Research Article

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Constant sign and nodal solutions for superlinear (p, q) –equations with indefinite potential and a concave boundary term

<https://doi.org/10.1515/anona-2020-0101>

Received February 11, 2020; accepted March 15, 2020.

Abstract: We consider a nonlinear elliptic equation driven by the (p, q) –Laplacian plus an indefinite potential. The reaction is $(p - 1)$ –superlinear and the boundary term is parametric and concave. Using variational tools from the critical point theory together with truncation, perturbation and comparison techniques and critical groups, we show that for all small values of the parameter, the problem has at least five nontrivial smooth solutions, all with sign information and which are linearly ordered.

Keywords: Superlinear reaction, AR–condition, concave boundary term, constant sign and nodal solutions, critical groups, nonlinear regularity theory, indefinite potential

MSC: 35J20, 35J60

1 Introduction

In this paper we study the following parametric (p, q) –equation:

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)|u(z)|^{p-2}u(z) = f(z, u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{pq}} = \lambda \beta(z)|u|^{\tau-2}u & \text{on } \partial\Omega, \quad 1 < \tau < q < p, \lambda > 0. \end{cases} \quad (P_\lambda)$$

In this problem $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2 –boundary $\partial\Omega$. For every $1 < r < \infty$ by Δ_r we denote the r –Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|Du|^{r-2}Du)$$

for all $u \in W^{1,r}(\Omega)$.

In problem (P_λ) we have the sum of two such operators. Therefore the differential operator driving the equation, is not homogeneous. In addition there is a potential term $u \rightarrow \xi(z)|u|^{p-2}u$ which is indefinite since the potential function $\xi \in L^\infty(\Omega)$ is in general sign–changing. So, the left hand side of problem (P_λ) is not coercive. The source (reaction) term $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \rightarrow f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \rightarrow f(z, x)$ is continuous). We assume that $f(z, \cdot)$ exhibits $(p - 1)$ –superlinear growth as $x \rightarrow \pm\infty$. In fact we assume that $f(z, \cdot)$ satisfies the well–known Ambrosetti–Rabinowitz condition (the AR–condition) which is common in the literature for superlinear problems. It is an interesting open problem if we can relax the AR–condition and use a more general one, as for example in Mugnai–Papageorgiou

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[12] or alternatively in Papageorgiou-Rădulescu [13]. Near zero we assume that $f(z, \cdot)$ has a kind of oscillatory behavior. In the boundary condition, $\frac{\partial u}{\partial n_{pq}}$ denotes conormal derivative corresponding to the (p, q) -Laplace differential operator. We interpret this directional derivative using the nonlinear Green's identity (see Corollary 1.5.17 of Papageorgiou-Rădulescu-Repovš [19, p. 35]). We know that for all $u \in C^1(\overline{\Omega})$,

$$\frac{\partial u}{\partial n_{pq}} \equiv \left[|Du|^{p-2} + |Du|^{q-2} \right] (Du, n)_{\mathbb{R}^N} = \left[|Du|^{p-2} + |Du|^{q-2} \right] \frac{\partial u}{\partial n},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta(\cdot)$ is Hölder continuous on $\partial\Omega$ and strictly positive, and $\lambda > 0$ is a parameter. Since $1 < \tau < q < p$, we see that the boundary condition contributes a concave term in the energy functional of the problem. Therefore, in problem we have the competing effects of two items which are of different nature. One is the “convex (superlinear) source term $f(z, \cdot)$ and the other is the parametric “concave” (sublinear) boundary term. By restricting appropriately the parameter, we will be able to balance the different effects of these two terms. Problem (P_λ) is a variant of the classical “concave–convex” problem. The different feature of our problem here is that the concave contribution comes from the boundary condition.

For the classical concave–convex problem, where the parametric concave term is part of the reaction of the problem, we refer to the works of Ambrosetti-Brezis-Cerami [1], Garcia Azorero-Manfredi-Peral Alonso [4], Guo-Zhang [6], Leonardi-Papageorgiou [9], Marano-Marino-Papageorgiou [11], Papageorgiou-Rădulescu-Repovš [18]. Analogous nonlinear parametric problems but with different competition phenomena, can be found in Bai-Motreanu-Zeng [2], Papageorgiou-Scapellato [20], Papageorgiou-Zhang [22].

Problems with a superlinear source term and a concave boundary condition were studied by Sabina de Lis-Segura de León [24], Hu-Papageorgiou [8], Papageorgiou-Rădulescu [15], Papageorgiou-Scapellato [21]. All these works deal with parametric problems, with the parameter appearing in the boundary term. They focus on the existence of positive solutions and they prove bifurcation-type results describing the changes in the set of positive solutions as the parameter $\lambda > 0$ varies. We mention that in all these works the potential coefficient $\xi(\cdot)$ is strictly positive, making the left side of the equation coercive. In particular Sabina de Lis-Segura de León [24] consider equations driven by the p -Laplacian plus a potential term with $\xi \equiv 1$. So, in their equation the left hand side is both homogeneous and coercive and these strong properties are exploited in their proof. In Hu-Papageorgiou [8], the problem is semilinear driven by the Laplacian plus an indefinite potential term. In Papageorgiou-Rădulescu [14], the equation is driven by the p -Laplacian and only existence of positive solutions is proved. Finally, in the recent work of Papageorgiou-Scapellato [21], the authors deal with equations driven by the $(p, 2)$ -Laplacian plus a positive potential term and $\beta \equiv 1$. As we already mentioned, all the aforementioned works focus on the existence and multiplicity of positive solutions as the parameter $\lambda > 0$ varies.

In the present paper, using the variational tools from the critical point theory, together with suitable truncation and perturbation techniques and the use of critical groups (Morse theory), we show that for all small values of the parameter $\lambda > 0$, problem (P_λ) has at least five nontrivial smooth solutions which can be ordered linearly and we can provide sign information for all of them. More precisely, we produce two positive solutions, two negative solutions and a fifth one which is nodal (sign-changing). This is the first result on the existence of nodal solutions for problems with concave boundary condition.

2 Auxiliary results and hypotheses

The main spaces which will be used in the study of problem (P_λ) are the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^s(\partial\Omega)$, $1 \leq s \leq \infty$.

By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$. We know that

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p \right]^{1/p}$$

for all $u \in W^{1,p}(\Omega)$.

The Banach space $C^1(\overline{\Omega})$ is ordered with positive (order) cone

$$C_+ = \left\{ u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega} \right\}.$$

This cone has a nonempty interior given by

$$\text{int } C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega} \right\}.$$

On $\partial\Omega$ we consider the $(N-1)$ -dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the boundary Lebesgue space $L^s(\partial\Omega)$. From the theory of Sobolev spaces (see [19]), we know that there exists a unique continuous linear map $y_0 : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$, known as the “trace map”, such that

$$y_0(u) = u|_{\partial\Omega}$$

for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.

So, the trace map extends the notion of boundary values to all Sobolev functions. We know that the trace map $y_0(\cdot)$ is compact into $L^s(\partial\Omega)$ for all $s \in \left[1, \frac{(N-1)p}{N-p}\right)$ if $p < N$ and into $L^s(\partial\Omega)$ for all $1 \leq s < \infty$ if $N \leq p$. Moreover, we have

$$\ker y_0 = W_0^{1,p}(\Omega) \text{ and } \text{im } y_0 = W^{\frac{1}{p'}, p}(\partial\Omega) \left(\frac{1}{p} + \frac{1}{p'} = 1 \right).$$

In the sequel for the sake of notational simplicity, we drop the use of the trace map $y_0(\cdot)$. All restrictions of the Sobolev functions on $\partial\Omega$, are understood in the sense of traces.

For $x \in \mathbb{R}$, we set $x^+ = \max\{\pm x, 0\}$. Then given $u \in W^{1,p}(\Omega)$, we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We have

$$\begin{aligned} u^\pm &\in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-, \\ (u^\pm)|_{\partial\Omega} &= (u|_{\partial\Omega})^\pm, \quad |u|_{\partial\Omega} = |u|_{\partial\Omega}. \end{aligned}$$

Given $u, v \in W^{1,p}(\Omega)$ with $u \leq v$, we define the following sets

$$\begin{aligned} [u, v] &= \left\{ h \in W^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega \right\}, \\ [u] &= \left\{ h \in W^{1,p}(\Omega) : u(z) \leq h(z) \text{ for a.a. } z \in \Omega \right\}, \\ [v] &= \left\{ h \in W^{1,p}(\Omega) : h(z) \leq v(z) \text{ for a.a. } z \in \Omega \right\}. \end{aligned}$$

Given a set $S \subseteq W^{1,p}(\Omega)$, we say that S is “downward directed” (resp. “upward directed”), if for every pair $u_1, u_2 \in S$, we can find $u \in S$ such that $u \leq u_1, u \leq u_2$ (resp. for every $v_1, v_2 \in S$, we can find $v \in S$ such that $v_1 \leq v, v_2 \leq v$).

Let $1 < r < \infty$ and consider the map $A_r : W^{1,r}(\Omega) \rightarrow W^{1,r}(\Omega)^*$ defined by

$$\langle A_r(u), h \rangle = \int_{\Omega} |Du|^{r-2} (Du, Dh)_{\mathbb{R}^N} dz$$

for all $u, h \in W^{1,r}(\Omega)$.

From Problem 2.192 of Gasinski-Papageorgiou [5, p. 279], we have :

Proposition 1. *The map $A_r(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$ that is, $u_n \xrightarrow{w} u$ in $W^{1,r}(\Omega)$ and $\limsup_{n \rightarrow +\infty} \langle A_r(u_n), u_n - u \rangle \leq 0 \Rightarrow u_n \rightarrow u$ in $W^{1,r}(\Omega)$.*

If X is a Banach space and $\varphi \in C^1(X, \mathbb{R})$, we say that $\varphi(\cdot)$ satisfies the “C-condition”, if it has the following property:

“Every sequence $\{u_n\}_{n \geq 1} \subseteq X$ which satisfies

$\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded,
 $(1 + \|u_n\|_X)\varphi'(u_n) \rightarrow 0$ in X^* as $n \rightarrow \infty$,
 admits a strongly convergent subsequence".

This is essentially a compactness-type condition on the function $\varphi(\cdot)$. It compensates for the fact that the ambient space X is not in general locally compact (since in almost all situations of interest, X is infinite dimensional). Using this property, one can prove a deformation theorem, from which follows the minimax theory of the critical values of φ (see [19], Chapter 5).

We define

$$K_\varphi = \left\{ u \in X : \varphi'(u) = 0 \right\} \text{ (the critical set of } \varphi \text{),}$$

$$\varphi^c = \left\{ u \in X : \varphi(u) \leq c \right\}, \quad c \in \mathbb{R}.$$

For (Y_1, Y_2) a topological pair such $Y_2 \subseteq Y_1 \subseteq X$. For every $k \in \mathbb{N}$, by $H_k(Y_1, Y_2)$ we denote the k -th relative singular homology group with coefficients in $\mathbb{R}$. Then the relative singular homology groups $H_k(Y_1, Y_2)$ are real vector spaces. Let $u \in K_\varphi$ isolated and $c = \varphi(u)$. The "critical groups" of $\varphi(\cdot)$ at u , are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\})$$

for all $k \in \mathbb{N}$, with U a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition is independent of the choice the isolating neighborhood U .

Now we will introduce our hypotheses on the data of problem (P_λ) :

H_0 : $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0, 1)$ and $\beta(z) > 0$ for all $z \in \partial\Omega$.

H_1 : $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z)[1 + |x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p < r < p^*$ with p^* being the critical Sobolev exponent for $p > 1$, that is,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N; \\ +\infty & \text{if } N \leq p \end{cases};$$

(ii) If $F(z, x) = \int_0^x f(z, s)ds$, then there exist $\mu > p$ and $M > 0$ such that

$$0 < \mu F(z, x) \leq f(z, x)x \text{ for a.a. } z \in \Omega, \text{ all } |x| \geq M,$$

$$0 < \operatorname{ess\,inf}_\Omega F(\cdot, \pm M);$$

(iii) there exist constants $\theta_- < 0 < \theta_+$ such that

$$f(z, \theta_+) - \xi(z)\theta_+^{p-1} \leq -c_+ < 0 < c_- \leq f(z, \theta_-) + \xi(z)|\theta_-|^{p-1} \text{ for a.a. } z \in \Omega;$$

(iv) there exists $\delta_0 > 0$ and $d \in [\tau, q]$ such that

$$0 < f(z, x)x \leq dF(z, x) \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta_0$$

and for every $\rho > 0$, we can find $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \rightarrow f(z, x) + \widehat{\xi}_\rho |x|^{p-2}x$$

is nondecreasing on $[-\rho, \rho]$.

Remark 1. Hypothesis H_1 (ii) is the AR-condition. An integration of this condition implies that

$$c_1 |x|^\mu \leq F(z, x) \text{ for a.a. } z \in \Omega, \text{ all } |x| \geq M, \text{ some } c_1 > 0, \quad (2.1)$$

$$\Rightarrow c_2 |x|^\mu \leq f(z, x)x \text{ for a.a. } z \in \Omega, \text{ all } |x| \geq M, \text{ some } c_2 > 0. \quad (2.2)$$

On account of hypotheses $H_1(i)$, (iv) we have

$$f(z, x)x \geq -c_3|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_3 > 0. \quad (2.3)$$

Then we consider the following auxiliary (p, q) -equation:

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) + |\xi(z)||u(z)|^{p-2}u(z) = -c_3|u(z)|^{r-2}u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{pq}} = \lambda\beta(z)|u|^{\tau-2}u & \text{on } \partial\Omega, \lambda > 0, 1 < \tau < q < p. \end{cases} \quad (2.4)$$

Proposition 2. *If hypotheses H_0 hold and $\lambda > 0$, then problem (2.4) admits a unique positive solution $\bar{u}_\lambda \in \text{int } C_+$ and since (2.4) is odd, $\bar{v}_\lambda = -\bar{u}_\lambda \in \text{int } C_+$ is the unique negative solution of (2.4).*

Proof. First we show the existence of a positive solution. To this end, we introduce the C^1 -functional $l_+ : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$l_+(u) = \frac{1}{p}\tilde{y}_p(u) + \frac{1}{p}\|Du\|_q^q + \frac{c_3}{r}\|u^+\|_r^r - \frac{1}{p}\|u^+\|_p^p - \frac{\lambda}{\tau} \int_{\partial\Omega} \beta(z)(u^+)^{\tau} d\sigma$$

for all $u \in W^{1,p}(\Omega)$, with $\tilde{y}_p(\cdot)$ being the functional

$$\tilde{y}_p(u) = \|Du\|_p^p + \int_{\Omega} |\xi(z)||u|^p dz$$

for all $u \in W^{1,p}(\Omega)$.

Since $1 < \tau < q < p < r$, it follows that

$l_+(\cdot)$ is coercive.

Also, from the Sobolev embedding theorem and the compactness of the trace map, we see that

$l_+(\cdot)$ is sequentially weakly lower semicontinuous.

So, by the Weierstrass-Tonelli theorem, we can find $\bar{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$l_+(\bar{u}_\lambda) = \min \{l_+(u) : u \in W^{1,p}(\Omega)\}. \quad (2.5)$$

Let $u \in \text{int } C_+$ and $t \in (0, 1)$. We have

$$l_+(tu) = \frac{t^p}{p}\tilde{y}_p(u) + \frac{t^q}{q}\|Du\|_q^q + \frac{c_3 t^r}{r}\|u\|_r^r - \frac{t^\tau \lambda}{\tau} \int_{\partial\Omega} \beta(z)(u)^\tau d\sigma.$$

Note that $\lambda \int_{\partial\Omega} \beta(z)(u)^\tau d\sigma > 0$. Since $q < p < r$ and $\xi \in L^\infty(\Omega)$, we obtain

$$l_+(tu) \leq c_4 t^q - c_5 t^\tau$$

for some $c_4 = c_4(u) > 0$, $c_5 = c_5(u) > 0$.

Since $\tau < q$, taking $t \in (0, 1)$ even smaller, we have

$$\begin{aligned} l_+(tu) &< 0, \\ \Rightarrow l_+(\bar{u}_\lambda) &< 0 = l_+(0) \text{ (see (2.5))}, \\ \Rightarrow \bar{u}_\lambda &\neq 0. \end{aligned}$$

From (2.5) we have

$$l'_+(\bar{u}_\lambda) = 0 \Rightarrow \langle A_p(\bar{u}_\lambda), h \rangle + \langle A_q(\bar{u}_\lambda), h \rangle + \int_{\Omega} [|\xi(z)| + 1] |\bar{u}_\lambda|^{p-2} \bar{u}_\lambda h dz$$

$$= \int_{\Omega} \left[-c_3(\bar{u}_{\lambda}^+)^{r-1} + (\bar{u}_{\lambda}^+)^{p-1} \right] h dz + \int_{\partial\Omega} \beta(z)(\bar{u}_{\lambda}^+)^{r-1} h d\sigma \quad (2.6)$$

for all $h \in W^{1,p}(\Omega)$.

In (2.6) we choose $h = -\bar{u}_{\lambda} \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} c_6 \|\bar{u}_{\lambda}\|^p &\leq 0, \\ \Rightarrow \bar{u}_{\lambda} &\geq 0, \quad \bar{u}_{\lambda} \neq 0. \end{aligned}$$

From (2.6) we have

$$\begin{cases} -\Delta_p \bar{u}_{\lambda} - \Delta_q \bar{u}_{\lambda} + \xi(z) \bar{u}_{\lambda}^{p-1} = -c_3 \bar{u}_{\lambda}^{r-1} \text{ for a.a. } z \in \Omega, \\ \frac{\partial \bar{u}_{\lambda}}{\partial n_{pq}} = \lambda \beta(z) \bar{u}_{\lambda}^{r-1} & \text{on } \partial\Omega. \end{cases} \quad (2.7)$$

From (2.7) and Proposition 2.10 of Papageorgiou-Rădulescu [14], we have $\bar{u}_{\lambda} \in L^{\infty}(\Omega)$. Then the nonlinear regularity theory of Lieberman [10] implies that $\bar{u}_{\lambda} \in C_+ \setminus \{0\}$. From (2.7) we have

$$\begin{aligned} \Delta_p \bar{u}_{\lambda}(z) + \Delta_q \bar{u}_{\lambda}(z) &\leq \left[\|\xi\|_{\infty} + c_3 \|\bar{u}_{\lambda}\|_{\infty}^{r-p} \right] \bar{u}_{\lambda}(z) \text{ for a.a. } z \in \Omega, \\ \Rightarrow \bar{u}_{\lambda} &\in \text{int } C_+ \quad (\text{see Pucci-Serrin [23, pp. 111, 120]}). \end{aligned}$$

Next we show the uniqueness of this positive solution. For this purpose, we introduce the integral functional $j : L^1(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p} \|Du^{1/q}\|_p^p + \frac{1}{q} \|Du^{1/q}\|_q^q & \text{if } u \geq 0, \quad u^{1/q} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\text{dom } j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of $j(\cdot)$) and consider the function $G_0(t) = \frac{1}{p} t^p + \frac{1}{q} t^q$ for all $t \geq 0$. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Note that the function $G_0(\cdot)$ is increasing and $t \rightarrow G_0(t^{1/q})$ is convex (recall that $q < p$).

Let $u_1, u_2 \in \text{dom } j$ and set $v = [tu_1 + (1-t)u_2]^{1/q}$ with $0 \leq t \leq 1$. From Diaz-Saa [3] (see the proof of Lemma 1), we have

$$\begin{aligned} |Dv| &\leq \left[t |Du_1^{1/q}|^q + (1-t) |Du_2^{1/q}|^q \right]^{1/q} \\ &\text{for a.a. } z \in \Omega, \\ \Rightarrow G_0(|Dv|) &\leq G_0 \left(\left[t |Du_1^{1/q}|^q + (1-t) |Du_2^{1/q}|^q \right]^{1/q} \right) \\ &\quad (\text{since } G_0(\cdot) \text{ is increasing}), \\ \Rightarrow G_0(|Dv|) &\leq t G_0(|Du_1^{1/q}|) + (1-t) G_0(|Du_2^{1/q}|) \\ &\quad (\text{since } t \rightarrow G_0(t^{1/q}) \text{ is convex}) \\ \Rightarrow G(Dv) &\leq t G(Du_1^{1/q}) + (1-t) G(Du_2^{1/q}), \\ \Rightarrow j(\cdot) &\text{ is convex.} \end{aligned}$$

Suppose that $\tilde{u}_{\lambda} \in W^{1,p}(\Omega)$ is another solution of problem (2.4). Again we show that $\tilde{u}_{\lambda} \in \text{int } C_+$. We set $h = \bar{u}_{\lambda}^q - \tilde{u}_{\lambda}^q \in C^1(\bar{\Omega})$. For $|t| \leq 1$ small, we have

$$\bar{u}_{\lambda}^q + th \in \text{dom } j \text{ and } \tilde{u}_{\lambda}^q + th \in \text{dom } j.$$

Hence the functional $j(\cdot)$ is Gâteaux differentiable at $\bar{u}_{\lambda}^q$ and $\tilde{u}_{\lambda}^q$ in the direction h . Using the nonlinear Green's identity (see Corollary 1.5.17 of Papageorgiou-Rădulescu-Repovš [19, p. 35]), we obtain

$$\begin{aligned} j'(\bar{u}_{\lambda}^q)(h) &= \frac{1}{q} \int_{\Omega} \frac{-\Delta_p \bar{u}_{\lambda} - \Delta_q \bar{u}_{\lambda}}{\bar{u}_{\lambda}^{q-1}} h dz + \frac{1}{q} \int_{\partial\Omega} \beta(z) \bar{u}_{\lambda}^{r-q} h d\sigma \\ &= \frac{1}{q} \int_{\Omega} \left[-c_3 \bar{u}_{\lambda}^{r-q} - |\xi(z)| \bar{u}_{\lambda}^{p-q} \right] h dz + \frac{1}{q} \int_{\partial\Omega} \frac{\beta(z)}{\bar{u}_{\lambda}^{q-r}} h d\sigma, \end{aligned}$$

$$\begin{aligned} j'(\tilde{u}_\lambda^q)(h) &= \frac{1}{q} \int_{\Omega} \frac{-\Delta_p \tilde{u}_\lambda - \Delta_q \tilde{u}_\lambda}{\tilde{u}_\lambda^{q-1}} h dz + \frac{1}{q} \int_{\partial\Omega} \beta(z) \tilde{u}_\lambda^{\tau-q} h d\sigma \\ &= \frac{1}{q} \int_{\Omega} \left[-c_3 \tilde{u}_\lambda^{r-q} - |\xi(z)| \tilde{u}_\lambda^{p-q} \right] h dz + \frac{1}{q} \int_{\partial\Omega} \frac{\beta(z)}{\tilde{u}_\lambda^{q-\tau}} h d\sigma. \end{aligned}$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. So, we have

$$\begin{aligned} 0 &\leq - \int_{\Omega} c_3 \left[\bar{u}_\lambda^{r-q} - \tilde{u}_\lambda^{r-q} \right] \left[\bar{u}_\lambda^q - \tilde{u}_\lambda^q \right] dz - \int_{\Omega} |\xi(z)| \left[\bar{u}_\lambda^{p-q} - \tilde{u}_\lambda^{p-q} \right] \left[\bar{u}_\lambda^q - \tilde{u}_\lambda^q \right] dz \\ &\quad + \int_{\partial\Omega} \beta(z) \left[\frac{1}{\bar{u}_\lambda^{q-\tau}} - \frac{1}{\tilde{u}_\lambda^{q-\tau}} \right] \left[\bar{u}_\lambda^q - \tilde{u}_\lambda^q \right] d\sigma \\ &\leq 0 \text{ (recall that } \tau < q < p), \\ &\Rightarrow \bar{u}_\lambda = \tilde{u}_\lambda. \end{aligned}$$

This proves the uniqueness of the positive solution $\bar{u}_\lambda \in \text{int } C_+$ for problem (2.4). Since the problem is odd, it follows that $\bar{v}_\lambda = -\bar{u}_\lambda \in -\text{int } C_+$ is the unique negative solution of (2.4). This completes the proof. $\square$

3 Constant sign solutions

We start by proving the existence of two constant sign solutions located in the order intervals $[0, \theta_+]$ and $[\theta_-, 0]$ respectively. To do this we do not need all the conditions in hypotheses H_1 . More precisely we do not need hypothesis $H_1(ii)$ which describes the asymptotic behavior as $x \rightarrow \pm\infty$ of the source term. Using truncation and perturbation techniques we focus on the intervals $[0, \theta_+]$ and $[\theta_-, 0]$ and so the behavior of $f(z, \cdot)$ near $\pm\infty$ becomes irrelevant.

Proposition 3. *If hypotheses $H_0, H_1(i), (iii), (iv)$ hold, then we can find $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*]$ problem (P_λ) has at least two constant sign solutions*

$$\begin{aligned} u_0 &\in \text{int } C_+, \quad u_0(z) < \theta_+ \text{ for all } z \in \Omega, \quad \frac{\partial u_0}{\partial n} \Big|_{\partial\Omega \cap \{u_0=\theta_+\}} > 0, \\ v_0 &\in -\text{int } C_+, \quad \theta_- < v_0(z) \text{ for all } z \in \Omega, \quad \frac{\partial v_0}{\partial n} \Big|_{\partial\Omega \cap \{v_0=\theta_-\}} < 0. \end{aligned}$$

Proof. First we produce the positive solution.

Let $\eta > \|\xi\|_\infty$ and consider the Carathéodory functions $\hat{k}_+ : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $\hat{b}_+ : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\hat{k}_+(z, x) = \begin{cases} f(z, x^+) + \eta(x^+)^{p-1} & \text{if } x \leq \theta_+, \\ f(z, \theta_+) + \eta\theta_+^{p-1} & \text{if } \theta_+ < x, \end{cases} \quad (z, x) \in \Omega \times \mathbb{R}, \quad (3.1)$$

$$\text{and } \hat{b}_+(z, x) = \begin{cases} \beta(z)(x^+)^{\tau-1} & \text{if } x \leq \theta_+, \\ \beta(z)\theta_+^{\tau-1} & \text{if } \theta_+ < x, \end{cases} \quad (z, x) \in \partial\Omega \times \mathbb{R}. \quad (3.2)$$

We set

$$\hat{K}_+(z, x) = \int_0^x \hat{k}_+(z, s) ds \text{ and } \hat{B}_+(z, x) = \int_0^x \hat{b}_+(z, s) ds.$$

Also, let

$$\hat{y}_p(u) = \|Du\|_p^p + \int_{\Omega} \xi(z)|u|^p dz$$

for all $u \in W^{1,p}(\Omega)$.

We introduce the C^1 -functional $\widehat{\psi}_+^\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\psi}_+^\lambda(u) = \frac{1}{p}\widehat{y}_p(u) + \frac{\eta}{p}\|u\|_p^p + \frac{1}{q}\|Du\|_q^q - \int_{\Omega} \widehat{K}_+(z, u)dz - \int_{\partial\Omega} \lambda \widehat{B}_+(z, u)d\sigma$$

for all $u \in W^{1,p}(\Omega)$.

From (3.1), (3.2) and since $\eta > \|\xi\|_\infty$, we see that

$$\widehat{\psi}_+^\lambda(\cdot) \text{ is coercive.}$$

Also the Sobolev embedding theorem and the compactness of the trace map, imply that

$$\widehat{\psi}_+^\lambda \text{ is sequentially weakly lower semicontinuous.}$$

By the Weierstrass-Tonelli theorem, there exists $u_0 \in W^{1,p}(\Omega)$ such that

$$\widehat{\psi}_+^\lambda(u_0) = \min \left\{ \widehat{\psi}_+^\lambda(u) : u \in W^{1,p}(\Omega) \right\}. \quad (3.3)$$

Let $u \in \text{int } C_+$ and choose $t \in (0, 1)$ small such that $tu \leq \min \{ \theta_+, \delta_0 \}$ (here δ_0 is as in hypothesis $H_1(iv)$). We have

$$\begin{aligned} \widehat{\psi}_+^\lambda(tu) &= \frac{t^p}{p}\widehat{y}_p(u) + \frac{t^q}{q}\|u\|_q^q - \int_{\Omega} F(z, tu)dz \\ &\leq c_7 t^q - \lambda c_8 t^\tau \end{aligned}$$

for all $c_7 = c_7(u) > 0$, $c_8 = c_8(u) > 0$ (since $t \in (0, 1)$, $q < p$ and by hypothesis $H_1(iv)$, $F(z, tu) \geq 0$ for a.a. $x \in \Omega$).

Recall that $1 < \tau < q$. So, choosing $t \in (0, 1)$ even smaller if necessary, we have

$$\begin{aligned} \widehat{\psi}_+^\lambda(tu) &< 0, \\ \Rightarrow \widehat{\psi}_+^\lambda(u_0) &< 0 = \widehat{\psi}_+^\lambda(0) \text{ (see (3.3))}, \\ \Rightarrow u_0 &\neq 0. \end{aligned}$$

From (3.3) we have

$$\begin{aligned} (\widehat{\psi}_+^\lambda)'(u_0) &= 0, \\ \Rightarrow \langle A_p(u_0), h \rangle + \langle A_q(u_0), h \rangle + \int_{\Omega} [\xi(z) + \eta] |u_0|^{p-2} u_0 h dz \\ &= \int_{\Omega} \widehat{k}_+(z, u_0) h dz + \int_{\partial\Omega} \lambda \widehat{b}_+(z, u_0) h d\sigma \end{aligned} \quad (3.4)$$

for all $h \in W^{1,p}(\Omega)$.

In (3.4) we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Then we have

$$\begin{aligned} \widehat{y}_p(u_0^-) + \eta \|u_0^-\|_p^p + \|Du_0^-\|_q^q &= 0, \\ \Rightarrow c_9 \|u_0^-\|^p &\leq 0, \text{ for some } c_9 > 0 \text{ (since } \eta > \|\xi\|_\infty), \\ \Rightarrow u_0 &\geq 0, u_0 \neq 0. \end{aligned}$$

Next in (3.4) we choose $h = [u_0 - \theta_+]^+ \in W^{1,p}(\Omega)$. We have

$$\langle A_p(u_0), (u_0 - \theta_+)^+ \rangle + \langle A_q(u_0), (u_0 - \theta_+)^+ \rangle + \int_{\Omega} [\xi(z) + \eta] u_0^{p-1} (u_0 - \theta_+)^+ dz$$

$$\begin{aligned}
&= \int_{\Omega} \left[f(z, \theta_+) + \eta \theta_+^{p-1} \right] (u_0 - \theta_+)^+ dz + \int_{\partial\Omega} \lambda \beta(z) \theta_+^{\tau-1} (u_0 - \theta_+)^+ d\sigma \quad (\text{see (3.1), (3.2)}) \\
&\leq \int_{\Omega} \left[\xi(z) + \eta \right] \theta_+^{p-1} (u_0 - \theta_+)^+ dz - c_+ \int_{\Omega} (u_0 - \theta_+)^+ dz \\
&\quad + \int_{\partial\Omega} \lambda \beta(z) \theta_+^{\tau-1} (u_0 - \theta_+)^+ d\sigma \quad (\text{see hypothesis } H_1(\text{iii})).
\end{aligned} \tag{3.5}$$

We choose $\lambda_+^* > 0$ so that

$$\lambda \int_{\partial\Omega} \eta(z) (u_0 - \theta_+)^+ d\sigma \leq c_+ \int_{\Omega} ((u_0 - \theta_+)^+) dz$$

for all $0 < \lambda \leq \lambda_+^*$.

Then from (3.5) we have

$$\begin{aligned}
&\langle A_p(u_0), (u_0 - \theta_+)^+ \rangle + \langle A_q(u_0), (u_0 - \theta_+)^+ \rangle + \int_{\Omega} \left[\xi(z) + \eta \right] u_0^{p-1} (u_0 - \theta_+)^+ dz \\
&\leq \langle A_p(\theta_+), (u_0 - \theta_+)^+ \rangle + \langle A_q(\theta_+), (u_0 - \theta_+)^+ \rangle + \int_{\Omega} \left[\xi(z) + \eta \right] \theta_+^{p-1} (u_0 - \theta_+)^+ dz \\
&\Rightarrow \int_{\Omega} \left[\xi(z) + \eta \right] (u_0^{p-1} - \theta_+^{p-1}) (u_0 - \theta_+)^+ dz \leq 0, \\
&\Rightarrow u_0 \leq \theta_+ \quad (\text{since } \eta > \|\xi\|_{\infty}).
\end{aligned}$$

So, we have proved that

$$u_0 \in [0, \theta_+] \text{ for all } \lambda \in (0, \lambda_+^*].$$

Let $\widehat{\xi}_{\theta_+} > 0$ be as postulated by hypothesis $H_1(\text{iv})$. We have

$$\begin{aligned}
-\Delta_p u_0 - \Delta_q u_0 + \left[\xi(z) + \widehat{\xi}_{\theta_+} \right] u_0^{p-1} &= f(z, u_0) + \widehat{\xi}_{\theta_+} u_0^{p-1} \\
&\leq f(z, \theta_+) + \widehat{\xi}_{\theta_+} \theta_+^{p-1} \quad (\text{see hypothesis } H_1(\text{iv})) \\
&\leq \left[\xi(z) + \widehat{\xi}_{\theta_+} \right] \theta_+^{p-1} \\
&= -\Delta_p \theta_+ - \Delta_q \theta_+ + \left[\xi(z) + \widehat{\xi}_{\theta_+} \right] \theta_+^{p-1}
\end{aligned} \tag{3.6}$$

for a.a. $z \in \Omega$.

Hypothesis $H_1(\text{iii})$ implies that

$$0 < c_+ \leq \xi(z) \theta_+^{p-1} - f(z, \theta_+) \text{ for a.a. } z \in \Omega.$$

Theorem 2.10 of Papageorgiou-Rădulescu-Repovš [17], implies that

$$0 < u_0(z) < \theta_+ \text{ for all } z \in \Omega \text{ and } \frac{\partial u_0}{\partial n} \Big|_{\partial\Omega \cap u_0^{-1}(0)} > 0.$$

For the negative solution, we consider the following Carathéodory functions (as before $\eta > \|\xi\|_{\infty}$)

$$\widehat{k}_-(z, x) = \begin{cases} f(z, \theta_-) - \eta |\theta_-|^{p-1} & \text{if } x < \theta_-, \\ f(z, -x^-) - \eta (x^-)^{p-1} & \text{if } \theta_- \leq x, \end{cases} \quad (z, x) \in \Omega \times \mathbb{R}, \tag{3.7}$$

$$\text{and } \widehat{b}_-(z, x) = \begin{cases} -\beta(z) |\theta_-|^{\tau-1} & \text{if } x < \theta_-, \\ -\beta(z) (x^-)^{\tau-1} & \text{if } \theta_- \leq x, \end{cases} \quad (z, x) \in \partial\Omega \times \mathbb{R}. \tag{3.8}$$

We set

$$\widehat{K}_-(z, x) = \int_0^x \widehat{k}_-(z, s) ds, \quad \widehat{B}_-(z, x) = \int_0^x \widehat{b}_-(z, s) ds$$

and introduce the C^1 -functional $\widehat{\psi}_-^\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\psi}_-^\lambda(u) = \frac{1}{p} \widehat{y}_p(u) + \frac{\eta}{p} \|u\|_p^p + \frac{1}{q} \|Du\|_q^q - \int_{\Omega} \widehat{K}_-(z, u) dz - \int_{\partial\Omega} \lambda \widehat{B}_-(z, u) d\sigma$$

for all $u \in W^{1,p}(\Omega)$.

Using (3.7), (3.8) and the direct method of the calculus of variation, we can find $\lambda^* > 0$ and $v_0 \in W^{1,p}(\Omega)$ such that

$$v_0 \in -\text{int } C_+ \text{ is a negative solution of } (P_\lambda), \lambda \in (0, \lambda^*],$$

$$\theta_- < v_0(z) < 0 \text{ for all } z \in \Omega \text{ and } \frac{\partial v_0}{\partial n} \Big|_{\partial\Omega \cap v_0^{-1}(0)} < 0.$$

Finally let $\lambda^* = \min \{ \lambda_+^*, \lambda_-^* \}$. This completes the proof. $\square$

Next we will produce two more sign smooth solutions, localized with respect to u_0 and v_0 respectively (see Proposition 3). With $\eta > \|\xi\|_\infty$ as before, we consider the following truncation perturbations of the reaction

$$g_+(z, x) = \begin{cases} f(z, u_0(z)) + \eta u_0(z)^{p-1} & \text{if } x < u_0(z), \\ f(z, x) + \eta x^{p-1} & \text{if } u_0(z) \leq x, \end{cases} \quad (z, x) \in \Omega \times \mathbb{R}, \quad (3.9)$$

$$g_-(z, x) = \begin{cases} f(z, x) + \eta |x|^{p-2} x & \text{if } x \leq v_0(z), \\ f(z, v_0(z)) + \eta |v_0(z)|^{p-2} v_0(z) & \text{if } v_0(z) < x, \end{cases} \quad (z, x) \in \Omega \times \mathbb{R}. \quad (3.10)$$

Also we consider the following truncations of the boundary term

$$b_+(z, x) = \begin{cases} \beta(z) u_0(z)^{\tau-1} & \text{if } x < u_0(z), \\ \beta(z) x^{\tau-1} & \text{if } u_0(z) \leq x, \end{cases} \quad (z, x) \in \partial\Omega \times \mathbb{R}, \quad (3.11)$$

$$b_-(z, x) = \begin{cases} \beta(z) |x|^{\tau-2} x & \text{if } x \leq v_0(z), \\ \beta(z) |v_0(z)|^{\tau-2} v_0(z) & \text{if } v_0(z) < x, \end{cases} \quad (z, x) \in \partial\Omega \times \mathbb{R}. \quad (3.12)$$

We set

$$G_\pm(z, x) = \int_0^x g_\pm(z, s) ds, \quad B_\pm(z, x) = \int_0^x b_\pm(z, s) ds$$

and introduce the C^1 -functional $\psi_\pm^\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\psi_\pm^\lambda(u) = \frac{1}{p} \widehat{y}_p(u) + \frac{\eta}{p} \|u\|_p^p + \frac{1}{q} \|Du\|_q^q - \int_{\Omega} G_\pm(z, u) dz - \int_{\partial\Omega} \lambda B_\pm(z, u) d\sigma$$

for all $u \in W^{1,p}(\Omega)$, all $0 < \lambda \leq \lambda^*$.

Also we consider the following truncations of $g_\pm(z, \cdot)$ and $b_\pm(z, \cdot)$

$$\widetilde{g}_+(z, x) = \begin{cases} g_+(z, x) & \text{if } x \leq \theta_+, \\ g_+(z, \theta_+) & \text{if } \theta_+ < x, \end{cases} \quad (z, x) \in \Omega \times \mathbb{R}, \quad (3.13)$$

$$\widetilde{g}_-(z, x) = \begin{cases} g_-(z, \theta_-) & \text{if } x < \theta_-, \\ g_-(z, x) & \text{if } \theta_- \leq x, \end{cases} \quad (z, x) \in \Omega \times \mathbb{R}, \quad (3.14)$$

$$\tilde{b}_+(z, x) = \begin{cases} b_+(z, x) & \text{if } x \leq \theta_+, \\ b_+(z, \theta_+) & \text{if } \theta_+ < x, \end{cases} \quad (z, x) \in \partial\Omega \times \mathbb{R}, \quad (3.15)$$

$$\tilde{b}_-(z, x) = \begin{cases} b_-(z, \theta_-) & \text{if } x < \theta_-, \\ b_-(z, x) & \text{if } \theta_- \leq x, \end{cases} \quad (z, x) \in \partial\Omega \times \mathbb{R}. \quad (3.16)$$

We set

$$\tilde{G}_\pm(z, x) = \int_0^x \tilde{g}_\pm(z, s) ds \text{ and } \tilde{B}_\pm(z, x) = \int_0^x \tilde{b}_\pm(z, s) ds$$

and introduce the C^1 -functional $\tilde{\psi}_\pm^\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\tilde{\psi}_\pm^\lambda(u) = \frac{1}{p} \gamma_p(u) + \frac{\eta}{p} \|u\|_p^p - \int_\Omega \tilde{G}_\pm(z, u) dz - \int_{\partial\Omega} \lambda \tilde{B}_\pm(z, u) d\sigma$$

for all $u \in W^{1,p}(\Omega)$, all $0 < \lambda \leq \lambda^*$.

From (3.9)–(3.16) it is clear that

$$\psi_+^\lambda \Big|_{[0, \theta_+]} = \tilde{\psi}_+^\lambda \Big|_{[0, \theta_+]} \text{ and } \psi_-^\lambda \Big|_{[\theta_-, 0]} = \tilde{\psi}_-^\lambda \Big|_{[\theta_-, 0]}, \quad (3.17)$$

$$(\psi_+^\lambda)' \Big|_{[0, \theta_+]} = (\tilde{\psi}_+^\lambda)' \Big|_{[0, \theta_+]} \text{ and } (\psi_-^\lambda)' \Big|_{[\theta_-, 0]} = (\tilde{\psi}_-^\lambda)' \Big|_{[\theta_-, 0]}. \quad (3.18)$$

Proposition 4. *If hypotheses H_0, H_1 hold and $0 < \lambda \leq \lambda^*$, then $K_{\psi_+^\lambda} \subseteq [u_0] \cap \text{int } C_+$, $K_{\psi_-^\lambda} \subseteq [v_0] \cap (-\text{int } C_+)$ and we may assume that $u_0 \in \text{int } C_+$ (resp. $v_0 \in -\text{int } C_+$) is a local minimizer of ψ_+^λ (resp. of ψ_-^λ).*

Proof. We do the proof for the functional $\psi_+^\lambda(\cdot)$, the proof for the functional $\psi_-^\lambda(\cdot)$ being similar.

So, let $u \in K_{\psi_+^\lambda}$. We have

$$\langle A_p(u), h \rangle + \langle A_q(u), h \rangle + \int_\Omega [\xi(z) + \eta] |u|^{p-2} u h dz = \int_\Omega g_+(z, u) h dz + \int_{\partial\Omega} \lambda b_+(z, u) h d\sigma \quad (3.19)$$

for all $h \in W^{1,p}(\Omega)$.

In (3.19) we choose $h = (u_0 - u)^+ \in W^{1,p}(\Omega)$. Then we have

$$\begin{aligned} & \langle A_p(u), (u_0 - u)^+ \rangle + \langle A_q(u), (u_0 - u)^+ \rangle + \int_\Omega [\xi(z) + \eta] u^{p-2} u (u_0 - u)^+ dz \\ &= \int_\Omega [f(z, u_0) + \eta u_0^{p-1}] (u_0 - u)^+ dz + \int_{\partial\Omega} \lambda \beta(z) u_0^{\tau-1} (u_0 - u)^+ d\sigma \quad (\text{see (3.9), (3.11)}) \\ &= \langle A_p(u_0), (u_0 - u)^+ \rangle + \langle A_q(u_0), (u_0 - u)^+ \rangle + \int_\Omega [\xi(z) + \eta] u_0^{\tau-1} (u_0 - u)^+ dz, \\ &\Rightarrow \int_\Omega [\xi(z) + \eta] (u_0^{p-1} - |u|^{p-2} u) (u_0 - u)^+ dz \leq 0, \\ &\Rightarrow u_0 \leq u \quad (\text{since } \eta > \|\xi\|_\infty). \end{aligned} \quad (3.20)$$

From (3.20), (3.9), (3.11) and (3.19) we have

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z) u(z)^{p-1} = f(z, u(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial u}{\partial n_{pq}} = \lambda \beta(z) u^{\tau-1} & \text{on } \partial\Omega. \end{cases} \quad (3.21)$$

From (3.21) and Proposition 2.10 of Papageorgiou-Rădulescu [14], we have $u \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [10] we have $u \in \text{int } C_+$ (recall $u_0 \in \text{int } C_+$). So, finally we can say that

$$K_{\psi_+^\lambda} \subseteq [u_0] \cap \text{int } C_+.$$

Similarly, using this time (3.10) and (3.12), we show that

$$K_{\psi_-^\lambda} \subseteq (v_0] \cap (-\text{int } C_+).$$

Using the functional $\tilde{\psi}_+^\lambda(\cdot)$ we will show that we may assume that $u_0 \in \text{int } C_+$ is a local minimizer of $\psi_+^\lambda(\cdot)$ and $v_0 \in -\text{int } C_+$ is a local minimizer of $\psi_-^\lambda(\cdot)$.

From (3.13) and (3.15) and since $\eta > \|\xi\|_\infty$ we see that

$$\tilde{\psi}_+^\lambda(\cdot) \text{ is coercive.}$$

Also the Sobolev embedding theorem and the compactness of the trace map, imply that

$$\tilde{\psi}_+^\lambda(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

So, by the Weierstrass-Tonelli theorem, we can find $\tilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} \tilde{\psi}_+^\lambda(\tilde{u}_0) &= \min \left\{ \tilde{\psi}_+^\lambda(u) : u \in W^{1,p}(\Omega) \right\} \\ &< 0 = \tilde{\psi}_+^\lambda(0) \quad (\text{recall that } 1 < \tau < q < p), \\ &\Rightarrow \tilde{u}_0 \neq 0. \end{aligned} \tag{3.22}$$

From (3.22) we see that $\tilde{u}_0 \in K_{\tilde{\psi}_+^\lambda}$. Moreover, on account of (3.13) and (3.15), as the proof of Proposition 3, we have that

$$K_{\tilde{\psi}_+^\lambda} \subseteq [u_0, \theta_+] \cap \text{int } C_+.$$

This and (3.18) imply that $\tilde{u}_0 \in K_{\tilde{\psi}_+^\lambda}$, hence $u_0 \leq \tilde{u}_0 \in \text{int } C_+$. If $u_0 \neq \tilde{u}_0$, then $\tilde{u}_0 \in \text{int } C_+$ is desired second solution of (P_λ) (see (3.9), (3.11)) bigger than u_0 . So, we are done. Therefore we may assume that

$$\tilde{u}_0 = u_0 \in [0, \theta_+] \cap \text{int } C_+.$$

Consider the open cone

$$\mathcal{D}_+ = \left\{ y \in C^1(\overline{\Omega}) : y(z) > 0 \text{ for all } z \in \Omega, \quad \frac{\partial y}{\partial n} \Big|_{\partial\Omega \cap \gamma^{-1}(0)} < 0 \right\}.$$

On account of Proposition 3, we have

$$\begin{aligned} \theta_+ - u_0 &\in \mathcal{D}_+, \\ \Rightarrow u_0 &\text{ is a local } C^1(\overline{\Omega}) - \text{minimizer of } \psi_+^\lambda(\cdot) \text{ (see (3.17))}, \\ \Rightarrow u_0 &\text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } \psi_+^\lambda(\cdot) \\ &\text{(see Papageorgiou-Rădulescu [14], Proposition 2.12).} \end{aligned}$$

In a similar fashion using this time the functional $\psi_-^\lambda(\cdot)$, we show that

$$v_0 \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } \psi_-^\lambda(\cdot).$$

This completes the proof. □

On account of Proposition 4, we see that we may assume that

$$K_{\psi_+^\lambda} \text{ and } K_{\psi_-^\lambda} \text{ are finite.} \tag{3.23}$$

Otherwise we already have an infinity of positive smooth solutions bigger than u_0 and an infinity of negative smooth solutions smaller than v_0 and so we are done.

From (3.23), Proposition 4 and Theorem 5.7.6 of Papageorgiou-Rădulescu[19, p. 449], we see that we find $\rho \in (0, 1)$ small such that

$$\psi_+^\lambda(u_0) < \inf \left\{ \psi_+^\lambda(u) : \|u - u_0\| = \rho \right\} = m_+^\lambda, \quad (3.24)$$

$$\psi_-^\lambda(v_0) < \inf \left\{ \psi_-^\lambda(v) : \|v - v_0\| = \rho \right\} = m_-^\lambda. \quad (3.25)$$

Proposition 5. *If hypotheses H_0, H_1 hold and $0 < \lambda \leq \lambda^*$, then the functionals $\psi_\pm^\lambda(\cdot)$ satisfy the C-condition.*

Proof. We do the proof for the functional $\psi_+^\lambda(\cdot)$, the proof for the functional $\psi_-^\lambda(\cdot)$ being similar.

Consider a sequence $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$|\psi_+^\lambda(u_n)| \leq c_{10} \text{ for some } c_{10} > 0, \text{ all } n \in \mathbb{N}, \quad (3.26)$$

$$(1 + \|u_n\|)(\psi_+^\lambda)'(u_n) \rightarrow 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \rightarrow \infty. \quad (3.27)$$

From (3.27) we have

$$\begin{aligned} & \left| \langle A_p(u_n), h \rangle + \langle A_q(u_n), h \rangle + \int_{\Omega} [\xi(z) + \eta] |u_n|^{p-2} u_n h dz \right. \\ & \quad \left. - \int_{\Omega} g_+(z, u_n) h dz - \int_{\partial\Omega} \lambda b_+(z, u_n) h d\sigma \right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \end{aligned} \quad (3.28)$$

for all $h \in W^{1,p}(\Omega)$, with $\varepsilon_n \rightarrow 0^+$.

In (3.28) we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Using (3.9) and (3.11), we obtain

$$\begin{aligned} & \|Du_n^-\|_p^p + \int_{\Omega} [\xi(z) + \eta] (u_n^-)^p dz \leq c_{11} \text{ for all } c_{11} > 0, \text{ all } n \in \mathbb{N}, \\ \Rightarrow & \|u_n^-\|_p^p \leq c_{12} \text{ for all } c_{12} > 0, \text{ all } n \in \mathbb{N} \text{ (recall that } \eta > \|\xi\|_{\infty}), \\ \Rightarrow & \{u_n^-\}_{n \geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \end{aligned} \quad (3.29)$$

In (3.28) we choose $h = u_n^+ \in W^{1,p}(\Omega)$ and using (3.9) and (3.11), we obtain

$$\begin{aligned} & -\|Du_n^+\|_p^p - \|Du_n^+\|_q^q - \int_{\Omega} \xi(z)(u_n^+)^p dz + \int_{\Omega} f(z, u_n^+) u_n^+ dz \\ & \quad + \int_{\partial\Omega} \lambda \beta(z)(u_n^+)^{\tau} d\sigma \leq c_{13} \end{aligned} \quad (3.30)$$

for some $c_{13} > 0$, all $n \in \mathbb{N}$.

On the other hand from (3.26), (3.29), (3.9) and (3.11), we see that

$$\begin{aligned} & \frac{\mu}{p} \|Du_n^+\|_p^p + \frac{\mu}{q} \|Du_n^+\|_q^q + \frac{\mu}{p} \int_{\Omega} \xi(z)(u_n^+)^p dz - \int_{\Omega} \mu F(z, u_n^+) dz \\ & \quad - \frac{\mu}{\tau} \int_{\partial\Omega} \lambda \beta(z)(u_n^+)^{\tau} d\sigma \leq c_{14} \end{aligned} \quad (3.31)$$

for some $c_{14} > 0$, all $n \in \mathbb{N}$.

We add (3.30) and (3.31) and using hypothesis $H_1(ii)$ (the AR-condition), we obtain

$$\left(\frac{\mu}{p} - 1 \right) \left[\|Du_n^+\|_p^p + \int_{\Omega} \xi(z)(u_n^+)^p dz \right] \leq \left(\frac{\mu}{\tau} - 1 \right) \int_{\partial\Omega} \lambda \beta(z)(u_n^+)^{\tau} d\sigma \quad (3.32)$$

(recall $q < p < \mu$).

From (3.30) and (2.2), we have

$$c_2 \|u_n\|_\mu^\mu \leq c_{15} + \|Du_n^+\|_p^p + \|\xi\|_\infty \|u_n^+\|_p^p + \|Du_n^+\|_q^q \quad (3.33)$$

for some $c_{15} > 0$, all $n \in \mathbb{N}$ (see hypotheses H_0).

Suppose that $\{u_n^+\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ is not bounded. We may assume that

$$\|u_n^+\| \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (3.34)$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \rightarrow y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \quad (3.35)$$

We multiply (3.33) with $\frac{1}{\|y_n^+\|^\mu}$ and obtain

$$\begin{aligned} c_2 \|y_n\|_\mu^\mu &\leq \frac{c_{15}}{\|y_n^+\|^\mu} + \frac{1}{\|u_n^+\|^{\mu-p}} \|Dy_n\|_p^p + \frac{\|\xi\|_\infty}{\|u_n^+\|^{\mu-p}} \|y_n\|_p^p + \frac{1}{\|u_n^+\|^{\mu-q}} \|Dy_n\|_q^q, \\ \Rightarrow y_n &\rightarrow 0 \text{ in } L^\mu(\Omega) \text{ (see (3.34) and recall that } q < p < \mu). \end{aligned} \quad (3.36)$$

From (3.32) we have

$$\begin{aligned} \left(\frac{\mu}{p} - 1\right) \left[\|Dy_n\|_p^p + \int_\Omega \xi(z) y_n^p dz \right] &\leq \frac{\frac{\mu}{p} - 1}{\|u_n\|^{p-\tau}} \int_{\partial\Omega} \beta(z) y_n^\tau d\sigma, \\ \Rightarrow \|Dy_n\|_p &\rightarrow 0 \text{ (see (3.36), (3.34) and recall that } \tau < q < p). \end{aligned} \quad (3.37)$$

From (3.36) and (3.37) it follows that

$$y_n \rightarrow 0 \text{ in } W^{1,p}(\Omega),$$

a contradiction to the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$. This proves that

$$\begin{aligned} \{u_n^+\}_{n \geq 1} &\subseteq W^{1,p}(\Omega) \text{ is bounded,} \\ \Rightarrow \{u_n\}_{n \geq 1} &\subseteq W^{1,p}(\Omega) \text{ is bounded (see (3.29)).} \end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \rightarrow u \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \quad (3.38)$$

We return to (3.28), choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ use (3.38). Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[\langle A_p(u_n), u_n - u \rangle + \langle A_q(u_n), u_n - u \rangle \right] &= 0, \\ \Rightarrow \limsup_{n \rightarrow \infty} \left[\langle A_p(u_n), u_n - u \rangle + \langle A_q(u), u_n - u \rangle \right] &\leq 0 \text{ (see Proposition 1),} \\ \Rightarrow \limsup_{n \rightarrow \infty} \langle A_p(u_n), u_n - u \rangle &\leq 0, \\ \Rightarrow u_n &\rightarrow u \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1).} \end{aligned}$$

This proves that $\psi_+^\lambda(\cdot)$ satisfies the C -condition.

Similarly we show that $\psi_-^\lambda(\cdot)$ satisfies the C -condition. This completes the proof. $\square$

From (2.1) we see that

$$\lim_{x \rightarrow \pm\infty} \frac{F(z, x)}{|x|^p} = +\infty \text{ uniformly for a.a. } z \in \Omega.$$

From this asymptotic property of the primitive, we infer the following result.

Proposition 6. *If hypotheses H_0, H_1 hold, $0 < \lambda \leq \lambda^*$ and $u \in \text{int } C_+$, then $\psi_\pm^\lambda(tu) \rightarrow -\infty$ as $t \rightarrow \pm\infty$.*

Now we are ready to produce two more constant sign smooth solutions.

Proposition 7. *If hypotheses H_0, H_1 hold, $0 < \lambda \leq \lambda^*$, then problem (P_λ) has two more constant sign solutions*

$$\begin{aligned}\widehat{u} &\in \text{int } C_+, \quad u_0 \leq \widehat{u}, \quad \widehat{u} \neq u_0, \\ \widehat{v} &\in -\text{int } C_+, \quad \widehat{v} \leq v_0, \quad \widehat{v} \neq v_0.\end{aligned}$$

Proof. Proposition 5 and 6 and (3.24), permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in W^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\psi_\pm^\lambda} \subseteq [u_0] \cap \text{int } C_+ \text{ and } m_\pm^\lambda \leq \psi_\pm^\lambda(\widehat{u}), \quad (3.39)$$

(see Proposition 4 and (3.24)). From (3.39), (3.9), (3.11) we infer that $\widehat{u} \in \text{int } C_+$ is a positive solution of (P_λ) , $u_0 \leq \widehat{u}$, $u_0 \neq \widehat{u}$.

Similarly, using the functional ψ_-^λ and (3.25), we produce a second negative solution of (P_λ) such that

$$\widehat{v} \in -\text{int } C_+, \quad \widehat{v} \leq v_0, \quad \widehat{v} \neq v_0.$$

This completes the proof. $\square$

4 Extremal constant sign solutions

In this section we produce extremal constant sign solutions for problem (P_λ) ($\lambda \in (0, \lambda^*]$), that is, we obtain a smallest positive solution and a biggest negative solution. These extremal solutions will be used in Section 5 to produce a nodal (sign-changing) solution.

For $\lambda \in (0, \lambda^*]$ we define the following two sets

$$\begin{aligned}S_\lambda^+ &= \text{set of positive solutions of } (P_\lambda), \\ S_\lambda^- &= \text{set of negative solutions of } (P_\lambda).\end{aligned}$$

In Section 3, we proved that

$$\emptyset \neq S_\lambda^+ \subseteq \text{int } C_+ \text{ and } \emptyset \neq S_\lambda^- \subseteq \text{int } C_-.$$

Proposition 8. *If hypotheses H_0, H_1 hold and $0 < \lambda \leq \lambda^*$, then $\bar{u}_\lambda \leq u$ for all $u \in S_\lambda^+$ and $\bar{v}_\lambda \leq v$ for all $v \in S_\lambda^-$.*

Proof. Let $u \in S_\lambda^+ \subseteq \text{int } C_+$. We consider the following Carathéodory functions

$$e_+(z, x) = \begin{cases} -c_3(x^+)^{r-1} + (x^+)^{p-1} & \text{if } x \leq u(z), \\ -c_3u(z)^{r-1} + u(z)^{p-1} & \text{if } u(z) < x, \end{cases} \quad (z, x) \in \Omega \times \mathbb{R}, \quad (4.1)$$

$$d_+(z, x) = \begin{cases} \beta(z)(x^+)^{r-1} & \text{if } x \leq u(z), \\ \beta(z)u(z)^{r-1} & \text{if } u(z) < x, \end{cases} \quad (z, x) \in \partial\Omega \times \mathbb{R}. \quad (4.2)$$

We set

$$E_+(z, x) = \int_0^x e_+(z, s) ds \text{ and } D_+(z, x) = \int_0^x d_+(z, s) ds$$

and then introduce the C^1 -functional $j_+^\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$j_+^\lambda(u) = \frac{1}{p} \widetilde{y}_p(u) + \frac{1}{p} \|u\|_p^p + \frac{1}{q} \|Du\|_q^q - \int_\Omega E_+(z, u) dz - \int_{\partial\Omega} \lambda D_+(z, u) d\sigma$$

for all $u \in W^{1,p}(\Omega)$.

Evidently $j_+^\lambda(\cdot)$ is coercive (see (4.1) and (4.2)). Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$j_+^\lambda(\tilde{u}_\lambda) = \min \left\{ j_+^\lambda(u) : u \in W^{1,p}(\Omega) \right\}. \quad (4.3)$$

As in the proof of Proposition 2, if $y \in \text{int } C_+$, then for $t \in (0, 1)$ small (so that at least we have $ty \leq u$, see Proposition 4.1.22 of the Papageorgiou-Rădulescu-Repovš [19, p. 274] and recall that $u \in \text{int } C_+$), we obtain

$$\begin{aligned} j_+^\lambda(tu) &< 0, \\ \Rightarrow j_+^\lambda(\tilde{u}_\lambda) &< 0 = j_+^\lambda(0) \quad (\text{see (4.3)}), \\ \Rightarrow \tilde{u} &\neq 0. \end{aligned}$$

From (4.3), we have

$$\begin{aligned} (j_+^\lambda)'(\tilde{u}_\lambda) &= 0, \\ \Rightarrow \langle A_p(\tilde{u}_\lambda), h \rangle + \langle A_q(\tilde{u}_\lambda), h \rangle + \int_\Omega \left[|\xi(z)| + 1 \right] |\tilde{u}_\lambda|^{p-2} \tilde{u}_\lambda h dz \\ &= \int_\Omega e_+(z, \tilde{u}_\lambda) h dz + \int_{\partial\Omega} \lambda d_+(z, \tilde{u}_\lambda) h d\sigma \end{aligned} \quad (4.4)$$

for all $h \in W^{1,p}(\Omega)$.

In (4.4) we choose $h = -\tilde{u}_\lambda^- \in W^{1,p}(\Omega)$ and obtain

$$\begin{aligned} \tilde{y}_p(\tilde{u}_\lambda) + \|\tilde{u}_\lambda^-\|_p^p &\leq 0, \quad (\text{see (4.1), (4.2)}), \\ \Rightarrow \|\tilde{u}_\lambda^-\|_p^p &\leq 0, \\ \Rightarrow \tilde{u}_\lambda &\geq 0, \tilde{u}_\lambda \neq 0. \end{aligned}$$

Next in (4.4) we choose $h = (\tilde{u}_\lambda - u)^+ \in W^{1,p}(\Omega)$. Then we have

$$\begin{aligned} &\langle A_p(\tilde{u}_\lambda), (\tilde{u}_\lambda - u)^+ \rangle + \langle A_q(\tilde{u}_\lambda), (\tilde{u}_\lambda - u)^+ \rangle + \int_\Omega \left[|\xi(z)| + 1 \right] |\tilde{u}_\lambda|^{p-1} (\tilde{u}_\lambda - u)^+ dz \\ &= \int_\Omega \left[-c_3 u^{r-1} + u^{p-1} \right] (\tilde{u}_\lambda - u)^+ dz + \int_{\partial\Omega} \lambda \beta(z) u^{r-1} (\tilde{u}_\lambda - u)^+ d\sigma \quad (\text{see (4.1), (4.2)}) \\ &\leq \int_\Omega \left[f(z, u) + u^{p-1} \right] (\tilde{u}_\lambda - u)^+ dz + \int_{\partial\Omega} \lambda \beta(z) u^{r-1} (\tilde{u}_\lambda - u)^+ d\sigma \quad (\text{see (2.3)}) \\ &= \langle A_p(u), (\tilde{u}_\lambda - u)^+ \rangle + \langle A_q(u), (\tilde{u}_\lambda - u)^+ \rangle \\ &\quad + \int_\Omega \left[|\xi(z)| + 1 \right] u^{p-1} (\tilde{u}_\lambda - u)^+ dz \quad (\text{since } u \in S_\lambda^+), \\ &\Rightarrow \int_\Omega \left[|\xi(z)| + 1 \right] (\tilde{u}_\lambda^{p-1} - u^{p-1}) (\tilde{u}_\lambda - u)^+ dz \leq 0 \quad (\text{see Proposition 1}), \\ &\Rightarrow \tilde{u}_\lambda \leq u. \end{aligned}$$

So, we have proved that

$$\tilde{u}_\lambda \in [0, u], \quad \tilde{u}_\lambda \neq 0. \quad (4.5)$$

From (4.5), (4.1), (4.2) and (4.4), we infer that

$$\begin{aligned} \tilde{u}_\lambda &\text{ is a positive solution of problem (2.4),} \\ \Rightarrow \tilde{u}_\lambda = \bar{u}_\lambda &\in \text{int } C_+ \quad (\text{see Proposition 2}), \end{aligned}$$

$$\Rightarrow \bar{u}_\lambda \leq u \text{ for all } u \in S_\lambda^+.$$

For the upper bound for the set S_λ^- , given $v \in S_\lambda^- \subseteq -\text{int } C_+$, we consider the Carathéodory functions $e_-(z, x)$ and $d_-(z, x)$ defined by

$$e_-(z, x) = \begin{cases} -c_3 |v_0(z)|^{r-2} v_0(z) + |v_0(z)|^{p-2} v_0(z) & \text{if } x < v_0(z), \\ c_3 (x^-)^{r-1} - (x^-)^{p-1} & \text{if } v_0(z) \leq x, \end{cases} \quad (z, x) \in \Omega \times \mathbb{R}, \quad (4.6)$$

$$d_-(z, x) = \begin{cases} \beta(z) |v_0(z)|^{r-2} v_0(z) & \text{if } x < v_0(z), \\ -\beta(z) (x^-)^{r-1} & \text{if } v_0(z) \leq x, \end{cases} \quad (z, x) \in \partial\Omega \times \mathbb{R}. \quad (4.7)$$

We set

$$E_-(z, x) = \int_0^x e_-(z, s) ds \text{ and } D_-(z, x) = \int_0^x d_-(z, s) ds$$

and then introduce the C^1 -functional $j^\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$j^\lambda(u) = \frac{1}{p} \tilde{y}_p(u) + \frac{1}{p} \|u\|_p^p + \frac{1}{q} \|Du\|_q^q - \int_\Omega E_-(z, u) dz - \int_{\partial\Omega} \lambda D_-(z, u) d\sigma$$

for all $u \in W^{1,p}(\Omega)$.

Reasoning as above, using this time the functional $j^\lambda(\cdot)$ and (4.6), (4.7), we obtain that

$$v \leq \bar{v}_\lambda \text{ for all } v \in S_\lambda^-.$$

This completes the proof. $\square$

From Papageorgiou-Rădulescu-Repovš [16] (see the proof Proposition 7) we have that

$$\begin{aligned} S_\lambda^+ &\text{ is downward directed,} \\ S_\lambda^- &\text{ is upward directed.} \end{aligned}$$

Now we are ready to produce extremal constant sign solutions.

Proposition 9. *If hypotheses H_0, H_1 hold and $0 < \lambda \leq \lambda^*$, then problem (P_λ) has*

- *a smallest positive solution $u_\lambda^* \in \text{int } C_+$ (that is, $u_\lambda^* \leq u$ for all $u \in S_\lambda^+$);*
- *a biggest negative solution $v_\lambda^* \in -\text{int } C_+$ (that is, $v \leq v_\lambda^*$ for all $u \in S_\lambda^-$).*

Proof. Since S_λ^+ is downward directed, by Lemma 3.10 of Hu-Papageorgiou [7, p. 178], we can find a decreasing sequence $\{u_n\}_{n \geq 1} \subseteq S_\lambda^+$ such that

$$\begin{aligned} \inf_{n \geq 1} u_n &= \inf_{n \geq 1} S_\lambda^+, \\ \bar{u}_\lambda &\leq u_n \leq u_1 \end{aligned} \quad (4.8)$$

for all $n \in \mathbb{N}$ (see Proposition (8)).

We have

$$\begin{aligned} &\langle A_p(u_n), h \rangle + \langle A_q(u_n), h \rangle + \int_\Omega \xi(z) u_n^{p-1} h dz \\ &= \int_\Omega f(z, u_n) h dz + \int_{\partial\Omega} \lambda \beta(z) u_n^{r-1} h d\sigma \end{aligned} \quad (4.9)$$

for all $h \in W^{1,p}(\Omega)$.

In (4.9) we choose $h \in W^{1,p}(\Omega)$. Using hypotheses $H_0, H_1(i)$ and (4.8) we infer that

$$\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \rightarrow u_\lambda^* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \quad (4.10)$$

In (4.9) we choose $h = u_n - u_\lambda^* \in W^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ use (4.10). We obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u_\lambda^* \rangle + \langle A_q(u_n), u_n - u_\lambda^* \rangle] = 0, \\ \Rightarrow & \limsup_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u_\lambda^* \rangle + \langle A_q(u), u_n - u_\lambda^* \rangle] \leq 0 \\ & \text{(see Proposition 1),} \\ \Rightarrow & \limsup_{n \rightarrow \infty} \langle A_p(u_n), u_n - u_\lambda^* \rangle \leq 0 \text{ (see (4.10))}, \\ \Rightarrow & u_n \rightarrow u_\lambda^* \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1)}. \end{aligned}$$

From (4.8) we have

$$\begin{aligned} & \bar{u}_\lambda \leq u_\lambda^*, \\ \Rightarrow & u_\lambda^* \in S_\lambda^+ \subseteq \text{int } C_+, \quad u_\lambda^* = \inf S_\lambda^+. \end{aligned}$$

For the biggest negative solution, we use the fact that S_λ^- is upward directed and so we can find an increasing sequence $\{v_n\}_{n \geq 1} \subseteq S_\lambda^-$ such that

$$\begin{aligned} & \sup_{n \geq 1} v_n = \sup_{n \geq 1} S_\lambda^-, \\ & v_1 \leq v_n \leq \bar{v}_\lambda \end{aligned}$$

for all $n \in \mathbb{N}$ (see Proposition (8) and [7, p. 178]).

Reasoning as above we produce $v_\lambda^* \in S_\lambda \in \text{int } C_+$ such that $v_\lambda^* = \sup S_\lambda^-$. The proof is finished. $\square$

5 Nodal solutions

In this section, we use the extremal constant sign solutions of (P_λ) in order to produce a nodal (sign-changing) solution.

So, let $u_\lambda^* \in \text{int } C_+$ and $v_\lambda^* \in -\text{int } C_+$ be the two extremal constant sign solutions from Proposition 9. With $\eta > \|\xi\|_\infty$, we consider the following Carathéodory function

$$i(z, x) = \begin{cases} f(z, v_\lambda^*(z)) + \eta |v_\lambda^*(z)|^{p-2} v_\lambda^*(z) & \text{if } x < v_\lambda^*(z), \\ f(z, x) + \eta |x|^{p-2} x & \text{if } v_\lambda^*(z) \leq x \leq u_\lambda^*(z), \quad (z, x) \in \Omega \times \mathbb{R}, \\ f(z, u_\lambda^*(z)) + \eta u_\lambda^*(z)^{p-1} & \text{if } u_\lambda^*(z) < x, \end{cases} \quad (5.1)$$

Also we consider the correspond truncation of the boundary term, namely the Carathéodory function $t(z, x)$ defined by

$$t(z, x) = \begin{cases} \beta(z) |v_\lambda^*(z)|^{\tau-2} v_\lambda^*(z) & \text{if } x < v_\lambda^*(z), \\ \beta(z) |x|^{\tau-2} x & \text{if } v_\lambda^*(z) \leq x \leq u_\lambda^*(z), \quad (z, x) \in \partial\Omega \times \mathbb{R}, \\ \beta(z) u_\lambda^*(z)^{\tau-1} & \text{if } u_\lambda^*(z) < x, \end{cases} \quad (5.2)$$

We also consider the positive and negative truncations of these functions, namely the Carathéodory function $i_\pm(z, x)$ and $t_\pm(z, x)$ defined by

$$i_\pm(z, x) = i(z, \pm x^\pm) \text{ and } t_\pm(z, x) = t(z, \pm x^\pm). \quad (5.3)$$

We set

$$I(z, x) = \int_0^x i(z, s) dz, \quad T(z, x) = \int_0^x t(z, s) dz,$$

$$I_{\pm}(z, x) = \int_0^x i_{\pm}(z, s) dz, \quad T_{\pm}(z, x) = \int_0^x t_{\pm}(z, s) dz,$$

and then introduce the C^1 -functionals $J^{\lambda}, J_{\pm}^{\lambda} : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$J^{\lambda}(u) = \frac{1}{p} \widehat{y}_p(u) + \frac{\eta}{p} \|u\|_p^p + \frac{1}{q} \|Du\|_q^q - \int_{\Omega} I(z, u) dz - \int_{\partial\Omega} \lambda T(z, u) d\sigma,$$

$$J_{\pm}^{\lambda}(u) = \frac{1}{p} \widehat{y}_p(u) + \frac{\eta}{p} \|u\|_p^p + \frac{1}{q} \|Du\|_q^q - \int_{\Omega} I_{\pm}(z, u) dz - \int_{\partial\Omega} \lambda T_{\pm}(z, u) d\sigma$$

for all $u \in W^{1,p}(\Omega)$.

Also let $\varphi_{\lambda} : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the energy (Euler) functional for problem (P_{λ}) defined by

$$\varphi_{\lambda}(u) = \frac{1}{p} \widehat{y}_p(u) + \frac{1}{q} \|Du\|_q^q - \int_{\Omega} F(z, u) dz - \frac{1}{\tau} \int_{\partial\Omega} \lambda \beta(z) |u|^{\tau} d\sigma$$

for all $u \in W^{1,p}(\Omega)$. Evidently $\varphi_{\lambda} \in C^1(W^{1,p}(\Omega), \mathbb{R})$.

Proposition 10. *If hypotheses H_0, H_1 hold and $0 < \lambda \leq \lambda^*$, then $K_{J^{\lambda}} \subseteq [v_{\lambda}^*, u_{\lambda}^*] \cap C^1(\overline{\Omega})$, $K_{J_{+}^{\lambda}} \subseteq \{0, u_{\lambda}^*\}$, $K_{J_{-}^{\lambda}} \subseteq \{0, v_{\lambda}^*\}$.*

Proof. Let $u \in K_{J^{\lambda}}$. We have

$$\begin{aligned} (J^{\lambda})'(u) &= 0, \\ \Rightarrow \langle A_p(u), h \rangle + \langle A_q(u), h \rangle + \int_{\Omega} [\xi(z) + \eta] |u|^{p-2} u h dz \\ &= \int_{\Omega} i(z, u) h dz + \int_{\partial\Omega} \lambda t(z, u) h d\sigma \end{aligned} \quad (5.4)$$

for all $h \in W^{1,p}(\Omega)$.

In (5.4) we choose $h = (u - u_{\lambda}^*)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned} &\langle A_p(u), (u - u_{\lambda}^*)^+ \rangle + \langle A_q(u), (u - u_{\lambda}^*)^+ \rangle \\ &\quad + \int_{\Omega} [\xi(z) + \eta] u^{p-1} (u - u_{\lambda}^*)^+ dz \\ &= \int_{\Omega} [f(z, u_{\lambda}^*) + \eta (u_{\lambda}^*)^{p-1}] (u - u_{\lambda}^*)^+ dz \\ &\quad + \int_{\partial\Omega} \lambda \beta(z) (u_{\lambda}^*)^{\tau-1} (u - u_{\lambda}^*)^+ d\sigma \quad (\text{see (5.1), (5.2)}) \\ &= \langle A_p(u_{\lambda}^*), (u - u_{\lambda}^*)^+ \rangle + \langle A_q(u_{\lambda}^*), (u - u_{\lambda}^*)^+ \rangle \\ &\quad + \int_{\Omega} [\xi(z) + \eta] (u_{\lambda}^*)^{p-1} (u - u_{\lambda}^*)^+ dz \quad (\text{since } u_{\lambda}^* \in S_{\lambda}^+), \end{aligned}$$

$$\begin{aligned} &\Rightarrow \int_{\Omega} [\xi(z) + \eta] (u^{p-1} - (u_{\lambda}^*)^{p-1}) (u - u_{\lambda}^*)^+ dz = 0, \\ &\Rightarrow u \leq u_{\lambda}^* \quad (\text{recall } \eta > \|\xi\|_{\infty}). \end{aligned}$$

Similarly, if in (5.4) we choose $h = (v_{\lambda}^* - u)^+ \in W^{1,p}(\Omega)$, then we obtain

$$v_{\lambda}^* \leq u.$$

So, we have proved that

$$u \in [v_{\lambda}^*, u_{\lambda}^*].$$

The nonlinear regularity theory of Lieberman [10], implies that $u \in C^1(\overline{\Omega})$. So, finally we have

$$K_{J^{\lambda}} \subseteq [v_{\lambda}^*, u_{\lambda}^*] \cap C^1(\overline{\Omega}).$$

Similarly we show that

$$K_{J_{+}^{\lambda}} \subseteq [0, u_{\lambda}^*] \cap C_{+} \text{ and } K_{J_{-}^{\lambda}} \subseteq [v_{\lambda}^*, 0] \cap (-C_{+}) \quad (\text{see (5.3)}).$$

The extremality of u_{λ}^* and v_{λ}^* , implies that

$$K_{J_{+}^{\lambda}} = \{0, u_{\lambda}^*\} \text{ and } K_{J_{-}^{\lambda}} = \{0, v_{\lambda}^*\}.$$

This completes the proof. $\square$

From this proposition and (5.1), (5.2), we see that every nontrivial element of $K_{J^{\lambda}}$ distinct from u_{λ}^* and v_{λ}^* , is a nodal solution of (P_{λ}) . Therefore we may assume that

$$K_{J^{\lambda}} \text{ is finite.} \quad (5.5)$$

Otherwise we already have an infinity of nodal solutions localized in $[v_{\lambda}^*, u_{\lambda}^*]$ and so we are done.

Proposition 11. *If hypotheses H_0, H_1 hold and $0 < \lambda \leq \lambda^*$, then $C_k(\varphi_{\lambda}, 0) = C_k(J^{\lambda}, 0)$ for all $k \in \mathbb{N}$.*

Proof. We consider the homotopy $\widehat{h}(t, u)$ defined by

$$\widehat{h}(t, u) = tJ^{\lambda}(u) + (1-t)\varphi_{\lambda}(u) \text{ for all } (t, u) \in [0, 1] \times W^{1,p}(\Omega).$$

Suppose we could find $\{t_n\}_{n \geq 1} \subseteq [0, 1]$ and $\{v_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$t_n \rightarrow t \text{ in } [0, 1], \quad u_n \rightarrow 0 \text{ in } W^{1,p}(\Omega), \quad \widehat{h}'_u(t_n, u_n) = 0 \quad (5.6)$$

for all $n \in \mathbb{N}$.

From the equation in (5.6), we have

$$\begin{aligned} &\langle A_p(u_n), h \rangle + \langle A_q(u_n), h \rangle + \int_{\Omega} [\xi(z) + t_n \eta] |u_n|^{p-2} u_n h dz \\ &= [t_n i(z, u_n) + (1-t_n) f(z, u_n)] h dz \\ &\quad + \int_{\partial \Omega} \lambda [t_n t(z, u_n) + (1-t_n) \beta(z) |u_n|^{r-2} u_n] h d\sigma \end{aligned} \quad (5.7)$$

for all $h \in W^{1,p}(\Omega)$, all $n \in \mathbb{N}$.

From (5.7) and Proposition 2.10 of Papageorgiou-Rădulescu [14], we know that we can find $c_{16} > 0$ such that

$$\|u_n\|_{\infty} \leq c_{16} \text{ for all } n \in \mathbb{N}.$$

Then the nonlinear regularity theory of Lieberman [10] implies that there exist $\alpha \in (0, 1)$ and $c_{17} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \text{ and } \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_{17} \text{ for all } n \in \mathbb{N}.$$

The compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and (5.6) imply that

$$\begin{aligned} u_n &\rightarrow 0 \text{ in } C^1(\overline{\Omega}), \\ \Rightarrow u_n &\in [v_\lambda^*, u_\lambda^*] \text{ for all } n \geq n_0, \\ \Rightarrow \{u_n\}_{n \geq n_0} &\subseteq K_{J^\lambda} \text{ (see Proposition 10).} \end{aligned}$$

This contradicts (5.5). Therefore (5.6) can not occur and then from the homotopy invariance property of critical groups (see Theorem 6.3.6 of Papageorgiou-Rădulescu-Repovš [19, p. 505]), we have

$$C_k(\varphi_\lambda, 0) = C_k(J^\lambda, 0)$$

for all $k \in \mathbb{N}$. This completes the proof. $\square$

Proposition 12. *If hypotheses H_0, H_1 hold and $\lambda > 0$, then $C_k(\varphi_\lambda, 0) = 0$ for all $k \in \mathbb{N}$.*

Proof. Let $u \in W^{1,p}(\Omega)$, $u \neq 0$ and $t > 0$. We have

$$\varphi_\lambda(tu) \leq \frac{t^p}{p} \tilde{y}_p(u) + \frac{t^q}{q} \|Du\|_q^q + \frac{t^r c_3}{r} \|u\|_r^r - \frac{t^\tau \lambda}{\tau} \int_{\partial\Omega} \beta(z) |u|^\tau d\sigma$$

(see (2.3) and note that $\hat{y}_p \leq \tilde{y}_p$). By hypotheses H_0 , we have

$$\int_{\partial\Omega} \beta(z) |u|^\tau d\sigma > 0.$$

Since $1 < \tau < q < p < r$, we can find $t^* = t^*(u) \in (0, 1)$ small such that

$$\varphi_\lambda(tu) < 0 \text{ for all } t \in (0, t^*). \quad (5.8)$$

Let $u \in W^{1,p}(\Omega)$ with $0 < \|u\| \leq 1$ and $\varphi_\lambda(u) = 0$. Then

$$\begin{aligned} &\left. \frac{d}{dt} \varphi_\lambda(tu) \right|_{t=1} \\ &= \langle \varphi'_\lambda(u), u \rangle - s \varphi_\lambda(u) \text{ with } s \in (d, q) \\ &\quad (\text{by the chain rule and since } \varphi_\lambda(u) = 0), \\ &\geq \left[1 - \frac{s}{p}\right] \hat{y}_p(u) + \int_{\Omega} [sF(z, u) - f(z, u)u] dz \\ &= \left[1 - \frac{s}{p}\right] \|Du\|_p^p + \left[1 - \frac{s}{p}\right] \int_{\Omega} \xi(z) |u|^p dz \\ &\quad + \int_{\Omega} [dF(z, u) - f(z, u)u] dz + (s - d) \int_{\Omega} F(z, u) dz, \\ &\quad (\text{see hypothesis } H_0 \text{ and recall that } \tau \leq d < s). \end{aligned} \quad (5.9)$$

Note that hypotheses $H_1(i)$, (iv) imply that

$$dF(z, x) - f(z, x)x \geq -c_{18}|x|^r \quad (5.10)$$

for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, some $c_{18} > 0$.

Also from hypothesis $H_1(iv)$ we have

$$\begin{aligned} F(z, x) &\geq c_{19}|x|^d \quad \text{for a.a. } z \in \Omega, \quad \text{all } |x| \leq \delta_0, \\ \Rightarrow F(z, x) &\geq \frac{c_{19}}{\delta^{p-d}}|x|^p \quad \text{for a.a. } z \in \Omega, \quad \text{all } |x| \leq \delta \leq \delta_0. \end{aligned}$$

Therefore using also hypothesis $H_1(i)$, we can say that

$$F(z, x) \geq \frac{c_{19}}{\delta^{p-d}}|x|^p - c_{20}|x|^r \quad (5.11)$$

for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, some $c_{20} > 0$.

We return to (5.9) and use (5.10) and (5.11). Then

$$\begin{aligned} \frac{d}{dt} \varphi_\lambda(tu) \Big|_{t=1} &\geq \left[1 - \frac{s}{p} \right] \|Du\|_p^p \\ &\quad + \int_{\Omega} \left[(s-d) \frac{c_{19}}{\delta^{p-d}} - (1 - \frac{s}{p}) \|\xi\|_{\infty} \right] |u|^p dz - c_{21} \|u\|^r \end{aligned}$$

for all $c_{21} > 0$.

Recall that $\delta \in (0, \delta_0]$ is arbitrary. Hence we choose $\delta > 0$ small so that

$$\frac{(s-d)p}{p-s} \frac{c_{19}}{\|\xi\|_{\infty}} > \delta^{p-d}.$$

Then we have

$$\frac{d}{dt} \varphi_\lambda(tu) \Big|_{t=1} \geq c_{22} \|u\|^p - c_{21} \|u\|^r$$

for some $c_{22} > 0$.

Since $p < r$, we can find $\rho \in (0, 1)$ small such that

$$\frac{d}{dt} \varphi_\lambda(tu) \Big|_{t=1} > 0 \quad (5.12)$$

for all $u \in W^{1,p}(\Omega)$ with $0 < \|u\| \leq \rho$, $\varphi_\lambda(u) = 0$.

Now let $u \in W^{1,p}(\Omega)$ with $0 < \|u\| \leq \rho$ and $\varphi_\lambda(u) = 0$. We show that

$$\varphi_\lambda(tu) \leq 0 \quad (5.13)$$

for all $t \in [0, 1]$.

We argue indirectly. So, suppose we can find $t_0 \in (0, 1]$ such that $\varphi_\lambda(t_0 u) > 0$. The function $t \rightarrow \varphi_\lambda(tu)$ is continuous and by hypothesis we have $\varphi_\lambda(u) = 0$. So, we can find $t_* \in (0, 1]$ such that $\varphi_\lambda(t_* u) = 0$. We choose the first time instant after $t_0 > 0$ for which this is true, that is,

$$t_* = \min \left\{ t \in [t_0, 1] : \varphi_\lambda(tu) = 0 \right\} > t_0 > 0.$$

Evidently we have

$$\varphi_\lambda(tu) > 0 \quad (5.14)$$

for all $t \in [t_0, t_*)$.

Let $y = t_* u$. We have

$$0 < \|y\| \leq \|u\| \leq \rho \text{ and } \varphi_\lambda(y) = 0$$

.

Then according to (5.12), we have

$$\frac{d}{dt} \varphi_\lambda(ty) \Big|_{t=1} > 0. \quad (5.15)$$

From (5.14) we have

$$\varphi_\lambda(y) = \varphi_\lambda(t_* u) = 0 < \varphi_\lambda(tu) \text{ for all } t_0 \leq t < t_*,$$

$$\Rightarrow \frac{d}{dt} \varphi_\lambda(ty) \Big|_{t=1} = t_\star \frac{d}{dt} \varphi_\lambda(tu) \Big|_{t=t_\star} = t_\star \lim_{t \rightarrow t_\star} \frac{\varphi_\lambda(tu)}{t - t_\star} \leq 0,$$

which contradicts (5.15). Therefore (5.13) is true.

Note that on account of (5.5) and since $\varphi'_\lambda|_{[v_\lambda^*, u_\lambda^*]} = (J^\lambda)'|_{[v_\lambda^*, u_\lambda^*]}$ (see (5.1), (5.2)), we see that $0 \in K_{\varphi_\lambda}$ is isolated. So by taking $\rho \in (0, 1)$ even smaller if necessary, we can have

$$K_{\varphi_\lambda} \cap \bar{B}_\rho = \{0\}, \quad \left(\bar{B}_\rho = \{u \in W^{1,p}(\Omega) : \|u\| \leq \rho\} \right).$$

From (5.13) we see that $h(\cdot, \cdot)$ is a deformation of $\varphi_\lambda^0 \cap \bar{B}_\rho$ into itself. It follows that

$$\varphi_\lambda^0 \cap \bar{B}_\rho \text{ is contractible.} \quad (5.16)$$

We fix $u \in \bar{B}_\rho$ with $\varphi_\lambda(u) > 0$. We show that there exists a unique $t(u) \in (0, 1)$ such that

$$\varphi_\lambda(t(u)u) = 0 \quad (5.17)$$

The continuity of $t \rightarrow \varphi_\lambda(tu)$ and (5.8) imply via Bolzano's theorem, the existence of $t(u) \in (0, 1)$ such that (5.17) holds. Next we show the uniqueness of this time instant. Let $0 < \hat{t}_1 < \hat{t}_2 < 1$ be two such time instants. We have $\varphi_\lambda(\hat{t}_1 u) = \varphi_\lambda(\hat{t}_2 u) = 0$. From (5.13) we have

$$\theta(t) = \varphi_\lambda(t\hat{t}_2 u) \leq 0$$

for all $t \in [0, 1]$.

We have

$$\begin{aligned} \frac{d}{dt} \theta(t) \Big|_{t=\frac{\hat{t}_1}{\hat{t}_2}} &= 0, \\ \Rightarrow \frac{\hat{t}_1}{\hat{t}_2} \frac{d}{dt} \varphi_\lambda(t\hat{t}_2 u) \Big|_{t=\frac{\hat{t}_1}{\hat{t}_2}} &= \frac{d}{dt} \varphi_\lambda(t\hat{t}_1 u) \Big|_{t=1} = 0. \end{aligned}$$

But this contradicts (5.12). So, we conclude that $t(u) \in (0, 1)$ satisfying (5.17) is unique. Therefore we have

$$\varphi_\lambda(tu) < 0 \text{ for } t \in (0, t(u)) \text{ and } \varphi_\lambda(tu) > 0 \text{ for } t \in (t(u), 1]. \quad (5.18)$$

We consider the map $v : \bar{B}_\rho \setminus \{0\} \rightarrow (0, 1]$ defined by

$$v(u) = \begin{cases} 1 & \text{if } u \in \bar{B}_\rho \setminus \{0\}, \quad \varphi_\lambda(u) \leq 0, \\ t(u) & \text{if } u \in \bar{B}_\rho \setminus \{0\}, \quad \varphi_\lambda(u) > 0. \end{cases} \quad (5.19)$$

Using (5.18) and (5.19), we can easily verify that $v(\cdot)$ is continuous.

Then we introduce the map $\hat{\rho} : \bar{B}_\rho \setminus \{0\} \rightarrow (\varphi_\lambda^0 \cap \bar{B}_\rho) \setminus \{0\}$ defined by

$$\hat{\rho}(u) = \begin{cases} u & \text{if } u \in \bar{B}_\rho \setminus \{0\}, \quad \varphi_\lambda(u) \leq 0, \\ v(u)u & \text{if } u \in \bar{B}_\rho \setminus \{0\}, \quad \varphi_\lambda(u) > 0. \end{cases} \quad (5.20)$$

The continuity of $v(\cdot)$ and (5.20) imply the continuity of $\hat{\rho}(\cdot)$. Note that

$$\hat{\rho} \Big|_{(\varphi_\lambda^0 \cap \bar{B}_\rho) \setminus \{0\}} = \text{id} \Big|_{(\varphi_\lambda^0 \cap \bar{B}_\rho) \setminus \{0\}} \quad (\text{see (5.20)}).$$

Therefore $(\varphi_\lambda^0 \cap \bar{B}_\rho)$ is a retract of $\bar{B}_\rho \setminus \{0\}$. The set $\bar{B}_\rho \setminus \{0\}$ is contractible (see Gasinski-Papageorgiou [5, pp. 677–678]). But a retract of a contractible space, is itself contractible. So, we have

$$(\varphi_\lambda^0 \cap \bar{B}_\rho) \setminus \{0\} \text{ is contractible.} \quad (5.21)$$

From (5.16) and (5.21), we have

$$\begin{aligned} H_k(\varphi_\lambda^0 \cap \bar{B}_\rho, (\varphi_\lambda^0 \cap \bar{B}_\rho) \setminus \{0\}) &= 0 \quad \text{for all } k \in \mathbb{N} \\ &(\text{ see Papageorgiou-Rădulescu-Repovš [19, p. 649] }, \\ &\Rightarrow C_k(\varphi_\lambda, 0) = 0 \end{aligned}$$

for all $k \in \mathbb{N}$. This completes the proof. $\square$

Now we are ready to prove the existence of a nodal solution.

Proposition 13. *If hypotheses H_0 , H_1 hold and $0 < \lambda \leq \lambda^*$, then problem (P_λ) admits a nodal solution y_0 such that*

$$y_0 \in [v_\lambda^*, u_\lambda^*] \cap C^1(\Omega).$$

Proof. From (5.1), (5.2) and (5.3) it is clear that $J_+^\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$J_+^\lambda(\tilde{u}_\lambda) = \min \{ J_+^\lambda(u) : u \in W^{1,p}(\Omega) \}. \quad (5.22)$$

As before, since $1 < \tau < q < p$, we have

$$\begin{aligned} J_+^\lambda(\tilde{u}_\lambda) &< 0 = J_+^\lambda(0), \\ &\Rightarrow \tilde{u}_\lambda \neq 0. \end{aligned} \quad (5.23)$$

From (5.22) we have

$$\begin{aligned} \tilde{u}_\lambda &\in K_{J_+^\lambda} = \{0, u_\lambda^*\} \quad (\text{ see Proposition 10 }), \\ &\Rightarrow \tilde{u}_\lambda = u_\lambda^* \in \text{int } C_+ \quad (\text{ see (5.23) }). \end{aligned}$$

From the (5.1), (5.2) and (5.3) it is clear that

$$\begin{aligned} J^\lambda|_{C_+} &= J_+^\lambda|_{C_+}, \\ &\Rightarrow u_\lambda^* \in \text{int } C_+ \text{ is a local } C^1(\bar{\Omega}) - \text{minimizer of } J^\lambda(\cdot), \\ &\Rightarrow u_\lambda^* \in \text{int } C_+ \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } J^\lambda(\cdot) \\ &\quad (\text{ see Papageorgiou-Rădulescu [14], Proposition 2.12}). \end{aligned} \quad (5.24)$$

Similarly, using this time the functional $J_-^\lambda(\cdot)$, we show that

$$v_\lambda^* \in \text{int } C_+ \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } J^\lambda(\cdot). \quad (5.25)$$

We may assume that

$$J^\lambda(v_\lambda^*) \leq J^\lambda(u_\lambda^*).$$

The analysis is similar if the opposite inequality holds, using this time (5.25) instead of (5.24).

Recall from (5.5) that K_{J^λ} is finite. This fact, (5.24) and Theorem 5.7.6 of Papageorgiou-Rădulescu-Repovš [19, p. 449], imply that we can find $\rho \in (0, 1)$ small such that

$$J^\lambda(v_\lambda^*) \leq J^\lambda(u_\lambda^*) < \inf \{ J^\lambda(u) : \|u - u_\lambda^*\| = \rho \}. \quad (5.26)$$

The functional $J^\lambda(\cdot)$ is coercive (see (5.1), (5.2)). Therefore

$$J^\lambda(\cdot) \text{ satisfies the } C\text{-condition} \quad (5.27)$$

(see Proposition 5.1.15 of Papageorgiou-Rădulescu-Repovš [19, p. 369]).

Then (5.26), (5.27) permit the use of the mountain pass theorem. So, we can find $y_0 \in W^{1,p}(\Omega)$ such that

$$y_0 \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}) \quad (\text{see Proposition 10}), \quad y_0 \notin \{u_\lambda^*, v_\lambda^*\}.$$

Therefore, if we can show $y_0 \neq 0$, then y_0 will be a nodal solution of (P_λ) (see (5.1), (5.2)). Since y_0 is a critical point of $J^\lambda(\cdot)$ of mountain pass type, from Corollary 6.6.9 of Papageorgiou-Rădulescu-Repovš [19, p. 533], we have

$$C_1(J^\lambda, y_0) \neq 0. \quad (5.28)$$

On the other hand, Propositions 11 and 12, imply that

$$C_k(J^\lambda, 0) = 0 \quad (5.29)$$

for all $k \in \mathbb{N}$.

Comparing (5.28) and (5.29), we conclude that $y_0 \neq 0$. Therefore $y_0 \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$ is a nodal solution of (P_λ) , $0 < \lambda \leq \lambda^*$. This completes the proof. $\square$

We can state the following multiplicity theorem for problem (P_λ) .

Theorem 1. *If hypotheses H_0, H_1 hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*]$ problem (P_λ) has at least five nontrivial solutions*

$$\begin{aligned} u_0, \hat{u} &\in \text{int } C_+, \quad u_0 \leq \hat{u}, \quad u_0 \neq \hat{u}, \\ v_0, \hat{v} &\in -\text{int } C_+, \quad \hat{v} \leq v_0, \quad v_0 \neq \hat{v}, \\ y_0 &\in [v_0, u_0] \cap C^1(\overline{\Omega}) \text{ nodal.} \end{aligned}$$

Remark 2. We emphasize that all the solutions produced have sign information and are linearly ordered. It is an interesting open problem, if in hypotheses H_1 , we can replace the AR-condition (see $H_1(ii)$) by another superlinearity condition which is less restrictive. Such alternative conditions can be found in the works of Mugnai-Papageorgiou [12] and Papageorgiou-Rădulescu [13]. With such an alternative superlinearity condition, we encounter serious difficulties in proving the C-condition for the functionals $\psi_\pm^\lambda(\cdot)$ (see Proposition 5), which we were unable to overcome.

Acknowledgement: The research of Youpei Zhang is partially supported by the Fundamental Research Funds for the Central Universities of Central South University (No.2019zzts211). This paper has been completed while Youpei Zhang was visiting University of Craiova (Romania) with the financial support of China Scholarship Council (No. 201906370079). Youpei Zhang would like to thank China Scholarship Council and Embassy of the People's Republic of China in Romania.

References

- [1] A. Ambrosetti, H. Brezis and G. Cerami: *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), 519–543.
- [2] Y. Bai, D. Motreanu and S. Zeng: *Continuity results for parametric nonlinear singular Dirichlet problems*, Adv. Nonlinear Anal. **9** (2020), 372–387.
- [3] J.J. Diaz and J.E. Saa: *Existence et unicité de solutions positives pour certaines équations elliptiques quasilineaires*, CRAS **305** (1987), 521–524.
- [4] J.P. Garcia Azorero, J.J. Manfredi and I. Peral Alonso: *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations*, Commun. Contemp. Math. **2** (2000), 385–404.
- [5] L. Gasinski and N.S. Papageorgiou: *Exercises in Analysis. Part 2: Nonlinear Analysis*, Springer, Cham, 2016.
- [6] Z. Guo and Z. Zhang: *$W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl. **286** (2003), 32–50.

- [7] S. Hu and N.S. Papageorgiou: *Handbook of Multivalued Analysis*. Volume I: Theory, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.
- [8] S. Hu and N.S. Papageorgiou: *Elliptic equations with indefinite and unbounded potential and a nonlinear concave boundary condition*, Commun. Contemp. Math. **19** (2017), No. 1, Article No. 1550090.
- [9] S. Leonardi and N.S. Papageorgiou: *Positive solutions for nonlinear Robin problems with indefinite potential and competing nonlinearities*, Positivity (2019), doi.org/10.1007/s11117-019-0681-5.
- [10] G. Lieberman: *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations **16** (1991), 311–361.
- [11] S. Marano, G. Marino and N.S. Papageorgiou: *On a Dirichlet problem with (p, q) -Laplacian and parametric concave-convex nonlinearity*, J. Math. Anal. Appl. **475** (2019), 1093–1107.
- [12] D. Mugnai and N.S. Papageorgiou: *Wang's multiplicity result for superlinear (p, q) -equations without the Ambrosetti–Rabinowitz condition*, Trans. Amer. Math. Soc. **366** (2014), 4919–4937.
- [13] N.S. Papageorgiou and V.D. Rădulescu: *Coercive and noncoercive nonlinear Neumann problems with indefinite potential*, Forum Math. **28** (2014), 545–571.
- [14] N.S. Papageorgiou and V.D. Rădulescu: *Nonlinear nonhomogeneous Robin problems with superlinear reaction term*, Adv. Nonlinear Stud. **16** (2016), 737–764.
- [15] N.S. Papageorgiou and V.D. Rădulescu: *Nonlinear elliptic problems with superlinear reaction and parametric boundary condition*, Israel J. Math. **212** (2016), 791–824.
- [16] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš: *Positive solutions for perturbation of the Robin problem plus an indefinite potential*, Discrete Contin. Dyn. Syst. **37** (2017), 2589–2618.
- [17] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš: *Positive solutions for nonhomogeneous parametric Robin problems*, Forum Math. **30** (2018), 553–580.
- [18] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš: *Nonlinear nonhomogeneous boundary value problems with competition phenomena*, Appl. Math. Optim. **80** (2018), 251–298.
- [19] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš: *Nonlinear Analysis—Theory and Methods*, Springer Nature, Switzerland, 2019.
- [20] N.S. Papageorgiou and A. Scapellato: *Constant sign and nodal solutions for parametric $(p, 2)$ -equations*, Adv. Nonlinear Anal. **8** (2020), 449–478.
- [21] N.S. Papageorgiou and A. Scapellato: *Positive solutions for parametric $(p, 2)$ -equations with superlinear reaction and a concave boundary term*, Electron. J. Qual. Theory Differ. Equ. **2020** (2020).
- [22] N.S. Papageorgiou and C. Zhang: *Noncoercive resonant $(p, 2)$ -equations with concave terms*, Adv. Nonlinear Anal. **9** (2020), 228–249.
- [23] P. Pucci and J. Serrin: *The Maximum Principle*, Birkhäuser, Basel, 2007.
- [24] J.C. Sabina de Lis and S. Segura de León: *Multiplicity of solutions to a concave-convex problem*, Adv. Nonlinear Studies **15** (2015), 61–90.