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Area Integral Characterization of Hardy space H_L^1 related to Degenerate Schrödinger Operators

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Abstract: Let

$$Lf(x) = -\frac{1}{\omega(x)} \sum_{i,j} \partial_i(a_{ij}(\cdot)\partial_j f)(x) + V(x)f(x)$$

be the degenerate Schrödinger operator, where ω is a weight from the Muckenhoupt class A_2 , V is a nonnegative potential that belongs to a certain reverse Hölder class with respect to the measure $\omega(x)dx$. For such an operator we define the area integral S_h^L associated with the heat semigroup and obtain the area integral characterization of H_L^1 , which is the Hardy space associated with L .

Keywords: Hardy space, Schrödinger operator, atom, area integral

MSC: Primary 42B30, 35J10, 42B25.

1 Introduction

As a suitable substitute of Lebesgue spaces $L^p(\mathbb{R}^n)$, the classical Hardy space $H^1(\mathbb{R}^n)$ plays an important role in various fields of analysis and partial differential equations. Let Δ be the Laplace operator on $\mathbb{R}^n$. It follows from [1] that $H^1(\mathbb{R}^n)$ can be characterized by the maximal function $\sup_{t>0} |e^{-t\Delta}f(x)|$. In fact, $H^1(\mathbb{R}^n)$ can be seen as a Hardy space associated with the operator $-\Delta$. We use L to denote a general differential operators, such as Schrödinger operators with nonnegative potential or second order elliptic self-adjoint operators in divergence form and so on. The Hardy spaces associated with L become one of the most concerned problems of the harmonic analysis. Readers can refer to [2–10] and the references therein. In recent years, [3] and [10] study the Hardy spaces associated with the degenerate Schrödinger operators.

As we know, the area integral is an important tool to characterize Hardy spaces. In [11], Fefferman and Stein obtain the area integral characterization of the classical Hardy spaces $H^p(\mathbb{R}^n)$. From then on, such characterization was extended to other settings. We refer the reader to [4, 5, 12] and the references therein. Let L be a degenerate Schrödinger operator L on $\mathbb{R}^n$. In this paper, motivated by the above literatures, we will prove that the Hardy space associated with L also has such a characterization. The degenerate Schrödinger operator L on $\mathbb{R}^n$ is defined as follows.

$$Lf(x) = -\frac{1}{\omega(x)} \sum_{i,j} \partial_i(a_{ij}(\cdot)\partial_j f)(x) + Vf(x),$$

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where $(a_{ij}(x))_{i,j}$ is a real symmetric matrix satisfying

$$C^{-1}\omega(x)|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\bar{\xi}_j \leq C\omega(x)|\xi|^2$$

with ω being a nonnegative weight from the Muckenhoupt class A_2 , and $V \geq 0$ belonging to a reverse Hölder class with respect to the measure $d\mu = \omega(x)dx$ (see Section 2 for their definitions). Denote by $\mathcal{E}(f, g)$ the Dirichlet form associated with L , that is,

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x) \partial_j f(x) \partial_i \bar{g}(x) dx + \int_{\mathbb{R}^n} V(x) f(x) d\mu(x).$$

The operator $-L$ is the infinitesimal generator of the heat semigroup $\{T_t\}_{t>0}$ of self-adjoint linear operators on $L^2(d\mu)$. Let $K_t(x, y)$ be the integral kernels of $\{T_t\}$, i.e.,

$$T_t f(x) = \int_{\mathbb{R}^n} K_t(x, y) f(y) d\mu(y).$$

In [3], Dziubański introduces the following Hardy space associated with the operator L .

Definition 1.1. We say a function f in $L^1(d\mu)$ belongs to $H_L^1(d\mu)$ if the heat maximal function $\mathcal{M}f$ is in $L^1(d\mu)$, where

$$\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)|.$$

The H_L^1 -norm of f is defined by $\|f\|_{H_L^1} = \|\mathcal{M}f\|_{L^1(d\mu)}$.

Dziubański in [3] has given the following atomic decomposition of $H_L^1(d\mu)$.

Definition 1.2. A function a is called an H_L^1 -atom associated with a ball $B(x, r)$ if

- (1) $r < \rho(x)$, $\text{supp } a \subset B(x, r)$, $\|a\|_{L^\infty} \leq \mu(B(x, r))^{-1}$;
- (2) If $r \leq \rho(x)/4$, then $\int a(y) d\mu(y) = 0$,

where $\rho(x)$ is the auxiliary function that defined in (2.3).

The atomic norm $\|\cdot\|_{H_L^1\text{-atom}}$ is defined by

$$\|f\|_{H_L^1\text{-atom}} = \inf \sum |\lambda_j|,$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$, where $\{a_j\}$ is a sequence of H_L^1 -atoms and $\{\lambda_j\}$ is a sequence of scalars.

Proposition 1.3. ([3, Theorem 2.1]) Assume that $\omega \in (RD)_\nu \cap D_\gamma \cap A_2$ with $2 < \nu \leq \gamma$. Let $V \in B_{q,\mu}$, $q > \gamma/2$. Then there exists a constant $C > 0$ such that

$$\frac{1}{C} \|f\|_{H_L^1\text{-atom}} \leq \|f\|_{H_L^1} \leq C \|f\|_{H_L^1\text{-atom}}.$$

The first main result of this paper can be stated as follows. Let S_h^L be the area integral associated with the heat semigroup generated by L , see (3.1) below. We have the following area integral characterization of $H_L^1(d\mu)$:

Assume that $\omega \in (RD)_\nu \cap D_\gamma \cap A_2$, $2 < \nu \leq \gamma$, and $V \in B_{q,\mu}$ with $q > \gamma/2$. Then

$$f \in H_L^1(d\mu) \Leftrightarrow S_h^L(f) \in L^1(d\mu),$$

see Theorems 3.9 & 3.14 for the details.

Following the classical case, we need a reproducing formula related to L in the distributional sense to divide the elements of $H_L^1(d\mu)$ into atoms. As the dual space of $H_L^1(d\mu)$ (cf. Definition 3.10), the BMO type

spaces $BMO_L(d\mu)$ are introduced by Yang-Yang-Zhou in [13, 14]. Suppose $f \in (BMO_L(d\mu))^*$, we obtain the desired reproducing formula which can be seen from Theorem 3.13. Since $(BMO_L(d\mu))^*$ is a subclass of the Schwartz temperate distribution space $\mathcal{S}'$, so we know that our reproducing formula is valid for the elements in $(BMO_L(d\mu))^*$ due to the fact that for a general potential V , the kernel of e^{-tL} only satisfies some Lipschitz condition, see Proposition 3.3. Also, the reproducing formula can be extended to all temperate distributions under this assumption if the high order derivatives of the kernel of e^{-tL} still have a Gaussian upper bound.

Remark 1.4.

(i) The Hardy space $H_L^1(d\mu)$ in this paper is a special case of the localized Hardy space $H_\rho^1(\mathcal{X})$ associated with the admissible function ρ , which has been investigated by Yang and Zhou in [10], where $\mathcal{X}$ is a RD-space. However, the authors give several maximal function characterizations of $H_\rho^1(\mathcal{X})$ without the area integral characterization in [10]. We will focus on the latter in this paper.

(ii) Our main results can be seen as the generalization of the classical case. In fact, if $\omega(x)dx = dx$ and $L = \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, the space $H_\Delta^1(dx)$ is exactly the classical space $H^1(\mathbb{R}^n)$. It is well-known that the Hardy space $H^1(\mathbb{R}^n)$ has the area integral characterization associated with heat semigroup $e^{-t\Delta}$.

Throughout this article, we will use c and C to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leq \frac{B_1}{B_2} \leq C$.

2 Preliminaries

A nonnegative function ω is an element of the Muckenhoupt class A_2 if there exists a constant $C > 0$ such that for every ball B ,

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega^{-1}(x) dx \right) \leq C. \quad (2.1)$$

Here and subsequently, $|B|$ denotes the volume of the ball B with respect to the Lebesgue measure dx . It is well-known that (2.1) implies that the measure $d\mu(x) = \omega(x)dx$ satisfies the doubling condition, that is, there exists a constant $C_0 > 0$ such that for every $x \in \mathbb{R}^n$, $r > 0$,

$$\mu(B(x, 2r)) \leq C_0 \mu(B(x, r)). \quad (2.2)$$

Using the notation of [15], we say that $\omega \in D_\gamma$, $\gamma > 0$, if there is a constant $C > 0$ such that for every $t > 1$,

$$\mu(B(x, tr)) \leq Ct^\gamma \mu(B(x, r)).$$

Notice that (2.2) guarantees the existence of such a γ .

Similarly, $\omega \in (RD)_\nu$ if for every $t > 1$,

$$t^\nu \mu(B(x, r)) \leq C \mu(B(x, tr)).$$

A nonnegative potential V belongs to the reverse Hölder class $B_{q,\mu}$, $q > 1$, with respect to the measure $d\mu$ if there exists a constant $C > 0$ such that for every Euclidean ball B , one has

$$\left(\frac{1}{\mu(B)} \int_B V^q(y) d\mu(y) \right)^{1/q} \leq C \left(\frac{1}{\mu(B)} \int_B V(y) d\mu(y) \right).$$

From now on we shall assume that $\omega \in A_2 \cap D_\gamma \cap (RD)_\nu$, $2 < \nu < \gamma$, $d\mu(x) = \omega(x)dx$ and $V \in B_{q,\mu}$, $q > \gamma/2$. We set $\delta = 2 - \gamma/q$.

In Definitions 1.2 & 3.10, we have used the following auxiliary function $m(x, V)$ which is defined by

$$\rho(x) = m(x, V)^{-1} = \sup \left\{ r > 0 : \frac{r^2}{\mu(B(x, r))} \int_{B(x, r)} V(y) d\mu(y) \leq 1 \right\}. \quad (2.3)$$

It is easy to see that, via a perturbation formula,

$$0 \leq K_t(x, y) \leq h_t(x, y).$$

Here $h_t(x, y)$ denotes the integral kernels of the semigroup $\{S_t\}_{t>0}$ on $L^2(d\mu)$ generated by $-L_0$, where

$$L_0 f(x) = -\frac{1}{\omega(x)} \sum_{i,j} \partial_i (a_{ij} \partial_j f)(x).$$

It is known that the kernels $h_t(x, y)$ satisfy the Gaussian estimates:

$$\frac{C_1}{\mu(B(x, \sqrt{t}))} \exp(-|x - y|^2 / c_2 t) \leq h_t(x, y) \leq \frac{C_1}{\mu(B(x, \sqrt{t}))} \exp(-|x - y|^2 / C_2 t), \quad (2.4)$$

which indicates that the kernels $K_t(x, y)$ have a Gaussian upper bound. Furthermore, Dziubański in [3] proves that

Lemma 2.1. *There exists a constant $C > 0$ such that for every $N > 0$ there exists a constant C_N such that*

$$K_t(x, y) \leq \frac{C_N}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \exp(-C|x - y|^2 / t).$$

In [16], Hebisch and Saloff-Coste have proved the following estimates for the heat kernels of L_0 :

$$|h_t(x, y) - h_t(x, z)| \leq C\mu(B(x, \sqrt{t}))^{-1} (|y - z|/\sqrt{t})^\alpha \exp(-(|x - y| - 2|y - z|)_+^2 / ct) \quad (2.5)$$

with constants $\alpha > 0$, $c > 0$, $C > 0$, and

$$|\partial_t^k h_t(x, y)| \leq \frac{C_k}{t^k \mu(B(x, \sqrt{t}))} \exp\left(-\frac{|x - y|^2}{ct}\right).$$

In the rest of this section, we state some properties of the function $m(x, V)$ which will be used in the sequel.

Lemma 2.2. ([15, Lemma 2]) *Assume that $\omega \in D_y$, $V \in B_{q,\mu}$ with $q > y/2$. Then there exists a constant $C > 0$ such that for every $0 < r < R < \infty$ and $y \in \mathbb{R}^n$, we have*

$$\frac{r^2}{\mu(B(y, r))} \int_{B(y, r)} V(x) d\mu(x) \leq C \left(\frac{r}{R}\right)^\delta \frac{R^2}{\mu(B(y, R))} \int_{B(y, R)} V(x) d\mu(x).$$

Lemma 2.3. ([15, Lemma 3]) *Under the assumptions of Lemma 2.2, for every constant $C_1 > 1$, there exists a constant $C_2 > 1$ such that if*

$$\frac{1}{C_1} \leq \frac{r^2}{\mu(B(x, r))} \int_{B(x, r)} V(y) d\mu(y) \leq C_1,$$

then $C_2^{-1} \leq rm(x, V) \leq C_2$.

Lemma 2.4. ([15, Lemma 4]) *Under the assumptions of Lemma 2.2, for every constant $C_1 \geq 1$ there is a constant $C_2 \geq 1$ such that $\frac{1}{C_2} \leq \frac{m(x, V)}{m(y, V)} \leq C_2$ for $|x - y| \leq C_1 r(x)$. Moreover, there exist constants $k_0, C, c > 0$ such that*

$$m(y, V) \leq C(1 + |x - y|m(x, V))^{k_0} m(x, V)$$

and

$$m(y, V) \geq cm(x, V)(1 + |x - y|m(x, V))^{-k_0/(1+k_0)}.$$

Lemma 2.5. ([3, Lemma 4.4]) *There exist constants $l, C > 0$ such that*

$$\frac{R^2}{\mu(B(x, R))} \int_{B(x, R)} V(y) d\mu(y) \leq C(Rm(x, V))^l \quad \text{provided } R \geq m(x, V)^{-1}.$$

Lemma 2.6. ([3, Corollary 4.5]) *For any constants $c, C' > 0$ there exists a constant $C > 0$ such that*

$$\int \frac{e^{-c|x-y|^2/t} V(y)}{\mu(B(x, \sqrt{t}))} d\mu(y) \leq Ct^{-1} (\sqrt{t} m(x, V))^\delta \quad \text{for } \sqrt{t} \leq C' m(x, V)^{-1}.$$

Lemma 2.7. *For $V \in B_{q,\mu}$ and $l > 0$, there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} \frac{1}{\mu(B(x, \sqrt{t}))} V(z) e^{-|x-z|^2/t} d\mu(z) \leq \frac{C}{t} \left(\frac{\sqrt{t}}{\rho(x)} \right)^l, \quad t \geq \rho(x)^2.$$

Proof.

$$\begin{aligned} & \int_{\mathbb{R}^n} V(z) \frac{1}{\mu(B(x, \sqrt{t}))} e^{-c|x-z|^2/t} d\mu(z) \\ & \leq \left(\int_{|x-z| < \sqrt{t}} + \int_{|x-z| \geq \sqrt{t}} \right) V(z) \frac{1}{\mu(B(x, \sqrt{t}))} e^{-c|x-z|^2/t} d\mu(z) \\ & =: I_1 + I_2. \end{aligned}$$

For I_1 , using Lemma 2.5, we have

$$I_1 \leq \frac{t^2}{t\mu(B(x, \sqrt{t}))} \int_{B(x, \sqrt{t})} V(z) d\mu(z) \leq \frac{C}{t} \left(\frac{\sqrt{t}}{\rho(x)} \right)^l.$$

Similarly, for I_2 , we have

$$\begin{aligned} I_2 & \leq \sum_{j=0}^{\infty} \frac{1}{\mu(B(x, \sqrt{t}))} \int_{2^j \sqrt{t} \leq |x-z| < 2^{j+1} \sqrt{t}} V(z) \left(1 + |x-z|^2/t \right)^{-N} d\mu(z) \\ & \leq \sum_{j=0}^{\infty} \frac{1}{\mu(B(x, \sqrt{t}))} \frac{1}{(1 + 2^{2j})^N} \int_{|x-z| < 2^{j+1} \sqrt{t}} V(z) d\mu(z) \\ & \leq \frac{1}{t} \sum_{j=0}^{\infty} 2^{-j(2N-y+2)} (2^{j+1} \sqrt{t} m(x, V))^l \leq \frac{C}{t} \left(\frac{\sqrt{t}}{\rho(x)} \right)^l. \end{aligned}$$

□

3 Area integral characterization associated to the heat semigroup

3.1 Smoothness estimates associated with $\{T_t\}$

In this section, by use of the area integral associated to the heat semigroup $\{T_t\}_{t>0}$, we characterize the Hardy type space $H_L^1(d\mu)$. The area integral associated to $\{T_t\}_{t>0}$ is defined as follows.

Definition 3.1. For $(x, t) \in \mathbb{R}_+^{n+1}$, let $(Q_t f)(x) = t^2 \left(\frac{dT_s}{ds} \Big|_{s=t^2} f \right)(x)$. The area integral associated to the heat semigroup $\{T_t\}_{t>0}$ is defined by

$$S_h^L f(x) = \left(\int_0^\infty \int_{B(x,t)} |Q_t f(y)|^2 \frac{d\mu(y) dt}{t\mu(B(x, t))} \right)^{1/2}. \quad (3.1)$$

Moreover, the Littlewood-Paley g -function associated to the heat semigroup $\{T_t\}_{t>0}$ is defined by

$$g_h^L f(x) = \left(\int_0^\infty |Q_t f(x)|^2 \frac{dt}{t} \right)^{1/2}. \quad (3.2)$$

To prove our main results, we need some estimates for the integral kernels of the operators Q_t :

$$Q_t(x, y) = t^2 \frac{\partial K_s(x, y)}{\partial s} \Big|_{s=t^2}.$$

Proposition 3.2. Assume that $\omega \in D_y$, $V \in B_{q,\mu}$ with $q > y/2$. Set $q_t(x, y) = h_t(x, y) - K_t(x, y)$. Then for $|h| \leq \min\{|x - y|/4, \rho(y)\}$, we have

$$|q_t(x, y + h) - q_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} e^{-c|x-y|^2/t} \left(\frac{|h|}{\rho(x)} \right)^{\delta'},$$

where $0 < \delta' < \min(\delta, 1)$.

Proof. Since $0 \leq K_t(x, y) \leq h_t(x, y)$, so we have $0 \leq q_t(x, y) \leq \frac{1}{\mu(B((x, \sqrt{t})))} e^{-c|x-y|^2/t}$. We divide the proof into two cases.

Case I: $|h| \geq \rho(x)$. For $\delta' > 0$, $(|h|/\rho(x))^{\delta'} \geq 1$. Because $|h| \leq |x - y|/4$, we can see that $|x - y - h| \geq 3|x - y|/4$ and $e^{-c|x-y-h|^2/t} \leq e^{-c'|x-y|^2/t}$, which imply that

$$|q_t(x, y)| + |q_t(x, y + h)| \leq \frac{1}{\mu(B(x, \sqrt{t}))} e^{-c|x-y|^2/t} \left(\frac{|h|}{\rho(x)} \right)^{\delta'}.$$

The above two estimates give

$$|q_t(x, y + h) - q_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} e^{-c'|x-y|^2/t} \left(\frac{|h|}{\rho(x)} \right)^{\delta'}.$$

Case II: $|h| < \rho(x)$. We further divide Case II into two subcases.

(1) $t \leq 2|h|^2$. By [3, Proposition 5.1], we conclude that the desired estimate is valid.

(2) $t > 2|h|^2$. We only need to prove:

$$|q_t(x, y + h) - q_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\rho(x)} \right)^{\delta'} e^{\varepsilon|x-y|^2/t}.$$

Via (2.5), we have

$$|h_t(x, y + h) - h_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}} \right)^\alpha \exp(-(|x - y| - 2|h|)_+^2/ct).$$

If $|h| < |x - y|/4$, then $|x - y| - 2|h| \geq |x - y|/2$ and

$$|h_t(x, y + h) - h_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}} \right)^\alpha e^{-|x-y|^2/c't}.$$

Because

$$q_t(x, y) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x, z) V(z) h_s(z, y) d\mu(z) ds,$$

we can use the change of variable to obtain

$$\begin{aligned} |q_t(x, y + h) - q_t(x, y)| &\leq \int_0^t \int_{\mathbb{R}^n} |K_{t-s}(x, z)| V(z) |h_s(z, y + h) - h_s(z, y)| d\mu(z) ds \\ &= \int_0^{t/2} \int_{\mathbb{R}^n} |K_s(x, z)| V(z) |h_{t-s}(z, y + h) - h_{t-s}(z, y)| d\mu(z) ds \\ &\quad + \int_{t/2}^t \int_{\mathbb{R}^n} |K_{t-s}(x, z)| V(z) |h_s(z, y + h) - h_s(z, y)| d\mu(z) ds \\ &=: J_1 + J_2. \end{aligned}$$

When $t < 2\rho(x)^2$, note that $0 < s < t/2$, $t/2 < t-s < t$. Because $B(x, \sqrt{t}) \subset B(z, |x-z| + \sqrt{t})$, a direct computation gives

$$\begin{aligned} \frac{1}{\mu(B(z, \sqrt{t}))} e^{-c|x-z|^2/s} &\leq \frac{1}{\mu(B(x, \sqrt{t}))} \frac{\mu(B(z, \sqrt{t} + |x-z|))}{\mu(B(z, \sqrt{t}))} e^{-c_1|x-z|^2/s} e^{-c_2|x-z|^2/s} \\ &\leq \frac{C}{\mu(B(x, \sqrt{t}))} e^{-c_1|x-z|^2/s}. \end{aligned} \quad (3.3)$$

We obtain, by Lemma 2.6 and (3.3),

$$\begin{aligned} J_1 &\leq \int_0^{t/2} \int_{\mathbb{R}^n} \frac{1}{\mu(B(x, \sqrt{s}))} e^{-c|x-z|^2/s} \frac{V(z)}{\mu(B(z, \sqrt{t-s}))} \left(\frac{|h|}{\sqrt{t-s}} \right)^\alpha d\mu(z) ds \\ &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}} \right)^\alpha \int_0^{t/2} \int_{\mathbb{R}^n} \frac{1}{\mu(B(z, \sqrt{s}))} e^{-c_1|x-z|^2/s} V(z) d\mu(z) ds \\ &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}} \right)^\alpha \int_0^{t/2} \frac{1}{s} \left(\frac{\sqrt{s}}{\rho(x)} \right)^\delta ds \\ &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}} \right)^\alpha \left(\frac{\sqrt{t}}{\rho(x)} \right)^\delta. \end{aligned}$$

Taking $\delta_0 = \min\{\alpha, \delta\} \leq \alpha$, for $\sqrt{t} \leq \sqrt{2}\rho(x)$, we have $(\sqrt{t}/\sqrt{2}\rho(x))^{\delta-\delta_0} \leq 1$. For $\sqrt{t} > \sqrt{2}|h|$, $(|h|/\sqrt{t})^\delta \leq (|h|/\sqrt{t})^{\delta_0}$. Then

$$J_1 \leq \frac{c}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}} \right)^\alpha \left(\frac{\sqrt{t}}{\rho(x)} \right)^{\delta_0} \left(\frac{\sqrt{t}}{\rho(x)} \right)^{\delta-\delta_0} \leq \frac{c}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\rho(x)} \right)^{\delta_0}.$$

When $t > 2\rho(x)^2$, note that $0 < s < t/2$, $t/2 < t-s < t$. Hence,

$$\begin{aligned} J_1 &\leq \int_0^{t/2} \int_{\mathbb{R}^n} \frac{C_N}{\mu(B(x, \sqrt{s}))} \left(1 + \frac{\sqrt{s}}{\rho(x)} \right)^{-N} \left(1 + \frac{\sqrt{s}}{\rho(z)} \right)^{-N} e^{-c|x-z|^2/s} V(z) \\ &\quad \frac{c}{\mu(B(z, \sqrt{t-s}))} \left(\frac{|h|}{\sqrt{t-s}} \right)^\alpha d\mu(z) ds \\ &\leq \left(\frac{|h|}{\sqrt{t}} \right)^\alpha \int_0^{t/2} \int_{\mathbb{R}^n} \frac{C_N}{\mu(B(x, \sqrt{s}))} \left(1 + \frac{\sqrt{s}}{\rho(x)} \right)^{-N} \left(1 + \frac{\sqrt{s}}{\rho(z)} \right)^{-N} \frac{V(z) e^{-c|x-z|^2/s}}{\mu(B(x, \sqrt{t}))} d\mu(z) ds. \end{aligned}$$

The term J_1 can be further divided as follows:

$$\begin{aligned} J_1 &\leq \left(\frac{|h|}{\sqrt{t}} \right)^\alpha \frac{C_N}{\mu(B(x, \sqrt{t}))} \int_0^{t/2} \int_{\mathbb{R}^n} \frac{1}{\mu(B(x, \sqrt{s}))} \left(1 + \frac{\sqrt{s}}{\rho(x)} \right)^{-N} e^{-c|x-z|^2/s} V(z) d\mu(z) ds \\ &=: J_{1,1} + J_{1,2}, \end{aligned}$$

where

$$J_{1,1} = \left(\frac{|h|}{\sqrt{t}} \right)^\alpha \frac{C_N}{\mu(B(x, \sqrt{t}))} \int_0^{\rho(x)^2} \int_{\mathbb{R}^n} \frac{V(z)}{\mu(B(x, \sqrt{s}))} \left(1 + \frac{\sqrt{s}}{\rho(x)} \right)^{-N} e^{-c|x-z|^2/s} d\mu(z) ds$$

and

$$J_{1,2} = \left(\frac{|h|}{\sqrt{t}} \right)^\alpha \frac{C_N}{\mu(B(x, \sqrt{t}))} \int_{\rho(x)^2}^{t/2} \int_{\mathbb{R}^n} \frac{V(z)}{\mu(B(x, \sqrt{s}))} \left(1 + \frac{\sqrt{s}}{\rho(x)} \right)^{-N} e^{-c|x-z|^2/s} d\mu(z) ds.$$

For $J_{1,1}$, since $|h| \leq \rho(x)$, then $(|h|/\rho(x))^\alpha \leq (|h|/\rho(x))^{\delta_0}$ for $\alpha \geq \delta_0$. So we can get

$$\begin{aligned} J_{1,1} &\leq \left(\frac{|h|}{\sqrt{t}}\right)^\alpha \frac{C_N}{\mu(B(x, \sqrt{t}))} \int_0^{\rho(x)^2} \frac{1}{\mu(B(x, \sqrt{s}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \int_{\mathbb{R}^n} e^{-c|x-z|^2/s} V(z) d\mu(z) ds \\ &\leq \left(\frac{|h|}{\sqrt{t}}\right)^\alpha \frac{C_N}{\mu(B(x, \sqrt{t}))} \int_0^{\rho(x)^2} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \frac{1}{s} \left(\frac{\sqrt{s}}{\rho(x)}\right)^\delta ds \\ &\leq \left(\frac{|h|}{\rho(x)}\right)^{\delta_0} \frac{C_N}{\mu(B(x, \sqrt{t}))}. \end{aligned}$$

For $J_{1,2}$, choose N large enough such that $(\frac{\sqrt{t/2}}{\rho(x)})^{l-N} \leq 1$, where l is the constant in Lemma 2.5. Using Lemma 2.5, we have

$$\begin{aligned} J_{1,2} &\leq \left(\frac{|h|}{\sqrt{t}}\right)^\alpha \frac{C_N}{\mu(B(x, \sqrt{t}))} \int_{\rho(x)^2}^{t/2} \frac{1}{s} \left(\frac{\sqrt{s}}{\rho(x)}\right)^{l_0} \left(1 + \frac{\sqrt{s}}{\rho(x)}\right)^{-N} ds \\ &\leq \left(\frac{|h|}{\sqrt{t}}\right)^\alpha \frac{C_N}{\mu(B(x, \sqrt{t}))} \int_{\rho(x)^2}^{t/2} \frac{1}{s} \left(\frac{\sqrt{s}}{\rho(x)}\right)^{l_0-N} ds \\ &\leq \frac{C_N}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\rho(x)}\right)^{\delta_0}, \end{aligned}$$

which implies $J_1 \leq \frac{C_N}{\mu(B(x, \sqrt{t}))} (|h|/\rho(x))^{\delta_0}$.

In what follows, we consider J_2 . In fact,

$$\begin{aligned} J_2 &= \int_0^{\frac{|h|^2}{2}} \int_{\mathbb{R}^n} |K_{t-s}(x, z)| V(z) |h_s(z, y+h) - h_s(z, y)| d\mu(z) ds \\ &\quad + \int_{\frac{|h|^2}{2}}^{t/2} \int_{|z-y| < 2|h|} |K_{t-s}(x, z)| V(z) |h_s(z, y+h) - h_s(z, y)| d\mu(z) ds \\ &\quad + \int_{\frac{|h|^2}{2}}^{t/2} \int_{|z-y| \geq 2|h|} |K_{t-s}(x, z)| V(z) |h_s(z, y+h) - h_s(z, y)| d\mu(z) ds \\ &=: J_{2,1} + J_{2,2} + J_{2,3}. \end{aligned}$$

For $J_{2,1}$, by the symmetry, we have

$$\begin{aligned} J_{2,1} &\leq \int_0^{\frac{|h|^2}{2}} \int_{\mathbb{R}^n} \left[\frac{e^{-c|z-y|^2/s}}{\mu(B(y, \sqrt{s}))} + \frac{e^{-c|z-y-h|^2/s}}{\mu(B(y+h, \sqrt{s}))} \right] \frac{V(z)}{\mu(B(x, \sqrt{t-s}))} \left(1 + \frac{\sqrt{t-s}}{\rho(x)}\right)^{-N} \\ &\quad \times \left(1 + \sqrt{t-s}/\rho(z)\right)^{-N} e^{-c|x-z|^2/t-s} d\mu(z) ds. \end{aligned}$$

Because $0 < s < |h|^2 < t/2$, we know $\sqrt{s} < |h| < \rho(y) \sim \rho(y+h)$. Then we obtain

$$\begin{aligned} J_{2,1} &\leq \frac{c}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \int_0^{\frac{|h|^2}{2}} \int_{\mathbb{R}^n} \left[\frac{e^{-c|z-y|^2/s}}{\mu(B(y, \sqrt{s}))} + \frac{e^{-c|z-y-h|^2/s}}{\mu(B(y+h, \sqrt{s}))} \right] V(z) d\mu(z) ds \\ &\leq \frac{c}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \int_0^{\frac{|h|^2}{2}} \int_{\mathbb{R}^n} \frac{1}{s} \left(\frac{\sqrt{s}}{\rho(y)}\right)^\delta ds \\ &\leq \frac{c}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \left(\frac{|h|}{\rho(x)}\right)^\delta \left(\frac{\rho(x)}{\rho(y)}\right)^\delta. \end{aligned}$$

Similarly, we can deal with $J_{2,2}$. It is easy to see that $B(y, 2|h|) \subset B(z, 4|h|)$. Using $2 < v \leq y$, we have

$$\frac{1}{|h|^2} \int_{|h|^2}^{t/2} \left(\frac{|h|}{\sqrt{s}} \right)^{\alpha+v} ds = |h|^{\alpha+v-2} \int_{|h|^2}^{t/2} \frac{1}{s^{(\alpha+v)/2}} ds \leq 1.$$

So

$$\begin{aligned} J_{2,2} &\leq \frac{1}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \int_{|h|^2}^{t/2} \int_{|z-y| < 2|h|} \left(\frac{|h|}{\sqrt{s}} \right)^{\alpha} \frac{1}{\mu(B(z, \sqrt{s}))} V(z) d\mu(z) ds \\ &\leq \frac{1}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \frac{1}{(2|h|)^2} \left(\frac{2|h|}{\rho(y)} \right)^{\delta} \int_{|h|^2}^{t/2} \left(\frac{|h|}{\sqrt{s}} \right)^{\alpha+v} ds \\ &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \left(\frac{|h|}{\rho(x)} \right)^{\delta} \left(\frac{\rho(x)}{\rho(y)} \right)^{\delta}. \end{aligned}$$

For $J_{2,3}$, we divide the estimate into two cases.

(1) If $t < 2\rho(y)^2$, $s < t/2 < \rho(y)^2$. Because $B(y, \sqrt{s}) \subset B(z, |y-z| + \sqrt{s})$, the doubling property of μ implies that

$$\frac{1}{\mu(B(z, \sqrt{s}))} \leq \left(1 + \frac{|y-z|}{\sqrt{s}} \right)^{\gamma} \frac{1}{\mu(B(y, \sqrt{s}))}.$$

We use (2.5) to obtain that

$$\begin{aligned} J_{2,3} &\leq \frac{1}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \int_{|h|^2}^{t/2} \int_{|z-y| > 2|h|} |h_s(z, y+h) - h_s(z, y)| V(z) d\mu(z) ds \\ &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \int_{|h|^2}^{t/2} \int_{|z-y| > 2|h|} \frac{e^{-c|y-z|^2/s} V(z)}{\mu(B(z, \sqrt{s}))} \left(\frac{|h|}{\sqrt{s}} \right)^{\alpha} d\mu(z) ds \\ &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \int_{|h|^2}^{t/2} \frac{1}{s} \left(\frac{|h|}{\rho(y)} \right)^{\delta} \left(\frac{\sqrt{s}}{|h|} \right)^{\delta} \left(\frac{|h|}{\sqrt{s}} \right)^{\alpha} ds. \end{aligned}$$

For $\sqrt{s} < \rho(y)$ and $\delta_0 = \min\{\alpha, \delta\}$, $(\sqrt{s}/\rho(y))^{\delta} \leq (\sqrt{s}/\rho(y))^{\delta_0}$. Moreover, $|h| < \rho(y)$, $(|h|/\sqrt{s})^{\alpha} \leq (|h|/\sqrt{s})^{\delta_0}$. Finally, we have

$$\begin{aligned} J_{2,3} &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \left(\frac{|h|}{\rho(y)} \right)^{\delta_0} \int_{|h|^2}^{t/2} \frac{1}{s} ds \\ &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \left(\frac{|h|}{\rho(x)} \right)^{\delta_0} \left(\frac{\rho(x)}{\rho(y)} \right)^{\delta_0} \left(1 + \left| \log \frac{|h|}{\rho(x)} \right| \right) \left(1 + \left| \log \frac{\rho(x)}{\rho(y)} \right| \right). \end{aligned}$$

(2) If $t \geq 2\rho(y)^2$, we have

$$\begin{aligned} J_{2,3} &\leq \left(\int_{|h|^2}^{\rho(y)^2} + \int_{\rho(y)^2}^{t/2} \right) \int_{|z-y| \geq 2|h|} |K_{t-s}(x, z)| V(z) |h_s(z, y+h) - h_s(z, y)| d\mu(z) ds \\ &=: J_{2,3}^{(1)} + J_{2,3}^{(2)}. \end{aligned}$$

For $J_{2,3}^{(1)}$, because $|h|^2 < s < \rho(y)^2$,

$$J_{2,3}^{(1)} \leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \left(\frac{|h|}{\rho(x)} \right)^{\delta_0} \left(\frac{\rho(x)}{\rho(y)} \right)^{\delta_0} \left(1 + \left| \log \frac{|h|}{\rho(x)} \right| \right) \left(1 + \left| \log \frac{\rho(x)}{\rho(y)} \right| \right).$$

For $J_{2,3}^{(2)}$, we can deduce that

$$\begin{aligned} J_{2,3}^{(2)} &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \int_{\rho(y)^2}^{t/2} \int_{|z-y| \geq 2|h|} \left(\frac{|h|}{\sqrt{s}}\right)^\alpha \frac{e^{-c|z-y|^2/s} V(z)}{\mu(B(z, \sqrt{s}))} d\mu(z) ds \\ &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \int_{\rho(y)^2}^{t/2} \left(\frac{|h|}{\sqrt{s}}\right)^\alpha \left(\frac{\sqrt{s}}{\rho(y)}\right)^{l_0} \frac{ds}{s} \\ &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N+l_0+\alpha} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{l_0-\alpha} \left(\frac{|h|}{\rho(x)}\right)^\alpha \left(\frac{\rho(x)}{\rho(y)}\right)^{l_0}. \end{aligned}$$

For any $\rho(x)$ and $\rho(y)$, we have

$$\frac{\rho(x)}{\rho(y)} \leq C \left(1 + \frac{|x-y|}{\sqrt{t}} \frac{\sqrt{t}}{\rho(x)}\right)^{k_0} \leq C_\varepsilon e^{\varepsilon|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{k_0}.$$

Taking N large enough, we get the desired result. This completes the proof of Proposition 3.2. $\square$

In what follows, we begin to estimate the smoothness of the kernel $K_t(\cdot, \cdot)$.

Proposition 3.3. *For every $0 < \delta' < \delta_0 = \min\{\alpha, \delta, \nu\}$, there exists a constant $C > 0$ such that for every $M > 0$ and $|h| < \sqrt{t}$,*

$$|K_t(x, y+h) - K_t(x, y)| \leq C_M \left(\frac{|h|}{\sqrt{t}}\right)^{\delta'} \frac{1}{\mu(B(x, \sqrt{t}))} e^{-c|x-y|^2/t} \left(1 + \sqrt{t}/\rho(x)\right)^{-M} \left(1 + \sqrt{t}/\rho(y)\right)^{-M}.$$

Proof. In the following proof, we denote by N the positive number, which may be different in different places. The proof is divided into four cases.

Case 1: $\sqrt{t}/2 \leq |h| \leq \sqrt{t}$. By Lemma 2.1

$$K_t(x, y) \leq \frac{C_N}{\mu(B(x, \sqrt{t}))} \left(1 + \sqrt{t}/\rho(x)\right)^{-N} \left(1 + \sqrt{t}/\rho(y)\right)^{-N} e^{-c|x-y|^2/t}.$$

Using the triangle inequality, we have $|x-y|^2/2t \leq 1 + |x-y-h|^2/t$. Hence, $e^{-c|x-y-h|^2/t} \leq C e^{-c|x-y|^2/2t}$. Then

$$\begin{aligned} &|K_t(x, y+h) - K_t(x, y)| \\ &\leq \frac{C_M}{\mu(B(x, \sqrt{t}))} (1 + \sqrt{t}/\rho(x))^{-N} [(1 + \sqrt{t}/\rho(y))^{-N} + (1 + \sqrt{t}/\rho(y+h))^{-N}] e^{-c|x-y|^2/t}. \end{aligned}$$

Notice that $(1 + |h|m(y, V))^{k_0/k_0+1} \geq 1$. We can use Lemma 2.4 to deduce that

$$(1 + \sqrt{t}/\rho(y+h))^{-N} \leq \frac{(1 + |h|m(y, V))^{k_0N/k_0+1}}{(1 + \sqrt{t}m(y, V))^N} \leq \frac{1}{(1 + \sqrt{t}m(y, V))^{N/k_0+1}},$$

where we have used the fact $|h| \leq \sqrt{t}$ in the last inequality. For any N , we can get

$$|K_t(x, y+h) - K_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} e^{-c|x-y|^2/t}.$$

Because $\sqrt{t}/2 \leq |h| \leq \sqrt{t}$, $(|h|/\sqrt{t})^{\delta'} \sim 1$ for $\delta' > 0$. If $\sqrt{t}/2 \leq |h| \leq \sqrt{t}$,

$$|K_t(x, y+h) - K_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}}\right)^{\delta'} e^{-c|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

Case 2: $|h| < |x-y|/4$. Similar to Case 1, for any N , we have

$$|K_t(x, y+h) - K_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} e^{-c|x-y|^2/t}.$$

If $|h| > \rho(y)$, we have

$$|K_t(x, y+h) - K_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \left(\frac{|h|}{\sqrt{t}}\right)^\delta e^{-c|x-y|^2/t}.$$

Case 3: $|h| < |x-y|/4$ and $|h| \leq \rho(y)$. On the one hand, we have

$$\begin{aligned} |q_t(x, y+h) - q_t(x, y)| &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\rho(x)}\right)^{\delta'} e^{-c|x-y|^2/t} \\ &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}}\right)^{\delta'} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{\delta'} e^{-c|x-y|^2/t}. \end{aligned}$$

On the other hand, because $|h| < |x-y|/4$, $|x-y| - 2|h| > |x-y|/2$, then $e^{-c(|x-y|-2|h|)^2/ct} \leq e^{-c|x-y|^2/t}$. And $|h| \leq \sqrt{t}$, $\delta' \leq \delta_0 < \min(\alpha, \delta)$, $(|h|/\sqrt{t})^\alpha \leq (|h|/\sqrt{t})^{\delta'}$. Therefore,

$$\begin{aligned} |h_t(x, y+h) - h_t(x, y)| &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}}\right)^\alpha e^{-c(|x-y|-2|h|)^2/ct} \\ &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}}\right)^{\delta'} e^{-c|x-y|^2/t} \\ &\leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}}\right)^{\delta'} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{\delta'} e^{-c|x-y|^2/t}. \end{aligned}$$

So we have

$$|K_t(x, y+h) - K_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} (|h|/\sqrt{t})^{\delta'} (1 + \sqrt{t}/\rho(x))^{\delta'} e^{-c|x-y|^2/t}.$$

Finally, we get

$$|K_t(x, y+h) - K_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}}\right)^{\delta'} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{\delta'} e^{-c|x-y|^2/t}$$

and

$$|K_t(x, y+h) - K_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} e^{-c|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

Combining with the above two estimates, we get, for N large enough,

$$|K_t(x, y+h) - K_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}}\right)^{\delta''} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} e^{-c|x-y|^2/t}.$$

Case 4: $|x-y|/4 < |h| \leq \sqrt{t}/2$. We divide $|K_t(x, y+h) - K_t(x, y)|$ into two parts. Precisely, we have

$$\begin{aligned} |K_t(x, y+h) - K_t(x, y)| &\leq \left(\int_{|z-y| \leq 4|h|} + \int_{|z-y| > 4|h|} \right) K_{t/2}(x, z) |K_{t/2}(z, y+h) - K_{t/2}(z, y)| d\mu(z) \\ &=: S_1 + S_2. \end{aligned}$$

For S_1 , due to Lemma 2.1, we have

$$\begin{aligned} S_1 &\leq \frac{C_N}{\mu(B(x, \sqrt{t/2}))} \left(1 + \frac{\sqrt{t/2}}{\rho(x)}\right)^{-N} \int_{|z-y| \leq 4|h|} \frac{1}{\mu(B(z, \sqrt{t/2}))} d\mu(z) \\ &\leq \frac{C_N}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{\sqrt{t/2}}{\rho(x)}\right)^{-N} \frac{1}{\mu(B(y, \sqrt{t/2}))} \int_{|z-y| \leq 4|h|} \frac{\mu(B(y, \sqrt{t/2}))}{\mu(B(z, \sqrt{t/2}))} d\mu(z). \end{aligned}$$

For any $u \in B(y, \sqrt{t/2})$, we can see that $B(y, \sqrt{t/2}) \subset B(z, 5\sqrt{t/2})$. By the doubling property of the measure μ , we obtain

$$\begin{aligned} S_1 &\leq \frac{C_y}{\mu(B(x, \sqrt{t/2}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \frac{\mu(B(y, |h|))}{\mu(B(y, \sqrt{t/2}))} \\ &\leq \frac{C_y}{\mu(B(x, \sqrt{t/2}))} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \left(\frac{|h|}{\sqrt{t/2}}\right)^\nu. \end{aligned}$$

Next, we consider S_2 . For $|z - y| > 4|h|$, similar to Case 2, we have

$$|K_{t/2}(z, y + h) - K_{t/2}(z, y)| \leq \frac{Ce^{-c|z-y|^2/t}}{\mu(B(z, \sqrt{t/2}))} \left(\frac{|h|}{\sqrt{t/2}}\right)^{\delta''} \left(1 + \frac{\sqrt{t/2}}{\rho(z)}\right)^{-N} \left(1 + \frac{\sqrt{t/2}}{\rho(z)}\right)^{-N}.$$

The above estimate gives

$$S_2 \leq C \left(\frac{|h|}{\sqrt{t/2}}\right)^{\delta''} \left(1 + \frac{\sqrt{t/2}}{\rho(z)}\right)^{-N} \int_{|z-y|>4|h|} K_{t/2}(x, z) \frac{e^{-c|z-y|^2/t}}{\mu(B(z, \sqrt{t/2}))} \left(1 + \frac{\sqrt{t/2}}{\rho(z)}\right)^{-N} d\mu(z).$$

Because $|x - y|/4 < |h|$ and $|z - y| > 4|h|$, then $|x - y| < |z - y|$. Using the estimate of $K_{t/2}(x, z)$ in Lemma 2.1, we know

$$\begin{aligned} S_2 &\leq C \left(\frac{|h|}{\sqrt{t}}\right)^{\delta''} \left(1 + \frac{\sqrt{t/2}}{\rho(y)}\right)^{-N} \int_{|z-y|>4|h|} \frac{e^{-c_1|z-y|^2/t} e^{-c_2|z-y|^2/t}}{\mu(B(z, \sqrt{t}))} \left(1 + \frac{\sqrt{t/2}}{\rho(z)}\right)^{-N} K_{t/2}(x, z) d\mu(z) \\ &\leq C \left(\frac{|h|}{\sqrt{t}}\right)^{\delta''} \left(1 + \frac{\sqrt{t/2}}{\rho(y)}\right)^{-N} \left(1 + \frac{\sqrt{t/2}}{\rho(x)}\right)^{-N} e^{-c_2|x-y|^2/t} \left\{ \int_{|z-y|>4|h|} \frac{e^{-c_1|z-y|^2/t}}{\mu(B(z, \sqrt{t}))} d\mu(z) \right\}. \end{aligned}$$

Set

$$I = \int_{|z-y|>4|h|} \frac{1}{\mu(B(z, \sqrt{t/2}))} e^{-c_1|z-y|^2/t} d\mu(z).$$

Because $B(y, \sqrt{t/2}) \subset B(z, |z - y| + \sqrt{t/2})$, we could get

$$\begin{aligned} I &\leq \frac{C}{\mu(B(y, \sqrt{t/2}))} \int_{|z-y|>4|h|} \frac{\mu(B(z, |y - z| + \sqrt{t/2}))}{\mu(B(z, \sqrt{t/2}))} e^{-c_1|z-y|^2/t} d\mu(z) \\ &\leq \frac{C}{\mu(B(y, \sqrt{t/2}))} \int_{|z-y|>4|h|} (1 + |y - z|/\sqrt{t})^y e^{-c_1|z-y|^2/t} d\mu(z) \\ &\leq \frac{C}{\mu(B(y, \sqrt{t/2}))} \left(\int_{|z-y|>4\sqrt{t/2}} + \int_{4|h|<|z-y|<4\sqrt{t/2}} \right) e^{-c_3|z-y|^2/t} d\mu(z) \\ &=: I_1 + I_2, \end{aligned}$$

where we have used the following fact again: $(1 + |y - z|/\sqrt{t})^y e^{-c_1|z-y|^2/t} \leq Ce^{-c_3|z-y|^2/t}$. For I_2 , a direct computation gives

$$I_2 = \frac{C}{\mu(B(y, \sqrt{t/2}))} \int_{4|h|<|z-y|<4\sqrt{t/2}} e^{-c_3|z-y|^2/t} d\mu(z) \leq C_y.$$

For I_1 , we have

$$\begin{aligned} I_1 &\leq \frac{C}{\mu(B(y, \sqrt{t/2}))} \sum_{k=2}^{\infty} \int_{2^k\sqrt{t/2}<|z-y|\leq 2^{k+1}\sqrt{t/2}} \frac{1}{(1 + |z - y|/\sqrt{t/2})^l} d\mu(z) \\ &\leq \frac{C}{\mu(B(y, \sqrt{t/2}))} \sum_{k=2}^{\infty} \frac{1}{(1 + 2^k\sqrt{t/2}/\sqrt{t/2})^l} \mu(B(y, 2^{k+1}\sqrt{t/2})) \\ &\leq \sum_{k=2}^{\infty} 2^{(k+1)y} \frac{1}{(1 + 2^k)^l} \leq C_y. \end{aligned}$$

So we have proved that $I \leq C_y$. Finally, we get

$$S_2 \leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{|h|}{\sqrt{t}}\right)^{\delta''} \left(1 + \frac{\sqrt{t/2}}{\rho(x)}\right)^{-N} \left(1 + \frac{\sqrt{t/2}}{\rho(x)}\right)^{-N} e^{-c|x-y|^2/t}.$$

This completes the proof of Proposition 3.3. $\square$

Proposition 3.4. Let $Q_t(x, y) = t^2 \frac{\partial}{\partial s} K_s(x, y)|_{s=t^2}$.

(a) For $N > 0$, there exists a constant $C_N > 0$ such that

$$|Q_t(x, y)| \leq \frac{C_N}{\mu(B(x, t))} e^{-|x-y|^2/t^2} \left(1 + t/\rho(x)\right)^{-N} \left(1 + t/\rho(y)\right)^{-N}.$$

(b) Let $0 < \delta' \leq \delta_0$ and $|h| < t$, where δ_0 appears in Proposition 3.3. For any $N > 0$, there exists a constant $C_N > 0$ such that

$$|Q_t(x+h, y) - Q_t(x, y)| \leq \frac{C_N}{\mu(B(x, t))} e^{-c|x-y|^2/t^2} \left(\frac{|h|}{t}\right)^{\delta'} \left(1 + \frac{t}{\rho(x)}\right)^{-N} \left(1 + \frac{t}{\rho(y)}\right)^{-N}.$$

(c) For any $N > 0$, there exists a constant $C_N > 0$ such that

$$\left| \int_{\mathbb{R}^n} Q_t(x, y) d\mu(y) \right| \leq C_N \left(\frac{t}{\rho(x)}\right)^{\delta} \left(1 + \frac{t}{\rho(x)}\right)^{-N}.$$

Remark 3.5. In fact, we could assume $0 < \delta < 1$ in (c) of Proposition 3.4. Because for $\delta > 1$, by use of the arbitrariness of N , we can choose a $0 < \delta' < 1 < \delta$ such that

$$(t/\rho(x))^{\delta} \frac{1}{(1 + t/\rho(x))^{N+\delta-\delta'}} \leq (t/\rho(x))^{\delta'} \frac{1}{(1 + t/\rho(x))^N}.$$

In order to prove Proposition 3.4, we need the following lemmas. Similar to [17, Corollary 6.2], we can use (2.4) to obtain

Lemma 3.6. The semigroup has the (unique) extension to a holomorphic semigroup T_{ξ} on $L^2(e^{\eta|x-y|} dx)$ in the sector $\triangle_{\pi/4} = \{\xi : |\arg \xi| < \pi/4\}$. Moreover, there exist constants $C, c' > 0$ such that for every $\eta > 0$ we have

$$\|T_{\xi}\|_{L^2(e^{\eta|x-y|} dx) \rightarrow L^2(e^{\eta|x-y|} dx)} \leq C e^{c' \eta^2 \operatorname{Re} \xi}.$$

Lemma 3.7. There exists a constant $c > 0$ such that for every $M > 0$, there exists a constant $C > 0$ such that for every $\eta > 0$ and $y \in \mathbb{R}^n$, we have

$$\int |K_{\xi}(x, y)|^2 e^{\eta|x-y|} d\mu(x) \leq C e^{c \eta^2 \operatorname{Re} \xi} \frac{1}{\mu(B(y, \sqrt{\operatorname{Re} \xi}))} (1 + \sqrt{\operatorname{Re} \xi}/\rho(y))^{-M}.$$

Proof. Let $t = \operatorname{Re} \xi$, we have $K_{\xi}(x, y) = [T_{\xi-t/10} K_{t/10}(\cdot, y)](x)$. Using Lemma 2.1 we have

$$\begin{aligned} \int |K_{\xi}(x, y)|^2 e^{\eta|x-y|} d\mu(x) &\leq C e^{c' \eta^2 \operatorname{Re}(\xi-t/10)} \|K_{t/10}(\cdot, y)\|_{L^2(e^{\eta|x-y|} d\mu(x))}^2 \\ &\leq C e^{c' \eta^2 t} \int_{\mathbb{R}^n} \frac{e^{-c|u-y|^2/t}}{\mu(B(u, \sqrt{t}))^2} \left(1 + \frac{\sqrt{t}}{\rho(u)}\right)^{-2M} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-2M} e^{\eta|u-y|} d\mu(u) \\ &\leq C e^{c' \eta^2 t} \left(1 + \sqrt{t}/\rho(y)\right)^{-2M} \int_{\mathbb{R}^n} e^{-c|u-y|^2/t+\eta|u-y|} d\mu(u). \end{aligned}$$

For every $\omega \in B(y, \sqrt{t})$, $|u - \omega| \leq |y - u| + |y - \omega| \leq |y - u| + \sqrt{t}$, that is, $B(y, \sqrt{t}) \subset B(u, |y - u| + \sqrt{t})$. Set

$$\begin{cases} B_0 = \{u : |u - y| < 2\sqrt{t} + \eta t\}; \\ B_k = \{u : 2^k \sqrt{t} + \eta t \leq |u - y| < 2^{k+1} \sqrt{t} + \eta t\}, k = 1, 2, \dots \end{cases}$$

We get

$$\begin{aligned}
 & \int |K_{\xi}(x, y)|^2 e^{\eta|x-y|} d\mu(x) \\
 & \leq C e^{c'\eta^2 t} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-2M} \frac{1}{\mu(B(y, \sqrt{t}))^2} \int_{\mathbb{R}^n} \left(1 + \frac{|y-u|}{\sqrt{t}}\right)^{2y} e^{-c|u-y|^2/t + \eta|u-y|} d\mu(u) \\
 & \leq C e^{c'\eta^2 t} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-2M} \frac{1}{\mu(B(y, \sqrt{t}))^2} \sum_{k=0}^{\infty} e^{-c_1(2^k \sqrt{t} + \eta t)^2/t} \frac{\mu(B(y, 2^{k+1}\sqrt{t} + \eta t))}{(1+2^k)^l} \\
 & \leq C \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-2M} \frac{e^{c'\eta^2 t}}{\mu(B(y, \sqrt{t}))} \sum_{k=0}^{\infty} \left(1 + 2^{k+1} + \eta\sqrt{t}\right)^y \frac{e^{-c_1(2^k + \eta\sqrt{t})^2}}{(1+2^k)^l} \\
 & \leq C e^{c'\eta^2 t} \frac{1}{\mu(B(y, \sqrt{t}))} \left(1 + \sqrt{t}/\rho(y)\right)^{-2M}.
 \end{aligned}$$

□

Lemma 3.8. *There exists a constant $c > 0$ such that for every $M > 0$ there is a constant $C_M > 0$ such that for any $\xi \in \triangle_{\pi/5}$,*

$$|K_{\xi}(x, y)| \leq \frac{C_M}{\mu(B(y, \sqrt{\text{Re}\xi}))} \left(1 + \sqrt{\text{Re}\xi}/\rho(x)\right)^{-M} \left(1 + \sqrt{\text{Re}\xi}/\rho(y)\right)^{-M} e^{-c|x-y|^2/\text{Re}\xi}.$$

Proof. We have

$$\begin{aligned}
 & |K_{\xi}(x, y)| e^{\eta|x-y|} = \left| \int K_{\xi/2}(x, u) K_{\xi/2}(u, y) d\mu(u) \right| e^{\eta|x-y|} \\
 & \leq \left(\int |K_{\xi/2}(x, u)|^2 e^{2\eta|x-u|} d\mu(u) \right)^{1/2} \left(\int |K_{\xi/2}(u, y)|^2 e^{2\eta|y-u|} d\mu(u) \right)^{1/2} \\
 & \leq \frac{1}{\mu(B(x, \sqrt{\text{Re}\xi}))^{1/2}} \frac{1}{\mu(B(y, \sqrt{\text{Re}\xi}))^{1/2}} e^{c\eta^2 \text{Re}\xi} \left(1 + \frac{\sqrt{\text{Re}\xi}}{\rho(x)}\right)^{-M} \left(1 + \frac{\sqrt{\text{Re}\xi}}{\rho(y)}\right)^{-M}.
 \end{aligned}$$

Set $\eta = c''|x-y|(\text{Re}\xi)^{-1}$, where c'' is a sufficiently small constant. Then we have

$$\begin{aligned}
 & |K_{\xi}(x, y)| e^{c''|x-y|^2/\text{Re}\xi} \\
 & \leq \frac{C_M}{\mu(B(x, \sqrt{\text{Re}\xi}))^{1/2}} \frac{e^{-c|x-y|^2/\text{Re}\xi}}{\mu(B(y, \sqrt{\text{Re}\xi}))^{1/2}} \left(1 + \frac{\sqrt{\xi}}{\rho(x)}\right)^{-M} \left(1 + \frac{\sqrt{\xi}}{\rho(y)}\right)^{-M} \\
 & \leq \frac{C_M}{\mu(B(x, \sqrt{\text{Re}\xi}))} \left(1 + \frac{|x-y|}{\sqrt{\text{Re}\xi}}\right)^{y/2} e^{-c|x-y|^2/\text{Re}\xi} \left(1 + \frac{\sqrt{\xi}}{\rho(x)}\right)^{-M} \left(1 + \frac{\sqrt{\xi}}{\rho(y)}\right)^{-M} \\
 & \leq \frac{C_M}{\mu(B(x, \sqrt{\text{Re}\xi}))} e^{-c'|x-y|^2/\text{Re}\xi} \left(1 + \frac{\sqrt{\xi}}{\rho(x)}\right)^{-M} \left(1 + \frac{\sqrt{\xi}}{\rho(y)}\right)^{-M}.
 \end{aligned}$$

Similarly, we can prove

$$|K_{\xi}(x, y)| \leq \frac{C_M}{\mu(B(y, \sqrt{\text{Re}\xi}))} e^{-c'|x-y|^2/\sqrt{\text{Re}\xi}} \left(1 + \frac{\sqrt{\xi}}{\rho(x)}\right)^{-M} \left(1 + \frac{\sqrt{\xi}}{\rho(y)}\right)^{-M}.$$

This completes the proof of Lemma 3.8. □

Proof of Proposition 3.4: Now we are in a position to complete the proof of Proposition 3.4.

(a) By the Cauchy integral formula and Lemma 3.8, we have

$$\begin{aligned}
 |Q_t(x, y)| &= \left| \frac{1}{2\pi i} \int_{|\xi-t^2|=t^2/2} \frac{t^2 K_{\xi}(x, y)}{(\xi-t^2)^2} d\xi \right| \\
 &\leq \frac{C}{\mu(B(x, t))} e^{-c|x-y|^2/t^2} \left(1 + \frac{t}{\rho(x)}\right)^{-M} \left(1 + \frac{t}{\rho(x)}\right)^{-M}.
 \end{aligned}$$

(b) By the definition of $Q_t(x, y)$, we have

$$|Q_t(x+h, y) - Q_t(x, y)| \leq C_n \int_{\mathbb{R}^n} |K_{t^2/2}(x+h, \eta) - K_{t^2/2}(x, \eta)| |Q_{t/2}(\eta, y)| d\mu(\eta).$$

It can be deduced from Proposition 3.3 that

$$|K_{t^2/2}(x+h, \eta) - K_{t^2/2}(x, \eta)| \leq C_M \left(\frac{|h|}{t}\right)^{\delta'} \frac{e^{-c|x-\eta|^2/t^2}}{\mu(B(x, t))} \left(1 + \frac{t}{\rho(x)}\right)^{-M} \left(1 + \frac{t}{\rho(y)}\right)^{-M}$$

and

$$\begin{aligned} & |Q_t(x+h, y) - Q_t(x, y)| \\ & \leq C_M \int_{\mathbb{R}^n} \left(\frac{|h|}{t}\right)^{\delta'} \frac{e^{-c|x-\eta|^2/t^2}}{\mu(B(x, t))} \left(1 + \frac{t}{\rho(x)}\right)^{-M} \frac{e^{-c|\eta-y|^2/t^2}}{\mu(B(\eta, t))} \left(1 + \frac{t}{\rho(y)}\right)^{-M} \left(1 + \frac{t}{\rho(\eta)}\right)^{-2M} d\mu(\eta) \\ & \leq C_M \frac{1}{\mu(B(x, t))} \left(1 + \frac{t}{\rho(x)}\right)^{-M} \left(1 + \frac{t}{\rho(y)}\right)^{-M} \left(\frac{|h|}{t}\right)^{\delta'} \int_{\mathbb{R}^n} \frac{e^{-c|x-\eta|^2/t^2 - c|\eta-y|^2/t^2}}{\mu(B(\eta, t))} d\mu(\eta). \end{aligned}$$

The integral in the last inequality is divided as follows:

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{e^{-c|x-\eta|^2/t^2 - c|\eta-y|^2/t^2}}{\mu(B(\eta, t))} d\mu(\eta) &= \left(\int_{|x-\eta|>t} + \int_{|x-\eta|\leq t} \right) \frac{e^{-c|x-\eta|^2/t^2 - c|\eta-y|^2/t^2}}{\mu(B(\eta, t))} d\mu(\eta) \\ &=: I_1 + I_2. \end{aligned}$$

Next, we consider I_1 and I_2 separately. For I_1 , because $|x-\eta|^2 + |\eta-y|^2 \geq |x-y|^2/2$, we have

$$\begin{aligned} I_1 &\leq e^{-c_2|x-y|^2/t^2} \frac{1}{\mu(B(x, t))} \int_{|x-\eta|\leq t} \frac{\mu(B(\eta, |x-\eta|+t))}{\mu(B(\eta, t))} e^{-c_1|x-\eta|^2/t^2} d\mu(\eta) \\ &\leq e^{-c_2|x-y|^2/t^2} \frac{1}{\mu(B(x, t))} \int_{|x-\eta|\leq t} \left(1 + \frac{|x-\eta|}{t}\right)^y e^{-c_1|x-\eta|^2/t^2} d\mu(\eta) \\ &\leq C e^{-c_2|x-y|^2/t^2}. \end{aligned}$$

For I_2 , we obtain

$$\begin{aligned} I_2 &\leq \sum_{k=1}^{\infty} \int_{2^k t \leq |x-\eta| < 2^{k+1} t} \frac{e^{-c|x-\eta|^2/t^2 - c|\eta-y|^2/t^2}}{\mu(B(\eta, t))} d\mu(\eta) \\ &\leq \frac{1}{\mu(B(x, t))} \sum_{k=1}^{\infty} \left(1 + \frac{2^{k+1}t}{t}\right)^y \frac{\mu(B(x, 2^{k+1}t))}{(1 + 2^k t/t)^l} e^{-c_2|x-y|^2/t^2} \\ &\leq \sum_{k=1}^{\infty} (1 + 2^{k+1})^y \frac{1}{(1 + 2^k)^l} 2^{(k+1)y} e^{-c_2|x-y|^2/t^2} \\ &\leq C e^{-c_2|x-y|^2/t^2}. \end{aligned}$$

The estimates for I_1 and I_2 imply that

$$|Q_t(x+h, y) - Q_t(x, y)| \leq \frac{C_N}{\mu(B(x, t))} e^{-c|x-y|^2/t^2} \left(\frac{|h|}{t}\right)^{\delta'} \left(1 + t/\rho(x)\right)^{-N} \left(1 + t/\rho(y)\right)^{-N}.$$

(c) It is easy to see that

$$\begin{aligned} I &= \left| \int_{\mathbb{R}^n} Q_t(x, y) d\mu(y) \right| \\ &\leq s \int_{\mathbb{R}^n} \frac{C_N}{\mu(B(x, t))} \left(1 + \frac{t}{\rho(x)}\right)^{-N} \left(1 + \frac{t}{\rho(y)}\right)^{-N} V(y) e^{-c|x-y|^2/t^2} d\mu(y) \\ &\leq s \left(1 + \frac{t}{\rho(x)}\right)^{-N} \int_{\mathbb{R}^n} \frac{e^{-c|x-y|^2/t^2} V(y)}{\mu(B(x, t))} d\mu(y). \end{aligned}$$

If $t \leq c'\rho(x)$, a direct computation gives

$$I \leq t^2 (1 + t/\rho(x))^{-N} \frac{1}{t^2} (t/\rho(x))^\delta = C_N (1 + t/\rho(x))^{-N} (t/\rho(x))^\delta.$$

If $t > c'\rho(x)$, as we have proved,

$$I \leq (1 + t/\rho(x))^{-N} (t/\rho(x))^{l_0} \leq C_N (1 + t/\rho(x))^{-N} (t/\rho(x))^\delta.$$

This completes the proof of Proposition 3.4.

3.2 The area integral characterization of $H_L^1(d\mu)$

Now we give the area integral characterization for Hardy spaces associated with the degenerate Schrödinger operator L . We will divide the proof into two steps. At first, we prove that for any $f \in H_L^1$, $S_h^L f$ belongs to $L^1(d\mu)$.

Theorem 3.9. Suppose $V \in B_{q,\mu}$, $q > 1$. Let $L = -\frac{1}{\omega(x)} \sum_{i,j} \partial_i(a_{ij}(\cdot)\partial_j)(x) + V(x)$ be the degenerate Schrödinger operator. For $f \in H_L^1(d\mu)$, we have $S_h^L f \in L^1(d\mu)$, where S_h^L is defined in (3.1).

Proof. At first, we can show that the Littlewood-Paley g -function g_L^h is bounded on $L^2(d\mu)$, where g_L^h is defined in (3.2). In fact, using the reproducing formula on $L^2(d\mu)$ and the spectral theorem, $\|g_L^h f\|_{L^2(d\mu)}^2 = \frac{1}{8} \|f\|_{L^2(d\mu)}^2$. To prove Theorem 3.9, we only need to verify that $S_h^L(a)$ is uniformly in $L^1(d\mu)$ for any H_L^1 -atom a . For $y \in \Gamma(x)$, we have for $z \in B(y, t)$, $|x - z| \leq |x - y| + |y - z| < 2t$, that is, $B(y, t) \subset B(x, 2t)$. So we can get

$$\begin{aligned} \|S_h^L a\|_{L^2(\mathbb{R}^n, d\mu)}^2 &= \int_{\mathbb{R}^n} \left[\int_0^\infty \int_{\mathbb{R}^n} |Q_t a(y)|^2 \chi_{\Gamma(x)}(y, t) \frac{d\mu(y) dt}{t\mu(B(x, t))} \right] d\mu(x) \\ &= \int_0^\infty \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \chi_{\Gamma(x)}(y, t) d\mu(x) \right] |Q_t a(y)|^2 \frac{d\mu(y) dt}{t\mu(B(x, t))} \\ &\leq \int_0^\infty \int_{\mathbb{R}^n} \left[\frac{1}{\mu(B(y, t))} \int_{\mathbb{R}^n} \chi_{\Gamma(x)}(y, t) d\mu(x) \right] |Q_t a(y)|^2 \frac{\mu(B(x, 2t))}{\mu(B(x, t))} \frac{d\mu(y) dt}{t} \\ &\leq 2^y \int_0^\infty \int_{\mathbb{R}^n} |Q_t a(y)|^2 \frac{d\mu(y) dt}{t} \\ &\leq C_y \mu(B(x, r))^{-1}, \end{aligned}$$

where in the last inequality we have used the condition: $\|a\|_{L^\infty} \leq \mu(B(x, r))^{-1}$. Then we write

$$\|S_h^L a\|_{L^1} = \int_{B(x_0, 4r)} |S_h^L a(x)| d\mu(x) + \int_{B^c(x_0, 4r)} |S_h^L a(x)| d\mu(x) := I + II.$$

For I , it is easy to see that

$$\begin{aligned} I &\leq \mu(B(x_0, 4r))^{1/2} \left(\int_{B(x_0, 4r)} |S_h^L a(x)|^2 d\mu(x) \right)^{1/2} \\ &\leq C_Y \mu(B(x_0, 4r))^{1/2} \mu(B(x, r))^{-1/2} \leq C. \end{aligned}$$

For the estimate of II , the following two cases are considered.

Case I: $r < \rho(x_0)/4$. By the canceling property of the atom a , we have

$$\begin{aligned} S_h^L a(x) &= \left[\int_0^\infty \int_{|x-y|<t} |Q_t a(y)|^2 \frac{d\mu(y)dt}{t\mu(B(x, t))} \right]^{1/2} \\ &\leq \left[\int_0^\infty \int_{|x-y|<t} \left(\int_{B(x_0, r)} |Q_t(y, z) - Q_t(y, x_0)| |a(z)| d\mu(z) \right)^2 \frac{d\mu(y)dt}{t\mu(B(x, t))} \right]^{1/2} \\ &\leq \left[\left(\int_0^{\frac{|x-x_0|}{2}} \int_{|x-y|<t} + \int_{\frac{|x-x_0|}{2}}^\infty \int_{|x-y|<t} \right) \left(\int_{B(x_0, r)} |Q_t(y, z) - Q_t(y, x_0)| \frac{d\mu(z)}{\mu(B(x_0, r))} \right)^2 \frac{d\mu(y)dt}{t\mu(B(x, t))} \right]^{1/2} \\ &=: II_1 + II_2. \end{aligned}$$

For II_1 , because $0 < t < |x - x_0|/2$ and $|x - y| < t$, we can get $|y - x_0| \sim |x - x_0|$. For $z \in B(x_0, r)$ and $x \in B^c(x_0, 4r)$, we have $|x_0 - z| < r \leq c|x_0 - y|/4$. Using (b) of Proposition 3.4, for $|h| < t$ and symmetry, we have

$$|Q_t(x, y+h) - Q_t(x, y)| \leq \frac{C_M}{\mu(B(y, t))} e^{-c|x-y|^2/t^2} \left(|h|/t \right)^{\delta'} \left(1 + t/\rho(x) \right)^{-M} \left(1 + t/\rho(y) \right)^{-M}$$

and

$$|Q_t(y, z) - Q_t(y, x_0)| \leq \frac{C_M}{\mu(B(x_0, t))} e^{-c|y-x_0|^2/t^2} \left(|z - x_0|/t \right)^{\delta'} \left(1 + t/\rho(x_0) \right)^{-M} \left(1 + t/\rho(y) \right)^{-M}.$$

By the fact that $|x - x_0| \sim |y - x_0|$, we obtain

$$\begin{aligned} II_1 &\leq \left(\int_0^{\frac{|x-x_0|}{2}} \int_{|x-y|<t} \left(\int_{B(x_0, r)} \frac{C_M e^{-c|y-x_0|^2/t^2}}{\mu(B(x_0, t))} \left(\frac{|z - x_0|}{t} \right)^{\delta'} \frac{d\mu(z)}{\mu(B(x_0, r))} \right)^2 \frac{d\mu(y)dt}{t\mu(B(x, t))} \right)^{1/2} \\ &\leq C_M \left(\int_0^{|x-x_0|/2} \int_{|x-y|<t} \frac{e^{-2|y-x_0|^2/t^2}}{\mu(B(x_0, t))^2} \left(\frac{r}{t} \right)^{2\delta'} \frac{d\mu(y)dt}{t\mu(B(x, t))} \right)^{1/2} \\ &\leq C_M r^{\delta'} \left(\int_0^{|x-x_0|/2} \frac{1}{\mu(B(x_0, t))^2} e^{-2|x-x_0|^2/t^2} \frac{dt}{t^{2\delta'+1}} \right)^{1/2}. \end{aligned}$$

For $0 < t < |x - x_0|/2$, $\frac{1}{B(x_0, t)} \leq \frac{1}{B(x_0, |x-x_0|)} \left(\frac{|x-x_0|}{t} \right)^Y$. We can choose l large enough such that

$$\begin{aligned} II_1 &\leq C_M r^{\delta'} \left(\int_0^{|x-x_0|/2} \frac{1}{\mu(B(x_0, |x-x_0|))^2} \left(\frac{|x-x_0|}{t} \right)^{2Y} e^{-c|x-x_0|^2/t^2} \frac{dt}{t^{2\delta'+1}} \right)^{1/2} \\ &\leq C_M \frac{r^{\delta'} |x-x_0|^Y}{\mu(B(x_0, |x-x_0|))} \left(\int_0^{|x-x_0|/2} \frac{1}{t^{2Y+2\delta'+1}} \frac{1}{(1+|x-x_0|^2/t^2)^l} dt \right)^{1/2} \\ &\leq C_M \left(\frac{r}{|x-x_0|} \right)^{\delta'} \frac{1}{\mu(B(x_0, |x-x_0|))}. \end{aligned}$$

The above estimate for II_1 implies

$$\begin{aligned} \int_{B^c(x_0, 4r)} II_1 d\mu(x) &\leq C_M \int_{|x-x_0| \geq 4r} \left(\frac{r}{|x-x_0|} \right)^{\delta'} \frac{1}{\mu(B(x_0, |x-x_0|))} d\mu(x) \\ &\leq C_M \sum_{k=2}^{\infty} \int_{2^k r \leq |x-x_0| < 2^{k+1} r} \left(\frac{r}{|x-x_0|} \right)^{\delta'} \frac{1}{\mu(B(x_0, |x-x_0|))} d\mu(x) \\ &\leq C_M \sum_{k=2}^{\infty} \left(\frac{r}{2^k r} \right)^{\delta'} \frac{1}{\mu(B(x_0, 2^k r))} \mu(B(x_0, 2^{k+1} r)) \leq C. \end{aligned}$$

In what follows, we estimate II_2 . Since $|z-x_0| \leq r < |x-x_0|/2 \leq t$,

$$\begin{aligned} |Q_t(y, z) - Q_t(y, x_0)| &\leq \frac{C_M}{\mu(B(x_0, t))} \left(\frac{|z-x_0|}{t} \right)^{\delta'} e^{-|y-x_0|^2/t^2} \left(1 + \frac{t}{\rho(x)} \right)^{-M} \left(1 + \frac{t}{\rho(x)} \right)^{-M} \\ &\leq \frac{C_M}{\mu(B(x_0, t))} \left(\frac{|z-x_0|}{t} \right)^{\delta'}. \end{aligned}$$

Due to the fact that $|x-x_0|/2 < t$, $B(x_0, t) \subset B(x_0, |x-x_0|)$. We could get

$$\begin{aligned} II_2 &\leq \left[\int_{|x-x_0|/2 \leq |x-y| < t} \int_{B(x_0, r)} \left(\frac{|z-x_0|/t}{\mu(B(x_0, t))} \frac{d\mu(z)}{\mu(B(x_0, r))} \right)^2 \frac{d\mu(y)dt}{t\mu(B(x, t))} \right]^{1/2} \\ &\leq \left(\int_{|x-x_0|/2 \leq |x-y| < t} \int \frac{1}{\mu(B(x_0, t))^2} \left(\frac{r}{t} \right)^{2\delta'} \frac{d\mu(y)dt}{t\mu(B(x, t))} \right)^{1/2} \\ &\leq \frac{r^{\delta'}}{\mu(B(x_0, |x-x_0|))} \left(\int_{|x-x_0|/2}^{\infty} (|x-x_0|/t)^{2\gamma} \frac{1}{t^{2\delta'+1}} dt \right)^{1/2} \\ &\leq \frac{C}{\mu(B(x_0, |x-x_0|))} \left(\frac{r}{|x-x_0|} \right)^{\delta'}. \end{aligned}$$

Similarly, we have

$$\int_{B^c(x_0, 4r)} II_2 d\mu(x) \leq \int_{|x-x_0| > 4r} \frac{C}{\mu(B(x_0, |x-x_0|))} \left(\frac{r}{|x-x_0|} \right)^{\delta'} d\mu(x) \leq C.$$

Case II: $\rho(x_0)/4 \leq r < \rho(x_0)$. In this case, the atom a has no canceling property. So

$$\begin{aligned} (S_h^L a(x))^2 &= \left(\int_0^{r/2} \int_{|x-y| < t} + \int_{r/2}^{|x-x_0|/4} \int_{|x-y| < t} + \int_{|x-x_0|/4}^{\infty} \int_{|x-y| < t} \right) |Q_t a(y)|^2 \frac{d\mu(y)dt}{t\mu(B(x, t))} \\ &=: II'_1 + II'_2 + II'_3. \end{aligned}$$

We first estimate

$$II'_1 = \int_0^{r/2} \int_{|x-y| < t} \left| \int_{B(x_0, r)} Q_t(y, z) a(z) d\mu(z) \right|^2 \frac{d\mu(y)dt}{t\mu(B(x, t))}.$$

Because $|x-x_0| > 4r$, $|x_0-z| < r$ and $|x-y| < t < r/2$, we have $|y-x_0| > 7r/2$. Set $s = t^2$. By (a) of Proposition 3.4,

$$|Q_t(x, y)| \leq \frac{C}{\mu(B(x, t))} e^{-c|x-y|^2/t^2} \left(1 + t/\rho(x) \right)^{-M} \left(1 + t/\rho(y) \right)^{-M}.$$

For $z \in B(x_0, r)$, $|z - x_0| < r < |x - x_0|/4$. The triangular inequality implies that $|y - z| \sim |x - x_0|$. Then we could estimate II'_1 as follows.

$$\begin{aligned} II'_1 &\leq C \int_0^{r/2} \int_{|x-y|<t} \left| \int_{B(x_0,r)} \frac{e^{-c|x-x_0|^2/t^2}}{\mu(B(y,t))} \frac{d\mu(z)}{\mu(B(x_0,r))} \right|^2 \frac{d\mu(y)dt}{t\mu(B(x,t))} \\ &\leq C \int_0^{r/2} \int_{|x-y|<t} \frac{1}{\mu(B(y,t))^2} e^{-2|x-x_0|^2/t^2} \frac{d\mu(y)dt}{t\mu(B(x,t))} \\ &\leq C \int_0^{r/2} e^{-2|x-x_0|^2/t^2} \frac{dt}{t\mu(B(x,t))^2}, \end{aligned}$$

where we have used the following facts: $B(x, t) \subset B(y, |x - y| + t)$ and $\mu(B(x, t))/\mu(B(y, t)) \leq (1 + \frac{|x-y|}{t})^y$. Note that $r < |x - x_0|/4$. Taking l large enough, we get

$$\begin{aligned} II'_1 &\leq \frac{c}{\mu(B(x_0, |x - x_0|))^2} \int_0^{r/2} \frac{1}{t} \left(\frac{|x - x_0|}{t} \right)^{2y} \frac{1}{(1 + |x - x_0|^2/t^2)^{l+1}} dt \\ &\leq \frac{cr^2}{|x - x_0|^2 \mu(B(x_0, |x - x_0|))^2}. \end{aligned}$$

For II'_2 , we have

$$II'_2 \leq \int_{r/2}^{\frac{|x-x_0|}{4}} \int_{|x-y|<t} \left[\int_{B(x_0,r)} \frac{ce^{-|y-z|^2/t^2}}{\mu(B(y,t))} \left(1 + \frac{t}{\rho(y)}\right)^{-M} \left(1 + \frac{t}{\rho(z)}\right)^{-M} \frac{d\mu(z)}{\mu(B(x_0,r))} \right]^2 \frac{d\mu(y)dt}{t\mu(B(x,t))}.$$

Because $z \in B(x_0, r)$, $|z - x_0| < r < \rho(x_0)$, then $\rho(z) \sim \rho(x_0) \sim r$. It follows from this fact that

$$\begin{aligned} II'_2 &\leq \int_{r/2}^{|x-x_0|/4} \int_{|x-y|<t} e^{-2c|x-x_0|^2/t^2} \left(1 + \frac{t}{\rho(x_0)}\right)^{-2M} \frac{1}{\mu(B(y,t))^2} \frac{d\mu(y)dt}{t\mu(B(x,t))} \\ &\leq \int_{r/2}^{|x-x_0|/4} \frac{1}{t\mu(B(x,t))^2} e^{-2c|x-x_0|^2/t^2} \left(1 + \frac{t}{\rho(x_0)}\right)^{-2M} dt \\ &\leq \int_{r/2}^{|x-x_0|/4} \left(\frac{r}{t}\right)^{2M} \left(\frac{t}{|x-x_0|}\right)^{2l} \frac{dt}{t\mu(B(x,t))^2}, \end{aligned}$$

where in the last inequality we have used the fact: $\rho(x_0)/4 \leq r < \rho(x_0)$. We can deduce from the doubling property of μ that

$$\begin{aligned} II'_2 &\leq \frac{cr^{2M}}{|x - x_0|^{2l}} \int_{r/2}^{|x-x_0|/4} \frac{t^{2l}}{t^{2M+1}} \frac{1}{\mu(B(x_0, |x - x_0|))^2} \frac{\mu(B(x_0, |x - x_0|))^2}{\mu(B(x_0, t))^2} dt \\ &\leq \frac{r^{2M}}{|x - x_0|^{2M} \mu(B(x_0, |x - x_0|))^2}. \end{aligned}$$

At last, we estimate II'_3 . For $|z - x_0| < r < \rho(x_0)$, we have $\rho(x_0) \sim \rho(z)$ and

$$\begin{aligned} II'_3 &\leq \int_{|x-x_0|/4}^{\infty} \int_{|x-y|<t} \left[\int_{B(x_0,r)} \frac{ce^{-c|y-z|^2/t^2}}{\mu(B(y,t))} \left(1 + \frac{t}{\rho(y)}\right)^{-M} \left(1 + \frac{t}{\rho(z)}\right)^{-M} \frac{d\mu(z)}{\mu(B(x_0,r))^2} \right]^2 \frac{d\mu(y)dt}{t\mu(B(x,t))} \\ &\leq \int_{|x-x_0|/4}^{\infty} \int_{|x-y|<t} \frac{c}{\mu(B(y,t))^2} \left(1 + \frac{t}{\rho(x_0)}\right)^{-2M} \frac{d\mu(y)dt}{t\mu(B(x,t))} \\ &\leq \int_{|x-x_0|/4}^{\infty} (\rho(x_0)/t)^{2M} \frac{1}{t\mu(B(x,t))} \left[\frac{1}{\mu(B(x,t))^2} \int_{|x-y|<t} \left(1 + \frac{|x-y|}{t}\right)^{2y} d\mu(y) \right] dt \\ &\leq \frac{1}{\mu(B(x_0, |x-x_0|))^2} \int_{|x-x_0|/4}^{\infty} \frac{\rho(x_0)^{2M}}{t^{2M+1}} \left(\frac{|x-x_0|}{t}\right)^{2y} dt \\ &\leq \frac{r^{2M}}{|x-x_0|^{2M} \mu(B(x_0, |x-x_0|))^2}. \end{aligned}$$

Finally, we get

$$\begin{aligned} \int_{B^c(x_0, 4r)} S_h^L a(x) d\mu(x) &\leq \sum_{k=2}^{\infty} \int_{2^k r \leq |x-x_0| < 2^{k+1} r} \left(\frac{r}{|x-x_0|}\right)^M \frac{1}{\mu(B(x_0, |x-x_0|))} d\mu(x) \\ &\leq \sum_{k=2}^{\infty} \left(\frac{r}{2^k r}\right)^M \frac{\mu(B(x_0, 2^{k+1} r))}{\mu(B(x_0, 2^k r))} \\ &\leq \sum_{k=2}^{\infty} \frac{2^{ky}}{2^{kM}} \leq C. \end{aligned}$$

This completes the proof of Theorem 3.9. $\square$

Now we prove the converse of Theorem 3.9. Firstly, we need a reproducing formula associated with Q_t . In order to establish the reproducing formula, we need the following bounded mean oscillation spaces associated with L which are introduced by Yang-Yang-Zhou [13]. For any ball B , let f_B denote the mean of f on B , that is,

$$f_B = \frac{1}{\mu(B)} \int_B f(y) d\mu(y).$$

Definition 3.10. A function $f \in L^1_{loc}(d\mu)$ is said to be in the space $BMO_L(d\mu)$ if

$$\begin{aligned} \|f\|_{BMO_L(d\mu)} &:= \sup_{B(x,r): r < \rho(x)} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f_{B(x,r)}| d\mu(y) \\ &\quad + \sup_{B(x,r): r \geq \rho(x)} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| d\mu(y) < \infty. \end{aligned}$$

We refer the reader to Yang-Yang-Zhou [13] for further information on the space $BMO_L(d\mu)$. A direct computation implies the following result.

Proposition 3.11. For any $t > 0$ and $x \in \mathbb{R}^n$, we have $Q_t(x, \cdot) \in BMO_L(d\mu)$.

Proof. For any ball $B(y_0, r)$, if $r < \rho(y_0)$, by Proposition 3.4,

$$\begin{aligned} & \frac{1}{\mu(B(y_0, r))} \int_{B(y_0, r)} |Q_t(x, y) - (Q_t(x, \cdot))_{B(x, r)}| d\mu(y) \\ & \leq \frac{1}{\mu(B(y_0, r))} \int_{B(y_0, r)} |Q_t(x, y) - Q_t(x, y_0)| d\mu(y) \\ & \leq C\mu(B(x, t))^{-1}, \end{aligned}$$

where $(Q_t(x, \cdot))_{B(x, r)} = \frac{1}{\mu(B)} \int_B Q_t(x, y) d\mu(y)$. If $r \geq \rho(y_0)$, using Proposition 3.4 again, we have

$$\frac{1}{\mu(B(y_0, r))} \int_{B(y_0, r)} |Q_t(x, y)| d\mu(y) \leq C\mu(B(x, t))^{-1}.$$

Therefore, $Q_t(x, \cdot) \in BMO_L(d\mu)$. □

In the characterizations of the Hardy space $H^1(\mathbb{R}^n)$, one of main tools is a Calderón type reproducing formula generated by a family of Littlewood-Paley functions $\{\phi_t, t > 0\}$. In the literature, since the functions $\{\phi_t, t > 0\} \subset \mathcal{S}(\mathbb{R}^n)$, such reproducing formula holds for $f \in \mathcal{S}'(\mathbb{R}^n)$ which equals to zero weakly at ∞ . To obtain the area integral characterization of $H_L^1(d\mu)$, we need a reproducing formula associated with Q_t . However, for a degenerate Schrödinger operator L , $Q_t(x, \cdot)$ may not belong to $\mathcal{S}(\mathbb{R}^n)$. By Proposition 3.11, we know that $Q_t(x, \cdot) \in BMO_L(d\mu)$ and $Q_t(f)$ is well defined for $f \in (BMO_L(d\mu))^*$. Now we introduce the definition of the functions which equal to zero weakly at ∞ associated with L .

Definition 3.12. We say $f \in (BMO_L(d\mu))^*$ equals to zero weakly at ∞ associated with L , if

$$\lim_{A \rightarrow \infty} \int_A^\infty (Q_t)^2 f(x) \frac{dt}{t} = 0,$$

where the above limit holds in the sense of $(BMO_L(d\mu))^*$.

Therefore, we have the following reproducing formula associated with Q_t .

Theorem 3.13. Suppose that f equals to zero weakly at ∞ associated with L . We have

$$f(x) = 8 \int_0^\infty (Q_t)^2 f(x) \frac{dt}{t}, \quad (3.4)$$

where the integral means that

$$\lim_{\epsilon \rightarrow 0} \lim_{A \rightarrow \infty} 8 \int_\epsilon^A (Q_t)^2 f(x) \frac{dt}{t} = f(x)$$

holds in $(BMO_L(d\mu))^*$.

Proof. It is easy to see

$$8 \int_\epsilon^A (Q_t)^2 f(x) \frac{dt}{t} = 8 \int_\epsilon^\infty (Q_t)^2 f(x) \frac{dt}{t} - 8 \int_A^\infty (Q_t)^2 f(x) \frac{dt}{t} = I_1 - I_2.$$

Because f equals to zero weakly at ∞ associated with L , we have $\lim_{A \rightarrow \infty} I_2 = 0$. For any $\phi \in BMO_L(d\mu)$,

$$\lim_{\epsilon \rightarrow 0} \left\langle 8 \int_\epsilon^\infty (Q_t)^2 f \frac{dt}{t}, \phi \right\rangle = \lim_{\epsilon \rightarrow 0} \left\langle f, 8 \int_\epsilon^\infty (Q_t)^2 \phi \frac{dt}{t} \right\rangle = \langle f, \phi \rangle,$$

where the last equality holds, since we have

$$\lim_{\epsilon \rightarrow 0} 8 \int_{\epsilon}^{\infty} (Q_t)^2 \frac{dt}{t} = I$$

in the sense of $(BMO_L(d\mu))^*$ and I is the identity operator in $(BMO_L(d\mu))^*$. $\square$

For the converse of Theorem 3.9, we assume that $f \in (BMO_L(d\mu))^* \cap L^1(d\mu)$. On the one hand, if $f \in H^1(d\mu)$, it is obvious that $f \in (BMO_L(d\mu))^* \cap L^1(d\mu)$. Theorem 3.9 guarantees that $S_L^h(f) \in L^1(d\mu)$. Conversely, if $f \in (BMO_L(d\mu))^* \cap L^1(d\mu)$ and $S_L^h(f) \in L^1(d\mu)$, we will use the reproducing formula (3.4) to derive that f can be represented as the linear combination of H^1 -atoms and the scalars. By Proposition 1.3, this means that $f \in H_L^1(d\mu)$. Precisely, we have the following theorem.

Theorem 3.14. Suppose $V \in B_{q,\mu}$, $q > 1$. Let $L = -\frac{1}{\omega(x)} \sum_{i,j} \partial_i(a_{ij}(\cdot)\partial_j)(x) + V(x)$ be the degenerate Schrödinger operator. For every $f \in (BMO_L(d\mu))^* \cap L^1(d\mu)$ and equals to zero weakly at ∞ associated with L . If $S_h^L f \in L^1(d\mu)$, we have $f \in H_L^1(d\mu)$.

Proof. Since

$$\int_{\mathbb{R}^n} |S_h^L f(x)| d\mu(x) = \int_{\mathbb{R}^n} \left(\int_0^{\infty} \int_{B(x,t)} |Q_t f(y)|^2 \frac{d\mu(y)dt}{t\mu(B(x,t))} \right)^{1/2} d\mu(x) < \infty,$$

we have $Q_t f \in T_2^1(\mathbb{R}^n, d\nu)$, where $T_2^1(\mathbb{R}^n, d\nu)$ is the weighted tent space. Then we have $Q_t f(x) = \sum_i \lambda_i a_i(x, t)$, where $a_i(x, t)$ are the atoms of T_2^1 and $\sum_i |\lambda_i| < \infty$. We assume $a_i(x, t)$ is supported in $\widehat{B(x_0, r)}$ and set $\alpha_i(x) = \int_0^{\infty} Q_t a_i(x, t) \frac{dt}{t}$. Using (3.4), we have

$$\begin{aligned} f(x) &= 8 \int_0^{\infty} Q_t \left(\sum_i \lambda_i a_i(x, t) \right) \frac{dt}{t} \\ &= c \sum_i \lambda_i \int_0^{\infty} Q_t a_i(x, t) \frac{dt}{t} \\ &=: c \sum_i \lambda_i \alpha_i(x). \end{aligned}$$

We need to prove that the H_L^1 -norm of α_i is bounded uniformly. For simplicity, we denote by α the function α_i . In fact, we have

$$\left\| \sup_{t>0} |e^{-tL} \alpha| \right\|_{L^1} \leq \left\| \left(\sup_{t>0} |e^{-tL} \alpha| \right) \chi_{B^*} \right\|_{L^1} + \left\| \left(\sup_{t>0} |e^{-tL} \alpha| \right) \chi_{(B^*)^c} \right\|_{L^1} =: I_1 + I_2.$$

For I_1 , we use Hölder's inequality to get

$$I_1 = \int_{B^*} \sup_{t>0} |e^{-tL} \alpha(x)| d\mu(x) \leq \mu(B^*)^{1/2} \|\alpha\|_2.$$

For the L^2 -norm of α , by the self-adjointness of Q_t^L and Hölder's inequality, we have

$$\begin{aligned} \|\alpha\|_2 &= \sup_{\|\beta\|_2 \leq 1} \int_{\mathbb{R}^n} \alpha(x) \bar{\beta}(x) d\mu(x) \\ &\leq \sup_{\|\beta\|_2 \leq 1} \left(\int_{\mathbb{R}^n} \int_0^{\infty} |a(x, t)|^2 \frac{d\mu(x)dt}{t} \right)^{1/2} \left(\int_{\mathbb{R}^n} \int_0^{\infty} |Q_t \bar{\beta}(x)|^2 \frac{d\mu(x)dt}{t} \right)^{1/2} \\ &\leq \sup_{\|\beta\|_2 \leq 1} \mu(B)^{-1/2} \|\beta\|_2 \\ &\leq C \mu(B)^{-1/2}, \end{aligned}$$

which gives $I_1 \leq C\mu(B^*)^{1/2}\mu(B)^{-1/2} \leq C$.

Next, we estimate I_2 . For $x \in B^c(x_0, 2r)$ and $y \in B(x_0, r)$, we have $|x - y| \sim |x - x_0|$. Since

$$\left(\int_0^r \int_{B(x_0, r)} |a(y, t)|^2 \frac{d\mu(y)dt}{t} \right)^{1/2} \leq \mu(B(x_0, r))^{-1/2},$$

we can use the fact that $t^2/(s+t^2) \leq 1$ and Hölder's inequality to obtain

$$\begin{aligned} & \sup_{s>0} \left| e^{-sL} \int_0^\infty Q_t a(x, t) \frac{dt}{t} \right| \\ & \leq \sup_{s>0} \int_0^\infty \frac{t^2}{s+t^2} \int_{\mathbb{R}^n} \frac{1}{\mu(B(x, \sqrt{s+t^2}))} e^{-c|x-y|^2/s+t^2} |a(y, t)| \frac{d\mu(y)dt}{t} \\ & \leq \mu(B(x_0, r))^{-1/2} \sup_{s>0} \left(\int_0^r \int_{B(x_0, r)} \left(\frac{t^2}{s+t^2} \right)^2 \frac{e^{-c|x-x_0|^2/s+t^2}}{\mu(B(x, \sqrt{s+t^2}))^2} \frac{d\mu(y)dt}{t} \right)^{1/2} \\ & \leq C \sup_{s>0} \frac{1}{\mu(B(x_0, |x-x_0|))} \left(\int_0^r \frac{t^2}{s+t^2} \frac{\mu(B(x_0, |x-x_0|))^2}{\mu(B(x, \sqrt{s+t^2}))} e^{-c|x-x_0|^2/s+t^2} \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

For $u \in B(x_0, |x-x_0|)$, it is easy to see that $\mu(B(x_0, |x-x_0|)) \leq \mu(B(x, 2|x-x_0| + 2\sqrt{s+t^2}))$. Set $l = y + 1$. So

$$\begin{aligned} & \sup_{s>0} \left| e^{-sL} \int_0^\infty Q_t a(x, t) \frac{dt}{t} \right| \\ & \leq C \sup_{s>0} \frac{1}{\mu(B(x_0, |x-x_0|))} \left(\int_0^r \frac{t}{s+t^2} \frac{(1+|x-x_0|/\sqrt{s+t^2})^{2y}}{(1+|x-x_0|^2/(s+t^2))^l} dt \right)^{1/2} \\ & \leq C \frac{r}{|x-x_0|} \frac{1}{\mu(B(x_0, |x-x_0|))}. \end{aligned}$$

Finally, we have

$$\begin{aligned} I_2 & \leq C \int_{B^c(x_0, 2r)} \frac{r}{|x-x_0|} \frac{1}{\mu(B(x_0, |x-x_0|))} d\mu(x) \\ & \leq \sum_{k=1}^\infty \frac{r}{2^k r} \frac{\mu(B(x_0, 2^{k+1}r))}{\mu(B(x_0, 2^k r))} \leq C. \end{aligned}$$

This completes the proof of Theorem 3.14. $\square$

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