Research article

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On variational nonlinear equations with monotone operators

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Abstract: Using monotonicity methods and some variational argument we consider nonlinear problems which involve monotone potential mappings satisfying condition (S) and their strongly continuous perturbations. We investigate when functional whose minimum is obtained by a direct method of the calculus of variations satisfies the Palais-Smale condition, relate minimizing sequence and Galerkin approximations when both exist, then provide structure conditions on the derivative of the action functional under which bounded Palais-Smale sequences are convergent. Finally, we make some comment concerning the convergence of Palais-Smale sequence obtained in the mountain pass theorem due to Rabier.

Keywords: monotone operator, direct variational method, Palais-Smale condition, minimizing sequence

MSC: 49J40, 47J05, 47J30

1 Introduction

Let E be a real, separable, reflexive Banach space and let $J : E \rightarrow \mathbb{R}$ be a Gâteaux differentiable functional. When we are interested in solving equation

$$J'(x) = 0 \text{ (in } E^*) \quad (1.1)$$

several methods can be used. We mention the following:

- i) the direct variational approach for which we require the action functional J to be sequentially weakly lower semicontinuous and coercive;
- ii) the Ekeland Variational Principle for which to be applied we need to assume that J is bounded from below and satisfies the Palais-Smale condition;
- iii) the Minty-Browder Theorem (monotonicity approach) for which we require operator J' to be coercive and monotone (then its potential J is necessarily demicontinuous).

The application of the direct method of the calculus of variations provides that the argument of a minimum is approximated by the weakly convergent minimizing sequence. In case of the application of the Ekeland Variational Principle the argument of a minimum is approximated by the strongly convergent sequence of almost critical points. The monotonicity approach on the other hand provides that the sequence of Galerkin type approximations to equation (1.1) converges strongly - under the additional assumption that operator J' satisfies condition (S). Thus it becomes interesting to know whether there are some natural conditions on J' which would provide that, disregarding of the approach which we use, we can approximate the solution by some strongly convergent sequence corresponding to the method that we use. We give positive answer to this question thereby proving that the minimizing sequence obtained by the direct variational approach is strongly convergent. This observation has several other implications like the result concerning the conver-

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gence of bounded Palais-Smale sequences and the nature of the sequence of Galerkin type approximations in case the solution is additionally the argument of a minimum. Moreover, we show that, under some conditions, existence results denoted above as i) and ii) overlap which means that the minimizing sequence obtained by a direct variational method is strongly convergent and consists of almost critical points. We obtain our results linking monotonicity and variational approaches.

Concerning the existing research connected to the area of our investigations it is known for semilinear problems involving the Laplacian, see for example Proposition 2.2 in [7], that if a derivative of a functional is a sum of an invertible linear bounded operator and a continuous and compact operator, then a bounded Palais-Smale sequence is necessarily convergent. The simplification of checking of the Palais-Smale condition which we provide works for boundary value problems related to the p -Laplacian and the $p(x)$ -Laplacian as well, see [11] for the background on the variable exponent laplacian. We use also this observation to derive a corollary from the version of the mountain pass theorem introduced by Rabier in [10]. Our improvement is that the Palais-Smale sequence is convergent at the expense of some additional assumptions. Some applications to potential problems are also given.

There have been already some results in the area of nonlinear abstract equations, see for example [2, 3] but the papers mentioned concerned mainly existence results for problem involving a duality mapping relative to function $t \rightarrow t^{p-1}$ for a suitable p .

Paper is organized as follows. Firstly we recall some background from the monotonicity theory. Then we provide some results for uniformly monotone operators due to the fact that we use a bit different definition than this in [6]. In the next section, the main topics are addressed with some additional comment on minimizing sequences and relations to Galerkin type approximations. As applications of our results we provide a corollary from Rabier theorem then, see [10] for the version of a mountain pass lemma. Some Dirichlet boundary value problem is also considered.

2 Preliminaries

For some background from the theory of monotone operators we refer to [6] and also [9]. Symbol $\langle \cdot, \cdot \rangle$ stands for a duality pairing between E^* and E in what follows. The norm in E is denoted by $\|\cdot\|$ and in E^* by $\|\cdot\|_*$. Operator $A : E \rightarrow E^*$ is called:

i) *uniformly monotone* if there exists an increasing function $\rho : [0, +\infty) \rightarrow [0, +\infty)$ such that $\rho(0) = 0$ and for all $u, v \in E$

$$\langle A(u) - A(v), u - v \rangle \geq \|u - v\| \rho(\|u - v\|);$$

ii) *d-monotone* if for some increasing function $\rho : [0, +\infty) \rightarrow \mathbb{R}$ it holds for all $u, v \in E$

$$\langle A(u) - A(v), u - v \rangle \geq (\rho(\|u\|) - \rho(\|v\|))(\|u\| - \|v\|). \quad (2.2)$$

When E is strictly convex, it follows that a d -monotone operator is strictly monotone, i.e. for all $u, v \in E$, $u \neq v$ it holds from (2.2)

$$\langle A(u) - A(v), u - v \rangle > 0.$$

Operator $A : E \rightarrow E^*$ satisfies condition (S), if:

$$u_n \rightharpoonup u_0 \text{ in } E \text{ and } \langle A(u_n) - A(u_0), u_n - u_0 \rangle \rightarrow 0 \text{ imply } u_n \rightarrow u_0 \text{ in } E.$$

While a uniformly monotone operator satisfies condition (S), a d -monotone operator does so in case E is additionally uniformly convex. Strongly continuous (strong - to - weak) perturbations of operators satisfying condition (S) also fulfill this condition. Note that a strongly continuous operator is necessarily compact, that is continuous and sending bounded sets into compact ones.

As far as the continuity is concerned there are a few notions which are equivalent for monotone operators. Operator $A : E \rightarrow E^*$ is called:

(i) *demicontinuous* if $u_n \rightarrow u_0$ in E implies $A(u_n) \rightarrow A(u_0)$ in E^* ;

(ii) *hemicontinuous* if for any $u, v, h \in E$ function

$$s \rightarrow \langle A(u + sv), h \rangle$$

is continuous $[0, 1]$;

(iii) *radially continuous* if for all $u, v \in E$ function

$$s \rightarrow \langle A(u + sv), v \rangle$$

is continuous on $[0, 1]$.

For any operator each former continuity notion implies the latter. We say that $A : E \rightarrow E^*$ is coercive when

$$\lim_{\|v\| \rightarrow +\infty} \frac{\langle A(v), v \rangle}{\|v\|} \rightarrow +\infty$$

or else if there is a function $\gamma : [0, +\infty) \rightarrow \mathbb{R}$ with $\lim_{x \rightarrow +\infty} \gamma(x) = +\infty$ such that

$$\langle A(v), v \rangle \geq \gamma(\|v\|) \|v\|.$$

A d -monotone operator is coercive, when $\rho(x) \rightarrow +\infty$ as $x \rightarrow +\infty$, where function ρ is from (2.2).

Lemma 1. If $A : E \rightarrow E^*$ is a radially continuous monotone operator and $u_0 \in E, g \in E^*$ are fixed, then relation

$$\langle g - A(v), u_0 - v \rangle \geq 0 \text{ for all } v \in E,$$

implies that

$$A(u_0) = g.$$

Moreover, operator A is also hemicontinuous and demicontinuous.

Operator $A : E \rightarrow E^*$ is called potential, if there exists a Gâteaux differentiable functional $f : E \rightarrow \mathbb{R}$, called the potential of A , such that $f' = A$. For a radially continuous potential operator $A : E \rightarrow E^*$ its potential $f : E \rightarrow \mathbb{R}$ satisfies that

$$f(v) = f(0) + \int_0^1 \langle A(sv), v \rangle ds \text{ for } v \in E. \quad (2.3)$$

When $A : E \rightarrow E^*$ is potential and monotone then its potential $f : E \rightarrow \mathbb{R}$ is sequentially weakly lower semicontinuous and operator A is demicontinuous.

We will use the following corollary from the Minty-Browder Theorem:

Theorem 2. Assume that operator $A : E \rightarrow E^*$ is radially continuous, strictly monotone and coercive. Then A is invertible and $A^{-1} : E^* \rightarrow E$ is strictly monotone, bounded and demicontinuous. If additionally A satisfies condition (S), then A^{-1} is continuous.

Remark 3. We see that a continuous and strictly monotone operator which fulfills condition (S) is a homeomorphism between E and E^* . As an example of such a mapping we can invoke a normalized duality mapping in case when both spaces E and E^* are uniformly convex.

We will need also some variational arguments, see [5]. We say that a Gâteaux differentiable functional J satisfies the Palais-Smale condition, denoted (PS) condition, if any sequence $(u_n) \subset E$ such that

i) $|J(u_n)| \leq M$ for all $n \in \mathbb{N}$ and some fixed $M > 0$,

ii) $\lim_{n \rightarrow +\infty} J'(u_n) = 0$ in E^*

admits a norm convergent subsequence.

Theorem 4 (Ekeland Variational Principle -differentiable form). *Let functional $J : E \rightarrow \mathbb{R}$ be Gâteaux differentiable, lower semicontinuous and bounded from below. Then there exists a minimizing sequence (u_n) consisting of almost critical points, i.e. such that*

$$J(u_n) \rightarrow \inf_{u \in E} J(u) \text{ and } J'(u_n) \rightarrow 0 \text{ (in } E^* \text{)}.$$

Theorem 5. *Let $J : E \rightarrow \mathbb{R}$ be a Gâteaux differentiable and a lower semicontinuous functional which satisfies the (PS) condition. Suppose in addition that J is bounded from below. Then J is coercive and the infimum of J is achieved at some $u_0 \in E$ which is a critical point of J .*

3 Auxiliary results for uniformly monotone operators

In [6] it is proved that a uniformly monotone operator is coercive, however a definition of a uniformly monotone operator is a bit different, i.e. it is said that if there exists an increasing function $\rho : [0, +\infty) \rightarrow [0, +\infty)$ such that $\rho(0) = 0$ and for all $u, v \in E$

$$\langle A(u) - A(v), u - v \rangle \geq \rho(\|u - v\|).$$

With our definition similar result holds.

Lemma 6. *A uniformly monotone operator $A : E \rightarrow E^*$ is coercive.*

Proof. We follow [6]. Indeed, take $u \in E$, $u \neq 0$, and define $v = \frac{u}{\|u\|}$, $n = \lceil \|u\| \rceil$. Note that when $\|u\| > n$

$$\langle A(\|u\|v) - A(nv), v \rangle = \frac{1}{\|u\| - n} \langle A(\|u\|v) - A(nv), \|u\|v - nv \rangle > 0,$$

while $\langle A(\|u\|v) - A(nv), v \rangle = 0$ when $\|u\| = n$. Then we have

$$\begin{aligned} \langle A(u), u \rangle &= \|u\| \langle A(u), v \rangle \\ &= \|u\| (\langle A(\|u\|v) - A(nv), v \rangle + \langle A(nv) - A(0), v \rangle + \langle A(0), v \rangle) \\ &\geq \|u\| (\langle A(nv) - A(0), v \rangle - \|A(0)\|_*) \\ &= \|u\| \left(\sum_{i=1}^n \langle A(iv) - A((i-1)v), v \rangle - \|A(0)\|_* \right) \\ &\geq \|u\| (n\rho(1) - \|A(0)\|_*) \geq \|u\| ((\|u\| - 1)\rho(1) - \|A(0)\|_*). \end{aligned}$$

If we put $\gamma(x) = (x - 1)\rho(1) - \|A(0)\|_*$, we see that operator A is coercive. □

There is a sufficient conditions for an operator to be uniformly monotone:

Lemma 7. *Assume that $A : E \rightarrow E^*$ has a Gâteaux derivative at every point and that for any $z, w \in E$ function*

$$t \rightarrow \langle A'(z + tw)w, w \rangle$$

is continuous on $[0, 1]$. Let there exists an increasing function $\rho : [0, +\infty) \rightarrow [0, +\infty)$ such that for all $u, v \in E$ it holds

$$\langle A'(v)u, u \rangle \geq \|u\| \rho(\|u\|).$$

Then A is uniformly monotone.

Proof. Observe that for any $u, v \in E$ it holds

$$\begin{aligned} \langle A(u) - A(v), u - v \rangle &= \int_0^1 \langle A'(v + t(u - v))(u - v), (u - v) \rangle dt \\ &\geq \int_0^1 \|u - v\| \rho(\|u - v\|) dt. \end{aligned}$$

□

Remark 8. The assumptions of the above lemma may seem awkward, but we recall that for mappings between Banach spaces the Gâteaux differentiability might be hard to be achieved, see for example Theorem 2.7 in [5].

Unfortunately it seems that one cannot find sufficient condition for A to be d -monotone in terms of the derivative of A .

Remark 9. We note that the potential of a monotone operator which equals 0 at 0 is bounded from below. From the example of $A(x) = \exp(x)$ we see that the potential of a monotone mapping need not be (weakly) coercive.

Problems mentioned above do not appear when A is coercive. In fact we have the following known result for which we provide a simple proof.

Lemma 10. Assume that $A : E \rightarrow E^*$ is radially continuous, potential and coercive. Then its potential $f : E \rightarrow \mathbb{R}$ defined by (2.3) is weakly coercive.

Proof. For any $t \in (0, 1)$ and any $u \in E$ we have by the coercivity of A :

$$\langle A(tu), u \rangle \geq \|u\| \rho(t\|u\|) \quad (3.4)$$

for some weakly coercive function $\rho : [0, +\infty) \rightarrow \mathbb{R}$. Therefore by (2.3) it follows from (3.4) by integration that

$$f(u) - f(0) \geq \int_0^1 \|u\| \rho(t\|u\|) dt = \int_0^{\|u\|} \rho(s) ds \text{ for any } u \in E.$$

Therefore we see that $\int_0^{\|u\|} \rho(s) ds \rightarrow +\infty$ as $\|u\| \rightarrow +\infty$. □

Corollary 11. Assume that $A : E \rightarrow E^*$ is potential and d -monotone with respect to some coercive function or else that A is uniformly monotone. Then its potential f given by (2.3) is coercive.

4 Main results

4.1 The direct method and the convergence of the minimizing sequence

We consider the existence of a solution to the following problem

$$A(u) = h \quad (4.5)$$

where A is a potential mapping and where element $h \in E^*$ is assumed to be fixed in what follows. Functional $J : E \rightarrow \mathbb{R}$ is defined by

$$J(u) = \int_0^1 \langle A(su), u \rangle ds - \langle h, u \rangle. \quad (4.6)$$

The followings two lemmas are known.

Lemma 12. Assume that $A : E \rightarrow E^*$ is a radially continuous and potential operator. Then a minimizer $u_0 \in E$ of the action functional (4.6) is a solution to (4.5).

Lemma 13. Assume that operator $A : E \rightarrow E^*$ is monotone, coercive and potential. Then there is a solution u_0 to (4.5) which minimizes functional (4.6), i.e. there is a sequence $(u_n) \subset E$ such that $u_n \rightharpoonup u_0$ and for which

$$\liminf_{n \rightarrow +\infty} J(u_n) = J(u_0) = \inf_{u \in E} J(u).$$

A solution is unique in case operator A is strictly monotone.

In order to formulate conditions leading to the conclusion that the minimizing sequence is strongly convergent we need some structure condition on operator A in (4.5). This is however common in case of the application of the direct method, see for example [4, 8]. The main result now follows:

Theorem 14. Let operator $A : E \rightarrow E^*$ be monotone, radially continuous, potential and satisfying condition (S). Let operator $T : E \rightarrow E^*$ be potential and strongly continuous. Let also $A + T$ be coercive. Then there is a solution u_0 to

$$A(u) + T(u) = h \quad (4.7)$$

which minimizes functional $J : E \rightarrow \mathbb{R}$ defined by

$$J(u) = \int_0^1 \langle A(su), u \rangle ds + \int_0^1 \langle T(su), u \rangle ds - \langle h, u \rangle. \quad (4.8)$$

Moreover there is a sequence $(u_n) \subset E$, $u_n \rightarrow u_0$ (strongly) in E and

$$J(u_n) \rightarrow \inf_{u \in E} J(u) \text{ and } J'(u_n) \rightarrow 0 \text{ in } E^*. \quad (4.9)$$

Proof. We see that functional

$$v \mapsto \int_0^1 \langle T(sv), v \rangle ds$$

is weakly continuous. Indeed, since T is strongly continuous, it follows that for any weakly convergent sequence $v_n \rightharpoonup v_0$ it holds $T(v_n) \rightarrow T(v_0)$ which implies that:

$$\int_0^1 \langle T(sv_n), v_n \rangle ds \rightarrow \int_0^1 \langle T(sv_0), v_0 \rangle ds. \quad (4.10)$$

Since functional

$$v \mapsto \int_0^1 \langle A(sv), v \rangle ds$$

is sequentially weakly l.s.c., it follows that J is sequentially weakly l.s.c. Now, by Lemma 10 it follows that functional J is coercive. Since J is also Gâteaux differentiable, it has an argument of a minimum u_0 which is a critical point, i.e. a solution to (4.7).

Since obviously functional J is bounded from below, it follows by Theorem 4 that there is a minimizing sequence (u_n) such that conditions (4.9) are satisfied. By the coercivity of J we can assume, taking a subsequence if necessary, that $u_n \rightharpoonup u_0$. Moreover, by Theorem 4, we see that the minimizing sequence can be chosen so that it consists of almost critical points.

Now we prove that $u_n \rightarrow u_0$. Recall that

$$J'(v) = A(v) + T(v) - h.$$

Since T is strongly continuous, it follows from condition $J'(u_n) \rightarrow 0$ that $A(u_n) \rightarrow g$, where $g \in E^*$ is some fixed element. We will show that $g = A(u_0)$. From the monotonicity of A we have that

$$\langle A(u_n) - A(v), u_n - v \rangle \geq 0 \text{ for any } v \in E.$$

Passing to a limit we see that

$$\langle g - A(v), u_0 - v \rangle \geq 0 \text{ for any } v \in E$$

which by Lemma 1 means that $g = A(u_0)$. Thus $A(u_n) \rightarrow A(u_0)$. Therefore we see that

$$\langle A(u_n) - A(u_0), u_n - u_0 \rangle \rightarrow 0$$

Since operator A satisfies condition (S), we obtain that $\|u_n - u_0\| \rightarrow 0$ as $n \rightarrow +\infty$. $\square$

The above result has a direct consequence that J satisfies the (PS) condition.

Corollary 15. *Under assumptions of Theorem 14, functional J defined by (4.8) satisfies the (PS) condition at level $\inf_E J$.*

Proof. We see that the (PS) sequence (u_n) is a bounded and consists of almost critical points. Moreover, it is such that $J'(u_n) \rightarrow 0$ (strongly). Then we proceed as in the proof of the above result. $\square$

Since a d -monotone operator in a uniformly convex space satisfies condition (S) and since the same property is shared by a uniformly monotone operator in any Banach reflexive space, we have the following sufficient condition which in certain situations helps us to verify the assumptions of the above result.

Corollary 16. *Assume either that space E is additionally uniformly convex and operator $A : E \rightarrow E^*$ is d -monotone, continuous and potential or else that operator A is uniformly monotone. Let operator $T : E \rightarrow E^*$ be potential and strongly continuous. Assume further that operator $A + T$ is coercive. Then there is a solution u_0 to (4.7) which minimizes functional (4.8), i.e. there is a sequence $(u_n) \subset E$, $u_n \rightarrow u_0$ and (4.9) are satisfied. Moreover, J satisfies the (PS) condition.*

It remains to comment on the link between Theorem 5 and the direct method of the calculus of variations.

Remark 17. *We know that a bounded below functional which satisfies the (PS) condition is also coercive. Thus it is of interest if the above mentioned results are equivalent in some context. And from what we have proved above the answer is that both methods are equivalent in the context of Theorem 14.*

4.2 Galerkin type approximations and minimizing sequences

We are interested whether the Galerkin approximations can constitute a strongly convergent minimizing sequence for the action functional. We recall the following well known result:

Lemma 18. *Assume that $A : E \rightarrow E^*$ is monotone and potential. Let moreover $h \in E^*$ be fixed and let u_0 be the solution to (4.5), i.e. $A(u_0) = h$. Let $J : E \rightarrow \mathbb{R}$ be given by (4.6). If $(u_n) \subset E$ is such that $u_n \rightarrow u_0$, then*

$$J(u_n) \rightarrow J(u_0) = \min_{u \in E} J(u),$$

i.e. (u_n) is a minimizing sequence for J .

Note that we do not assume the coercivity of J in the above. However, the monotonicity of A is crucial in the proof. Thus it is interesting if there is a counterpart of the above result without the monotonicity. The assertion is obvious when J is C^1 since in this case A is continuous. However, when A is monotone we can still consider its strongly continuous perturbations under assumption that a minimizer exists. Then any sequence approximating a minimizer is minimizing.

Proposition 19. Assume that operator $A : E \rightarrow E^*$ is monotone, potential and $T : E \rightarrow E^*$ is potential and strongly continuous. Let u_0 be a solution to (4.7) minimizing (4.8), i.e. $A(u_0) + T(u_0) = 0$ and $J(u_0) = \min_{u \in E} J(u)$. If $(u_n) \subset E$ is such that $u_n \rightarrow u_0$, then

$$J(u_n) \rightarrow J(u_0) = \min_{u \in E} J(u).$$

Proof. By the convexity of the potential of A we see that

$$\int_0^1 \langle A(su_n), u_n \rangle ds \leq \int_0^1 \langle A(su_0), v \rangle ds - \langle A(u_n), u_n - u_0 \rangle.$$

Since A is potential and monotone, it is demicontinuous. Since A is demicontinuous and monotone, it is locally bounded. This means that there is a constant $M > 0$ such that for all $n \in \mathbb{N}$ it holds

$$\|A(u_n)\|_* \leq M.$$

Moreover, since T is strongly continuous, we see that (4.10) holds. Then it follows that

$$\begin{aligned} J(u_0) &\leq \int_0^1 \langle T(su_n), u_n \rangle ds + \int_0^1 \langle A(su_n), u_n \rangle ds \leq \\ &\int_0^1 \langle T(su_n), u_n \rangle ds + \int_0^1 \langle A(su_0), v \rangle ds - \langle A(u_n), u_0 - u_n \rangle \leq \\ &\int_0^1 \langle T(su_n), u_n \rangle ds + \int_0^1 \langle A(su_0), v \rangle ds + M \|u_n - u_0\| \end{aligned}$$

The above estimation means that $J(u_n) \rightarrow J(u_0)$ as $n \rightarrow +\infty$. $\square$

We see from Theorem 14 that the sequence of Galerkin type approximation for equation (4.7) constitutes a candidate (finite dimensional) minimizing sequence, possibly up to a suitably chosen subsequence. Namely, we have

Theorem 20. Assume that operator $A : E \rightarrow E^*$ is radially continuous, potential, monotone and satisfies condition (S). Let operator $T : E \rightarrow E^*$ be strongly continuous, potential and such that $A + T$ is coercive. Then there is at least one solution u_0 to (4.7). For any $n \in \mathbb{N}$ there exists at least one n -Galerkin type solution u_n to (4.7) and $u_{n_k} \rightarrow u_0$ for some subsequence (u_{n_k}) of (u_n) . Moreover, (u_{n_k}) is a minimizing sequence for J .

Proof. We need to demonstrate that (u_{n_k}) is a minimizing sequence. This follows directly from Proposition 19. $\square$

5 Applications

5.1 Applications to the mountain geometry

Theorem 14 provides some clue as far as checking the (PS) condition is concerned in case the action functional has some special structure.

Theorem 21. Assume that operator $A : E \rightarrow E^*$ is monotone, continuous, potential and satisfies condition (S). Let operator $T : E \rightarrow E^*$ be potential and strongly continuous. Then any bounded (PS) sequence for functional J given by (4.8) admits a strongly convergent subsequence.

Proof. Let (u_n) be a bounded (PS) sequence for J . We assume (u_n) to be weakly convergent to some u_0 , up to a subsequence, and proceed as in the proof of Theorem 14. $\square$

The applicable version of the above is as follows

Corollary 22. *Assume either that E is additionally a uniformly convex space and operator $A : E \rightarrow E^*$ is d -monotone, continuous and potential or else that operator A is uniformly monotone. Assume that operator $T : E \rightarrow E^*$ is potential and strongly continuous. Then any bounded (PS) sequence for functional J given by (4.8) admits a strongly convergent subsequence.*

The above arguments allow us to make a comment on a Rabier Theorem in which bounded Palais–Smale sequences in problems without parameter are obtained, see [10]. We recall the following result:

Theorem 23. *Let E be a Banach space. Let $\mathcal{A}, \mathcal{B} : E \rightarrow \mathbb{R}$ be C^1 functionals with derivatives A, B respectively. Set $I : E \rightarrow \mathbb{R}$ by the formula*

$$I(u) = \mathcal{A}(u) - \mathcal{B}(u).$$

Assume that

- (i) $\mathcal{A}$ is homogeneous of degree $p > 1$;
- (ii) There exist $v_1, v_2 \in E$ such that

$$\max\{I(v_1), I(v_2)\} < c := \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)), \quad (5.11)$$

where $\Gamma = \{g \in C([0, 1], E) : g(0) = v_1, g(1) = v_2\}$;

- (iii) $\lim_{\|u\| \rightarrow +\infty} \mathcal{A}(u) = \infty$;
- (iv) $B(u) - p\mathcal{B}(u) \geq 0$ for every $u \in E$;
- (v) $\lim_{B(u) \rightarrow +\infty} (B(u) - p\mathcal{B}(u)) = \infty$;

Then functional I possesses a bounded Palais–Smale sequence at level c .

Using our observation we see that

Theorem 24. *Assume that E is a real reflexive separable Banach space. Assume that conditions (i)–(v) are satisfied and that*

- (vi) A is monotone and satisfies condition (S) and B is strongly continuous.

Then functional I possesses a convergent Palais–Smale sequence at level c .

Remark 25. *Observe that in Theorem 23 space E is just assumed to be Banach and it is not assumed anything neither about the monotonicity of a derivative of $\mathcal{A}$, nor about the compactness of a derivative of $\mathcal{B}$. Now we assume additionally that E is reflexive and separable and assumption (vi) is added to the original ones. At this expense we obtain instead of a bounded Palais–Smale sequence the convergent one. This seems to change nothing as far as the range of problems is concerned for which the applications are meant. But at the same time it simplifies the applications since there is no further need to investigate the PS-sequences. In fact, we obtain in this case the existence of a critical point at level c .*

We have the following corollary taking into account that I , number c and Γ are as above. We provide all necessary assumptions since we change some of them suitably.

Corollary 26. *Assume that space E is uniformly convex. Let $\mathcal{A}, \mathcal{B} : E \rightarrow \mathbb{R}$ be C^1 functionals with derivatives A, B respectively. Set $I : E \rightarrow \mathbb{R}$ by the formula $I(u) = \mathcal{A}(u) - \mathcal{B}(u)$ and assume that*

- (i) $\mathcal{A}$ is homogeneous of degree $p > 1$;
- (ii) There exist $v_1, v_2 \in E$ such that (5.11) holds;
- (iii) $B(u) - p\mathcal{B}(u) \geq 0$ for every $u \in E$;
- (iv) $\lim_{B(u) \rightarrow +\infty} (B(u) - p\mathcal{B}(u)) = \infty$;

(v) Operator A is d -monotone with respect to some coercive function and operator B is strongly continuous. Then functional I possesses a convergent Palais–Smale sequence at level c .

Remark 27. There is also another applicable condition called $(S)_2$ which follows from condition (S) , i.e. $A : E \rightarrow E^*$ satisfies condition $(S)_2$ if $u_n \rightharpoonup u_0$ in E and $A(u_n) \rightarrow A(u_0)$ in E^* imply that $u_n \rightarrow u_0$ in E . Again strongly continuous perturbations does not violate this condition. Results mentioned above hold with this condition instead of (S) .

Remark 28. We mention that Rabier version of the Mountain Pass Theorem has been applied for problems with fractional laplacian in [1]. It follows from results contained there that the fractional setting complies with what we have imposed as structure conditions.

5.2 Applications to boundary value problems

It is apparent what types of problems can be considered as examples. Let

$$2 \leq p < N, \quad N \geq 3, \quad p^* = \frac{Np}{N-p}.$$

Assume that $\Omega \subset \mathbb{R}^N$ is a bounded region with locally Lipschitz boundary. Let $q \in (1, p^*)$. We may consider the following Dirichlet problem

$$\begin{aligned} -\operatorname{div} \left(\varphi \left(y, |\nabla u(y)|^{p-1} \right) |\nabla u(y)|^{p-2} \nabla u(y) \right) + f(y, u(y)) &= h(y), \\ u(y)|_{\partial\Omega} &= 0, \end{aligned} \quad (5.12)$$

where

F1 $\varphi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function and there exists a constant $M > 0$ such that

$$|\varphi(y, a)| \leq M$$

for a.e. $y \in \Omega$ and for all $a \in \mathbb{R}_+$; there exists a constant $\gamma > 0$ such that

$$\varphi(y, a)a - \varphi(y, b)b \geq \gamma(a - b) \quad (5.13)$$

for all $a \geq b$ and for a.e. $y \in \Omega$.

F2 $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is an L^1 -Caratheodory function with $f(y, 0) = 0$ for a.e. $y \in \Omega$, $h \in L^q(\Omega)$, $h \neq 0$.

We see that in this case $E = W_0^{1,p}(\Omega)$ is a uniformly convex, separable real Banach space. With assumption (5.13), it follows that the operator $A : W_0^{1,p}(\Omega) \rightarrow \left(W_0^{1,p}(\Omega)\right)^*$ defined by

$$\langle Au, v \rangle = \int_{\Omega} \varphi \left(y, |\nabla u(y)|^{p-1} \right) |\nabla u(y)|^{p-2} \nabla u(y) \nabla v(y) dy$$

is d -monotone with respect to $\rho(x) = \gamma x^{p-1}$. Moreover, A is potential and its potential $\mathcal{A} : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ is defined by

$$\mathcal{A}(u) = \int_{\Omega} \int_0^{|u(y)|} \varphi \left(y, s^{p-1} \right) s^{p-1} ds dy.$$

We define also a potential operator $T : W_0^{1,p}(\Omega) \rightarrow \left(W_0^{1,p}(\Omega)\right)^*$ with potential $\mathcal{T} : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ given by

$$\langle Tu, v \rangle = \int_{\Omega} f(y, u(y))v(y) dy, \quad \mathcal{T}(u) = \int_0^1 \int_{\Omega} f(y, tu(y))u(y) dt$$

for $u, v \in W_0^{1,p}(\Omega)$. A direct calculation shows that

$$\mathcal{T}(u) = \int_{\Omega} \int_0^{u(y)} f(y, s) ds dy = \int_{\Omega} F(y, u(y)) dy,$$

where $F(y, x) := \int_0^x f(y, s) ds$ for a.e. $y \in \Omega$ and all $x \in \mathbb{R}$. Then $J : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$

$$J(u) = \mathcal{A}(u) + \mathcal{T}(u) - \langle h, u \rangle \text{ for } u \in W_0^{1,p}(\Omega) \quad (5.14)$$

is a classical Euler action functional corresponding to (5.12) which is exactly (4.8). We put $A_1 = A + T$. Then problem (5.12) can be considered in the equivalent form

$$A_1(u) = h \quad (5.15)$$

understood in the sense of space E^* .

From the Rellich-Kondrashov Theorem we know that for $q \in (1, p^*)$ the embedding

$$W_0^{1,p}(\Omega) \subset L^q(\Omega)$$

is compact. Let C_1 be the constant from the Poincaré inequality

$$\|u\|_{L^p} \leq C_1 \|\nabla u\|_{L^p}, \quad \|u\|_{W_0^{1,p}} := \|\nabla u\|_{L^p}.$$

To assumptions **F1**, **F2** we add the following:

F3 there exists a constant $a_1 < \gamma (C_1)^{-p}$ such that

$$f(y, x)x \geq -a_1 |x|^{p-1}$$

for all $x \in \mathbb{R}$ and a.e. $y \in \Omega$.

We have the following:

Proposition 29. Assume **F1**, **F2**, **F3**. Then problem (5.12) has a non-trivial weak solution $u_0 \in W_0^{1,p}(\Omega)$ such that $u_n \rightarrow u_0$ for any sequence (u_n) having the property that

$$\lim_{n \rightarrow +\infty} J(u_n) = \inf_{u \in W_0^{1,p}(\Omega)} J(u) = J(u_0) \text{ and } J'(u_0) = 0.$$

Moreover, functional J given by (5.14) satisfies the (PS) condition.

Proof. Observe that for a weakly convergent sequence (u_n)

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f(y, u_n(y))v(y) dy = \int_{\Omega} f(y, u_0(y))v(y) dy \text{ for all } v \in E.$$

Hence T is a strongly continuous mapping. Concerning the coercivity of A_1 we obtain by **F3** for $u \in E$:

$$\langle A_1 u, u \rangle \geq \gamma \int_{\Omega} |\nabla u(y)|^p dy - a_1 \int_{\Omega} |u(y)|^p dy \geq (\gamma - a_1 (C_1)^p) \|u\|_{W_0^{1,p}}^p.$$

Now, it follows that A_1 is coercive. Since (5.12) is equivalent to (5.15) and since operators appearing in (5.15) satisfy the assumptions of Theorem 14 and its Corollary 15, we get the assertion. This finishes the proof. $\square$

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