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$\frac{\pi}{4}$ -tangentiality of solutions for one-dimensional Minkowski-curvature problems

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Abstract: We analyze $\frac{\pi}{4}$ -tangentiality of solutions for several types of scalar equations and systems of one-dimensional Minkowski-curvature problems with two points Dirichlet boundary conditions, which is dependent on singularity of weight functions and the growth of nonlinear terms. One of the goals is to show non $\frac{\pi}{4}$ -tangentiality ($\|u'\|_\infty < 1$) of solutions for some of the above problems. We consider a larger class of weight functions and find out suitable nonlinear terms associated with it to keep non $\frac{\pi}{4}$ -tangentiality of solutions. Finally, we obtain Lyapunov-type inequalities for some nonlinear problems as by-products which also extend the results in some previous studies.

Keywords: Minkowski-curvature, singular weight, $\frac{\pi}{4}$ -tangentiality, Lyapunov-type inequality

MSC 2010: 34B16, 34B60, 34C40

1 Introduction

In this paper, we are first concerned with the one-dimensional Minkowski-curvature problem

$$\begin{cases} -\left(\phi(u'(t))\right)' = r(t)f(u(t)), & \text{in } (a, b), \\ u(a) = 0 = u(b), \end{cases} \quad (P)$$

where $\phi(y) = \frac{y}{\sqrt{1-|y|^2}}$, for $y \in (-1, 1)$, $r \in C((a, b), [0, \infty))$ and $f \in C(\mathbb{R}, \mathbb{R})$. The origin of problem (P) is related to mean curvature operator in the flat Minkowski space endowed with the Lorentzian metric in Differential Geometry, one may refer to [1-3].

We say u a solution of problem (P) if $u \in C[a, b] \cap C^1(a, b)$, $|u'(t)| < 1$ for $t \in (a, b)$, $\phi(u'(\cdot))$ is absolutely continuous in any compact subinterval of (a, b) , and u satisfies the equation and the boundary conditions in problem (P). To classify solutions of (P), let us first consider a couple of examples.

Example 1.1. Consider the problem

$$\begin{cases} -\left(\phi(u'(t))\right)' = r(t)u(t), & t \in (0, 1), \\ u(0) = 0 = u(1), \end{cases} \quad (1.1)$$

where $r(t) = \frac{2}{t(1-t)\left(\sqrt{\left(\frac{1}{2}+t\right)\left(\frac{3}{2}-t\right)}\right)^3}$. Taking $u(t) = \frac{1}{2}t(1-t)$, we see that u is a solution of problem (1.1) satisfying $u \in C^1[0, 1]$ and $\|u'\|_\infty < 1$.

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Example 1.2. Consider the problem

$$\begin{cases} -\left(\phi(u'(t))\right)' = \frac{3}{2}t^{-\frac{5}{2}}, & t \in (0, 1), \\ u(0) = 0 = u(1). \end{cases} \quad (1.2)$$

We note that a function $G : (0, 1) \rightarrow \mathbb{R}$ given by

$$G(\alpha) = \int_0^1 \phi^{-1} \left(\frac{1}{\sqrt{s^3}} \left(\frac{\sqrt{s^3}}{\sqrt{\alpha^3}} - 1 \right) \right) ds$$

has a unique zero $\alpha^* \in (0, 1)$ by strict monotonicity of G . Define

$$u(t) = \int_t^1 \phi^{-1} \left(\frac{1}{\sqrt{s^3}} \left(\frac{\sqrt{s^3}}{\sqrt{\alpha^{*3}}} - 1 \right) \right) ds.$$

We can see that u is a solution of problem (1.2) satisfying $u \in C^1[0, 1]$ and $u'(0^+) = 1$.

Bartnik-Simon [2] considered the following problem

$$\begin{cases} \frac{1}{\sqrt{1-|\nabla u|^2}} \sum_{i,j=1}^n \left(\delta_{ij} + \frac{1}{1-|\nabla u|^2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u(x)), \\ u = \varphi \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded, $C^{2,\alpha}$ domain for some $\alpha > 0$, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $\varphi : \partial\Omega \rightarrow \mathbb{R}$ are given bounded functions. The authors proved that a solution $u \in C^1(\Omega)$ satisfying $|Du| < 1$ in Ω belongs to $u \in C^1(\bar{\Omega})$ and $|Du| < 1$ on $\bar{\Omega}$.

Inspired by the above examples and the result in [2], we give the definition of $\frac{\pi}{4}$ -tangential solution as follows. We say that a solution $u \in C^1[a, b]$ of problem (P) is $\frac{\pi}{4}$ -tangential if either $|u'(a)| = 1$ or $|u'(b)| = 1$. On the other hand, we say that a solution $u \in C^1[a, b]$ of problem (P) is non $\frac{\pi}{4}$ -tangential if both $|u'(a)| < 1$ and $|u'(b)| < 1$.

It is worth noting that we do not know whether every solution of problem (P) is of class $C^1[a, b]$ or not (see Section 3). If $r \in L^1[a, b]$ and $f \in C(\mathbb{R})$, then it is not hard to see that every solution of problem (P) is of $C^1[a, b]$ and non $\frac{\pi}{4}$ -tangential. So it is natural to ask what happens when r is beyond L^1 .

Let us introduce a larger class than L^1 as follows:

$$\mathcal{B}_q \triangleq \left\{ r \in C((a, b), [0, \infty)) : \int_a^b (\tau - a)^q (b - \tau)^q r(\tau) d\tau < \infty \right\},$$

where $q \geq 1$, r satisfies that $r(t) \geq 0$, $r(t) \not\equiv 0$ in any compact subinterval of $[a, b]$. So far, most studies ([4-9] and references therein) for the type of problem (P) have been concerned with continuous or L^1 -class weight functions. Recently, using a bifurcation technique, Yang-Lee [10] and Yang-Lee-Sim [11] investigated the existence and multiplicity of positive radial solutions as well as nodal radial solutions for a prescribed mean curvature problem on an exterior domain which is transformed into one dimensional problems such as problem (P) with a singular weight in $\mathcal{B}_q$. Besides, when $q = 1$ and $f(u) = u$, Yang-Sim-Lee [12] estimated Lyapunov-type inequalities to give necessary conditions on weight functions in $\mathcal{B}_1$ for the existence of a positive solution under the assumption of non $\frac{\pi}{4}$ -tangentiality.

It is worth mentioning that our results in this paper explain why definitions of solutions in [12] are only for non $\frac{\pi}{4}$ -tangential solutions. Furthermore, applying non $\frac{\pi}{4}$ -tangentiality of solutions for problem (P_0) as well as systems (CS_0) and (SCS_0) (see Section 4 and 6), we get Lyapunov-type inequalities for weight functions in $\mathcal{B}_q$ class corresponding to nonlinear terms associated with q , which extend the results in [12].

The rest of this paper is organized as follows. For a fixed $r \in \mathcal{B}_q$, we give sufficient conditions on f to have non $\frac{\pi}{4}$ -tangentiality or $\frac{\pi}{4}$ -tangentiality of solutions for two types of scalar equations in Section 2 and

Section 3, respectively. In Section 4, we establish Lyapunov-type inequality of a scalar equation with $\mathcal{B}_q$ -weight ($q \geq 1$) as a consequence of non $\frac{\pi}{4}$ -tangentiality. In Section 5, we investigate non $\frac{\pi}{4}$ -tangentiality of solutions for a cycled and strongly coupled systems of one-dimensional Minkowski-curvature problems with $\mathcal{B}_q$ -weight. Section 6 is devoted to obtain Lyapunov-type inequalities for two systems by making use of non $\frac{\pi}{4}$ -tangentiality in Section 5.

2 Non $\frac{\pi}{4}$ -tangentiality of solution for scalar equation

In this section, for a fixed $r \in \mathcal{B}_q$, we give a sufficient condition on f for non $\frac{\pi}{4}$ -tangential solutions of problem (P). Throughout this section, $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $f(s)s > 0$ for $s \neq 0$. Assume

$$(F_q) \quad 0 \leq \lim_{|s| \rightarrow 0} \frac{f(s)}{|s|^{q-1}s} < \infty,$$

we can show non $\frac{\pi}{4}$ -tangentiality of all solutions of problem (P). The result is valid not only for positive solutions but also for every solution.

Theorem 2.1. *Assume (F_q) . Then every solution of problem (P) is non $\frac{\pi}{4}$ -tangential.*

Proof. Let u be a solution of problem (P). It suffices to prove that $u \in C^1[a, b]$, $|u'(a)| < 1$, and $|u'(b)| < 1$. We first prove $u \in C^1[a, b]$. Let $u \not\equiv 0$ on $[a, b]$ and $t^* = \inf\{t \in (a, b) : u'(t) = 0\}$. We claim $a < t^* < b$. Suppose $t^* = a$ (a similar argument can be applied for the case $t^* = b$). Then there exists a sequence $\{t_n\}$ with $u'(t_n) = 0$ satisfying $t_n \rightarrow a$ as $n \rightarrow \infty$. From the definition of solution, it is obvious that $u'(t) < 1$ for $t \in (a, b)$. By the mean value theorem, there exists $\xi \in (a, t)$ for any $t \in (a, b)$ such that

$$|u(t)| = |u(t) - u(a)| = |u'(\xi)|(t - a) < t - a, \text{ for } t \in (a, b).$$

Together with $u(a) = 0 = u(b)$, we have

$$|u(t)| \leq t - a, \text{ for } t \in [a, b], \quad (2.1)$$

and similarly

$$|u(t)| \leq b - t, \text{ for } t \in [a, b]. \quad (2.2)$$

By (F_q) , we obtain, for some $c \triangleq c(\|u\|_\infty) > 0$,

$$|f(u(t))| \leq c|u(t)|^q, \text{ for } t \in [a, b]. \quad (2.3)$$

Since $r \in \mathcal{B}_q$, we can take a small constant $y_0 > a$ satisfying

$$c \int_a^{y_0} (\tau - a)^q r(\tau) d\tau < 1. \quad (2.4)$$

We choose N_0 large enough such that $a < t_n < y_0$ for all $n \geq N_0$ and consider terms t_n for all $n \geq N_0$. Integrating the first equation in problem (P) on (t, t_n) for $t \in (a, t_n)$, we obtain

$$\phi(u'(t)) = \int_t^{t_n} r(\tau) f(u(\tau)) d\tau.$$

It follows that

$$u'(t) = \phi^{-1} \left(\int_t^{t_n} r(\tau) f(u(\tau)) d\tau \right).$$

Integrating the above equation on (a, t) for $t \in (a, t_n)$, we get

$$u(t) = \int_a^t \phi^{-1} \left(\int_s^{t_n} r(\tau) f(u(\tau)) d\tau \right) ds.$$

By (2.1) and (2.3), for $t \in (a, t_n)$, setting $\|u\|_{t_n, \infty} \triangleq \max_{a \leq t \leq t_n} |u(t)|$ and applying Fubini's Theorem, we have

$$\begin{aligned} |u(t)| &\leq \int_a^t c \int_s^{t_n} r(\tau) |u(\tau)|^q d\tau ds \\ &\leq c \int_a^{t_n} \int_s^{t_n} r(\tau) |u(\tau)|^q d\tau ds \\ &= c \int_a^{t_n} (\tau - a) r(\tau) |u(\tau)|^q d\tau \\ &\leq c \int_a^{t_n} (\tau - a)^q r(\tau) d\tau \|u\|_{t_n, \infty}^q. \end{aligned}$$

Using (2.4), we obtain $\|u\|_{t_n, \infty} = 0$ for all $n \geq N_0$, i.e., u is identically zero on $[a, t_n]$ for all $n \geq N_0$. Thus we may assume that $u(t_n) = u'(t_n) = 0$ for sufficiently large n . For $t \in [t_n, b]$, by (2.2) and (2.3), using Fubini's Theorem again, we obtain

$$\begin{aligned} |u(t)| &= \left| \int_{t_n}^t \phi^{-1} \left(\int_{t_n}^s r(\tau) f(u(\tau)) d\tau \right) ds \right| \\ &\leq \int_{t_n}^t \int_{t_n}^s r(\tau) |f(u(\tau))| d\tau ds \\ &= \int_{t_n}^t (t - \tau) r(\tau) |f(u(\tau))| d\tau \\ &\leq c \int_{t_n}^t (b - \tau) r(\tau) |u(\tau)|^q d\tau \\ &\leq c \int_{t_n}^t (b - \tau)^q r(\tau) |u(\tau)| d\tau. \end{aligned}$$

By Gronwall-Bellman inequality, we obtain $u(t) = 0$ for $t \in [t_n, b]$. Thus, $u \equiv 0$ on $[a, b]$ and this is a contradiction. Hence, $t^* \geq y > a$ with a constant y . By the continuity of u and definition of t^* , u has one sign on (a, t^*) . Without loss of generality, we may assume $u(t) > 0$ for $t \in (a, t^*)$. The first equation in problem (P) can be rewritten as

$$-\frac{u''(t)}{\left(\sqrt{1 - |u'(t)|^2}\right)^3} = r(t)f(u(t)).$$

Thus $u'' \leq 0$ on (a, t^*) . Hence, $u'(t)$ is increasing as $t \searrow a^+$. Since $u' \in C(a, b)$ and $|u'(t)| < 1$ for $t \in (a, b)$, we have $u'(a) = \lim_{t \rightarrow a^+} u'(t) \leq 1$ and so $u \in C^1[a, b)$. Similarly, $u \in C^1(a, b]$. This shows $u \in C^1[a, b]$.

We now prove $|u'(a)| < 1$. Noting

$$|u(t)| \leq (t - a) \|u'\|_{\infty} \leq \frac{2(t - a)(b - t)}{b - a} \|u'\|_{\infty}, \text{ for } t \in [a, \frac{a + b}{2}],$$

and

$$|u(t)| \leq (b-t)\|u'\|_\infty \leq \frac{2(t-a)(b-t)}{b-a}\|u'\|_\infty, \text{ for } t \in [\frac{a+b}{2}, b],$$

we have

$$|u(t)| \leq \frac{2(t-a)(b-t)}{b-a}\|u'\|_\infty, \text{ for } t \in [a, b].$$

Together with (2.3), we obtain

$$|f(u(t))| \leq c \left(\frac{2(t-a)(b-t)}{b-a} \right)^q \|u'\|_\infty^q, \text{ for } t \in [a, b].$$

It follows that

$$\int_a^{t^*} r(\tau) |f(u(\tau))| d\tau \leq \frac{2^q c}{(b-a)^q} \|u'\|_\infty^q \int_a^{t^*} r(\tau) (\tau-a)^q (b-\tau)^q d\tau < \infty. \quad (2.5)$$

Integrating the first equation of problem (P) over (t, t^*) for $t \in (a, t^*)$, we get

$$u'(t) = \phi^{-1} \left(\int_t^{t^*} r(\tau) f(u(\tau)) d\tau \right). \quad (2.6)$$

Noting $\phi^{-1}(\alpha) < 1$ for $0 \leq \alpha < \infty$, from (2.5) and (2.6), we obtain

$$\begin{aligned} |u'(a)| &= \left| \lim_{t \rightarrow a^+} \phi^{-1} \left(\int_t^{t^*} r(\tau) f(u(\tau)) d\tau \right) \right| \\ &\leq \phi^{-1} \left(\int_a^{t^*} r(\tau) |f(u(\tau))| d\tau \right) < 1. \end{aligned}$$

By a similar argument, we can show $|u'(b)| < 1$. This completes the proof. $\square$

3 $\frac{\pi}{4}$ -tangentiality of positive solution for a scalar equation

Consider the problem

$$\begin{cases} -(\phi(u'(t)))' = (t-a)^{-q} f(u(t)), & \text{in } (a, b), \\ u(a) = 0 = u(b), \end{cases} \quad (P_1)$$

where $q \geq 1$ and $f \in C(\mathbb{R})$. In this section, we focus on $\frac{\pi}{4}$ -tangentiality of nontrivial nonnegative solutions for problem (P_1) under a suitable condition on function f . A sufficient condition for $\frac{\pi}{4}$ -tangentiality of solutions is given as follows.

(A1) there exist two constants $c_0 > 0$ and $m \geq 0$ such that $|f(s)| \geq c_0 |s|^m$ for $-\frac{b-a}{2} \leq s \leq \frac{b-a}{2}$.

Remark 3.1. Note that $|u'(t)| < 1$, for $t \in (a, b)$ with $u(a) = 0 = u(b)$ implies $\|u\|_\infty < \frac{b-a}{2}$. Thus, condition (A1) is reasonable since every solution u of (P_1) satisfies $\|u\|_\infty < \frac{b-a}{2}$.

Example 3.1. $f(u) = |u|^{m-1}u$ with $0 \leq m \leq q-1$ satisfies condition (A1).

Theorem 3.2. Assume $f(s) > 0$ for $s > 0$ and (A1) with $0 \leq m \leq q - 1$. Then every nontrivial nonnegative solution of problem (P_1) is $\frac{\pi}{4}$ -tangential.

Proof. Let u be a nontrivial nonnegative solution of problem (P_1) . We claim that u has no double zero points in (a, b) . Suppose on the contrary that $t_* \in (a, b)$ is a double zero of u . Then

$$\phi(u'(t)) = \int_t^{t_*} (\tau - a)^{-q} f(u(\tau)) d\tau.$$

This implies that $u'(t) \leq 0$ for $t_* < t < b$ and $u'(t) \geq 0$ for $a < t < t_*$. Together with the fact $u(a) = u(t_*) = u(b) = 0$, we obtain $u \equiv 0$ on $[a, b]$. This contradiction implies that u has no double zero and as a consequence, $u > 0$ on (a, b) .

Now, we show that $u \in C^1[a, b]$. First, let us rewrite the first equation in problem (P_1) as

$$-\frac{u''(t)}{\left(\sqrt{1 - |u'(t)|^2}\right)^3} = (t - a)^{-q} f(u(t)).$$

Then we see that $u'' < 0$ on (a, b) . Thus u is concave on (a, b) and this implies that $u'(t)$ is increasing as $t \searrow a^+$ and decreasing as $t \nearrow b^-$. Moreover, by the fact $u' \in C(a, b)$ and $|u'(t)| < 1$ for $t \in (a, b)$, we get the existence of $\lim_{t \rightarrow a^+} u'(t)$. Defining $u'(a) = \lim_{t \rightarrow a^+} u'(t)$, we obtain $u \in C^1[a, b]$. By a similar argument near b , we finally obtain $u \in C^1[a, b]$.

Next, we show $u'(a^+) = 1$. By the mean value theorem, there exists $t_0 \in (a, b)$ such that $u'(t_0) = 0$. Since u is concave in (a, b) , $u'(t)$ is decreasing and $u'(t) \geq 0$ for $t \in [a, t_0]$. Thus,

$$u(t) \geq \frac{u(t_0)}{t_0 - a}(t - a), \text{ for } t \in [a, t_0].$$

It follows that

$$(t - a)^{-q} (u(t))^m \geq \left(\frac{u(t_0)}{t_0 - a}\right)^m (t - a)^{m-q} > 0, \text{ for } a < t \leq t_0.$$

Integrating the above inequality over (t, t_0) for $a < t \leq t_0$, we obtain

$$\int_t^{t_0} (\tau - a)^{-q} (u(\tau))^m d\tau \geq \left(\frac{u(t_0)}{t_0 - a}\right)^m \int_t^{t_0} (\tau - a)^{m-q} d\tau, \text{ for } a < t \leq t_0. \quad (3.1)$$

Using (3.1) and (A1) with $0 \leq m \leq q - 1$ and noting that $\phi^{-1} \rightarrow 1$ as $s \rightarrow \infty$, we deduce

$$\begin{aligned} u'(a^+) &= \lim_{t \rightarrow a^+} u'(t) = \lim_{t \rightarrow a^+} \phi^{-1} \left(\int_t^{t_0} (\tau - a)^{-q} f(u(\tau)) d\tau \right) \\ &\geq \lim_{t \rightarrow a^+} \phi^{-1} \left(c_0 \int_t^{t_0} (\tau - a)^{-q} (u(\tau))^m d\tau \right) \\ &\geq \lim_{t \rightarrow a^+} \phi^{-1} \left(c_0 \left(\frac{u(t_0)}{t_0 - a} \right)^m \int_t^{t_0} (\tau - a)^{m-q} d\tau \right) \\ &= \phi^{-1} \left(c_0 \left(\frac{u(t_0)}{t_0 - a} \right)^m \int_a^{t_0} (\tau - a)^{m-q} d\tau \right) \\ &= 1. \end{aligned}$$

Together with the fact $u'(a^+) \leq 1$, we get $u'(a^+) = 1$. Hence, u is a $\frac{\pi}{4}$ -tangential solution. $\square$

Remark 3.3. Assume that $f(s) > 0$ for $s \in \mathbb{R}$. Then f satisfies condition (A1) and we note that Example 1.2 follows the result of Theorem 3.2.

4 Lyapunov-type inequality for scalar equation

Let us consider the problem

$$\begin{cases} -\left(\phi(u'(t))\right)' = r(t)|u(t)|^{l-1}u(t), & \text{in } (a, b), \\ u(a) = 0 = u(b), \end{cases} \quad (P_0)$$

where $r \in \mathcal{B}_q$ and $l \geq 1$. As mentioned in Introduction, the authors estimated Lyapunov-type inequality to give necessary condition on $\mathcal{B}_1$ -class weight functions for the existence of non $\frac{\pi}{4}$ -tangential positive solution for problem (P_0) when $l = 1$ in [12]. In this section, we extend such a result to $\mathcal{B}_q$ -class weight function by providing a condition on $\mathcal{B}_q$ -class weight functions for the existence of non $\frac{\pi}{4}$ -tangential solution of problem (P_0) for any $l \geq q$.

Theorem 4.1. *Let $1 \leq q \leq l$. If problem (P_0) has a nontrivial solution, then one has*

$$\frac{(b-a)^{2q-l}}{2^{2q-l-1}} < \int_a^b (t-a)^q (b-t)^q r(t) dt. \quad (4.1)$$

Proof. Let u be a nontrivial solution of problem (P_0) . Note that by Theorem 2.1, u is non $\frac{\pi}{4}$ -tangential. Applying Hölder's inequality, we obtain

$$|u(t)| \leq \int_a^t |u'(\tau)| d\tau \leq (t-a)^{\frac{l}{l+1}} \left(\int_a^t |u'(\tau)|^{l+1} d\tau \right)^{\frac{1}{l+1}}.$$

Noting $(b-t)/(b-a) \geq 0$, we have

$$\left(\frac{b-t}{b-a} \right)^{\frac{l}{l+1}} |u(t)| \leq \left(\frac{(t-a)(b-t)}{b-a} \right)^{\frac{l}{l+1}} \left(\int_a^t |u'(\tau)|^{l+1} d\tau \right)^{\frac{1}{l+1}}.$$

Thus

$$\frac{b-t}{b-a} |u(t)|^{\frac{l+1}{l}} \leq \frac{(t-a)(b-t)}{b-a} \left(\int_a^t |u'(\tau)|^{l+1} d\tau \right)^{\frac{1}{l}}. \quad (4.2)$$

Similarly, we also have

$$\frac{t-a}{b-a} |u(t)|^{\frac{l+1}{l}} \leq \frac{(t-a)(b-t)}{b-a} \left(\int_t^b |u'(\tau)|^{l+1} d\tau \right)^{\frac{1}{l}}. \quad (4.3)$$

Adding (4.2) and (4.3), we get

$$|u(t)|^{\frac{l+1}{l}} \leq \frac{(t-a)(b-t)}{b-a} \left[\left(\int_a^t |u'(\tau)|^{l+1} d\tau \right)^{\frac{1}{l}} + \left(\int_t^b |u'(\tau)|^{l+1} d\tau \right)^{\frac{1}{l}} \right].$$

Thus,

$$|u(t)|^{l+1} \leq \left(\frac{(t-a)(b-t)}{b-a} \right)^l \left[\left(\int_a^t |u'(\tau)|^{l+1} d\tau \right)^{\frac{1}{l}} + \left(\int_t^b |u'(\tau)|^{l+1} d\tau \right)^{\frac{1}{l}} \right]^l. \quad (4.4)$$

Applying the inequality

$$(\alpha + \beta)^l \leq 2^{l-1}(\alpha^l + \beta^l), \quad \alpha \geq 0, \beta \geq 0,$$

to (4.4), we obtain

$$|u(t)|^{l+1} \leq \left(\frac{(t-a)(b-t)}{b-a} \right)^l 2^{l-1} \int_a^b |u'(\tau)|^{l+1} d\tau. \quad (4.5)$$

Multiplying both sides of (4.5) by $r(t)$, we obtain

$$\frac{(b-a)^l}{2^{l-1}} r(t) |u(t)|^{l+1} \leq (t-a)^l (b-t)^l r(t) \int_a^b |u'(\tau)|^{l+1} d\tau. \quad (4.6)$$

Since u is a nontrivial solution of problem (P_0) , we have

$$\int_a^b |u'(t)|^2 dt < \int_a^b \frac{|u'(t)|^2}{\sqrt{1-|u'(t)|^2}} dt = \int_a^b r(t) |u(t)|^{l+1} dt. \quad (4.7)$$

We note that the integration in the right-hand side of (4.7) makes sense due to $l \geq q$. Integrating (4.6) on $[a, b]$ and using $\|u'\|_\infty < 1$, we have

$$\begin{aligned} \frac{(b-a)^l}{2^{l-1}} \int_a^b r(t) |u(t)|^{l+1} dt &\leq \int_a^b (t-a)^l (b-t)^l r(t) dt \int_a^b |u'(\tau)|^{l+1} d\tau \\ &= \int_a^b (t-a)^l (b-t)^l r(t) dt \int_a^b |u'(\tau)|^2 |u'(\tau)|^{l-1} d\tau \\ &\leq \int_a^b (t-a)^l (b-t)^l r(t) dt \int_a^b |u'(\tau)|^2 d\tau. \end{aligned} \quad (4.8)$$

From (4.7) and (4.8), it follows that

$$\frac{(b-a)^l}{2^{l-1}} \int_a^b |u'(t)|^2 dt < \int_a^b (t-a)^l (b-t)^l r(t) dt \int_a^b |u'(\tau)|^2 d\tau.$$

Therefore

$$\frac{(b-a)^l}{2^{l-1}} < \int_a^b (t-a)^l (b-t)^l r(t) dt.$$

Together with the fact $(t-a)(b-t) \leq \frac{(b-a)^2}{2^2}$, we conclude (4.1). $\square$

5 Non $\frac{\pi}{4}$ -tangentiality of solution for cycled and strongly coupled systems

In this section, we study non $\frac{\pi}{4}$ -tangentiality of solution for two types of systems. First, we deal with the following cycled system

$$\begin{cases} \left(\phi(u'_1(t))\right)' + r_1(t)f_1(u_2(t)) = 0, & \text{in } (a, b), \\ \left(\phi(u'_2(t))\right)' + r_2(t)f_2(u_3(t)) = 0, & \text{in } (a, b), \\ \dots \\ \left(\phi(u'_n(t))\right)' + r_n(t)f_n(u_1(t)) = 0, & \text{in } (a, b), \\ u_1(a) = \dots = u_n(a) = 0 = u_1(b) = \dots = u_n(b), \end{cases} \quad (CS)$$

where $r_i \in \mathcal{B}_q$, $f_i : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $f_i(s)s > 0$ for $s \neq 0$, $1 \leq i \leq n$.

We say $(u_1, u_2, \dots, u_n)$ a solution of problem (CS) if each $u_i \in C[a, b] \cap C^1(a, b)$, $|u'_i(t)| < 1$ for $t \in (a, b)$, and $\phi(u'_i(t))$ is absolutely continuous in any compact subinterval of (a, b) , and u_i satisfies the equations and the boundary conditions in problem (CS). We say $(u_1, u_2, \dots, u_n)$ a $\frac{\pi}{4}$ -tangential solution of problem (CS) if $(u_1, u_2, \dots, u_n)$ is a solution satisfying $u_i \in C^1[a, b]$, $1 \leq i \leq n$ and there exists u_j with $|u'_j(a)| = 1$ or $|u'_j(b)| = 1$ for some $j \in \{1, \dots, n\}$. On the other hand, we say $(u_1, u_2, \dots, u_n)$ a non $\frac{\pi}{4}$ -tangential solution of problem (CS) if $(u_1, u_2, \dots, u_n)$ is a solution satisfying $u_i \in C^1[a, b]$ and $\|u'_i\|_\infty < 1$, $1 \leq i \leq n$.

Theorem 5.1. Assume that f_i satisfies (F_q) for $1 \leq i \leq n$. Then every solution of problem (CS) is non $\frac{\pi}{4}$ -tangential.

Proof. Let $(u_1, u_2, \dots, u_n)$ be a solution of problem (CS). Then we may assume that all u_i , $1 \leq i \leq n$, are nontrivial since if one of them is trivial then the others are also trivial obviously. For convenience, we prove for the case $n = 2$. In this case, problem (CS) is reduced to the following problem

$$\begin{cases} \left(\phi(u'_1(t))\right)' + r_1(t)f_1(u_2(t)) = 0, & \text{in } (a, b), \\ \left(\phi(u'_2(t))\right)' + r_2(t)f_2(u_1(t)) = 0, & \text{in } (a, b), \\ u_1(a) = u_2(a) = 0 = u_1(b) = u_2(b). \end{cases} \quad (5.1)$$

We claim that positive part and negative part of u_1 and u_2 coincide in any subinterval of $[a, b]$. Suppose on the contrary that there exist a subinterval $(a_1, b_1) \subset (a, b)$ and a point $a_1^* \in (a_1, b_1)$ such that $u_1(t) > 0$ for $t \in (a_1, b_1)$, $u_1(a_1) = u_1(b_1) = 0$, and $u_2(a_1^*) < 0$. Then, by the continuity of u_2 in (a, b) , there exists a maximum subinterval $(a_2, b_2) \subset (a, b)$ such that $a_1^* \in (a_2, b_2)$, $u_2 < 0$ in (a_2, b_2) , and $u_2(a_2) = u_2(b_2) = 0$. It follows from the first equation in problem (5.1) that $u_1'' \geq 0$ on $[a_2, b_2]$. Thus, by convexity and concavity argument, $a_1 \notin [a_2, b_2]$ and $b_1 \notin [a_2, b_2]$. Once again convexity and concavity argument implies that $a_2 \notin [a_1, b_1]$ and $b_2 \notin [a_1, b_1]$. This contradicts $a_1^* \in [a_1, b_1]$. Note that the claim also implies that convexity-concavity of u_1 and u_2 coincides in any subinterval of $[a, b]$ and thus u'_i has the unique zero in the interval (a_1, b_1) .

Define $t_i^* = \inf\{t \in (a, b) : u'_i(t) = 0\}$. We show $a < t_i^* < b$. Suppose $t_i^* = a$ (a similar argument can be shown for the case $t_i^* = b$). Then there exists a sequence $\{t_m^i\}$ with $u'_i(t_m^i) = 0$ satisfying $t_m^i \rightarrow a$ as $m \rightarrow \infty$. Thus by the claim, there exists a sequence $\{s_m\}$ such that u_i is one-sign in (s_{m+1}, s_m) with $u_i(s_{m+1}) = u_i(s_m) = 0$ and $s_{m+1} < t_m^i < s_m$. By the fact that each f_i satisfies (F_q) , there exist $c_1 \triangleq c_1(\|u_2\|_\infty) > 0$ and $c_2 \triangleq c_2(\|u_1\|_\infty) > 0$ such that

$$|f_1(u_2(t))| \leq c_1|u_2(t)|^q \text{ and } |f_2(u_1(t))| \leq c_2|u_1(t)|^q, \text{ for } t \in [a, b]. \quad (5.2)$$

Since $r_i \in \mathcal{B}_q$, we take a small constant $y_0 > a$ such that

$$c_i \int_a^{y_0} (\tau - a)^q r_i(\tau) d\tau < 1, i = 1, 2, \quad (5.3)$$

and N_1 so large that $a < s_{m+1} < s_m < y_0$ for all $m \geq N_1$. Integrating the first equation in (5.1) on $[t_m^1, t]$ for $t \in [t_m^1, s_m]$, we obtain

$$\phi(u_1'(t)) = - \int_{t_m^1}^t r_1(\tau) f_1(u_2(\tau)) d\tau.$$

It follows that

$$u_1'(t) = -\phi^{-1} \left(\int_{t_m^1}^t r_1(\tau) f_1(u_2(\tau)) d\tau \right).$$

Integrating the above equation on $[t, s_m]$ for $t \in [t_m^1, s_m]$, we get

$$u_1(t) = \int_t^{s_m} \phi^{-1} \left(\int_{t_m^1}^s r_1(\tau) f_1(u_2(\tau)) d\tau \right) ds.$$

By the similar argument to obtain (2.1) and (2.2), we have

$$|u_i(t)| \leq t - a, \text{ for } t \in [a, b], \quad (5.4)$$

and

$$|u_i(t)| \leq b - t, \text{ for } t \in [a, b]. \quad (5.5)$$

By (5.4) and (5.2), for $t \in [t_m^1, s_m]$, we have

$$\begin{aligned} |u_1(t)| &\leq \int_t^{s_m} c_1 \int_{t_m^1}^s r_1(\tau) |u_2(\tau)|^q d\tau ds \\ &\leq c_1 \int_{t_m^1}^{s_m} \int_{t_m^1}^s r_1(\tau) |u_2(\tau)|^q d\tau ds \\ &\leq c_1 \int_{t_m^1}^{s_m} (\tau - a) r_1(\tau) |u_2(\tau)|^q d\tau \\ &\leq c_1 \int_{t_m^1}^{s_m} (\tau - a)^q r_1(\tau) |u_2(\tau)| d\tau. \end{aligned}$$

Similarly for $t \in [s_{m+1}, t_m^1]$, we obtain

$$|u_1(t)| \leq c_1 \int_{s_{m+1}}^{t_m^1} (\tau - a)^q r_1(\tau) |u_2(\tau)| d\tau.$$

Thus

$$|u_1(t)| \leq c_1 \int_{s_{m+1}}^{s_m} (\tau - a)^q r_1(\tau) |u_2(\tau)| d\tau, \text{ for } t \in [s_{m+1}, s_m].$$

Similarly,

$$|u_2(t)| \leq c_2 \int_{s_{m+1}}^{s_m} (\tau - a)^q r_2(\tau) |u_1(\tau)| d\tau, \text{ for } t \in [s_{m+1}, s_m].$$

Combining the above two inequalities and setting $\|u_i\|_{s_m, \infty} \triangleq \max_{s_{m+1} \leq t \leq s_m} |u_i(t)|$, we get

$$|u_i(t)| \leq c_1 c_2 \int_{s_{m+1}}^{s_m} (\tau - a)^q r_1(\tau) d\tau \int_{s_{m+1}}^{s_m} (\tau - a)^q r_2(\tau) d\tau \|u_i\|_{s_m, \infty}. \quad (5.6)$$

Recalling (5.3), we have

$$c_1 c_2 \int_{s_{m+1}}^{s_m} (\tau - a)^q r_1(\tau) d\tau \int_{s_{m+1}}^{s_m} (\tau - a)^q r_2(\tau) d\tau < 1.$$

It follows from (5.6) that $u_1(t) = 0 = u_2(t)$ for $t \in [s_{m+1}, s_m]$. Thus, we may assume $u_1(t_m^1) = u_1'(t_m^1) = 0 = u_2(t_m^1) = u_2'(t_m^1)$ for $m \geq N_1$. For $t \in [t_m^1, b]$, using (5.2) and (5.5), we obtain

$$\begin{aligned} |u_1(t)| &= \left| \int_{t_m^1}^t \phi^{-1} \left(\int_{t_m^1}^s r_1(\tau) f_1(u_2(\tau)) d\tau \right) ds \right| \\ &\leq \int_{t_m^1}^t \int_{t_m^1}^s r_1(\tau) |f_1(u_2(\tau))| d\tau ds \\ &= \int_{t_m^1}^t (t - \tau) r_1(\tau) |f_1(u_2(\tau))| d\tau \\ &\leq \int_{t_m^1}^t (b - \tau) r_1(\tau) |f_1(u_2(\tau))| d\tau \\ &\leq c_1 \int_{t_m^1}^t (b - \tau)^q r_1(\tau) |u_2(\tau)| d\tau, \end{aligned} \quad (5.7)$$

and

$$|u_2(t)| \leq c_2 \int_{t_m^1}^t (b - \tau)^q r_2(\tau) |u_1(\tau)| d\tau. \quad (5.8)$$

Combining (5.7) and (5.8), we have

$$\begin{aligned} |u_1(t)| &\leq c_1 \int_{t_m^1}^t (b - \tau)^q r_1(\tau) \left(c_2 \int_{t_m^1}^{\tau} (b - s)^q r_2(s) |u_1(s)| ds \right) d\tau \\ &= c_1 c_2 \int_{t_m^1}^t (b - s)^q r_2(s) |u_1(s)| \left(\int_s^t (b - \tau)^q r_1(\tau) d\tau \right) ds \\ &\leq c_1 c_2 \int_{t_m^1}^t (b - s)^q r_2(s) |u_1(s)| \left(\int_{t_m^1}^b (b - \tau)^q r_1(\tau) d\tau \right) ds \\ &\leq c_3 \int_{t_m^1}^t (b - s)^q r_2(s) |u_1(s)| ds, \end{aligned}$$

and

$$|u_2(t)| \leq c_4 \int_{t_m^1}^t (b-s)^q r_1(s) |u_2(s)| ds.$$

By Gronwall-Bellman inequality, we obtain $u_1(t) = 0 = u_2(t)$ for $t \in [t_m^1, b]$. Similarly, we derive $u_1(t) = 0 = u_2(t)$ for $t \in [a, t_m^1]$. Thus, $u_1 \equiv 0 \equiv u_2$ on $[a, b]$, which is a contradiction and we conclude $t_i^* > a$.

Let $t^* = \min\{t_1^*, t_2^*\}$. Then, for $t \in (a, t^*)$, we have $u_1(t) > 0$, $u_1'(t) > 0$, $u_2(t) > 0$, $u_2'(t) > 0$ or $u_1(t) < 0$, $u_1'(t) < 0$, $u_2(t) < 0$, $u_2'(t) < 0$. We only treat the first case here. The equations in problem (5.1) can be rewritten as

$$-\frac{u_1''(t)}{\left(\sqrt{1-|u_1'(t)|^2}\right)^3} = r_1(t)f_1(u_2(t)),$$

and

$$-\frac{u_2''(t)}{\left(\sqrt{1-|u_2'(t)|^2}\right)^3} = r_2(t)f_2(u_1(t)).$$

From the above equations, we obtain that $u_i'' \leq 0$ on (a, t^*) . Thus, $u_i'(t)$ is increasing for $t \nearrow a^+$. Since $u_i' \in C(a, b)$ and $|u_i'(t)| < 1$ for $t \in (a, b)$, we have $u_i'(a) = \lim_{t \rightarrow a^+} u_i'(t) \leq 1$ and thus $u_i \in C^1[a, b]$. Similarly, $u_i \in C^1(a, b]$. Hence, $u_i \in C^1[a, b]$.

Now, we prove $u_i'(a) < 1$. By similar argument as we derive (2.5), we have

$$\int_a^{t^*} r_1(\tau)f_1(u_2(\tau)) d\tau \leq \frac{2^q c_1}{(b-a)^q} \|u_2'\|_\infty^q \int_a^{t^*} r_1(\tau)(\tau-a)^q (b-\tau)^q d\tau < \infty,$$

and

$$\int_a^{t^*} r_2(\tau)f_2(u_1(\tau)) d\tau \leq \frac{2^q c_2}{(b-a)^q} \|u_1'\|_\infty^q \int_a^{t^*} r_2(\tau)(\tau-a)^q (b-\tau)^q d\tau < \infty.$$

Therefore

$$\begin{aligned} u_1'(a) &= \lim_{t \rightarrow a^+} \phi^{-1} \left(\int_t^{t^*} r_1(\tau)f_1(u_2(\tau)) d\tau \right) \\ &= \phi^{-1} \left(\int_a^{t^*} r_1(\tau)f_1(u_2(\tau)) d\tau \right) < 1. \end{aligned}$$

Similarly, we get $u_2'(a) < 1$ and $|u_i'(b)| < 1$, $i = 1, 2$. This completes the proof. $\square$

Next, we consider a strongly coupled system of the form

$$\begin{cases} \left(\phi(u_1'(t))\right)' + r_1(t)f_1(u_1(t), \dots, u_n(t)) = 0, & \text{in } (a, b), \\ \dots \\ \left(\phi(u_n'(t))\right)' + r_n(t)f_n(u_1(t), \dots, u_n(t)) = 0, & \text{in } (a, b), \\ u_1(a) = \dots = u_n(a) = 0 = u_1(b) = \dots = u_n(b), \end{cases} \quad (\text{SCS})$$

where $r_i \in \mathcal{B}_q$, $f_i : \mathbb{R}_+ \times \dots \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\mathbb{R}_+ \triangleq [0, \infty)$ is a continuous function and $s_i f_i(s_1, s_2, \dots, s_n) > 0$ for $s_i \neq 0$, $i = 1, \dots, n$. The definition of a solution for (SCS) is the same as that of (CS) with one additional condition: $u_i \geq 0$, for $i = 1, \dots, n$. We give an assumption on f_i .

(F3) $f_i(u_1, \dots, u_n) \leq d_{i,1}u_1^{p_1} + d_{i,2}u_2^{p_2} + \dots + d_{i,n}u_n^{p_n}$ with $d_{i,j} \geq 0$, $\prod_{j=1}^n d_{i,j} \neq 0$, $p_j \geq q \geq 1$, $1 \leq i, j \leq n$.

Theorem 5.2. Assume (F3). Then every solution of problem (SCS) is non $\frac{\pi}{4}$ -tangential.

Proof. Let $(u_1, u_2, \dots, u_n)$ be a solution of problem (SCS) which has at least one nontrivial component u_i . We claim that u_i cannot have a double zero on $[a, b]$. Suppose on the contrary that $t_0^i \in [a, b]$ is a double zero of u_i . Then,

$$\phi(u_i'(t)) = \int_t^{t_0^i} r_i(\tau) f_i(u_1(\tau), \dots, u_n(\tau)) d\tau.$$

Thus, $u_i'(t) \leq 0$ for $t \geq t_0^i$ and $u_i'(t) \geq 0$ for $t \leq t_0^i$ which contradicts the fact that u_i is nontrivial. Hence $u_i > 0$ in (a, b) and it suffices to prove that, for u_i satisfying $u_i > 0$ in (a, b) , it has $u_i \in C^1[a, b]$, $|u_i'(a)| < 1$, and $|u_i'(b)| < 1$. To show $u_i \in C^1[a, b]$, rewriting (SCS) as we did for problem (CS) and multiplying u_i , we have

$$-\frac{u_i''(t)u_i(t)}{\left(\sqrt{1-|u_i'(t)|^2}\right)^3} = r_i(t)u_i(t)f_i(u_1(t), \dots, u_n(t)).$$

We can see that $u_i'' \leq 0$ in (a, b) . Thus, $u_i'(t)$ is increasing as $t \searrow a^+$ and decreasing as $t \nearrow b^-$. Since $u_i' \in C(a, b)$ and $|u_i'(t)| < 1$ for $t \in (a, b)$, we have $|u_i'(a)| = \lim_{t \rightarrow a^+} u_i'(t) \leq 1$ and $|u_i'(b)| = \lim_{t \rightarrow b^-} u_i'(t) \leq 1$. Hence, $u_i \in C^1[a, b]$.

Now, we prove $|u_i'(a)| < 1$. By the mean value theorem and concavity, u_i can not oscillate and u_i' has at least one zero in (a, b) . Let $t_i^* = \inf\{t \in (a, b) : u_i'(t) = 0\}$, then we know $a < t_i^* < b$ by a similar argument as the one in the proof of Theorem 5.1. Thus $u_i(t) > 0$, $u_i'(t) > 0$ for $t \in (a, t_i^*)$. It follows from (F3) with a similar argument for (2.5) that

$$\begin{aligned} & \int_a^{t_i^*} r_i(\tau) f_i(u_1(\tau), \dots, u_n(\tau)) d\tau \\ & \leq \frac{2^q}{(b-a)^q} \int_a^{t_i^*} r_i(\tau) (\tau-a)^q (b-\tau)^q \left(d_{i,1} \left(\frac{2(\tau-a)(b-\tau)}{b-a} \right)^{p_1-q} \|u_1'\|_\infty^{p_1} + \right. \\ & \quad \dots \\ & \quad \left. + d_{i,n} \left(\frac{2(\tau-a)(b-\tau)}{b-a} \right)^{p_n-q} \|u_n'\|_\infty^{p_n} \right) d\tau \\ & \leq \left(d_{i,1} \left(\frac{b-a}{2} \right)^{p_1-2q} \|u_1'\|_\infty^{p_1} + \dots + d_{i,n} \left(\frac{b-a}{2} \right)^{p_n-2q} \|u_n'\|_\infty^{p_n} \right) \int_a^{t_i^*} r_i(\tau) (\tau-a)^q (b-\tau)^q d\tau \\ & < \infty. \end{aligned}$$

Therefore

$$\begin{aligned} |u_i'(a)| &= \left| \lim_{t \rightarrow a^+} \phi^{-1} \left(\int_t^{t_i^*} r_i(\tau) f_i(u_1(\tau), \dots, u_n(\tau)) d\tau \right) \right| \\ &= \left| \phi^{-1} \left(\int_a^{t_i^*} r_i(\tau) f_i(u_1(\tau), \dots, u_n(\tau)) d\tau \right) \right| < 1. \end{aligned}$$

By a similar argument, we can show $|u_i'(b)| < 1$ and the proof is complete. $\square$

Remark 5.3. Compared with the following condition on $f_i = f_i(t, u_1, u_2, \dots, u_n)$ given in [13],

(F3') for each $M > 0$, there exists $d_{i,M} > 0$ such that
 $f_i(t, u_1, \dots, u_n) \leq d_{i,M} h_i(t)(u_1 + u_2 + \dots + u_n)^{p-1}$, for a.e. $t \in (0, 1)$
 and $0 \leq u_i \leq M$, $p > 1$, $1 \leq i \leq n$,

coefficients and powers in the right-hand side of the inequality in (F3) are more general.

6 Lyapunov-type inequalities for systems

In this section, we apply Theorem 5.1 and Theorem 5.2 to establish Lyapunov-type inequalities for the following two systems.

We first consider a cycled system

$$\begin{cases} \left(\phi(u'_1(t)) \right)' + r_1(t)|u_2(t)|^{l-1}u_2(t) = 0, & \text{in } (a, b), \\ \left(\phi(u'_2(t)) \right)' + r_2(t)|u_3(t)|^{l-1}u_3(t) = 0, & \text{in } (a, b), \\ \dots \\ \left(\phi(u'_{n-1}(t)) \right)' + r_{n-1}(t)|u_n(t)|^{l-1}u_n(t) = 0, & \text{in } (a, b), \\ \left(\phi(u'_n(t)) \right)' + r_n(t)|u_1(t)|^{l-1}u_1(t) = 0, & \text{in } (a, b), \\ u_1(a) = \dots = u_n(a) = 0 = u_1(b) = \dots = u_n(b), \end{cases} \quad (CS_0)$$

where $r_i \in \mathcal{B}_q$, $1 \leq i \leq n$.

Theorem 6.1. *Let $1 \leq q \leq l$. If problem (CS_0) has a solution with a nontrivial u_j for some $j \in \{1, \dots, n\}$, then one has*

$$\frac{(b-a)^{n(2q-l)}}{2^{n(2q-l-1)}} < \prod_{i=1}^n \int_a^b (t-a)^q (b-t)^q r_i(t) dt. \quad (6.1)$$

Proof. Let $(u_1, u_2, \dots, u_n)$ be a solution of problem (CS_0) with a nontrivial u_j for some $j \in \{1, \dots, n\}$. Then, as in the proof of Theorem 5.1, all u_i for $i = 1, \dots, n$ are nontrivial and non $\frac{\pi}{4}$ -tangential. Since $l \geq 1$, as in (4.5), we have

$$|u_i(t)| \leq \left(\frac{(t-a)(b-t)}{b-a} \right)^{\frac{l}{l+1}} 2^{\frac{l-1}{l+1}} \left(\int_a^b |u'_i(\tau)|^{l+1} d\tau \right)^{\frac{1}{l+1}}, \quad (6.2)$$

and thus

$$|u_i(t)|^l \leq \left(\frac{(t-a)(b-t)}{b-a} \right)^{\frac{l^2}{l+1}} 2^{\frac{(l-1)l}{l+1}} \left(\int_a^b |u'_i(\tau)|^{l+1} d\tau \right)^{\frac{l}{l+1}}, \quad (6.3)$$

for $t \in [a, b]$. Multiplying both sides of the first equation in problem (CS_0) by $u_1(t)$ and integrating it on $[a, b]$, we get

$$\int_a^b \frac{|u'_1(t)|^2}{\sqrt{1-|u'_1(t)|^2}} dt \leq \int_a^b r_1(t)|u_1(t)||u_2(t)|^l dt.$$

Since $l \geq 1$, applying (6.2), (6.3), and the fact $\|u'_i\|_\infty < 1$, we get

$$\begin{aligned} \int_a^b |u'_1(t)|^{l+1} dt &\leq \int_a^b |u'_1(t)|^2 dt < \int_a^b \frac{|u'_1(t)|^2}{\sqrt{1-|u'_1(t)|^2}} dt \\ &\leq \int_a^b r_1(t) |u_1(t)| (|u_1(t)|^l + \cdots + |u_n(t)|^l) dt \\ &\leq \frac{2^{l-1}}{(b-a)^l} \int_a^b (t-a)^l (b-t)^l r_1(t) dt \left(\int_a^b |u'_1(t)|^{l+1} dt \right)^{\frac{1}{l+1}} \left(\int_a^b |u'_1(t)|^{l+1} dt \right)^{\frac{l}{l+1}} \\ &\quad + \cdots + \left(\int_a^b |u'_n(t)|^{l+1} dt \right)^{\frac{l}{l+1}}, \end{aligned}$$

thus,

$$\begin{aligned} \left(\int_a^b |u'_1(t)|^{l+1} dt \right)^{\frac{l}{l+1}} &< \frac{2^{l-1}}{(b-a)^l} \int_a^b (t-a)^l (b-t)^l r_1(t) dt \left(\int_a^b |u'_1(t)|^{l+1} dt \right)^{\frac{l}{l+1}} \\ &\quad + \cdots + \left(\int_a^b |u'_n(t)|^{l+1} dt \right)^{\frac{l}{l+1}}. \end{aligned}$$

Using the fact $(t-a)(b-t) \leq \frac{(b-a)^2}{2^2}$, we obtain

$$\begin{aligned} \left(\int_a^b |u'_1(t)|^{l+1} dt \right)^{\frac{l}{l+1}} &< \frac{2^{2q-l-1}}{(b-a)^{2q-l}} \int_a^b (t-a)^q (b-t)^q r_1(t) dt \left(\int_a^b |u'_1(t)|^{l+1} dt \right)^{\frac{l}{l+1}} \\ &\quad + \cdots + \left(\int_a^b |u'_n(t)|^{l+1} dt \right)^{\frac{l}{l+1}}. \end{aligned}$$

Repeating this and adding all inequalities from $i = 1$ to n , we obtain (6.4) and the proof is done. $\square$

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