$, and $u_0 \in S^{2d}$ where

$$S^d := \{u \in H_\lambda : \text{dist}_\lambda(u, S) \leq d\}$$

and dist_λ denotes the distance in H_λ .

Proof. Since S is compact in H , we can choose $\{\bar{u}_n\} \subset S$ satisfy

$$\text{dist}_\lambda(u_n, \bar{u}_n) \leq d.$$

On the other hand, there exists $\bar{u} \in S$ such that, up to a subsequence, $\bar{u}_n \rightarrow \bar{u}$ in H . Hence, $\text{dist}(u_n, \bar{u}) \leq d$ for n large enough. Thus $\{u_n\}$ is bounded in H_λ . Up to a subsequence, $u_n \rightharpoonup u_0$ weakly in H_λ . Since $B_{2d}(u)$ is weakly closed in H_λ , so $u_0 \in B_{2d}(u) \subset S^{2d}$.

Lemma 5.3. Let $d \in (0, \tau_1)$, where τ_1 is given by (4.4). Suppose that there exist a sequence $\lambda_n > 0$ with $\lambda_n \rightarrow \infty$, and $\{u_n\} \subset S^d$ satisfying

$$\lim_{n \rightarrow \infty} \Phi_{\lambda_n}(u_n) \leq m, \quad \lim_{n \rightarrow \infty} \Phi'_{\lambda_n}(u_n) = 0.$$

Then, up to a subsequence, $\{u_n\}$ converges strongly in $H_r^1(\mathbb{R}^2)$ to an element $u \in S$.

Proof. Observe that, by $\lim_{n \rightarrow \infty} \Phi_{\lambda_n}(u_n) \leq m$ and $\lim_{n \rightarrow \infty} \Phi'_{\lambda_n}(u_n) = 0$ we deduce that $\{\|u_n\|_{\lambda_n}\}$ and $\{\Phi_{\lambda_n}(u_n)\}$ are bounded. Up to a subsequence, we may assume that $\Phi_{\lambda_n}(u_n) \rightarrow c \leq m$. By Proposition 3.1, there exists $u \in H_r^1(\mathbb{R}^2)$ such that $u_n \rightarrow u$ in $H_r^1(\mathbb{R}^2)$, $u = 0$ in $\mathbb{R}^2 \setminus \Omega$ and $\Phi_{\lambda_n}(u_n) \rightarrow J(u)$. Moreover, u is a solution of equation (1.9). Next we prove that $u \in S$. Since $\{u_n\} \subset S^d$ and $d \in (0, \tau_1)$, we can deduce that $(u|_{\Omega^1})^+ \neq 0$, $(u|_{\Omega^2})^- \neq 0$ and $(u|_{\Omega^3})^\pm \neq 0$. Indeed, if the conclusion is not correct, we can assume $(u|_{\Omega^1})^+ = 0$, we can choose $\{\tilde{u}_n\} \subset S$ satisfy

$$\text{dist}_\lambda(u_n, \tilde{u}_n) \leq d,$$

so

$$\tau_1 \leq \|\tilde{u}_{n,1}\| \leq \|\tilde{u}_{n,1} - u_{n,1}\|_{H_{\lambda_n}} + \|u_{n,1}\| \leq d,$$

which implies that a contradiction. Hence, by Proposition 3.1 again, we get $J'(u) = 0$, $u = 0$ in $\mathbb{R}^2 \setminus \Omega$. Then we get that $J(u) \geq m$. At the same time, $\Phi_{\lambda_n}(u_n) \rightarrow J(u) \leq m$, therefore $u \in S$.

Lemma 5.4. Let $\delta \in (0, \tau_1)$, where τ_1 is given by (4.4). Then there exist constants $0 < \sigma < 1$ and $\Lambda_1 > 0$ such that $\|\Phi'_\lambda(u)\|_{H_\lambda^*} \geq \sigma$ for any $u \in \Phi_\lambda^m \cap (S^\delta \setminus S^{\frac{\delta}{2}})$ and $\lambda \geq \Lambda_1$.

Proof. We prove it by contradiction. Suppose that there exist a number $\delta_0 \in (0, \tau_1)$, a positive sequence $\{\lambda_j\}$ with $\lambda_j \rightarrow 0$, and a sequence of function $\{u_j\} \subset \Phi_{\lambda_j}^m \cap (S^{\delta_0} \setminus S^{\frac{\delta_0}{2}})$ such that

$$\lim_{j \rightarrow +\infty} \Phi'_{\lambda_j}(u_j) = 0.$$

Up to a subsequence, we get $\{u_j\} \subset S^{\delta_0}$ and $\lim_{j \rightarrow \infty} \Phi_{\lambda_j}(u_j) \leq m$. By Lemma 5.3, we can deduce that there exists $u \in S$ such that $u_j \rightarrow u$ in $H_{\lambda_j}(\mathbb{R}^2)$. Therefore, $\text{dist}_{\lambda_j}(u_j, S) \rightarrow 0$ as $j \rightarrow +\infty$. This contradicts the assumption that $u_j \notin S^{\frac{\delta_0}{2}}$.

Lemma 5.5. There exist $\Lambda_2 \geq \Lambda_1$ and $\alpha > 0$ such that for any $\lambda \geq \Lambda_2$,

$$\Phi_\lambda(y_0(t_1, t_2, t_3, t_4)) \geq m_\lambda - \alpha$$

implies that $y_0(t_1, t_2, t_3, t_4) \in S^{\frac{\delta}{2}}$ for some $\delta \in (0, \tau_1)$.

Proof. We argue by contradiction. There exist $\lambda_n \rightarrow \infty$, $\alpha_n \rightarrow 0$ and $(t_1^{(n)}, t_2^{(n)}, t_3^{(n)}, t_4^{(n)}) \in [\frac{1}{2}, \frac{3}{2}]^4$ such that

$$\Phi_\lambda(y_0(t_1^{(n)}, t_2^{(n)}, t_3^{(n)}, t_4^{(n)})) \geq m_{\lambda_n} - \alpha_n \text{ and } y_0(t_1^{(n)}, t_2^{(n)}, t_3^{(n)}, t_4^{(n)}) \notin S^{\frac{\delta}{2}}.$$

We can choose a subsequence $(t_1^{(n)}, t_2^{(n)}, t_3^{(n)}, t_4^{(n)}) \rightarrow (\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4) \in [\frac{1}{2}, \frac{3}{2}]^4$. Then we have

$$J(y_0(\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4)) \geq \lim_{n \rightarrow \infty} (m_{\lambda_n} - \alpha_n) = m.$$

By the unique of 4-tuple, it is easy to have $(\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4) = (1, 1, 1, 1)$. It implies that

$$\lim_{n \rightarrow \infty} \|y_0(t_1^{(n)}, t_2^{(n)}, t_3^{(n)}, t_4^{(n)}) - y_0(1, 1, 1, 1)\| = 0.$$

Since $y_0(1, 1, 1, 1) = v \in S$, which contradicts the assumption.

Now, we define

$$\alpha_0 := \min\left\{\frac{\alpha}{2}, \frac{\epsilon_0}{2}, \frac{1}{8}\delta\sigma^2\right\}, \quad (5.1)$$

where $\delta, \sigma, \alpha, \epsilon_0$ are from Lemma 5.4, Lemma 5.5 and Lemma 4.3-(iii) respectively. Using Lemma 4.2, one has that there exists $\Lambda_3 \geq \Lambda_2$ such that

$$|m_\lambda - m| < \alpha_0 \text{ for all } \lambda \geq \Lambda_3. \quad (5.2)$$

Lemma 5.6. There exists a critical point u_λ of Φ_λ with $u_\lambda \in S^\delta \cap \Phi_\lambda^m$ for $\lambda \geq \Lambda_3$.

Proof. We argue by contradiction. Fix a $\lambda \geq \Lambda_3$, by the Lemma 5.3, we can assume that there exists $0 < \rho_\lambda < 1$ such that $\|\Phi'_\lambda(u)\| \geq \rho_\lambda$ on $S^\delta \cap \Phi_\lambda^m$. There exists a pseudo-gradient vector field K_λ in H_λ which is defined on a neighborhood Z_λ of $S^\delta \cap \Phi_\lambda^m$ such that for any $u \in Z_\lambda$ there holds

$$\begin{cases} \|K_\lambda(u)\| \leq 2 \min\{1, \|\Phi'_\lambda(u)\|\}, \\ \langle \Phi'_\lambda(u), K_\lambda(u) \rangle \geq \min\{1, \|\Phi'_\lambda(u)\|\} \|\Phi'_\lambda(u)\|. \end{cases}$$

Define ψ_λ be a Lipschitz function on H_λ such that $0 \leq \psi_\lambda \leq 1$, $\psi_\lambda \equiv 1$ on $S^\delta \cap \Phi_\lambda^m$ and $\psi_\lambda \equiv 0$ on $H_\lambda \setminus Z_\lambda$. Define ξ_λ be a Lipschitz function on $\mathbb{R}$ such that $0 \leq \xi_\lambda \leq 1$, $\xi_\lambda(t) \equiv 1$ if $|t - m_\lambda| \leq \frac{\alpha}{2}$ and $\xi_\lambda(t) \equiv 0$ if $|t - m_\lambda| \geq \alpha$. Let

$$e_\lambda(u) := \begin{cases} -\psi_\lambda(u)\xi_\lambda(\Phi_\lambda(u))K_\lambda(u), & \text{if } u \in Z_\lambda, \\ 0, & \text{if } u \in H_\lambda \setminus Z_\lambda. \end{cases}$$

Then there exists a global solution $\eta_\lambda : H_\lambda \times [0, +\infty) \rightarrow H_\lambda$ for the initial value problem

$$\begin{cases} \frac{d}{d\theta}\eta_\lambda(u, \theta) = e_\lambda(\eta_\lambda(u, \theta)), \\ \eta_\lambda(u, 0) = u. \end{cases}$$

We can deduce that η_λ has the following properties:

- (i) $\eta_\lambda(u, \theta) = u$ if $\theta = 0$ or $u \in H_\lambda \setminus Z_\lambda$ or $|\Phi_\lambda(u) - m_\lambda| \geq \alpha$;
- (ii) $\|\frac{d}{d\theta}\eta_\lambda(u, \theta)\| \leq 2$;
- (iii) $\frac{d}{d\theta}\Phi_\lambda(\eta_\lambda(u, \theta)) = \langle \Phi'_\lambda(\eta_\lambda(u, \theta)), e_\lambda(\eta_\lambda(u, \theta)) \rangle \leq 0$.

Assertion 1. For any $(t_1, t_2, t_3, t_4) \in [\frac{1}{2}, \frac{3}{2}]^4$, there exists $\bar{\theta} = \theta(t_1, t_2, t_3, t_4) \in [0, +\infty)$ such that $\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \bar{\theta}) \in \Phi_\lambda^{m_\lambda - \alpha_0}$.

Assuming by contradiction that there exists $(t_1, t_2, t_3, t_4) \in [\frac{1}{2}, \frac{3}{2}]^4$ such that

$$\Phi_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta)) > m_\lambda - \alpha_0$$

for any $\theta \geq 0$. By Lemma 5.5, we get $y_0(t_1, t_2, t_3, t_4) \in S^{\frac{\delta}{2}}$. Note $\Phi_\lambda(y_0(t_1, t_2, t_3, t_4)) \leq m_\lambda + \alpha_0$, due the property (3) of η_λ ,

$$m_\lambda - \alpha_0 < \Phi_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta)) < \Phi_\lambda(y_0(t_1, t_2, t_3, t_4)) \leq m_\lambda + \alpha_0.$$

So we can deduce that $\xi_\lambda(\Phi_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta))) \equiv 1$. If $\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta) \in S^\delta$ for all $\theta \geq 0$, so it imply that

$$\psi(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta)) \equiv 1 \text{ and } \|\Phi'_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta))\| \geq \rho_\lambda \text{ for all } \theta > 0.$$

It follows that

$$\Phi_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \frac{\alpha}{\rho_\lambda^2})) \leq m_\lambda + \frac{\alpha}{2} - \int_0^{\frac{\alpha}{\rho_\lambda^2}} \rho_\lambda^2 dt \leq m_\lambda - \frac{\alpha}{2} \leq m_\lambda - \alpha_0,$$

which is a contradiction. Thus, there exists $\theta_3 > 0$ such that $\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta_3) \notin S^\delta$. Note that $y_0(t_1, t_2, t_3, t_4) \in S^{\frac{\delta}{2}}$, there exist $0 < \theta_1 < \theta_2 < \theta_3$ such that

$$\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta_1) \in \partial S^{\frac{\delta}{2}}, \quad \eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta_2) \in \partial S^\delta$$

and $\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta) \in S^\delta \setminus S^{\frac{\delta}{2}}$ for all $\theta \in (\theta_1, \theta_2)$. By Lemma 5.4, one has

$$\|\Phi'_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta))\| \geq \sigma \text{ for all } \theta \in (\theta_1, \theta_2).$$

Using the property (2) of η_λ we get that

$$\frac{\delta}{2} \leq \|\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta_2) - \eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta_1)\| \leq 2|\theta_2 - \theta_1|.$$

This deduce that

$$\begin{aligned} \Phi_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta_2)) &\leq \Phi_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), 0)) + \int_0^{\theta_2} \frac{d}{d\theta} \Phi_\lambda(\eta_\lambda(u, v, \theta)) d\theta \\ &< \Phi_\lambda(y_0(t_1, t_2, t_3, t_4)) + \int_{\theta_1}^{\theta_2} \frac{d}{d\theta} \Phi_\lambda(\eta_\lambda(u, v, \theta)) d\theta \\ &\leq m_\lambda + \alpha_0 - \sigma^2(\theta_2 - \theta_1) \leq m_\lambda + \alpha_0 - \frac{1}{4}\delta\sigma^2 \\ &\leq m_\lambda - \alpha_0, \end{aligned}$$

which is a contradiction. Therefore, we prove the assertion 1.

Now, we define

$$\Gamma(t_1, t_2, t_3, t_4) := \inf\{\theta \geq 0 : \Phi_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta)) \leq m_\lambda - \alpha_0\}$$

and

$$\tilde{y}(t_1, t_2, t_3, t_4) := \eta_\lambda(y_0(t_1, t_2, t_3, t_4), \Gamma(t_1, t_2, t_3, t_4)).$$

Then $\Phi_\lambda(\tilde{y}(t_1, t_2, t_3, t_4)) \leq m_\lambda - \alpha_0$ for all $(t_1, t_2, t_3, t_4) \in [\frac{1}{2}, \frac{3}{2}]^4$.

Assertion 2. $\tilde{y}(t_1, t_2, t_3, t_4) = \eta_\lambda(y_0(t_1, t_2, t_3, t_4), \Gamma(t_1, t_2, t_3, t_4)) \in \Sigma_\lambda$.

For any $(t_1, t_2, t_3, t_4) \in \partial[\frac{1}{2}, \frac{3}{2}]^4$, we have

$$\Phi_\lambda(y_0(t_1, t_2, t_3, t_4)) \leq J(y_0(t_1, t_2, t_3, t_4)) < m - \epsilon_0 \leq m_\lambda + \alpha_0 - \epsilon_0 \leq m_\lambda - \alpha_0,$$

which implies that $\Gamma(t_1, t_2, t_3, t_4) = 0$. So $\tilde{y}(t_1, t_2, t_3, t_4) = y_0(t_1, t_2, t_3, t_4)$ for $(t_1, t_2, t_3, t_4) \in \partial[\frac{1}{2}, \frac{3}{2}]^4$. We also need to prove $\|\tilde{y}(t_1, t_2, t_3, t_4)\| \leq 6\tau_2 + \tau_1$ for all $[\frac{1}{2}, \frac{3}{2}]^4$ and $\Gamma(t_1, t_2, t_3, t_4)$ is continuous with respect to (t_1, t_2, t_3, t_4) .

For any $(t_1, t_2, t_3, t_4) \in [\frac{1}{2}, \frac{3}{2}]^4$, we have $\Gamma(t_1, t_2, t_3, t_4) = 0$ if $\Phi_\lambda(y_0(t_1, t_2, t_3, t_4)) \leq m_\lambda - \alpha_0$, so $\tilde{y}(t_1, t_2, t_3, t_4) = y_0(t_1, t_2, t_3, t_4)$. By the definition of $y_0(t)$, we have $\|\tilde{y}(t_1, t_2, t_3, t_4)\| \leq 6\tau_2 < 6\tau_2 + \tau_1$.

On the other hand, if $\Phi_\lambda(y_0(t_1, t_2, t_3, t_4)) > m_\lambda - \alpha_0$, it implies that $y_0(t_1, t_2, t_3, t_4) \in S^{\frac{\delta}{2}}$ and

$$m_\lambda - \alpha_0 < \Phi_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta)) < m_\lambda + \alpha_0, \text{ for all } \theta \in [0, \Gamma(t_1, t_2, t_3, t_4)).$$

So one has

$$\xi_\lambda(\Phi_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta))) \equiv 1 \text{ for all } \theta \in [0, \Gamma(t_1, t_2, t_3, t_4)).$$

Now, we will prove $\tilde{y}(t_1, t_2, t_3, t_4) \in S^\delta$. Otherwise, $\tilde{y}(t_1, t_2, t_3, t_4) \notin S^\delta$, similar to the proof of assertion 1, we can find two constants $0 < \theta_1 < \theta_2 < \Gamma(t_1, t_2, t_3, t_4)$ such that

$$\Phi_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \theta_2)) < m_\lambda - \alpha_0.$$

It contradicts to the definition of $\Gamma(t_1, t_2, t_3, t_4)$. Therefore

$$\tilde{y}(t_1, t_2, t_3, t_4) = \eta_\lambda(y_0(t_1, t_2, t_3, t_4), \Gamma(t_1, t_2, t_3, t_4)) \in S^\delta.$$

Thus, there exists $u \in S$, such that

$$\|\tilde{y}(t_1, t_2, t_3, t_4)\| \leq \|u\| + \tau_1 \leq 6\tau_2 + \tau_1.$$

To prove the continuity of $\Gamma(t_1, t_2, t_3, t_4)$, we fix arbitrarily $(t_1, t_2, t_3, t_4) \in [\frac{1}{2}, \frac{3}{2}]^4$. First, we assume that $\Phi_\lambda(\tilde{y}(t_1, t_2, t_3, t_4)) < m_\lambda - \alpha_0$. In this case, it is to see that $\Gamma(t_1, t_2, t_3, t_4) = 0$, which gives that $\Phi_\lambda(y_0(t_1, t_2, t_3, t_4)) < m_\lambda - \alpha_0$. By the continuity of y_0 , there exists $r > 0$ such that for any $(s_1, s_2, s_3, s_4) \in B_r(t_1, t_2, t_3, t_4) \cap [\frac{1}{2}, \frac{3}{2}]^4$, we have $\Phi_\lambda(y_0(s_1, s_2, s_3, s_4)) < m_\lambda - \alpha_0$, so $\Gamma(s_1, s_2, s_3, s_4) = 0$, and hence Γ is continuous at (t_1, t_2, t_3, t_4) . On the other hand, we assume that $\Phi_\lambda(\tilde{y}(t_1, t_2, t_3, t_4)) = m_\lambda - \alpha_0$. Similar to the proof of assertion 1, we can deduce that $\tilde{y}(t_1, t_2, t_3, t_4) = \eta_\lambda(y_0(t_1, t_2, t_3, t_4), \Gamma(t_1, t_2, t_3, t_4)) \in S^\delta$, thus

$$\|\Phi'_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \Gamma(t_1, t_2, t_3, t_4)))\| \geq \rho_\lambda > 0.$$

Therefore, for any $w > 0$, we have

$$\Phi_\lambda(\eta_\lambda(y_0(t_1, t_2, t_3, t_4), \Gamma(t_1, t_2, t_3, t_4) + w)) < m_\lambda - \alpha_0.$$

Using the continuity of η_λ , there exists $r > 0$ such that

$$\Phi_\lambda(\eta_\lambda(y_0(s_1, s_2, s_3, s_4), \Gamma(t_1, t_2, t_3, t_4) + w)) < m_\lambda - \alpha_0$$

for any $(s_1, s_2, s_3, s_4) \in B_r(t_1, t_2, t_3, t_4) \cap [\frac{1}{2}, \frac{3}{2}]^4$. Thus, $\Gamma(s_1, s_2, s_3, s_4) \leq \Gamma(t_1, t_2, t_3, t_4) + w$. It follows that

$$0 \leq \limsup_{(s_1, s_2, s_3, s_4) \rightarrow (t_1, t_2, t_3, t_4)} \Gamma(s_1, s_2, s_3, s_4) \leq \Gamma(t_1, t_2, t_3, t_4). \quad (5.3)$$

If $\Gamma(t_1, t_2, t_3, t_4) = 0$, we immediately obtain that

$$\lim_{(s_1, s_2, s_3, s_4) \rightarrow (t_1, t_2, t_3, t_4)} \Gamma(s_1, s_2, s_3, s_4) = \Gamma(t_1, t_2, t_3, t_4).$$

If $\Gamma(t_1, t_2, t_3, t_4) > 0$, we can similarly deduce that

$$\Phi_\lambda(\eta_\lambda(y_0(s_1, s_2, s_3, s_4), \Gamma(t_1, t_2, t_3, t_4) - w)) > m_\lambda - \alpha_0$$

for any $0 < w < \Gamma(t_1, t_2, t_3, t_4)$.

By the continuity of η_λ again, we see that

$$\liminf_{(s_1, s_2, s_3, s_4) \rightarrow (t_1, t_2, t_3, t_4)} \Gamma(s_1, s_2, s_3, s_4) \geq \Gamma(t_1, t_2, t_3, t_4). \quad (5.4)$$

Combining (5.3) and (5.4), it is easy to see Γ is continuous at (t_1, t_2, t_3, t_4) . This completes the proof of Assertion 2.

Thus, we have proved that $\tilde{y}(t_1, t_2, t_3, t_4) \in \Sigma_\lambda$ and

$$\max_{(t_1, t_2, t_3, t_4) \in [\frac{1}{2}, \frac{3}{2}]^4} \Phi_\lambda(\tilde{y}(t_1, t_2, t_3, t_4)) \leq m_\lambda - \alpha_0$$

which contradicts the definition of m_λ . This completes the proof.

Proof of Theorem 1.1 We still prove it with $T_1 = \{1\}$, $T_2 = \{2\}$ and $T_3 = \{3\}$. By Lemma 5.6, when $\lambda > \Lambda_3$, we can get that there exists a solution $u_\lambda \in S^\delta \cap \Phi_\lambda^m$ for equation (3.3). By Lemma 3.2, we can know that u_λ is a solution of equation (1.5) when $\lambda > \Lambda := \max\{\Lambda_0, \Lambda_3\}$. Moreover, combining with Lemma 5.3, $u_\lambda \rightarrow u \in S$ (up to subsequence) strongly in $H_r^1(\mathbb{R}^2)$. So, we complete the proof of Theorem 1.1.

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