Research article

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Positive Solutions for Resonant (p,q)-equations with convection

<https://doi.org/10.1515/anona-2020-0108>

Received April 7, 2020; accepted May 31, 2020.

Abstract: We consider a nonlinear parametric Dirichlet problem driven by the (p,q)-Laplacian (double phase problem) with a reaction exhibiting the competing effects of three different terms. A parametric one consisting of the sum of a singular term and of a drift term (convection) and of a nonparametric perturbation which is resonant. Using the frozen variable method and eventually a fixed point argument based on an iterative asymptotic process, we show that the problem has a positive smooth solution.

Keywords: Singular term, resonance, nonlinear regularity, Leray-Schauder alternative principle, minimal solution, iterative asymptotic process

MSC: 35J60, 35J91, 35J92, 35D30, 35D35

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 -boundary $\partial\Omega$. In this paper we study the following parametric singular double phase Dirichlet problem with gradient dependence (convection)

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = \lambda[u(z)^{-\eta} + r(z)|Du(z)|^{\tau-1}] + f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0, \quad \lambda > 0, \quad 1 < q, \tau < p, \quad 0 < \eta < 1. \end{cases} \quad (E_\lambda)$$

For every $r \in (0, \infty)$ by Δ_r we denote the r -Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|Du|^{r-2} Du) \quad \forall u \in W_0^{1,r}(\Omega).$$

In problem (E_λ) we have the sum of two such operators with different exponents. Hence the differential operator of the problem is not homogeneous. In the reaction (right hand side of (E_λ)), we have the combined effects of three terms, each of different nature. There is a parametric contribution which is the sum of a singular term and of a gradient dependent term (a drift term). Both are multiplied with the parameter $\lambda > 0$. So, we have dependence on the gradient of u (convection). The third term is the perturbation $f(z, x)$ which is a Carathéodory function (that is, for all $x \in \mathbb{R}$ $z \rightarrow f(z, x)$ is measurable and for a.a $z \in \Omega$ $x \rightarrow f(z, x)$ is continuous). We assume that $f(z, \cdot)$ exhibits (p-1)-linear growth as $x \rightarrow +\infty$ and it is resonant with respect to a nonprincipal variational eigenvalue of $(\Delta_p, W_0^{1,p}(\Omega))$. We are looking for positive solutions of the problem.

The presence of the drift term makes the problem nonvariational and so ultimately our method of proof will be topological. Our approach uses the so called "frozen variable method". More specifically, we fix (freeze)

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the gradient (drift) term and this way we have a variational problem. However, the presence of the singular term is a source of difficulties, since the energy (Euler) functional of this variational problem is not C^1 and so the results and methods of critical point theory can not be used directly on this function. We need to find a way to bypass the singularity and work with a C^1 -functional. This is done by introducing an auxiliary problem which we solve and then use its solution and suitable truncation techniques to neutralize the singularity. We are able to show that for all small parameter values, the "frozen problem" has a positive solution. Next, we need to find a canonical way to choose a solution of the "frozen problem". To this end, we show that each such problem has a minimal positive solution (a smallest positive solution). We choose this solution and we have the minimal positive solution map. We show that this map has a fixed point using the Leray-Schauder Alternative Principle (see Theorem 4). To show that the minimal solution map is compact (requirement in the Leray-Schauder theorem), we employ an iterative asymptotic process.

Singular problems with convection, were studied recently by Bai-Gasiński-Papageorgiou [1], Liu-Motreanu-Zeng [2] and Papageorgiou-Rădulescu-Repovš [3]. These works deal with equations driven by the p -Laplacian (hence the differential operator of the problem is homogeneous and this property is exploited in their proofs) and their hypotheses require that the perturbation term asymptotically as $x \rightarrow +\infty$ stays below $\hat{\lambda}_1(p)$, the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ and no interaction is allowed (uniform nonresonance). This way the "frozen problem" becomes coercive and the direct method of the calculus of variations can be used to produce a solution. In contrast here we have resonance with respect to a nonprincipal variational eigenvalue and so the frozen problem has an indefinite energy functional. Finally, we mention that resonant singular problems driven by the p -Laplacian only and with no convection, were investigated by Papageorgiou-Rădulescu-Repovš [4] and by Papageorgiou-Vetro-Vetro [5], while resonant (p,q)-equations were studied by Papageorgiou-Zhang [6]. Additional relevant results can be found in the works of Gasiński-Winkert [7], Marano-Winkert [8], Papageorgiou-Rădulescu-Repovš [9–12], Papageorgiou-Vetro-Vetro [13, 14], Ragusa-Tachikawa [15], Tang-Lin-Yu [16], Vetro [17–19].

2 Mathematical Background, Hypotheses

The main spaces in the study of problem (E_λ) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$. By $\|\cdot\|$, we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. From the Poincaré inequality, we have

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\bar{\Omega})$ is ordered with positive (order) cone $C_+ = \{u \in C_0^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}$. This cone has a nonempty interior given by

$$\text{int}C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \quad \frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} < 0\}$$

with $\mathbf{n}(\cdot)$ being the outward unit normal on $\partial\Omega$.

Let $r \in (1, \infty)$. By r' we denote the conjugate exponent (that is, $\frac{1}{r} + \frac{1}{r'} = 1$). Also by r^* , we denote the critical Sobolev exponent for r , that is,

$$r^* = \begin{cases} \frac{Nr}{N-r}, & \text{if } r < N, \\ +\infty, & \text{if } N \leq r. \end{cases}$$

By $A_r : W_0^{1,r}(\Omega) \rightarrow W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^*$, we denote the nonlinear map defined by

$$\langle A_r(u), h \rangle = \int_{\Omega} |Du|^{r-2} (Du, Dh)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,r}(\Omega).$$

The next proposition recalls the main properties of this map (see, for example, Gasiński-Papageorgiou [20], Problem 2.192, p.219)

Proposition 2.1. *The operator $A_r(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, that is*

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,r}(\Omega), \limsup_{n \rightarrow \infty} \langle A_r(u_n), u_n - u \rangle \leq 0 \Rightarrow u_n \rightarrow u \text{ in } W_0^{1,r}(\Omega).$$

For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We have

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

If $u \in W_0^{1,p}(\Omega)$, then

$$[u] = \{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \text{ for a.a. } z \in \Omega\}.$$

If X is a Banach space and $\varphi \in C^1(X, \mathbb{R})$, then we set

$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \text{ (the critical set of } \varphi \text{)}.$$

We say that $\varphi(\cdot)$ satisfies the "C-condition", if the following property holds:

"Every sequence $\{u_n\} \subseteq X$ such that $\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \rightarrow 0$ in X^* as $n \rightarrow \infty$, admits a strongly convergent subsequence."

Next, we recall some basic facts about the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. So, we consider the following nonlinear eigenvalue problem

$$-\Delta_p u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \quad (2.1)$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an "eigenvalue" of $(-\Delta_p, W_0^{1,p}(\Omega))$, if problem (2.1) admits a nontrivial solution $\hat{u}$, known as an "eigenfunction" corresponding to $\hat{\lambda}$. By $\hat{\sigma}(p)$ we denote the set of eigenvalues (the "spectrum") of $(-\Delta_p, W_0^{1,p}(\Omega))$. There is a smallest eigenvalue denoted by $\hat{\lambda}_1(p)$ which has the following properties:

- i) $\hat{\lambda}_1(p) > 0$ and it is isolated (that is, there exists $\varepsilon > 0$ such that $\hat{\sigma}(p) \cap (\hat{\lambda}_1(p), \hat{\lambda}_1(p) + \varepsilon) = \emptyset$).
- ii) $\hat{\lambda}_1(p)$ is simple (that is, if $\hat{u}, \hat{v} \in W_0^{1,p}(\Omega)$ are eigenfunctions for $\hat{\lambda}_1(p)$, then we have $\hat{u} = \theta\hat{v}$ for some $\theta \in \mathbb{R} \setminus \{0\}$).
- iii)

$$\hat{\lambda}_1(p) = \inf \left[\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0 \right]. \quad (2.2)$$

The infimum in (2.2) is realized on the one dimensional eigenspace corresponding to $\hat{\lambda}_1(p)$. By $\hat{u}_1(p)$ we denote the L^p -normalized (that is, $\|\hat{u}_1(p)\|_p = 1$), positive eigenfunction corresponding to $\hat{\lambda}_1(p)$. The nonlinear regularity and the nonlinear maximum principle (see, for example, Gasiński-Papageorgiou [21], pp 737-738), imply that $\hat{u}_1 \in \text{int}C_+$. Since $\hat{\sigma}(p) \subseteq (0, +\infty)$ is closed and $\hat{\lambda}_1(p)$ is isolated, the second eigenvalue is well-defined by

$$\hat{\lambda}_2(p) = \inf \left[\hat{\lambda} \in \hat{\sigma}(p) : \hat{\lambda} > \hat{\lambda}_1(p) \right].$$

Using the Ljusternik-Schnirelmann minimax scheme, we produce a whole strictly increasing sequence $\{\hat{\lambda}_k(p)\}_{k \in \mathbb{N}}$ of eigenvalues such that $\hat{\lambda}_k(p) \rightarrow +\infty$. These eigenvalues are known as "variational eigenvalues" of $(-\Delta_p, W_0^{1,p}(\Omega))$. We do not know if the variational eigenvalues exhaust $\hat{\sigma}(p)$. This is the case if $p = 2$ (linear eigenvalue problem) or if $N = 1$ (ordinary differential equations). By the nonlinear regularity theory, we know that every eigenfunction $\hat{u} \in C_0^1(\overline{\Omega})$. Moreover, if $\hat{\lambda} \in \hat{\sigma}(p) \setminus \{\hat{\lambda}_1(p)\}$, then every eigenfunction corresponding to $\hat{\lambda}$ is nodal (that is, sign changing).

We will also use a weighted version of the eigenvalue problem (2.1). So, let $m \in L^\infty(\Omega)$, $m(z) \geq 0$ for a.a. $z \in \Omega$, $m \neq 0$. We consider the following nonlinear eigenvalue problem

$$\Delta_p u(z) = \hat{\lambda}m(z)|u(z)|^{p-2}u(z) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \quad (2.3)$$

The same results hold for the eigenvalues $\hat{\lambda}(p, m)$ of problem (2.3). In this case the Rayleigh quotient in the variational characterization of $\hat{\lambda}_1(p, m) > 0$, is given by

$$R(u) = \frac{\|Du\|_p^p}{\int_\Omega m(z)|u|^p dz} \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Then using this fact, we can easily prove the following monotonicity property of the map $m \rightarrow \hat{\lambda}_1(p, m)$.

Proposition 2.2. If $m, \hat{m} \in L^\infty(\Omega)$, $0 \leq m(z) \leq \hat{m}(z)$ for a.a. $z \in \Omega$, $m \neq 0$, $m \neq \hat{m}$, then $\hat{\lambda}_1(p, \hat{m}) < \hat{\lambda}_1(p, m)$.

Remark 2.1. Evidently, if $m(z) = 1$ for a.a. $z \in \Omega$, then $\hat{\lambda}_1(p, m) = \hat{\lambda}_1(p)$.

Now, we introduce the hypotheses on the data of (E_λ) .

H_0 : $r \in L^\infty(\Omega)$, $r(z) \geq 0$ for a.a. $z \in \Omega$, $r \neq 0$.

H_1 : $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

- (i) $0 \leq f(z, x) \leq a(z)(1 + x^{p-1})$ for a.a. $z \in \Omega$, all $x \geq 0$, with $a \in L^\infty(\Omega)$;
- (ii) there exists $m \in \mathbb{N}$, $m \geq 2$ such that

$$\lim_{x \rightarrow +\infty} \frac{f(z, x)}{x^{p-1}} = \hat{\lambda}_m(p) \quad \text{uniformly for a.a. } z \in \Omega;$$

- (iii) if $F(z, x) = \int_0^x f(z, s) ds$, then there exists $\theta \in (q, p)$ such that

$$0 < c_0 \leq \liminf_{x \rightarrow +\infty} \frac{pF(z, x) - f(z, x)x}{x^\theta} \quad \text{uniformly for a.a. } z \in \Omega;$$

- (iv) $\lim_{x \rightarrow 0^+} \frac{f(z, x)}{x^{q-1}} = 0$, uniformly for a.a. $z \in \Omega$.

Remark 2.2. Since our goal is to find positive solutions and the above hypotheses concern the positive semi-axis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that

$$f(z, x) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \leq 0. \quad (2.4)$$

Hypothesis $H_1(ii)$ implies that the problem is resonant with respect to a nonprincipal variational eigenvalue $\hat{\lambda}_m(p)$.

As we already mentioned in the Introduction eventually our proof will be topological and to reach that point we will use the frozen variable method.

The topological tool which we will use is the so-called "Leray-Schauder Alternative Principle" (see Papageorgiou-Rădulescu-Repovš [22], Proposition 3.2.22, p.198). So, let X be a Banach space and $\xi : X \rightarrow X$ a map. We say that $\xi(\cdot)$ is "compact", if it is continuous and maps bounded sets to relatively compact sets. The Leray-Schauder Alternative Principle asserts the following:

Theorem 2.3. If X is a Banach space, $\xi : X \rightarrow X$ is compact and

$$D(\xi) = \{u \in X : u = t\xi(u) \text{ for some } t \in (0, 1)\},$$

then the following alternative holds

- (a) $D(\xi)$ is unbounded; or
- (b) $\xi(\cdot)$ has a fixed point.

We said in the Introduction that in order to solve the frozen problem using the critical point theory, we will need to use the solution of an auxiliary double phase Dirichlet problem. This is the following parametric purely singular problem

$$-\Delta_p u(z) - \Delta_q u(z) = \lambda u(z)^{-\eta} \text{ in } \Omega, \quad u|_{\partial\Omega} = 0, \quad u > 0, \quad \lambda > 0. \quad (Q_\lambda)$$

From Proposition 11 of Papageorgiou-Rădulescu-Repovš [4], we have the following result concerning problem (Q_λ) .

Proposition 2.3. For every $\lambda > 0$, problem (Q_λ) has a unique positive solution $\bar{u}_\lambda \in \text{int } C_+$.

Consider the Banach space $C_0(\bar{\Omega}) = \{u \in C(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$. This is an ordered Banach space with positive (order) cone $K_+ = \{u \in C_0(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}$. This cone has a nonempty interior given by

$$\text{int}K_+ = \{u \in K_+ : c_u \hat{d} \leq u \text{ for some } c_u > 0\},$$

with $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \bar{\Omega}$. According to Lemma 14.16, p. 355, of Gilbarg-Trudinger [23], there exists $\delta_0 > 0$ such that $\hat{d} \in C^2(\Omega_{\delta_0})$, where $\Omega_{\delta_0} = \{z \in \bar{\Omega} : d(z, \partial\Omega) < \delta_0\}$. It follows that $\hat{d} \in \text{int}K_+$ and so from Proposition 4.1.22, p.274, of Papageorgiou-Rădulescu-Repovš [22], we can find $0 < c_1 < c_2$ such that

$$c_1 \hat{d} \leq \bar{u}_\lambda \leq c_2 \hat{d} \quad (\text{recall that } \bar{u}_\lambda \in \text{int}K_+) \Rightarrow \bar{u}_\lambda \in \text{int}K_+.$$

Let $s > N$ and consider $\hat{u}_1(p)^{\frac{1}{s}} \in K_+$. Then using Proposition 4.1.22, p. 274, of Papageorgiou-Rădulescu-Repovš [22], we know that we can find $c_3 > 0$ such that

$$\hat{u}_1(p)^{\frac{1}{s}} \leq c_3 \bar{u}_\lambda, \Rightarrow \bar{u}_\lambda^{-\eta} \leq c_4 \hat{u}_1(p)^{-\frac{\eta}{s}} \quad \text{for some } c_4 > 0.$$

From the Lemma in Lazer-McKenna [24], we have $\hat{u}_1(p)^{-\frac{\eta}{s}} \in L^s(\Omega)$, $s > N$. So, we have

$$\bar{u}_\lambda^{-\eta} \in L^s(\Omega), \quad s > N. \quad (2.5)$$

In the next section, this fact will help us bypass the singular term and use variational tools on the "frozen" problem.

3 The "Frozen" Problem

In this section, we develop the "frozen variable method" for problem (E_λ) . So, we fix $v \in C_0^1(\bar{\Omega})$ and consider the Carathéodory function

$$g_v^\lambda(z, x) = \lambda \left[x^{-\eta} + r(z)|Dv(z)|^{\tau-1} \right] + f(z, x).$$

As we already mentioned in the Introduction, due to the singular term $\lambda x^{-\eta}$, the function $g_v^\lambda(z, x)$ leads to an energy function which is not C^1 . For this reason, we use (2.5) and consider the following truncation of $g_v^\lambda(z, \cdot)$

$$\hat{g}_v^\lambda(z, x) = \begin{cases} \lambda [\bar{u}_\lambda(z)^{-\eta} + r(z)|Dv(z)|^{\tau-1}] + f(z, x) & \text{if } x < \bar{u}_\lambda(z), \\ \lambda [x^{-\eta} + r(z)|Dv(z)|^{\tau-1}] + f(z, x) & \text{if } \bar{u}_\lambda(z) \leq x. \end{cases} \quad (3.6)$$

Then we consider the following parametric double phase Dirichlet problem ("the frozen problem")

$$-\Delta_p u(z) - \Delta_q u(z) = \hat{g}_v^\lambda(z, u(z)) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0, \quad u > 0, \lambda > 0. \quad (E_v^\lambda)$$

This problem is variational and its energy (Euler) function is C^1 . So, we can use the results of critical point theory of (E_v^λ) .

Let S_v^λ denote the set of positive solutions of (E_v^λ) . We set $\hat{G}_v^\lambda(z, x) = \int_0^x \hat{g}_v^\lambda(z, s) ds$ and consider the functional $\hat{\psi}_v^\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\hat{\psi}_v^\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{q} \|Du\|_q^q - \int_\Omega \hat{G}_v^\lambda(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

On account of (2.5) (see also Papageorgiou-Smyrlis [25], Proposition 3), we have that

$$\hat{\psi}_v^\lambda \in C^1(W_0^{1,p}(\Omega)).$$

Proposition 3.1. *If hypotheses H_0, H_1 hold and $\lambda > 0$, then the function $\hat{\psi}_v^\lambda(\cdot)$ satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\hat{\psi}_v^\lambda(u_n)| \leq c_5 \quad \text{for some } c_5 > 0, \quad \text{all } n \in \mathbb{N}, \quad (3.7)$$

$$(1 + \|u_n\|)(\hat{\psi}_v^\lambda)'(u_n) \rightarrow 0 \text{ in } W^{-1,p}(\Omega) = W_0^{1,p}(\Omega)^* \text{ as } n \rightarrow \infty. \quad (3.8)$$

From (3.8) we have

$$\begin{aligned} |\langle A_p(u_n), h \rangle + \langle A_q(u_n), h \rangle - \int_{\Omega} \hat{g}_v^\lambda(z, u_n) h dz| &\leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|}, \\ &\text{for all } h \in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n \rightarrow 0^+. \end{aligned} \quad (3.9)$$

In (3.9) we test with $h = -u_n^- \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} \|Du_n^-\|_p^p &\leq c_6 \text{ for some } c_6 > 0, \text{ all } n \in \mathbb{N} \text{ (see (2.4), (2.5) and (3.6))} \\ \Rightarrow \{u_n^-\}_{n \geq 1} &\subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \end{aligned} \quad (3.10)$$

Next, we show that $\{u_n^+\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. Suppose that this is not true. Passing to a subsequence if necessary, we may assume that

$$\|u_n^+\| \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (3.11)$$

We set $y_n = \frac{u_n^+}{\|u_n^+\|}$ for all $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \quad \text{and } y_n \rightarrow y \text{ in } L^p(\Omega) \quad \text{as } n \rightarrow \infty. \quad (3.12)$$

From (3.9) and (3.10), we have

$$\begin{aligned} |\langle A_p(u_n^+), h \rangle + \langle A_q(u_n^+), h \rangle - \int_{\Omega} \hat{g}_v^\lambda(z, u_n^+) h dz| &\leq c_7 \|h\| \\ &\text{for some } c_7 > 0, \text{ all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \\ \Rightarrow \left| \langle A_p(y_n), h \rangle + \frac{1}{\|u_n^+\|^{p-q}} \langle A_q(y_n), h \rangle - \int_{\Omega} \frac{\hat{g}_v^\lambda(z, u_n^+)}{\|u_n^+\|^{p-1}} h dz \right| &\leq \frac{c_7 \|h\|}{\|u_n^+\|^{p-1}}, \\ &\text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \quad (3.13)$$

From (2.5), (3.6) and hypothesis $H_1(i)$, we see that

$$\left\{ \frac{\hat{g}_v^\lambda(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \right\}_{n \in \mathbb{N}} \subseteq L^{p'}(\Omega) \text{ is bounded.} \quad (3.14)$$

In (3.13) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (3.11) (recall $q < p$), (3.12), (3.14), we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle A_p(y_n), y_n - y \rangle &= 0 \\ \Rightarrow y_n \rightarrow y \text{ in } W_0^{1,p}(\Omega) \quad \text{and so } \|y\| = 1, y \geq 0 \text{ (see Proposition 2.1)} \end{aligned} \quad (3.15)$$

From (3.14) and hypothesis $H_1(ii)$ and if we pass to a subsequence if necessary, we have

$$\frac{\hat{g}_v^\lambda(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \hat{\lambda}_m(p) y^{p-1} \text{ in } L^{p'}(\Omega) \text{ as } n \rightarrow \infty. \quad (3.16)$$

(see Aizicovici-Papageorgiou-Staicu [26], proof of Proposition 16)

We return to (3.13), pass to the limit as $n \rightarrow \infty$ and use (3.11) (recall $q < p$), (3.15), and (3.16). We obtain

$$\begin{aligned} \langle A_p(y), h \rangle &= \hat{\lambda}_m(p) \int_{\Omega} y^{p-1} h dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow -\Delta_p y(z) &= \hat{\lambda}_m(p) y(z)^{p-1} \quad \text{for a.a. } z \in \Omega, y|_{\partial\Omega} = 0, \\ \Rightarrow y &\text{ must be nodal (since } m \geq 2), \end{aligned}$$

a contradiction to (3.15).

Therefore, $\{u_n^+\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded and so from (3.10), it follows that

$$\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded.}$$

We may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and } u_n \rightarrow u \quad \text{in } L^p(\Omega) \quad \text{as } n \rightarrow \infty. \quad (3.17)$$

In (3.9) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (3.17), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \langle A_q(u_n), u_n - u \rangle] &= 0, \\ \Rightarrow \limsup_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \langle A_q(u), u_n - u \rangle] &\leq 0 \quad (\text{since } A_q(\cdot) \text{ is monotone}), \\ \Rightarrow \limsup_{n \rightarrow \infty} \langle A_p(u_n), u_n - u \rangle &\leq 0 \quad (\text{see (3.17)}), \\ \Rightarrow u_n \rightarrow u &\quad \text{in } W_0^{1,p}(\Omega) \quad \text{as } n \rightarrow \infty \quad (\text{see Proposition 2.1}). \end{aligned}$$

This proves that the functional $\hat{\psi}_V^\lambda(\cdot)$ satisfies the C-condition. $\square$

The next proposition will help us satisfy the mountain pass geometry for the functional $\hat{\psi}_V^\lambda(\cdot)$.

Proposition 3.2. *If Hypotheses H_0, H_1 hold and $\lambda > 0$, then $\hat{\psi}_V^\lambda(t\hat{u}_1(p)) \rightarrow -\infty$ as $t \rightarrow +\infty$.*

Proof. We have

$$\begin{aligned} \frac{d}{dx} \left(\frac{F(z, x)}{x^p} \right) &= \frac{f(z, x)x^p - px^{p-1}F(z, x)}{x^{2p}} \\ &= \frac{f(z, x)x - pF(z, x)}{x^{p+1}} \quad \text{for a.a. } z \in \Omega, \text{ all } x > 0. \end{aligned} \quad (3.18)$$

By hypothesis $H_1(iii)$, we can find $M > 0$ and $c_8 \in (0, c_0)$ such that

$$f(z, x)x - pF(z, x) \leq -c_8 x^\theta \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M. \quad (3.19)$$

Using (3.19) in (3.18), we have

$$\begin{aligned} \frac{d}{dx} \left(\frac{F(z, x)}{x^p} \right) &\leq -\frac{c_8}{x^{p+1-\theta}} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M, \\ \Rightarrow \frac{F(z, t)}{t^p} - \frac{F(z, x)}{x^p} &\leq \frac{c_8}{p-\theta} \left[\frac{1}{t^{p-\theta}} - \frac{1}{x^{p-\theta}} \right] \quad \text{for a.a. } z \in \Omega, \text{ all } t \geq x \geq M. \end{aligned} \quad (3.20)$$

Hypothesis $H_1(ii)$ implies that

$$\lim_{t \rightarrow +\infty} \frac{pF(z, x)}{t^p} = \hat{\lambda}_m(p) \quad \text{uniformly for a.a. } z \in \Omega. \quad (3.21)$$

In (3.20), we let $t \rightarrow +\infty$ and use (3.21). We obtain

$$\frac{\hat{\lambda}_m(p)}{p} - \frac{F(z, x)}{x^p} \leq -\frac{c_8}{p-\theta} \cdot \frac{1}{x^{p-\theta}}$$

$$\begin{aligned}
&\Rightarrow \frac{\hat{\lambda}_m(p)x^p - pF(z, x)}{x^\theta} \leq -\frac{pc_8}{p-\theta} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M, \\
&\Rightarrow \limsup_{x \rightarrow +\infty} \frac{\hat{\lambda}_m(p)x^p - pF(z, x)}{x^p} \leq -c_9 \text{ uniformly for a.a. } z \in \Omega \\
&\quad \text{with } c_9 = \frac{pc_8}{p-\theta}.
\end{aligned} \tag{3.22}$$

For $t > 0$, we have

$$\hat{\psi}_v^\lambda(t\hat{u}_1(p)) = \frac{t^p}{p}\hat{\lambda}_1(p) + \frac{t^q}{q}\|D\hat{u}_1(p)\|_q^q - \int_{\Omega} \hat{G}_v^\lambda(z, t\hat{u}_1(p))dz.$$

Recall that $\hat{u}_1(p) \in \text{int}C_+$. So, we can find $t \geq 1$ big such that $\bar{u}_\lambda \leq t\hat{u}_1(p)$ (see [22], p.274). Then from (3.6) and since $m \geq 2$, we have

$$\begin{aligned}
\hat{\psi}_v^\lambda(t\hat{u}_1(p)) &\leq \frac{t^p}{p}\hat{\lambda}_m(p) + \frac{t^q}{q}\|D\hat{u}_1(p)\|_q^q - \int_{\Omega} \hat{G}_v^\lambda(z, t\hat{u}_1(p))dz \\
&\leq \frac{1}{p} \int_{\Omega} [\hat{\lambda}_m(p)(t\hat{u}_1(p))^p - pF(z, t\hat{u}_1(p))] + \frac{1}{q}t^q\|D\hat{u}_1(p)\|_q^q, \\
\Rightarrow \frac{\hat{\psi}_v^\lambda(t\hat{u}_1(p))}{t^\theta} &\leq \frac{1}{p} \int_{\Omega} \frac{\hat{\lambda}_m(p)(t\hat{u}_1(p))^p - pF(z, t\hat{u}_1(p))}{t^\theta} dz \\
&\quad + \frac{1}{qt^{\theta-q}}\|D\hat{u}_1(p)\|_q^q \\
&\Rightarrow \limsup_{t \rightarrow +\infty} \frac{\hat{\psi}_v^\lambda(t\hat{u}_1(p))}{t^\theta} \leq -c_{10},
\end{aligned}$$

for some $c_{10} > 0$ (see (3.22), using Fatou's Lemma and recall $q < \theta$),

$$\Rightarrow \hat{\psi}_v^\lambda(t\hat{u}_1(p)) \rightarrow -\infty \quad \text{as } t \rightarrow +\infty.$$

□

Remark 3.1. The above proof reveals that the resonance at $\hat{\lambda}_m(p)$ occurs from the right of the eigenvalue in the sense that

$$pF(z, x) - \hat{\lambda}_m(p)x^p \rightarrow +\infty \quad \text{uniformly for a.a. } z \in \Omega, \text{ as } x \rightarrow +\infty.$$

Proposition 3.3. If hypotheses H_0, H_1 hold and $\lambda > 0$, then $K_{\hat{\psi}_v^\lambda} \subseteq [\bar{u}_\lambda) \cap \text{int}C_+$.

Proof. If $K_{\hat{\psi}_v^\lambda} = \emptyset$, then the result is trivially true. So, suppose that $K_{\hat{\psi}_v^\lambda} \neq \emptyset$ and let $u \in K_{\hat{\psi}_v^\lambda}$. We have

$$\langle A_p(u), h \rangle + \langle A_q(u), h \rangle = \int_{\Omega} \hat{g}_v^\lambda(z, u)h dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.23}$$

In (3.23), we choose $h = (\bar{u}_\lambda - u)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\langle A_p(u), (\bar{u}_\lambda - u)^+ \rangle + \langle A_q(u), (\bar{u}_\lambda - u)^+ \rangle \\
&= \int_{\Omega} [\lambda \bar{u}_\lambda^{-\eta} + \lambda r(z)|Dv|^{\tau-1} + f(z, u)] (\bar{u}_\lambda - u)^+ dz \quad (\text{see (3.6)}) \\
&\geq \int_{\Omega} \lambda \bar{u}_\lambda^{-\eta} (\bar{u}_\lambda - u)^+ dz \quad (\text{see } H_0, H_1(i)) \\
&= \langle A_p(\bar{u}_\lambda), (\bar{u}_\lambda - u)^+ \rangle + \langle A_q(\bar{u}_\lambda), (\bar{u}_\lambda - u)^+ \rangle \quad (\text{see Proposition 2.3}) \\
&\Rightarrow \bar{u}_\lambda \leq u.
\end{aligned} \tag{3.24}$$

From (3.24), (3.6), and (3.23), it follows that

$$-\Delta_p u(z) - \Delta_q u(z) = \lambda \left[u(z)^{-\eta} + r(z)|Dv(z)|^{\tau-1} \right] + f(z, u(z)) \quad \text{for a.a. } z \in \Omega. \quad (3.25)$$

Theorem 7.1, p.286, of Ladyzhenskaya-Ural'tseva [29], implies that $u \in L^\infty(\Omega)$. Let

$$K_\lambda(z) = \lambda \left[u(z)^{-\eta} + r(z)|Dv(z)|^{\tau-1} \right] + f(z, u(z)).$$

On account of (3.24), (2.5) and hypotheses $H_0, H_1(i)$, we have that

$$K_\lambda \in L^s(\Omega), \quad s > N.$$

We consider the following linear Dirichlet problem

$$-\Delta w(z) = K_\lambda(z) \quad \text{in } \Omega, \quad w|_{\partial\Omega} = 0. \quad (3.26)$$

Theorem 9.15, p.241, of Gilbarg-Trudinger [23] implies that problem (3.26) admits a unique solution $w \in W_0^{2,s}(\Omega)$. The Sobolev embedding theorem says that

$$\begin{aligned} W^{2,s}(\Omega) &\hookrightarrow C^{1,\alpha}(\bar{\Omega}) \quad \text{compactly with } \alpha = 1 - \frac{N}{s} \in (0, 1), \\ \Rightarrow w &\in C^{1,\alpha}(\bar{\Omega}) \cap C_0^1(\bar{\Omega}). \end{aligned}$$

Let $\sigma(z) = Dw(z)$. Then $\sigma \in C^{0,\alpha}(\bar{\Omega}, \mathbb{R}^N)$. Using this function, we can rewrite (3.25) as

$$-\operatorname{div} \left[|Du|^{p-2} Du + |Du|^{q-2} Du - \sigma \right] = 0 \quad \text{in } \Omega.$$

Then the nonlinear regularity theory of Lieberman [27] and (3.24) imply that $u \in \operatorname{int} C_+$. So, we conclude that

$$K_{\hat{\psi}_v^\lambda} \subseteq [\bar{u}_\lambda] \cap \operatorname{int} C_+.$$

□

Next, we are ready to show the nonemptiness of S_v^λ when $\lambda > 0$ is small.

Proposition 3.4. *If hypotheses H_0, H_1 hold and $v \in C_0^1(\bar{\Omega})$, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, we have*

$$\emptyset \neq S_v^\lambda \subseteq [\bar{u}_\lambda] \cap \operatorname{int} C_+.$$

Proof. Let $u \in W_0^{1,p}(\Omega)$ with $\|u\| \leq 1$. We have

$$\begin{aligned} \hat{\psi}_v^\lambda(u) &= \frac{1}{p} \|Du\|_p^p + \frac{1}{q} \|Du\|_q^q - \int_{\{u < \bar{u}_\lambda\}} \hat{G}_v^\lambda(z, u) dz - \int_{\{u \geq \bar{u}_\lambda\}} \hat{G}_v^\lambda(z, u) dz \\ &\quad - \lambda \int_{\{u \geq \bar{u}_\lambda\}} \bar{u}_\lambda^{1-\eta} dz - \int_{\{u \geq \bar{u}_\lambda\}} F(z, u(z)) dz \\ &\geq \frac{1}{p} \|Du\|_p^p + \frac{1}{q} \|Du\|_q^q - \frac{\lambda(2-\eta)}{1-\eta} \int_\Omega |u|^{1-\eta} dz - \lambda \int_\Omega r(z)|Dv(z)|^{\tau-1} u dz \\ &\quad - \int_\Omega F(z, u) dz \quad (\text{see hypotheses } H_1(i), (iv)). \end{aligned} \quad (3.27)$$

On account of hypotheses $H_1(i), (iv)$, we see that given $\varepsilon > 0$, we can find $c_{11} > 0$ such that

$$f(z, x) x \leq \varepsilon |x|^q + c_{11} |x|^p \quad \text{for a.a. } z \in \Omega, \quad \text{all } x \in \mathbb{R} \quad (\text{see (2.4)}). \quad (3.28)$$

Returning to (3.27) and using (3.28) with $\varepsilon > 0$ small, we obtain from the Poincaré inequality and $\|u\| \leq 1$

$$\hat{\psi}_v^\lambda(u) \geq \frac{1}{2q} \|Du\|_q^q - c_{12} \|Du\|_p^p - \lambda c_{13} \|Du\|_p^{1-\eta} \text{ for some } c_{12}, c_{13} > 0. \quad (3.29)$$

Let $m = \inf \left\{ \|Dv\|_q^q : v \in W_0^{1,p}(\Omega), \|v\| = \|Dv\|_p = 1 \right\}$. Obviously, $m > 0$. We choose $\rho \in (0, 1)$ such that for all $v \in W_0^{1,p}(\Omega)$ with $\|v\| = 1$ and $u = \rho v$

$$\begin{aligned} \frac{1}{2q} \|Du\|_q^q - c_{12} \|Du\|_p^p &= \frac{1}{2q} \rho^q \|Dv\|_q^q - c_{12} \rho^p \\ &\geq \rho^q \left\{ \frac{m}{2q} - c_{12} \rho^{p-q} \right\} \\ &\geq \hat{\eta}_0 > 0 \text{ (recall } q < p) \end{aligned}$$

and then choose $\lambda^* > 0$ small so that $\lambda c_{13} \rho^{1-\eta} < \hat{\eta}_0$ for all $\lambda \in (0, \lambda^*)$. From (3.29), it follows that

$$\hat{\psi}_v^\lambda(0) = 0 < \inf \left[\hat{\psi}_v^\lambda(u) : \|u\| = \rho \right] \text{ for all } \lambda \in (0, \lambda^*). \quad (3.30)$$

Then (3.30) together with Propositions 3.1 and 3.2, permit the use of the mountain pass theorem. So, we can find $\hat{u}_v^\lambda \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} \hat{u}_v^\lambda &\in K_{\hat{\psi}_v^\lambda} \subseteq [\bar{u}_\lambda) \cap \text{int} C_+ \text{ (see Proposition 3.3),} \\ \Rightarrow \hat{u}_v^\lambda &\in S_v^\lambda \neq \emptyset \text{ (see (3.6)) and } S_v^\lambda \subseteq [\bar{u}_\lambda) \cap \text{int} C_+. \end{aligned}$$

□

The next proposition suggests a canonical way to choose a solution from S_v^λ as $v \in C_0^1(\bar{\Omega})$ moves.

Proposition 3.5. *If hypotheses H_0, H_1 hold, $v \in C_0^1(\bar{\Omega})$ and $\lambda \in (0, \lambda^*)$, then S_v^λ admits a smallest element $\tilde{u}_v^\lambda \in \text{int} C_+$, that is*

$$\tilde{u}_v^\lambda \leq u \text{ for all } u \in S_v^\lambda.$$

Proof. From Proposition 19 of Papageorgiou-Rădulescu-Repovš [4], we know that the solution set S_v^λ is downward directed (that is given $u_1, u_2 \in S_v^\lambda$ we can find $u \in S_v^\lambda$ such that $u \leq u_1, u \leq u_2$). Then Lemma 3.10, p.178 of Hu-Papageorgiou [28] implies that we can find a decreasing sequence $\{u_n\}_{n \geq 1} \subseteq S_v^\lambda$ such that $\inf_{n \geq 1} u_n = \inf S_v^\lambda$. We have

$$\langle A_p(u_n), h \rangle + \langle A_q(u_n), h \rangle = \int_{\Omega} \hat{g}_v^\lambda(z, u_n) h dz \text{ for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \quad (3.31)$$

$$\bar{u}_\lambda \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N} \text{ (see Proposition 3.3).} \quad (3.32)$$

If we test (3.31) with $h = u_n \in W_0^{1,p}(\Omega)$ and use (3.32) and hypothesis $H_1(i)$, we infer that $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} \tilde{u}_v^\lambda \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \rightarrow \tilde{u}_v^\lambda \text{ in } L^p(\Omega) \text{ as } n \rightarrow \infty. \quad (3.33)$$

In (3.31), we choose $h = u_n - \tilde{u}_v^\lambda \in W_0^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (3.33). Reasoning as in the last part of the proof of Proposition 3.1, we obtain

$$u_n \rightarrow \tilde{u}_v^\lambda \text{ in } W_0^{1,p}(\Omega). \quad (3.34)$$

Pass to the limit as $n \rightarrow \infty$ in (3.31) and using (3.34), we obtain

$$\langle A_p(\tilde{u}_v^\lambda), h \rangle + \langle A_q(\tilde{u}_v^\lambda), h \rangle = \int_{\Omega} \hat{g}_v^\lambda(z, \tilde{u}_v^\lambda) h dz \text{ for all } h \in W_0^{1,p}(\Omega), \quad (3.35)$$

$$\bar{u}_\lambda \leq \tilde{u}_v^\lambda \quad (\text{see (3.32)}). \quad (3.36)$$

From (3.35) and (3.36), it follows that

$$\tilde{u}_v^\lambda \in S_v^\lambda \quad \text{and} \quad \tilde{u}_v^\lambda = \inf S_v^\lambda.$$

□

We can define the minimal solution map $\xi_\lambda : C_0^1(\bar{\Omega}) \rightarrow C_0^1(\bar{\Omega})$ by

$$\xi_\lambda(u) = \tilde{u}_v^\lambda \in \text{int} C_+ \quad (\lambda \in (0, \lambda^*)).$$

Clearly a fixed point of this map will be a positive smooth solution of (E_λ) $\lambda \in (0, \lambda^*)$. To produce a fixed point of $\xi_\lambda(\cdot)$, we will use Theorem 2.3 (Leray-Schauder Alternative Principle). This is done in the next section.

4 Positive Solution

According to Theorem 2.3, to produce a fixed point for the minimal solution map, we need to show that $\xi_\lambda(\cdot)$ is compact and that $D(\xi_\lambda)$ is bounded ($\lambda \in (0, \lambda^*)$).

First, we show that $\xi_\lambda(\cdot)$ is compact. To this end the following proposition will be helpful.

Proposition 4.1. *If hypotheses H_0, H_1 hold, $v_n \rightarrow v$ in $C_0^1(\bar{\Omega})$, $\lambda \in (0, \lambda^*)$ and $u \in S_v^\lambda$, then for $m \geq 2$ big we can find $u_n \in S_{v_n}^\lambda$, $n \in \mathbb{N}$ such that $u_n \rightarrow u$ in $C_0^1(\bar{\Omega})$.*

Proof. From Proposition 3.3, we know that $u \in [\bar{u}_\lambda] \cap \text{int} C_+$. We consider the following Dirichlet problem

$$-\Delta_p w(z) - \Delta_q w(z) = \hat{g}_{v_n}^\lambda(z, u(z)) \quad \text{in } \Omega, \quad w|_{\partial\Omega} = 0, \quad u > 0, \quad \lambda > 0.$$

We have $\hat{g}_{v_n}^\lambda(\cdot, u(\cdot)) \neq 0$, $\hat{g}_{v_n}^\lambda(\cdot, u(\cdot)) \geq 0$ (see hypothesis $H_1(i)$ and recall $\bar{u}_\lambda \in \text{int} C_+$, $\bar{u}_\lambda^{-\eta} \in L^s(\Omega)$). In fact we have

$$\hat{g}_{v_n}^\lambda(\cdot, u(\cdot)) \in L^s(\Omega) \quad \text{with } s > N.$$

Note that $s' < N' = \frac{N}{N-1} < p^*$. So, it follows that

$$\begin{aligned} L^s(\Omega) &\hookrightarrow W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))^* \quad \text{compactly} \\ &(\text{see Gasiński-Papageorgiou [21], Lemma 2.2.27, p.141}), \\ &\Rightarrow \hat{g}_{v_n}^\lambda(\cdot, u(\cdot)) \in W^{-1,p'}(\Omega) \setminus \{0\} \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

We consider the nonlinear map $V : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ defined by

$$V(u) = A_p(u) + A_q(u) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

According to Proposition 2.1, this map is continuous, strictly monotone (hence maximal monotone too) and coercive. It follows that $V(\cdot)$ is surjective (see Papageorgiou-Rădulescu-Repovš [22], Corollary 2.8.7, p.135). So, we can find a unique $w_n \in W_0^{1,p}(\Omega)$ (uniqueness is a consequence of the strict monotonicity of $V(\cdot)$), $w_n \geq 0$, $w_n \neq 0$ such that

$$V(w_n) = \hat{g}_{v_n}^\lambda(\cdot, u(\cdot)) \quad \text{for all } n \in \mathbb{N}. \quad (4.37)$$

Clearly, $\{\hat{g}_{v_n}^\lambda(\cdot, u(\cdot))\}_{n \in \mathbb{N}} \subseteq L^s(\Omega)$ is bounded. So, by Theorem 9.16, p.241 and Lemma 9.17, p.242, of Gilbarg-Trudinger [23], there exists $c_{15} > 0$ such that

$$w_n \in W^{2,s}(\Omega) \quad \text{and} \quad \|w_n\|_{W^{2,s}(\Omega)} \leq c_{15} \quad \text{for all } n \in \mathbb{N}.$$

By the Sobolev embedding theorem, we have

$$W^{2,s}(\Omega) \hookrightarrow C^{1,\alpha}(\bar{\Omega}) \quad \left(\alpha = 1 - \frac{N}{s} \in (0, 1) \right) \quad \text{compactly.}$$

So, we see that at least for a subsequence, we have

$$w_n \rightarrow \tilde{u} \quad \text{in } C^{1,\alpha}(\bar{\Omega}) \quad \text{as } n \rightarrow \infty. \quad (4.38)$$

Passing to the limit as $n \rightarrow \infty$ in (4.37) and using (4.38), we obtain

$$\begin{aligned} V(\tilde{u}) &= \hat{g}_v^\lambda(\cdot, u(\cdot)), \\ \Rightarrow -\Delta_p \tilde{u} - \Delta_q \tilde{u} &= \hat{g}_v^\lambda(z, \tilde{u}) \text{ in } \Omega, \tilde{u}|_{\partial\Omega} = 0, \\ \Rightarrow \tilde{u} &= u \\ &\text{(since the problem has a unique solution and } u \in S_V^\lambda). \end{aligned}$$

Therefore for the original sequence, we have

$$w_n \rightarrow u \quad \text{in } C_0^1(\bar{\Omega}).$$

Next, we consider the following Dirichlet problem

$$-\Delta_p u - \Delta_q u = \hat{g}_{v_n}^\lambda(z, w_n) \text{ in } \Omega, \tilde{u}|_{\partial\Omega} = 0.$$

Reasoning as above, we see that this problem has a unique solution $w_n^1 \in W^{2,s}(\Omega)$, $w_n^1 \geq 0$, $w_n^1 \neq 0$ and

$$w_n^1 \rightarrow u \quad \text{in } C_0^1(\bar{\Omega}) \text{ as } n \rightarrow \infty.$$

We continue this way and generate a sequence $\{w_n^k\}_{n,k \in \mathbb{N}}$ such that

$$A_p(w_n^k) + A_q(w_n^k) = \hat{g}_{v_n}^\lambda(\cdot, w_n^{k-1}(\cdot)) \quad \text{for all } n, k \in \mathbb{N} \text{ } (w_n^0 = w_n), w_n^k \geq 0, \quad (4.39)$$

$$w_n^k \rightarrow u \quad \text{in } C_0^1(\bar{\Omega}) \text{ as } n \rightarrow \infty \text{ for all } k \in \mathbb{N}. \quad (4.40)$$

For each $n \in \mathbb{N}$, we consider the sequence $\{w_n^k\}_{k \in \mathbb{N}}$. We claim that this sequence is bounded in $W_0^{1,p}(\Omega)$. Arguing by contradiction, we may assume that

$$\|w_n^k\| \rightarrow \infty \quad \text{as } k \rightarrow \infty \quad \text{and } \{\|w_n^k\|\}_{k \in \mathbb{N}} \text{ is increasing.} \quad (4.41)$$

We set $y_k = \frac{w_n^k}{\|w_n^k\|}$, $k \in \mathbb{N}$. Then $\|y_k\| = 1$, $y_k \geq 0$ for all $k \in \mathbb{N}$. So, we may assume that

$$y_k \xrightarrow{w} y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and } y_k \rightarrow y \quad \text{in } L^p(\Omega) \quad \text{as } k \rightarrow \infty, y \geq 0. \quad (4.42)$$

From (4.39) we have

$$A_p(y_k) + \frac{1}{\|w_n^k\|^{p-q}} A_q(y_k) = \frac{\hat{g}_{v_n}^\lambda(\cdot, w_n^{k-1}(\cdot))}{\|w_n^k\|^{p-1}} \quad \text{for all } k \in \mathbb{N}. \quad (4.43)$$

On account of hypothesis $H_1(i)$ and of (4.38), (4.39), we see that

$$\left\{ \frac{\hat{g}_{v_n}^\lambda(\cdot, w_n^{k-1}(\cdot))}{\|w_n^k\|^{p-1}} \right\}_{k \in \mathbb{N}} \subseteq L^{p'}(\Omega) \quad \text{is bounded.} \quad (4.44)$$

we act on (4.43) with $y_k - y \in W_0^{1,p}(\Omega)$ and then pass to the limit as $k \rightarrow \infty$ and use (4.41), (4.42), and (4.44). We obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \langle A_p(y_k), y_k - y \rangle &= 0 \quad (\text{recall } q < p) \Rightarrow y_k \rightarrow y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{as } k \rightarrow \infty \\ &\text{and } \|y\| = 1, y \geq 0 \quad (\text{see Proposition 2.1}). \end{aligned} \quad (4.45)$$

From (4.44) and hypothesis $H_1(ii)$, at least for a subsequence, we have

$$\frac{\hat{g}_{v_n}^\lambda(\cdot, w_n^{k-1}(\cdot))}{\|w_n^k\|^{p-1}} \xrightarrow{w} \hat{\lambda}_m(p)(\eta y)^{p-1} \text{ in } L^{p'}(\Omega), \eta \in (0, 1) \quad \text{as } k \rightarrow \infty \quad (4.46)$$

(see Aizicovici-Papageorgiou-Staicu [26], proof of Proposition 16).

In (4.43), we pass to the limit as $k \rightarrow \infty$ and use (4.45), (4.41) (recall $q < p$) and (4.46), we obtain

$$\begin{aligned} A_p(y) &= \hat{\lambda}_m(p)(\eta y)^{p-1}, \\ \Rightarrow -\Delta_p y(z) &= \hat{\lambda}_m(p)(\eta y(z))^{p-1} \quad \text{in } \Omega, \quad y|_{\partial\Omega} = 0. \end{aligned}$$

If $m \geq 2$ is big so that $\eta > (\frac{\hat{\lambda}_1(p)}{\hat{\lambda}_m(p)})^{\frac{1}{p-1}}$, then it follows that $y = 0$ or y is nodal, both possibilities leading to a contradiction (see (4.45)). This proves that for every $n \in \mathbb{N}$, $\{w_n^k\}_{k \in \mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. Then Theorem 7.1, p.286, of Ladyzhenskaya-Uraltseva [29] implies that $\{w_n^k\}_{k \in \mathbb{N}} \subseteq L^\infty(\Omega)$ is bounded. The nonlinear regularity theory of Lieberman [27] implies that for each $n \in \mathbb{N}$, we can find $\alpha \in (0, 1)$ and $c_{16} > 0$ such that

$$w_n^k \in C_0^{1,\alpha}(\bar{\Omega}), \|w_n^k\|_{C_0^{1,\alpha}(\bar{\Omega})} \leq c_{16} \quad \text{for all } k \in \mathbb{N}.$$

As before, the compact embedding of $C_0^{1,\alpha}(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$, implies that at least for a subsequence, we have

$$w_n^k \rightarrow u_n \quad \text{in } C_0^1(\bar{\Omega}) \quad \text{as } k \rightarrow \infty. \quad (4.47)$$

Passing to the limit as $k \rightarrow \infty$ in (4.39), we obtain

$$\begin{aligned} A_p(u_n) + A_q(u_n) &= \hat{g}_{v_n}^\lambda(\cdot, u_n(\cdot)) \quad k \in \mathbb{N}, \\ \Rightarrow -\Delta_p u_n(z) - \Delta_q u_n(z) &= \hat{g}_{v_n}^\lambda(z, u_n(z)) \quad \text{in } \Omega, \quad u_n|_{\partial\Omega} = 0. \end{aligned} \quad (4.48)$$

Moreover, as in the proof of Proposition 3.3, using (3.6) and the fact that $f \geq 0$ (see hypothesis $H_1(i)$), we have

$$\begin{aligned} \bar{u}_\lambda &\leq w_n^k \quad \text{for all } n, k \in \mathbb{N}, \\ \Rightarrow \bar{u}_\lambda &\leq u_n \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow u_n &\in S_{v_n}^\lambda \subseteq \text{int } C_+ \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

As above, using (4.48) and a contradiction argument, we show that $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded, hence relatively compact in $C_0^1(\bar{\Omega})$ (nonlinear regularity). Then the double limit lemma (see Gasiński-Papageorgiou [30], Problem 1.175, p.61), implies that

$$u_n \rightarrow u \quad \text{in } C_0^1(\bar{\Omega}) \quad \text{as } n \rightarrow \infty, \quad u_n \in S_{v_n}^\lambda \quad \text{for all } n \in \mathbb{N}.$$

□

Using this proposition, we can show that the minimal solution map $\xi_\lambda(\cdot)$ is compact.

Proposition 4.2. *If hypotheses H_0, H_1 hold, $m \geq 2$ is big and $\lambda \in (0, \lambda^*)$, then the map $\xi_\lambda : C_0^1(\bar{\Omega}) \rightarrow C_0^1(\bar{\Omega})$ is compact.*

Proof. First, we show that $\xi_\lambda(\cdot)$ maps bounded sets in $C_0^1(\bar{\Omega})$ to relatively compact sets in $C_0^1(\bar{\Omega})$. So, let $D \subseteq C_0^1(\bar{\Omega})$ be bounded. Then as in the proof of Proposition 4.1, with a contradiction argument and using the fact that $m \geq 2$ is big, we show that $\xi_\lambda(D) \subseteq W_0^{1,p}(\Omega)$ is bounded and from this by the nonlinear regularity theory (see [27, 29]), we obtain that $\xi_\lambda(D) \subseteq C_0^1(\bar{\Omega})$ relatively compact.

Next, we show that $\xi_\lambda(\cdot)$ is continuous. So, let $v_n \rightarrow v$ in $C_0^1(\bar{\Omega})$ and let $\tilde{u}_v^\lambda = \xi_\lambda(v)$. According to Proposition 4.1, we can find $u_n \in S_{v_n}^\lambda, n \in \mathbb{N}$ such that

$$u_n \rightarrow \tilde{u}_v^\lambda. \quad (4.49)$$

We know that

$$\xi_\lambda(v_n) \leq u_n \quad \text{for all } n \in \mathbb{N}, \quad (4.50)$$

and from the first part if the proof, we have that

$$\{\xi_\lambda(v_n)\}_{n \in \mathbb{N}} \subseteq C_0^1(\bar{\Omega}) \quad \text{is relatively compact.}$$

So, by passing to a subsequence if necessary, we have

$$\xi_\lambda(v_n) \rightarrow u_v^\lambda \quad \text{in } C_0^1(\bar{\Omega}) \quad \text{as } n \rightarrow \infty \quad (4.51)$$

Clearly, $u_v^\lambda \in S_v^\lambda$ and from (4.49), (4.50), (4.51), we have

$$u_v^\lambda \leq \tilde{u}_v^\lambda \Rightarrow u_v^\lambda = \tilde{u}_v^\lambda = \xi_\lambda(v).$$

So, for the original sequence, we have

$$\begin{aligned} \xi_\lambda(v_n) &\rightarrow \xi_\lambda(v) \quad \text{in } C_0^1(\bar{\Omega}) \quad \text{as } n \rightarrow \infty, \\ \Rightarrow \xi_\lambda(\cdot) &\text{ is continuous.} \end{aligned}$$

Therefore, we have proved that the minimal map $\xi_\lambda(\cdot)$ is compact. $\square$

We introduce the following set

$$D(\xi_\lambda) = \{u \in C_0^1(\bar{\Omega}) : u = t\xi_\lambda(u), 0 < t < 1\}.$$

Proposition 4.3. *If hypotheses H_0, H_1 hold, $m \geq 2$ is big and $\lambda \in (0, \lambda^*)$, then $D(\xi_\lambda) \subseteq C_0^1(\bar{\Omega})$ is bounded.*

Proof. As before (see the proof of Proposition 4.1), it suffices to show that $D(\xi_\lambda) \subseteq W_0^{1,p}(\Omega)$ is bounded. Then the nonlinear regularity theory (see [27, 29]) implies that $D(\xi_\lambda) \subseteq C_0^1(\bar{\Omega})$ is bounded (in fact, relatively compact).

Again, we argue indirectly. So, suppose we could find $\{u_n\}_{n \geq 1} \subseteq D(\xi_\lambda)$ such that

$$\|u_n\| \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (4.52)$$

we have

$$\frac{1}{t_n} u_n = \xi_\lambda(u_n) \quad \text{with } 0 < t_n < 1 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, we have

$$\begin{aligned} \frac{1}{t_n^{p-1}} A_p(u_n) + \frac{1}{t_n^{q-1}} A_q(u_n) &= t_n^\eta u_n^{-\eta} + r(z) |Du_n|^{\tau-1} + f(\cdot, \frac{1}{t_n} u_n(\cdot)) \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow A_p(u_n) + t_n^{p-q} A_q(u_n) &= t_n^{p+\eta-1} u_n^{-\eta} + t_n^{p-1} r(z) |Du_n|^{\tau-1} + t_n^{p-1} f(\cdot, \frac{1}{t_n} u_n(\cdot)) \\ &\quad \text{for all } n \in \mathbb{N}, \end{aligned} \quad (4.53)$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then we have $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and } y_n \rightarrow y \quad \text{in } L^p(\Omega) \quad \text{as } n \rightarrow \infty. \quad (4.54)$$

From (4.53) we have

$$\begin{aligned} A_p(y_n) + \left(\frac{t_n}{\|u_n\|} \right)^{p-q} A_q(y_n) &= \frac{t_n^\eta}{\|u_n\|^{p-1}} u_n^{-\eta} + \frac{t_n^{p-1}}{\|u_n\|^{p-\tau}} r(z) |Dy_n|^{\tau-1} \\ &\quad + \frac{f(\cdot, \frac{1}{t_n} u_n(\cdot))}{\|\frac{1}{t_n} u_n\|^{p-1}} \quad \text{for all } n \in \mathbb{N}. \end{aligned} \quad (4.55)$$

Note that

$$0 \leq \frac{t_n^\eta}{\|u_n\|^{p-1}} u_n^{-\eta} \leq \frac{1}{\|u_n\|^{p-1}} \tilde{u}_\lambda^{-\eta} \in L^s(\Omega) \quad s > N \quad (\text{see Proposition 3.3 and (2.5)}). \quad (4.56)$$

Also, from hypothesis $H_1(i)$, we have

$$\left\{ \frac{f(\cdot, \frac{1}{t_n} u_n(\cdot))}{\|\frac{1}{t_n} u_n\|^{p-1}} \right\}_{n \geq 1} \subseteq L^{p'}(\Omega) \quad \text{is bounded.} \quad (4.57)$$

We test (4.55) with $y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (4.52) (recall $q, \tau < p$), (4.54), (4.56) and (4.57). We obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle A_p(y_n), y_n - y \rangle &= 0, \\ \Rightarrow y_n \rightarrow y \text{ in } W_0^{1,p}(\Omega) \text{ as } n \rightarrow \infty \text{ and so } \|y\| &= 1, y \geq 0 \text{ (see Proposition 2.1).} \end{aligned} \quad (4.58)$$

From (4.57) and hypothesis $H_1(ii)$, we have

$$\frac{f(\cdot, \frac{1}{\epsilon_n} u_n(\cdot))}{\|\frac{1}{\epsilon_n} u_n\|^{p-1}} \xrightarrow{w} \hat{\lambda}_m(p) y^{p-1} \text{ in } L^{p'}(\Omega) \text{ as } n \rightarrow \infty \text{ (see [26]).} \quad (4.59)$$

If in (4.55) we pass to the limit as $n \rightarrow \infty$ and use (4.58), (4.59) and (4.52) (recall $q, \tau < p$), we obtain

$$\begin{aligned} A_p(y) &= \hat{\lambda}_m(p) y^{p-1}, \\ \Rightarrow -\Delta_p y(z) &= \hat{\lambda}_m(p) y(z)^{p-1} \text{ in } \Omega, y|_{\partial\Omega} = 0, \\ \Rightarrow y &\text{ is zero or nodal (since } m \geq 2), \end{aligned}$$

both contradicting (4.58). This proves that $D(\xi_\lambda) \subseteq W_0^{1,p}(\Omega)$ is bounded, from which it follows the boundedness in $C_0^1(\bar{\Omega})$ as explained in the beginning of the proof. $\square$

On account of Propositions 4.2 and 4.3, we can apply Theorem 2.3 and produce a fixed point for $\xi_\lambda(\cdot), \lambda \in (0, \lambda^*)$. So, we can state the following existence theorem for problem (E_λ) .

Theorem 4.1. *If hypotheses H_0, H_1 hold and $m \geq 2$ is big, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (E_λ) has a positive solution $u_\lambda \in \text{int } C_+$.*

Remark 4.2. There are three interesting open problems concerning problem (E_λ)

- (a) What can be said if we have resonance with respect to the principal eigenvalue $\hat{\lambda}_1(p) > 0$ (that is, $m = 1$)?
- (b) Is it possible to have $\tau = p$ in the drift term?
- (c) Is it possible to have an indefinite drift term (that is, drop the hypothesis that $r \geq 0$)? In the direction see Hu-Papageorgiou [31].

Acknowledgement: The authors wish to thank the two knowledgeable referees for their corrections and remarks.

The work was supported by NNSF of China Grant No. 11671101, NSF of Guangxi Grant No. 2018GXNSFDA138002.

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