Research article

Wen Zhang, Jian Zhang*, and Heilong Mi

Ground states and multiple solutions for Hamiltonian elliptic system with gradient term

<https://doi.org/10.1515/anona-2020-0113>

Received April 15, 2020; accepted June 17, 2020.

Abstract: This paper is concerned with the following nonlinear Hamiltonian elliptic system with gradient term

$$\begin{cases} -\Delta u + \vec{b}(x) \cdot \nabla u + V(x)u = H_v(x, u, v) & \text{in } \mathbb{R}^N, \\ -\Delta v - \vec{b}(x) \cdot \nabla v + V(x)v = H_u(x, u, v) & \text{in } \mathbb{R}^N. \end{cases}$$

Compared with some existing issues, the most interesting feature of this paper is that we assume that the nonlinearity satisfies a local super-quadratic condition, which is weaker than the usual global super-quadratic condition. This case allows the nonlinearity to be super-quadratic on some domains and asymptotically quadratic on other domains. Furthermore, by using variational method, we obtain new existence results of ground state solutions and infinitely many geometrically distinct solutions under local super-quadratic condition. Since we are without more global information on the nonlinearity, in the proofs we apply a perturbation approach and some special techniques.

Keywords: Hamiltonian elliptic system; ground state solutions; infinitely many solutions; local super-quadratic condition

MSC: 35J50, 58E05

1 Introduction and main results

We study the following nonlinear Hamiltonian elliptic systems with gradient term

$$\begin{cases} -\Delta u + \vec{b}(x) \cdot \nabla u + V(x)u = H_v(x, u, v) & \text{in } \mathbb{R}^N, \\ -\Delta v - \vec{b}(x) \cdot \nabla v + V(x)v = H_u(x, u, v) & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $z = (u, v) : \mathbb{R}^N \rightarrow \mathbb{R}^2$, $N \geq 3$, $\vec{b}(x) = (b_1(x), \dots, b_N(x)) \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $H \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$. In the present paper, our main goal is to establish some new existence results of ground state

Wen Zhang, School of Mathematics and Statistics, Hunan University of Technology and Business, Changsha, 410205 Hunan, P. R. China, Key Laboratory of Hunan Province for Statistical Learning and Intelligent Computation, Hunan University of Technology and Business, Changsha, 410205 Hunan, P. R. China, E-mail: zwmath2011@163.com

***Corresponding Author: Jian Zhang**, School of Mathematics and Statistics, Hunan University of Technology and Business, Changsha, 410205 Hunan, P. R. China, Key Laboratory of Hunan Province for Statistical Learning and Intelligent Computation, Hunan University of Technology and Business, Changsha, 410205 Hunan, P. R. China, School of Mathematics and Statistics, Central South University, Changsha, 410083 Hunan, P. R. China, E-mail: zhangjian433130@163.com

Heilong Mi, School of Mathematics and Statistics, Hunan University of Technology and Business, Changsha, 410205 Hunan, P. R. China

solutions and infinitely many geometrically distinct solutions of system (1.1) under some suitable conditions on the potential V and the nonlinearity H .

This type of systems arises when one is looking for the standing wave solutions to system of diffusion equations

$$\begin{cases} \partial_t \varphi - \Delta_x \varphi + \vec{b}(t, x) \cdot \nabla \varphi + V(x) \varphi = H_\psi(t, x, \varphi, \psi) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ -\partial_t \psi - \Delta_x \psi - \vec{b}(t, x) \cdot \nabla \psi + V(x) \psi = H_\varphi(t, x, \varphi, \psi) & \text{in } \mathbb{R} \times \mathbb{R}^N, \end{cases}$$

which comes from the time-space diffusion processes and is related to the Schrödinger equations. It appears in various fields, such as physics and chemistry, quantum mechanics, finance, dynamic programming, optimization and control theory and Brownian motions. For more details in the application backgrounds, we refer the readers to see the monographs [13] and [19].

In recent years, the Hamiltonian elliptic system has being extensively investigated in the literatures based on various assumptions on the potential and nonlinearity. But most of them focused on the case $\vec{b} = 0$, namely,

$$\begin{cases} -\Delta u + V(x)u = H_v(x, u, v) & \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = H_u(x, u, v) & \text{in } \mathbb{R}^N. \end{cases} \quad (1.2)$$

For instance, the papers [2, 4, 7, 9, 10, 23, 31, 35] studied the super-quadratic growth case, and the asymptotically quadratic case can be found in [18, 29, 34]. Moreover, the existence of nontrivial solutions, ground state solutions, multiple solutions and semiclassical solutions were obtained in these works by using various variational arguments, such as dual methods, reduction methods, generalized mountain pass theorem, generalized linking theorem and many others. For further related topics including the Hamiltonian systems, we refer the reader to [3, 11, 12, 21] and their references.

When $\vec{b} \neq 0$, as we all know, there are a few works devoted to the existence and multiplicity of solutions of system (1.1), see [15, 30, 32, 36, 40]. For this case, since the appearance of the gradient term in system itself, system (1.1) has some differences and difficulties comparing to system (1.2). For example, the variational framework for the case $\vec{b} = 0$ cannot work any longer in this case, then the first problem is how to establish a suitable variational framework. To solve this problem, Zhao and Ding [32] handled (1.1) as a generalized Hamiltonian system, and established a strongly indefinite variational framework by studying the structure of essential spectrum of Hamiltonian operator. In this framework, the existence and multiplicity of solutions were obtained by using critical point theorems of strongly indefinite functional and reduction method for system (1.1) with periodic and non-periodic asymptotically quadratic growth condition. After that, Zhang et al. [36] studied the periodic super-quadratic case and proved the existence of ground state solutions by means of the linking and concentration compactness arguments. Later, this result has been extended to more general nonlinearity model by Liao et al. [15]. An asymptotically periodic case was considered in [40], and some properties of ground state solutions were obtained by constructing linking levels and analyzing behavior of Cerami sequence. The existence of least energy solution for the non-periodic super-quadratic case was studied in [30]. In [37], the authors studied the Hamiltonian elliptic system with inverse square potential of the form

$$\begin{cases} -\Delta u + \vec{b}(x) \cdot \nabla u + V(x)u - \frac{\mu}{|x|^2} v = H_v(x, u, v) & \text{in } \mathbb{R}^N, \\ -\Delta v - \vec{b}(x) \cdot \nabla v + V(x)v - \frac{\mu}{|x|^2} u = H_u(x, u, v) & \text{in } \mathbb{R}^N, \end{cases}$$

and the ground state solutions was obtained by using non-Nehari manifold developed by Tang [25]. Moreover, some asymptotic behaviors of ground state solutions, such as the monotonicity and convergence property of ground state energy, were also explored as parameter μ tends to 0. In addition, the singularly perturbed problem has been considered in [38, 39]. More precisely, the authors proved the existence of semi-classical ground state solutions, and shown some new concentration phenomena of semi-classical states.

It is worth pointing out that, for the aforementioned papers about super-quadratic problems, the classical condition frequently used in the literature is due to Ambrosetti and Rabinowitz [1]

(AR) there exists $\theta > 2$ such that, for each $x \in \mathbb{R}^N$, $z \in \mathbb{R}^2 \setminus \{0\}$, there holds

$$0 < \theta H(x, z) \leq H_z(x, z)z.$$

It is well known that condition (AR) has been used in a technical but crucial way not only in establishing the geometry structure of the energy functional but also in proving the boundedness of Palais-Smale sequences. Via a straightforward calculation, we can see easily that $H(x, z) \geq c|z|^\theta$ for large values of $|z|$ under condition (AR). Clearly, it puts strict constraints on the growth at infinity, and therefore it is natural to consider a weaker condition. After that, there are many works devoted to replacing condition (AR) with a more natural super-quadratic condition

$$(SQ) \lim_{|z| \rightarrow +\infty} \frac{H(x, z)}{|z|^2} = \infty \text{ uniformly in } x \in \mathbb{R}^N.$$

Condition (SQ) was first introduced by Liu and Wang [16] in studying the superlinear problem of the elliptic equation. Moreover, it plays a crucial role in verifying the link geometry and in showing the boundedness of Cerami sequences for the energy functional. Indeed, condition (SQ) is essential to prove the existence of nontrivial solutions in all literature.

Recently, motivated by [27] where the authors introduced a local version of super-quadratic condition when studying the scalar field Schrödinger equation, Zhang and Liao [33] obtained a new existence result of nontrivial solution for system (1.1) under the following condition

$$(f_0) \text{ there exists a domain } \Omega \subset \mathbb{R}^N \text{ such that } \lim_{|z| \rightarrow +\infty} \frac{H(x, z)}{|z|^2} \rightarrow \infty \text{ a.e. } x \in \Omega.$$

Here condition (f_0) is called local super-quadratic condition which allows the nonlinearity to be super-quadratic at some domains and asymptotically quadratic at other domains. Hence it weakens the usual global super-quadratic condition (SQ). However, to the best of our knowledge, it seems that ground state solutions and multiplicity results for system (1.1) with local super-quadratic condition (f_0) have not been studied so far. As a complement, in this paper we will continue the work in [33] in this direction. More precisely, our purpose in this paper is twofold, one is to prove the existence of ground states, that is, the least energy nontrivial solutions; the other is to establish the existence of infinitely many geometrically distinct solutions.

In what follows, in order to state our statements we assume that the following conditions:

- (B) $\vec{b} \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ is 1-periodic in x_i for $i = 1, \dots, N$ and $\operatorname{div} \vec{b} = 0$.
- (V) $V \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x_i for $i = 1, \dots, N$ and $a := \min_{x \in \mathbb{R}^N} V(x) > 0$.
- (f₁) $H \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is 1-periodic in x_i for $i = 1, \dots, N$, and there exist $p \in (2, 2^*)$ and $c > 0$ such that

$$|H_z(x, z)| \leq c(1 + |z|^{p-1}) \text{ for all } (x, z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where $2^* = \frac{2N}{N-2}$ denotes the usual critical exponent for $N \geq 3$.

- (f₂) $|H_z(x, z)| = o(|z|)$ as $|z| \rightarrow 0$ uniformly in $x \in \mathbb{R}^N$.
- (f₃) there exists $h : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow [0, +\infty)$ non-decreasing in the second variable such that

$$H_z(x, z) = h(x, |z|)z \text{ for all } (x, z) \in \mathbb{R}^N \times \mathbb{R}^2.$$

- (f₄) there exist $c_0 > 0$, $R_0 > 0$ and $\sigma \in (0, 1)$ such that

$$\left(\frac{|H_z(x, z)|}{|z|^\sigma} \right)^{\frac{2^*}{2^*-1-\sigma}} \leq c_0 \mathcal{H}(x, z) \text{ for all } |z| \geq R_0,$$

where $\mathcal{H}(x, z) := \frac{1}{2} H_z(x, z)z - H(x, z)$.

- (f₅) $H(x, z) \geq 0$, $\mathcal{H}(x, z) \geq 0$, and there exist $c_0 > 0$, $\delta_0 \in (0, a)$ and $\sigma \in (0, 1)$ such that

$$|H_z(x, z)| \geq (a - \delta_0)|z| \text{ implies } \left(\frac{|H_z(x, z)|}{|z|^\sigma} \right)^{\frac{2^*}{2^*-1-\sigma}} \leq c_0 \mathcal{H}(x, z).$$

For the sake of convenience to describe our results, here we first need to give some notations. Let E be the Hilbert space with an orthogonal decomposition $E = E^- \oplus E^+$, and let Φ denote the energy functional of system (1.1), where E and Φ will be defined in Section 2. We define the generalized Nehari manifold

$$\mathcal{M} := \{z \in E \setminus E^- : \Phi'(z)z = 0 \text{ and } \Phi'(z)w = 0 \text{ for any } w \in E^-\}.$$

and the critical points set $\mathcal{N} := \{z \in E \setminus \{0\} : \Phi'(z) = 0\}$ of Φ . According to [20, 24], the set $\mathcal{M}$ is a natural constraint and it contains all nontrivial critical points of Φ . Obviously, $\mathcal{N} \subset \mathcal{M}$. We say that a nontrivial solution z_0 is ground state solution if its energy attains the minimum among all nontrivial critical points. Additionally, observe that, due to the periodicity of $\vec{b}$, V and H , if z is a solution of system (1.1), then so is $k * z$ for all $k \in \mathbb{Z}^N$, where $(k * z)(x) = z(x + k)$. Two solutions z_1 and z_2 are said to be geometrically distinct if $k * z_1 \neq z_2$ for all $k \in \mathbb{Z}^N$.

We are now in position to state the main results of this paper. On the existence of ground state solutions we have the following results.

Theorem 1.1. *Suppose that (B), (V) and (f_0) – (f_4) hold. Then system (1.1) has a ground state solution $\tilde{z}$ such that $\Phi(\tilde{z}) = \inf_{\mathcal{M}} \Phi > 0$.*

Theorem 1.2. *Suppose that (B), (V), (f_0) – (f_2) and (f_5) are satisfied. Then system (1.1) has a ground state solution $\tilde{z}$ such that $\Phi(\tilde{z}) = \inf_{\mathcal{N}} \Phi > 0$.*

On multiplicity results of solutions we have the following theorems.

Theorem 1.3. *Assume that (B), (V) and (f_0) – (f_4) hold, and if $H(x, z)$ is even in z , then system (1.1) has infinitely many geometrically distinct solutions.*

Theorem 1.4. *Assume that (B), (V), (f_0) – (f_2) and (f_5) are satisfied, and if $H(x, z)$ is even in z , then system (1.1) has infinitely many geometrically distinct solutions.*

As observed in [33], the first author and co-author only proved the existence result of nontrivial solution by using generalized linking theorem under the conditions of Theorem 1.2, and the other related results are all unknown. Compared to the result in [33], the results obtained in this paper can be viewed as a continuation of [33], and seem more delicate.

To prove Theorem 1.1 and Theorem 1.2, some arguments are in order. Here we first introduce the problem of ground state solution. For the Theorem 1.2, based on the result in [33], we can see that $\mathcal{N} \neq \emptyset$. By using minimization method and concentration compactness argument, the conclusion for ground state solution in Theorem 1.2 holds. However, Theorem 1.1 seems more complicated than Theorem 1.2, and there are many new difficulties. Generally speaking, $\mathcal{M}$ contains infinitely many elements of E , while $\mathcal{N}$ may contain only one element. So it becomes more difficult to find a ground state solution $\tilde{z}$ which satisfies $\Phi(\tilde{z}) = \inf_{\mathcal{M}} \Phi$ than one that satisfies $\Phi(\tilde{z}) = \inf_{\mathcal{N}} \Phi$. Additionally, we note that the variational structure of system (1.1) is strongly indefinite, thus the usual Nehari manifold method cannot be applied directly. To overcome these difficulties, in the spirit of [20, 24] we choose the generalized Nehari manifold $\mathcal{M}$ to work. More precisely, we intend to make use of the non-Nehari manifold method developed by Tang [25] to complete the proof of Theorem 1.1. The main idea of this method is to construct a minimizing Cerami sequence for energy functional Φ outside $\mathcal{M}$ by using the diagonal method and linking argument.

We would like to point out that the global super-quadratic (SQ) is indispensable in verifying the linking geometry structure and constructing Cerami sequence by the diagonal method and linking argument, see [37, 40]. Unfortunately, in the present paper we have no global information on the nonlinearity like (SQ), then the non-Nehari manifold method seems not work to our problem under the local super-quadratic condition. So, some new methods and techniques need to be introduced. Motivated by [26], we present a perturbation approach by adding a perturbation term of power function. More precisely, for $\mu \in (0, 1]$ and $p \in (2, 2^*)$, we consider the following perturbation problem

$$\begin{cases} -\Delta u + \vec{b}(x) \cdot \nabla u + V(x)u = H_v(x, u, v) + \mu|z|^{p-2}v & \text{in } \mathbb{R}^N, \\ -\Delta v - \vec{b}(x) \cdot \nabla v + V(x)v = H_u(x, u, v) + \mu|z|^{p-2}u & \text{in } \mathbb{R}^N, \end{cases} \quad (1.3)$$

and its associated functional is as follows

$$\Phi_\mu(z) = \Phi(z) - \frac{\mu}{p} \int_{\mathbb{R}^N} |z|^p dx.$$

In such a way, the modified nonlinearity satisfies the global super-quadratic condition (SQ), then, by using the non-Nehari method in [25] and concentration compactness principle, we can obtain a ground state solution z_μ of the perturbation problem. Finally, by passing to the limit and by some special techniques, we show the convergence as $\mu \rightarrow 0$ of $\{z_n\}$ towards to a ground state solution of the original problem.

For the sake of completeness, next we consider the multiplicity results of system (1.1). In view of Theorem 1.1 and Theorem 1.2, we know that $\mathcal{N} \neq \emptyset$. To prove the existence of infinitely many geometrically distinct solutions, we choose a subset $\mathcal{F}$ of $\mathcal{N}$ such that $\mathcal{F} = -\mathcal{F}$ and each orbit $\mathcal{O}(w) \subset \mathcal{N}$ has a unique representative in $\mathcal{F}$ due to $\Phi(z) = \Phi(-z)$, and then show that the set $\mathcal{F}$ is infinite. To do this, inspired by [24, 26], using some arguments about deformation type and Krasnoselskii genus, we find infinitely many geometrically distinct solutions.

The remainder of this paper is organized as follows. In Section 2, we introduce the variational setting of the problem and present some useful preliminaries. In Section 3, we prove that the perturbation problem has a ground state solution. In Section 4, we give the proof of the existence of ground state solutions in Theorem 1.1 and Theorem 1.2, respectively. At last, the existence of infinitely many geometrically distinct solutions is established in Section 5.

2 Variational setting and preliminaries

Below by $|\cdot|_q$ we denote the usual L^q -norm, $(\cdot, \cdot)_2$ denotes the usual L^2 inner product, c, c_i or C_i stand for different positive constants. For the sake of convenience, we need the following notations. Let

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{J}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $\mathcal{S} = -\Delta + V$. We denote

$$A := \mathcal{S}\mathcal{J}_0 + \vec{b} \cdot \nabla \mathcal{J} = \begin{pmatrix} 0 & -\Delta - \vec{b} \cdot \nabla + V \\ -\Delta + \vec{b} \cdot \nabla + V & 0 \end{pmatrix},$$

then system (1.1) can be rewritten as

$$Az = H_z(x, z).$$

According to [8], in this way, system (1.1) can be regarded as a Hamiltonian system.

Denote by $\sigma(A)$ and $\sigma_e(A)$ the spectrum and the essential spectrum of operator A , respectively. In order to establish a suitable variational framework for system (1.1), we need to utilize some properties of spectrum of operator A due to [32].

Lemma 2.1. Assume (B) and (V) are satisfied. Then operator A is a selfadjoint operator on $L^2 := L^2(\mathbb{R}^N, \mathbb{R}^2)$ with domain $\mathcal{D}(A) := H^2(\mathbb{R}^N, \mathbb{R}^2)$.

Lemma 2.2. Assume (B) and (V) are satisfied. The following two conclusions hold:

- (1) $\sigma(A) = \sigma_e(A)$, i.e., A has only essential spectrum;
- (2) $\sigma(A) \subset \mathbb{R} \setminus (-a, a)$ and $\sigma(A)$ is symmetric with respect to origin.

Observe that, it follows from Lemma 2.1 and 2.2 that the space L^2 possesses the following orthogonal decomposition

$$L^2 = L^- \oplus L^+, \quad z = z^- + z^+$$

such that A is negative definite (resp. positive definite) in L^- (resp. L^+). Let $|A|$ denote the absolute value of A and $|A|^{1/2}$ be the square root of $|A|$. Let $E = \mathcal{D}(|A|^{1/2})$ be the Hilbert space with the inner product

$$(z, w) = (|A|^{1/2}z, |A|^{1/2}w)_2$$

and norm $\|z\| = (z, z)^{1/2}$. Moreover, it is obvious that E possesses the following decomposition

$$E = E^- \oplus E^+, \text{ where } E^\pm = E \cap L^\pm,$$

which is orthogonal with respect to the inner products $(\cdot, \cdot)_2$ and $(\cdot, \cdot)$. Note that $E = H^1 := H^1(\mathbb{R}^N, \mathbb{R}^2)$ and $\|\cdot\|$ is equivalent to the usual norm of H^1 (see [32]). Then E embeds continuously into L^q for all $q \in [2, 2^*]$ and compactly into L^q_{loc} for all $q \in [1, 2^*)$. Moreover, there exists a positive constant $\pi_q > 0$ such that for all $z \in E$

$$\pi_q |z|_q \leq \|z\|, \quad q \in [2, 2^*]. \quad (2.1)$$

On the one hand, by virtue of (f_1) and (f_2) , for any $\epsilon > 0$, there exists a positive constant c_ϵ such that

$$\begin{cases} |H_z(x, z)| \leq \epsilon |z| + c_\epsilon |z|^{p-1} \\ |H(x, z)| \leq \epsilon |z|^2 + c_\epsilon |z|^p \end{cases} \text{ for all } (x, z) \in \mathbb{R}^N \times \mathbb{R}^2, \quad p \in (2, 2^*). \quad (2.2)$$

On the other hand, it follows from (f_3) that

$$\frac{1}{2} H_z(x, z) z \geq H(x, z) \geq 0, \text{ for all } z \neq 0. \quad (2.3)$$

In fact, given $z \neq 0$, (f_3) implies that

$$H(x, z) = \int_0^1 \frac{d}{dt} (H(x, tz)) dt = \int_0^1 H_z(x, tz) z dt = |z|^2 \int_0^1 h(x, t|z|) t dt \geq 0.$$

This, together with the monotonicity of $h(x, |z|)$, implies that

$$\frac{1}{2} H_z(x, z) z - H(x, z) = |z|^2 \int_0^1 (h(x, |z|) - h(x, t|z|)) t dt \geq 0.$$

On E we define the energy functional Φ corresponding to system (1.1) as follows

$$\Phi(z) = \frac{1}{2} (\|z^+\|^2 - \|z^-\|^2) - \int_{\mathbb{R}^N} H(x, z).$$

By the above facts and some standard arguments, we can easily see that $\Phi \in C^1(E, \mathbb{R})$ and the critical points of Φ are solutions of system (1.1) (see [6, 28]), and for $z, \varphi \in E$

$$\langle \Phi'(z), \varphi \rangle = (z^+, \varphi^+) - (z^-, \varphi^-) - \int_{\mathbb{R}^N} H_z(x, z) \varphi.$$

Now we discuss the linking geometry structure of the energy functional Φ .

Lemma 2.3. *Suppose that (f_1) and (f_2) are satisfied. Then there exists $\rho > 0$ such that $\kappa := \inf_{S_\rho} \Phi > 0$, where $S_\rho := \{z \in E^+, \|z\| = \rho\}$.*

The proof of Lemma 2.3 is standard, and the details can be seen in [36] and hence is omitted.

Without loss of generality, we can assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain. We choose $\tilde{e} \in C_0^\infty(\mathbb{R}^N) \cap C_0^\infty(\Omega)$ such that $\|\tilde{e}^+\|^2 - \|\tilde{e}^-\|^2 = (A\tilde{e}, \tilde{e})_2 \geq 1$, which implies that $\tilde{e}^+ \neq 0$. Based on this fact, using the special technique as in [33], we can obtain the following lemma, which is very critical in our arguments, the proof can be found in [33].

Lemma 2.4. *Suppose that (f_0) – (f_2) are satisfied. Then $\sup \Phi(E^- \oplus \mathbb{R}^+ \tilde{e}^+) < \infty$, and there is $R_{\tilde{e}} > 0$ such that*

$$\Phi(z) \leq 0, \quad \forall z \in E^- \oplus \mathbb{R}^+ \tilde{e}^+, \quad \|z\| \geq R_{\tilde{e}}.$$

Lemma 2.5. Suppose that (f_0) – (f_2) are satisfied. Then $\mathcal{M} \neq \emptyset$.

Proof. Let $E(\tilde{e}^+) = E^- \oplus \mathbb{R}^+ \tilde{e}^+$. It follows from Lemma 2.4 that there exists $R_{\tilde{e}} > 0$ such that $\Phi(z) \leq 0$ for $z \in E(\tilde{e}^+) \setminus B_{R_{\tilde{e}}}(0)$. Moreover, Lemma 2.3 implies that $\Phi(t\tilde{e}^+) > 0$ for small $t > 0$. Thus, $0 < \sup \Phi(E(\tilde{e}^+)) < \infty$. It is easy to see that Φ is weakly upper semi-continuous on $E(\tilde{e}^+)$, therefore, $\Phi(z_0) = \sup \Phi(E(\tilde{e}^+))$ for some $z_0 \in E(\tilde{e}^+)$. This z_0 is a critical point of $\Phi|_{E(\tilde{e}^+)}$, so $\Phi'(z_0)z_0 = \Phi'(z_0)w = 0$ for all $w \in E(\tilde{e}^+)$. Consequently, $z_0 \in \mathcal{M} \cap E(\tilde{e}^+)$, and $\mathcal{M} \neq \emptyset$. $\square$

We recall that a functional $\Phi \in C^1(E, \mathbb{R})$ is said to be weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ in E one has $\Phi(u) \leq \liminf_{n \rightarrow \infty} \Phi(u_n)$, and Φ' is said to be weakly sequentially continuous if $\lim_{n \rightarrow \infty} \Phi'(u_n)\varphi = \Phi'(u)\varphi$ for each $\varphi \in E$. We recall that a sequence $\{u_n\} \subset E$ is called Cerami sequence for Φ at the level c ($(C)_c$ -sequence in short) if

$$\Phi(u_n) \rightarrow c \text{ and } (1 + \|u_n\|)\|\Phi'(u_n)\| \rightarrow 0.$$

We say that Φ satisfy the $(C)_c$ -condition if any $(C)_c$ -sequence has a convergent subsequence.

To prove the main results, we need the following generalized linking theorem in [17].

Lemma 2.6. Let X be a real Hilbert space with $X = X^- \oplus X^+$, and let $\Phi \in C^1(X, \mathbb{R})$ be of the form

$$\Phi(u) = \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) - \Psi(u), \quad u = u^- + u^+ \in X^- \oplus X^+.$$

Suppose that the following assumptions are satisfied:

(A_1) $\Psi \in C^1(X, \mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous;

(A_2) Ψ' is weakly sequentially continuous;

(A_3) there exist $R > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that

$$\kappa := \inf \Phi(S_\rho^+) > \sup \Phi(\partial Q),$$

where

$$S_\rho^+ = \{u \in X^+ : \|u\| = \rho\}, \quad Q = \{v + se : v \in X^-, s \geq 0, \|v + se\| \leq R\}.$$

Then there exist a constant $c \in [\kappa, \sup \Phi(Q)]$ and a sequence $\{u_n\} \subset X$ satisfying

$$\Phi(u_n) \rightarrow c \text{ and } (1 + \|u_n\|)\|\Phi'(u_n)\| \rightarrow 0.$$

3 The perturbation problem

In this section, we will in the sequel focus on the perturbation problem (1.3) and study the existence of ground state solution. We define the perturbation functional Φ_μ of Φ

$$\Phi_\mu(z) = \frac{1}{2} (\|z^+\|^2 - \|z^-\|^2) - \int_{\mathbb{R}^N} H(x, z) - \frac{\mu}{p} \int_{\mathbb{R}^N} |z|^p$$

and the corresponding generalized Nehari manifold

$$\mathcal{M}_\mu := \{z \in E \setminus E^- : \Phi'_\mu(z)z = 0 \text{ and } \Phi'_\mu(z)w = 0 \text{ for any } w \in E^-\},$$

where $\mu \in (0, 1]$ and $p \in (2, 2^*)$. Let

$$m_\mu := \inf_{z \in \mathcal{M}_\mu} \Phi_\mu.$$

If m_μ is attained by $z_\mu \in \mathcal{M}_\mu$, then z_μ is a critical point of Φ_μ . Since m_μ is the lowest level for Φ_μ , then z_μ is called a ground state solution of the perturbation problem (1.3).

For convenience, let

$$\Psi_\mu(z) = \int_{\mathbb{R}^N} G_\mu(x, z) = \int_{\mathbb{R}^N} H(x, z) + \frac{\mu}{p} \int_{\mathbb{R}^N} |z|^p,$$

and $g_\mu(x, |z|) = h(x, |z|) + \mu|z|^{p-2}$. Plainly, the modified nonlinearity

$$G_\mu(x, z) = H(x, z) + \frac{\mu}{p} |z|^p$$

satisfies the global super-quadratic condition (SQ). Applying a standard argument (see [6, Lemma 5.2]), one can check easily the following lemma, and omit the details of the proof.

Lemma 3.1. Ψ_μ is weakly sequentially lower semi-continuous. Ψ'_μ is weakly sequentially continuous.

Lemma 3.2. Suppose that (f_0) – (f_3) are satisfied. Let $z \in E$, $w \in E^-$ and $t \geq 0$, we have

$$\Phi_\mu(z) \geq \Phi_\mu(tz + w) - \Phi'_\mu(z) \left(\frac{t^2 - 1}{2} z + tw \right). \quad (3.1)$$

In particular, let $z \in \mathcal{M}_\mu$, $w \in E^-$ and $t \geq 0$. There holds

$$\Phi_\mu(z) \geq \Phi_\mu(tz + w). \quad (3.2)$$

Proof. Observe that,

$$\Phi_\mu(tz + w) - \Phi_\mu(z) - \Phi'_\mu(z) \left(\frac{t^2 - 1}{2} z + tw \right) = -\frac{1}{2} \|w\|^2 + \int_{\mathbb{R}^N} \mathcal{G}_\mu(x, t),$$

where

$$\mathcal{G}_\mu(x, t) := g_\mu(x, |z|)z \left(\frac{t^2 - 1}{2} z + tw \right) + G_\mu(x, z) - G_\mu(x, tz + w).$$

Since $g_\mu(x, s)$ is increasing in s on $(0, +\infty)$ due to (f_3) , we can obtain $\mathcal{G}_\mu(x, t) \leq 0$ for $t \geq 0$ by using some arguments in [37, 40]. So, we get the first conclusion from the above formula. If $z \in \mathcal{M}_\mu$, then $\Phi'_\mu(z)z = \Phi'_\mu(z)w = 0$, then the second conclusion holds. $\square$

For convenience of notation, we write $E(z) := E^- \oplus \mathbb{R}^+ z = E^- \oplus \mathbb{R}^+ z^+$ for $z \in E \setminus E^-$. Let $z \in \mathcal{M}_\mu$, then Lemma 3.2 implies that z is the global maximum of $\Phi_\mu|_{E(z)}$. Next we shall verify that Φ_μ possesses the linking structure.

Lemma 3.3. Suppose that (f_0) – (f_3) are satisfied. Then there exist positive constants ρ and α both independent of $\mu \in (0, 1]$ such that

- (i) there holds: $m_\mu = \inf_{\mathcal{M}_\mu} \Phi_\mu \geq \kappa := \inf_{S_\rho} \Phi_\mu \geq \alpha$, where $S_\rho := \{z \in E^+, \|z\| = \rho\}$.
- (ii) $\|z^+\| \geq \max \{\|z^-\|, \sqrt{2m_\mu}\}$ for all $z \in \mathcal{M}_\mu$.

Proof. (i) For $z \in E^+$ and $\mu \in (0, 1]$, by (2.1) and (2.2), we obtain

$$\begin{aligned} \Phi_\mu(z) &= \frac{1}{2} \|z\|^2 - \int_{\mathbb{R}^N} H(x, z) - \frac{\mu}{p} \int_{\mathbb{R}^N} |z|^p \\ &\geq \frac{1}{2} \|z\|^2 - \epsilon |z|_2^2 - \left(c_\epsilon + \frac{1}{p} \right) |z|_p^p \\ &\geq \left(\frac{1}{2} - \epsilon \pi_2^{-2} \right) \|z\|^2 - \pi_p^{-p} C_\epsilon \|z\|^p. \end{aligned}$$

It is easy to see that there exist positive constants ρ and α both independent of μ such that $\inf_{S_\rho} \Phi \geq \alpha$ due to the arbitrariness of $\epsilon > 0$. So the second inequality holds. Note that for every $z \in \mathcal{M}_\mu$ there exists $s > 0$ such that $sz^+ \in E(z) \cap S_\rho$. Hence, by Lemma 3.2 we know that the first inequality holds.

(ii) It follows from (2.3) and $\mu \in (0, 1]$ that $\Psi_\mu(z) \geq 0$. For $z \in \mathcal{M}_\mu$, then we have $\Phi_\mu(z) \geq m_\mu$ and

$$m_\mu \leq \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \Psi_\mu(z) \leq \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2),$$

and this implies that $\|z^+\| \geq \max\{\|z^-\|, \sqrt{2m_\mu}\}$. $\square$

Lemma 3.4. Suppose that (f_0) – (f_3) are satisfied. Then for any $e \in E^+$, $\sup \Phi_\mu(E^- \oplus \mathbb{R}^+ e) < \infty$, and there is $R_e > 0$ such that

$$\Phi_\mu(z) < 0, \quad \forall z \in E^- \oplus \mathbb{R}^+ e, \quad \|z\| \geq R_e.$$

In particular, there is a $R_0 > \rho$ such that $\sup \Phi_\mu(\partial Q_R) \leq 0$ for $R \geq R_0$, where

$$Q_R = \{se + w : w \in E^-, s \geq 0, \|se + w\| \leq R\}.$$

Proof. Since the modified nonlinearity $G_\mu(x, z)$ satisfies the global super-quadratic condition (SQ), the proof is standard, see [36, 40]. So we omit it here. $\square$

Employing Lemmas 2.6, 3.1, 3.3 and 3.4, we have

Lemma 3.5. Suppose that (f_0) – (f_3) are satisfied. Then there exist a constant $\hat{c}_\mu \in [\kappa, \sup \Phi_\mu(Q_R)]$ and a $(C)_{\hat{c}_\mu}$ -sequence $\{z_n\} \subset E$ satisfying

$$\Phi_\mu(z_n) \rightarrow \hat{c}_\mu \quad \text{and} \quad \|\Phi'_\mu(z_n)\|(1 + \|z_n\|) \rightarrow 0.$$

To prove the existence of ground state solutions for the perturbation problem (1.3), next we construct a special Cerami sequence by using diagonal method (see [25]), which is very important in our arguments.

Lemma 3.6. Suppose that (f_0) – (f_3) are satisfied. Then there exist a constant $\tilde{c}_\mu \in [\kappa, m_\mu]$ and a $(C)_{\tilde{c}_\mu}$ -sequence $\{z_n\} \subset E$ such that

$$\Phi_\mu(z_n) \rightarrow \tilde{c}_\mu \quad \text{and} \quad \|\Phi'_\mu(z_n)\|(1 + \|z_n\|) \rightarrow 0.$$

Proof. Choose $\xi_k \in \mathcal{M}_\mu$ such that

$$m_\mu \leq \Phi_\mu(\xi_k) < m_\mu + \frac{1}{k}, \quad k \in \mathbb{N}. \quad (3.3)$$

By virtue of Lemma 3.3, $\|\xi_k^+\| \geq \sqrt{2m_\mu} > 0$. Set $e_k = \xi_k^+ / \|\xi_k^+\|$. Then $e_k \in E^+$ and $\|e_k\| = 1$. From Lemma 3.4, it follows that there exists R_k such that $\sup \Phi_\mu(\partial Q_k) \leq 0$, where

$$Q_k = \{se_k + w : w \in E^-, s \geq 0, \|se_k + w\| \leq R_k\}, \quad k \in \mathbb{N}. \quad (3.4)$$

Hence, using Lemma 3.5 to the above set Q_k , there exist a constant $c_{\mu,k} \in [\kappa, \sup \Phi_\mu(Q_k)]$ and a sequence $\{z_{k,n}\}_{n \in \mathbb{N}} \subset E$ satisfying

$$\Phi_\mu(z_{k,n}) \rightarrow c_{\mu,k} \quad \text{and} \quad \|\Phi'_\mu(z_{k,n})\|(1 + \|z_{k,n}\|) \rightarrow 0, \quad k \in \mathbb{N}. \quad (3.5)$$

On the other hand, by Lemma 3.2, one can get that

$$\Phi_\mu(\xi_k) \geq \Phi_\mu(t\xi_k + w), \quad \forall t \geq 0, w \in E^-. \quad (3.6)$$

Since $\xi_k \in Q_k$, it follows from (3.4) and (3.6) that $\Phi_\mu(\xi_k) = \sup \Phi_\mu(Q_k)$. Hence, by (3.3) and (3.5), one has

$$\Phi_\mu(z_{k,n}) \rightarrow c_{\mu,k} < m_\mu + \frac{1}{k} \quad \text{and} \quad \|\Phi'_\mu(z_{k,n})\|(1 + \|z_{k,n}\|) \rightarrow 0, \quad k \in \mathbb{N}.$$

We can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\Phi_\mu(z_{k,n_k}) < m_\mu + \frac{1}{k} \quad \text{and} \quad \|\Phi'_\mu(z_{k,n_k})\|(1 + \|z_{k,n_k}\|) < \frac{1}{k}, \quad k \in \mathbb{N}.$$

Let $z_k = z_{k,n_k}$, $k \in \mathbb{N}$. Then, up to a subsequence, we have

$$\Phi_\mu(z_k) \rightarrow \tilde{c}_\mu \in [\kappa, m_\mu] \text{ and } \|\Phi'_\mu(z_k)\|(1 + \|z_k\|) \rightarrow 0.$$

□

Similar to the proof of Lemma 2.5, we have the following result.

Lemma 3.7. *Suppose that (f_0) – (f_3) are satisfied. Then for any $z \in E \setminus E^-$, $\mathcal{M}_\mu \cap E(z) \neq \emptyset$, i.e., there exist $t_\mu > 0$ and $w_\mu \in E^-$ such that $t_\mu z + w_\mu \in \mathcal{M}_\mu$.*

Lemma 3.8. *Suppose that (f_0) – (f_3) are satisfied. If $\{z_n\} \subset E$ satisfies $(1 + \|z_n\|)\Phi'_\mu(z_n) \rightarrow 0$ and $\Phi_\mu(z_n)$ is bounded from above, then $\{z_n\}$ is bounded. In particular, any $(C)_c$ -sequence of Φ_μ at level $c \geq 0$ is bounded.*

Proof. Let $\{z_n\} \subset E$ be such that

$$(1 + \|z_n\|)\Phi'_\mu(z_n) \rightarrow 0 \text{ and } \Phi_\mu(z_n) \leq C \quad (3.7)$$

for some $C > 0$. Suppose to the contrary that $\|z_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Setting $w_n = z_n/\|z_n\|$, then $\|w_n\| = 1$. After passing to a subsequence, we assume that $w_n \rightharpoonup w$ in E , $w_n \rightarrow w$ in L^q_{loc} for $2 \leq q < 2^*$, and $w_n(x) \rightarrow w(x)$ a.e. on $\mathbb{R}^N$. Let

$$\delta := \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |w_n^+|^2.$$

If $\delta = 0$, by Lions' concentration compactness principle (see [14, 28]), then $w_n^+ \rightarrow 0$ in L^q for any $2 < q < 2^*$. It follows from (2.2) that for any $s > 0$,

$$\int_{\mathbb{R}^N} \left(H(x, sw_n^+) + \frac{\mu}{p} |sw_n^+|^p \right) = o(1). \quad (3.8)$$

Observe that, from (3.1) and (3.7), we deduce that

$$\begin{aligned} C &\geq \Phi_\mu(z_n) \geq \Phi_\mu(t_n z_n + (-t_n z_n^-)) + \Phi'_\mu(z_n) \left(\frac{t_n^2 - 1}{2} z_n - t_n^2 z_n^- \right) \\ &= \frac{t_n^2}{2} \|z_n^+\|^2 - \int_{\mathbb{R}^N} \left(H(x, t_n z_n^+) + \frac{\mu}{p} |t_n z_n^+|^p \right) + o(1). \end{aligned}$$

Let $t_n = s/\|z_n\|$, then by (3.8) we have

$$C \geq \frac{s^2}{2} \|w_n^+\|^2 - \int_{\mathbb{R}^N} \left(H(x, sw_n^+) + \frac{\mu}{p} |sw_n^+|^p \right) + o(1) = \frac{s^2}{2} \|w_n^+\|^2 + o(1). \quad (3.9)$$

On the other hand, by (2.3) we get $\langle \Phi'_\mu(z_n), z_n \rangle \leq \|z_n^+\|^2 - \|z_n^-\|^2$, which implies that

$$2\|z_n^+\|^2 \geq \|z_n\|^2 + \langle \Phi'_\mu(z_n), z_n \rangle.$$

This shows that $\|w_n^+\|^2 \geq c_1$ for some $c_1 > 0$. Hence, (3.9) yields a contradiction if s is large enough. Then $\delta > 0$. Up to a subsequence, we assume $k_n \in \mathbb{Z}^N$ such that

$$\int_{B(k_n, 1+\sqrt{N})} |w_n^+|^2 > \frac{\delta}{2}.$$

Let $\tilde{w}_n(x) = w_n(x + k_n)$. By the periodicity of $\vec{b}$ and V , we have that $\|w_n\| = \|\tilde{w}_n\| = 1$ and

$$\int_{B(0, 1+\sqrt{N})} |\tilde{w}_n^+|^2 > \frac{\delta}{2}.$$

Therefore, passing to a subsequence, $\tilde{w}_n^+ \rightarrow \tilde{w}^+$ in L_{loc}^2 and $\tilde{w}^+ \neq 0$. Note that if $\tilde{w} \neq 0$, then $|z_n(x + k_n)| = |\tilde{w}_n(x)| \|z_n\| \rightarrow \infty$. Since G_μ satisfies condition (SQ), then it follows from Fatou's lemma that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \frac{\Phi_\mu(z_n)}{\|z_n\|^2} \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{2} (\|w_n^+\|^2 - \|w_n^-\|^2) - \int_{\mathbb{R}^N} \frac{G_\mu(x + k_n, z_n(x + k_n))}{|z_n(x + k_n)|^2} |\tilde{w}_n|^2 \right) \\ &= -\infty, \end{aligned}$$

which is a contradiction. So, $\{z_n\}$ is bounded in E . $\square$

Next we show the existence of ground state solutions of the perturbation problem (1.3).

Lemma 3.9. *The perturbation problem (1.3) possesses a ground state solution, and m_μ is attained for all $\mu \in (0, 1]$.*

Proof. Applying Lemma 3.6, we deduce that there exists a $(C)_{\tilde{c}_\mu}$ -sequence $\{z_n\}$ of Φ_μ such that

$$\Phi_\mu(z_n) \rightarrow \tilde{c}_\mu \leq m_\mu \quad \text{and} \quad \|\Phi'_\mu(z_n)\|(1 + \|z_n\|) \rightarrow 0.$$

Lemma 3.8 shows that $\{z_n\}$ is bounded. Let

$$\delta := \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, 1)} |z_n|^2 = 0.$$

By Lions' concentration compactness principle (see [14, 28]), we have $z_n \rightarrow 0$ in L^q for $2 < q < 2^*$. Moreover, by (2.2) we get

$$\int_{\mathbb{R}^N} \left(\mathcal{H}(x, z_n) + \frac{p-2}{2p} \mu |z_n|^p \right) = o(1),$$

and consequently

$$\begin{aligned} \tilde{c}_\mu + o(1) &= \Phi_\mu(z_n) - \frac{1}{2} \langle \Phi'_\mu(z_n), z_n \rangle \\ &= \int_{\mathbb{R}^N} \left(\mathcal{H}(x, z_n) + \frac{p-2}{2p} \mu |z_n|^p \right) \\ &= o(1), \end{aligned}$$

which is a contradiction. Thus $\delta > 0$. Up to a subsequence, we assume $k_n \in \mathbb{Z}^N$ such that

$$\int_{B_{1+\sqrt{N}}(k_n)} |z_n|^2 \geq \frac{\delta}{2}.$$

Let us define $\tilde{z}_n(x) = z_n(x + k_n)$ so that

$$\int_{B_{1+\sqrt{N}}(0)} |\tilde{z}_n|^2 \geq \frac{\delta}{2}. \quad (3.10)$$

Since the periodicity of $\vec{b}$ and V , we have $\|\tilde{z}_n\| = \|z_n\|$ and

$$\Phi_\mu(\tilde{z}_n) \rightarrow \tilde{c}_\mu \leq m_\mu \quad \text{and} \quad (1 + \|\tilde{z}_n\|)\Phi'_\mu(\tilde{z}_n) \rightarrow 0. \quad (3.11)$$

Passing to a subsequence, we assume that $\tilde{z}_n \rightharpoonup \tilde{z}$ in E , $\tilde{z}_n \rightarrow \tilde{z}$ in L_{loc}^q for $2 \leq q < 2^*$, and $\tilde{z}_n(x) \rightarrow \tilde{z}(x)$ a.e. on $\mathbb{R}^N$. Hence it follows from (3.10) and (3.11) that $\tilde{z} \neq 0$ and $\Phi'_\mu(\tilde{z}) = 0$. This shows that $\tilde{z} \in \mathcal{M}_\mu$ and

$\Phi_\mu(\tilde{z}) \geq m_\mu$. On the other hand, by (2.3), (3.11) and Fatou's lemma, we get

$$\begin{aligned} m_\mu &\geq \tilde{c}_\mu = \lim_{n \rightarrow \infty} \left(\Phi_\mu(\tilde{z}_n) - \frac{1}{2} \langle \Phi'_\mu(\tilde{z}_n), \tilde{z}_n \rangle \right) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left(\mathcal{H}(x, \tilde{z}_n) + \frac{p-2}{2p} \mu |\tilde{z}_n|^p \right) \\ &\geq \int_{\mathbb{R}^N} \lim_{n \rightarrow \infty} \left(\mathcal{H}(x, \tilde{z}_n) + \frac{p-2}{2p} \mu |\tilde{z}_n|^p \right) \\ &= \Phi_\mu(\tilde{z}) - \frac{1}{2} \langle \Phi'_\mu(\tilde{z}), \tilde{z} \rangle = \Phi_\mu(\tilde{z}). \end{aligned}$$

This shows that $\Phi_\mu(\tilde{z}) \leq m_\mu$. Hence $\Phi_\mu(\tilde{z}) = m_\mu = \inf_{z \in \mathcal{M}_\mu} \Phi_\mu$, and $\tilde{z}$ is a ground state solution of the perturbation problem (1.3). $\square$

4 The proofs of Theorems 1.1 and 1.2

In this section, we give the proofs of Theorem 1.1 and Theorem 1.2. Indeed, for Theorem 1.1, we will make use of the conclusion of the perturbation problem (1.3), by passing to the limit as $\mu \rightarrow 0$ and some special techniques, to show that z_μ towards to a ground state solution z of the original problem. For Theorem 1.2, we will use a minimization method and concentration compactness argument to complete the proof.

Proof of Theorem 1.1. We note that $\mathcal{M} \neq \emptyset$ from Lemma 2.5. Let $z_0 \in \mathcal{M}$, then $\Phi(z_0) := c^* \geq 0$. In view of Lemma 3.7, there exist $t_\mu > 0$ and $w_\mu \in E^-$ such that $t_\mu z_0 + w_\mu \in \mathcal{M}_\mu$. On the other hand, (3.2) also holds for $\mu = 0$ by (f_3) . Thus, from Lemma 3.2 and Lemma 3.3, it follows that, for $\mu \in (0, 1]$,

$$\begin{aligned} c^* &= \Phi(z_0) = \Phi_0(z_0) \geq \Phi_0(t_\mu z_0 + w_\mu) \\ &\geq \Phi_\mu(t_\mu z_0 + w_\mu) \geq m_\mu \geq \kappa. \end{aligned} \quad (4.1)$$

Let $\{\mu_n\} \subset (0, 1]$ be a sequence such that $\mu_n \rightarrow 0^+$ as $n \rightarrow \infty$, and

$$z_{\mu_n} \in \mathcal{M}_{\mu_n}, \quad \Phi_{\mu_n}(z_{\mu_n}) = m_{\mu_n} \rightarrow \tilde{m} \in [\kappa, c^*], \quad \Phi'_{\mu_n}(z_{\mu_n}) = 0. \quad (4.2)$$

We denote $z_n := z_{\mu_n}$. In view of (4.2), we obtain

$$c^* \geq \Phi_{\mu_n}(z_n) - \frac{1}{2} \langle \Phi'_{\mu_n}(z_n), z_n \rangle = \int_{\mathbb{R}^N} \left(\mathcal{H}(x, z_n) + \frac{p-2}{2p} \mu_n |z_n|^p \right). \quad (4.3)$$

In what follows, we show that $\{z_n\}$ is bounded in E . Suppose to the contrary that $\|z_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Setting $w_n = z_n / \|z_n\|$, then $\|w_n\| = 1$. After passing to a subsequence, we assume that $w_n \rightarrow w$ in E , $w_n \rightarrow w$ in L^q_{loc} for $2 \leq q < 2^*$, and $w_n(x) \rightarrow w(x)$ a.e. on $\mathbb{R}^N$. Let

$$\delta := \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, 1)} |w_n^+|^2.$$

Similarly to the proof of Lemma 3.8, we can show that $\delta > 0$.

Passing if necessary to a subsequence, we assume $k_n \in \mathbb{Z}^N$ such that

$$\int_{B(k_n, 1 + \sqrt{N})} |w_n^+|^2 > \frac{\delta}{2}.$$

Let $\tilde{w}_n(x) = w_n(x + k_n)$, since the periodicity of $\vec{b}$ and V , then $\|w_n\| = \|\tilde{w}_n\| = 1$ and

$$\int_{B(0, 1 + \sqrt{N})} |\tilde{w}_n^+|^2 > \frac{\delta}{2}. \quad (4.4)$$

Up to a subsequence, we have $\tilde{w}_n \rightharpoonup \tilde{w}$ in E , $\tilde{w}_n \rightarrow \tilde{w}$ in L^q_{loc} for $2 \leq q < 2^*$ and $\tilde{w}_n \rightarrow \tilde{w}$ a.e. on $\mathbb{R}^N$. Moreover, (4.4) shows that $\tilde{w} \neq 0$. Let $\tilde{z}_n = z_n(x + k_n)$, then $\tilde{z}_n/\|z_n\| = \tilde{w}_n$ and $\tilde{w}_n \rightarrow \tilde{w}$ a.e. on $\mathbb{R}^N$. For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, let $\varphi_n = \varphi(x - k_n)$. By virtue of (4.2), we get

$$\begin{aligned} 0 &= \langle \Phi'_{\mu_n}(z_n), \varphi_n \rangle \\ &= (z_n^+, \varphi_n^+) - (z_n^-, \varphi_n^-) - \int_{\mathbb{R}^N} \left(H_z(x, z_n) + \mu_n |z_n|^{p-2} z_n \right) \varphi_n \\ &= (\tilde{z}_n^+, \varphi^+) - (\tilde{z}_n^-, \varphi^-) - \int_{\mathbb{R}^N} \left(H_z(x, \tilde{z}_n) + \mu_n |\tilde{z}_n|^{p-2} \tilde{z}_n \right) \varphi \\ &= \|z_n\| \left((\tilde{w}_n^+, \varphi^+) - (\tilde{w}_n^-, \varphi^-) \right) - \int_{\mathbb{R}^N} \left(H_z(x, \tilde{z}_n) + \mu_n |\tilde{z}_n|^{p-2} \tilde{z}_n \right) \varphi, \end{aligned}$$

which implies that

$$(\tilde{w}_n^+, \varphi^+) - (\tilde{w}_n^-, \varphi^-) = \frac{1}{\|z_n\|} \int_{\mathbb{R}^N} \left(H_z(x, \tilde{z}_n) + \mu_n |\tilde{z}_n|^{p-2} \tilde{z}_n \right) \varphi. \quad (4.5)$$

Before estimating the formula in the right hand side of (4.5), we need to show some estimates. On the one hand, for given $\sigma \in (0, 1)$, $R_0 > 0$ and $0 < |\tilde{z}_n| < R_0$, by (f_1) and (f_2) we obtain

$$\begin{aligned} \left(\frac{|H_z(x, \tilde{z}_n)|}{|\tilde{z}_n|^\sigma} + \mu_n |\tilde{z}_n|^{p-1-\sigma} \right) &= \left(\frac{|H_z(x, \tilde{z}_n)|}{|\tilde{z}_n|} |\tilde{z}_n|^{1-\sigma} + \mu_n |\tilde{z}_n|^{p-1-\sigma} \right) \\ &\leq c_1 |R_0|^{1-\sigma} + |R_0|^{p-1-\sigma} \\ &:= c_2. \end{aligned}$$

On the other hand, for $|\tilde{z}_n| \geq R_0$, since $2^*/(2^* - 1 - \sigma) > 1$ and $2^*(p - 1 - \sigma)/(2^* - 1 - \sigma) < p$, by (f_4) we have

$$\begin{aligned} \left(\frac{|H_z(x, \tilde{z}_n)|}{|\tilde{z}_n|^\sigma} + \mu_n |\tilde{z}_n|^{p-1-\sigma} \right)^{\frac{2^*}{2^*-1-\sigma}} &\leq c_3 \left(\left(\frac{|H_z(x, \tilde{z}_n)|}{|\tilde{z}_n|^\sigma} \right)^{\frac{2^*}{2^*-1-\sigma}} + \left(\mu_n |\tilde{z}_n|^{p-1-\sigma} \right)^{\frac{2^*}{2^*-1-\sigma}} \right) \\ &\leq c_3 \left(\frac{|H_z(x, \tilde{z}_n)|}{|\tilde{z}_n|^\sigma} \right)^{\frac{2^*}{2^*-1-\sigma}} + c_4 \left(\frac{p-2}{2p} \mu_n |\tilde{z}_n|^p \right) \\ &\leq c_5 \left(\mathcal{H}(x, \tilde{z}_n) + \frac{p-2}{2p} \mu_n |\tilde{z}_n|^p \right). \end{aligned}$$

Combining the above facts, (4.3) and the Hölder inequality, we deduce that

$$\begin{aligned}
 & \frac{1}{\|z_n\|} \int_{\mathbb{R}^N} \left| \left(H_z(x, \tilde{z}_n) + \mu_n |\tilde{z}_n|^{p-2} \tilde{z}_n \right) \varphi \right| \\
 & \leq \frac{1}{\|z_n\|^{1-\sigma}} \int_{\tilde{z}_n \neq 0} \left(\frac{|H_z(x, \tilde{z}_n)|}{|\tilde{z}_n|^\sigma} + \mu_n |\tilde{z}_n|^{p-1-\sigma} \right) |\tilde{w}_n|^\sigma |\varphi| \\
 & = \frac{1}{\|z_n\|^{1-\sigma}} \left(\int_{0 < |\tilde{z}_n| < R_0} + \int_{|\tilde{z}_n| \geq R_0} \right) \left(\frac{|H_z(x, \tilde{z}_n)|}{|\tilde{z}_n|^\sigma} + \mu_n |\tilde{z}_n|^{p-1-\sigma} \right) |\tilde{w}_n|^\sigma |\varphi| \\
 & \leq \frac{c_6 |\tilde{w}_n|_2^\sigma \|\varphi\|_{2/(2-\sigma)}}{\|z_n\|^{1-\sigma}} + \frac{|\tilde{w}_n|_{2^*}^\sigma \|\varphi\|_{2^*}}{\|z_n\|^{1-\sigma}} \left[\int_{|\tilde{z}_n| \geq R_0} \left(\frac{|H_z(x, \tilde{z}_n)|}{|\tilde{z}_n|^\sigma} + \mu_n |\tilde{z}_n|^{p-1-\sigma} \right)^{\frac{2^*}{2^*-1-\sigma}} \right]^{\frac{2^*-1-\sigma}{2^*}} \\
 & \leq \frac{c_7 \|\varphi\|_{2/(2-\sigma)}}{\|z_n\|^{1-\sigma}} + \frac{c_8 \|\varphi\|_{2^*}}{\|z_n\|^{1-\sigma}} \left[\int_{|\tilde{z}_n| \geq R_0} \left(\mathcal{H}(x, \tilde{z}_n) + \frac{p-2}{2p} \mu_n |\tilde{z}_n|^p \right) \right]^{\frac{2^*-1-\sigma}{2^*}} \\
 & \leq \frac{c_7 \|\varphi\|_{2/(2-\sigma)}}{\|z_n\|^{1-\sigma}} + \frac{c_8 \|\varphi\|_{2^*}}{\|z_n\|^{1-\sigma}} \left[\int_{\mathbb{R}^N} \left(\mathcal{H}(x, \tilde{z}_n) + \frac{p-2}{2p} \mu_n |\tilde{z}_n|^p \right) \right]^{\frac{2^*-1-\sigma}{2^*}} \\
 & \leq c_9 \|z_n\|^{\sigma-1} (\|\varphi\|_{2/(2-\sigma)} + \|\varphi\|_{2^*}) = o(1).
 \end{aligned}$$

Then (4.5) and the above estimate imply that

$$(\tilde{w}_n^+, \varphi^+) - (\tilde{w}_n^-, \varphi^-) = o(1), \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \quad (4.6)$$

According to the fact $\tilde{w}_n \rightharpoonup \tilde{w}$ in E , it follows from (4.6) that

$$(A\tilde{w}, \varphi)_2 = (\tilde{w}^+, \varphi^+) - (\tilde{w}^-, \varphi^-) = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

This shows that $A\tilde{w} = 0$ and hence $\tilde{w}$ is an eigenfunction of the operator A , which contradicts the fact that A has only continuous spectrum. Therefore, $\{z_n\}$ is bounded in E .

Next, we prove that there exists $\tilde{z} \in E$ such that $\Phi'(\tilde{z}) = 0$ and $\Phi(\tilde{z}) \geq m_0 = \inf_{\mathcal{M}_0} \Phi_0 = \inf_{\mathcal{M}} \Phi$. Indeed, similarly to the proof of Lemma 3.9, we claim that the vanishing does not occur. Then there exist a constant $\delta_1 > 0$ and a sequence $x_n \in \mathbb{Z}^N$ such that

$$\int_{B_1(x_n)} |z_n|^2 \geq \delta_1.$$

Let $\hat{z}_n(x) = z_n(x + x_n)$. Then $\|\hat{z}_n\| = \|z_n\|$ by the periodicity condition, and

$$\hat{z}_n \in \mathcal{M}_{\mu_n}, \quad \Phi_{\mu_n}(\hat{z}_n) = m_{\mu_n} \rightarrow \tilde{m} \in [\kappa, c^*], \quad \Phi'_{\mu_n}(\hat{z}_n) = 0. \quad (4.7)$$

Since $\{z_n\}$ is bounded in E , there exists $\tilde{z} \in E$ such that, passing to a subsequence if necessary, $\hat{z}_n \rightharpoonup \tilde{z}$ in E , $\hat{z}_n \rightarrow \tilde{z}$ in L^q_{loc} for $q \in [2, 2^*)$ and $\hat{z}_n(x) \rightarrow \tilde{z}(x)$ a.e. on $\mathbb{R}^N$. For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, it follows from (4.7) that

$$\begin{aligned}
 \langle \Phi'(\tilde{z}), \varphi \rangle &= (\tilde{z}^+, \varphi^+) - (\tilde{z}^-, \varphi^-) - \int_{\mathbb{R}^N} H_z(x, \tilde{z}) \varphi \\
 &= \lim_{n \rightarrow \infty} \left((\hat{z}_n^+, \varphi^+) - (\hat{z}_n^-, \varphi^-) - \int_{\mathbb{R}^N} (H_z(x, \hat{z}_n) + \mu_n |\hat{z}_n|^{p-2} \hat{z}_n) \varphi \right) \\
 &= \lim_{n \rightarrow \infty} \langle \Phi'_{\mu_n}(\hat{z}_n), \varphi \rangle = 0.
 \end{aligned}$$

This shows $\Phi'(\tilde{z}) = 0$. So, $\tilde{z} \in \mathcal{M}$ and $\Phi(\tilde{z}) \geq m_0$.

Finally, we verify that $\Phi(\tilde{z}) = m_0$. Indeed, by (2.3), (4.7) and Fatou's lemma, we obtain

$$\begin{aligned} \tilde{m} &= \lim_{n \rightarrow \infty} m_{\mu_n} = \lim_{n \rightarrow \infty} \left(\Phi_{\mu_n}(\hat{z}_n) - \frac{1}{2} \langle \Phi'_{\mu_n}(\hat{z}_n), \hat{z}_n \rangle \right) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left(\mathcal{H}(x, \hat{z}_n) + \frac{p-2}{2p} \mu_n |\hat{z}_n|^p \right) \\ &\geq \int_{\mathbb{R}^N} \mathcal{H}(x, \tilde{z}) \\ &= \Phi(\tilde{z}) - \frac{1}{2} \langle \Phi'(\tilde{z}), \tilde{z} \rangle \geq m_0. \end{aligned} \quad (4.8)$$

Let ϵ be any positive number. There exists $z_\epsilon \in \mathcal{M}$ such that $\Phi(z_\epsilon) < m_0 + \epsilon$. By Lemma 3.7, there exist $t_n > 0$ and $w_n \in E^-$ such that $t_n z_\epsilon + w_n \in \mathcal{M}_{\mu_n}$. Moreover, from (3.2), we have

$$m_0 + \epsilon > \Phi(z_\epsilon) = \Phi_0(z_\epsilon) \geq \Phi_0(t_n z_\epsilon + w_n) \geq \Phi_{\mu_n}(t_n z_\epsilon + w_n) \geq m_{\mu_n} \geq \kappa,$$

and this yields $\tilde{m} = \lim_{n \rightarrow \infty} m_{\mu_n} \leq m_0 + \epsilon$. According to the arbitrariness of ϵ , we have $\tilde{m} \leq m_0$. By (4.8), we have $\Phi(\tilde{z}) = \tilde{m} = m_0 > 0$. So, $\tilde{z}$ is a ground state solution of system (1.1). $\square$

Before proving Theorem 1.2, we need to show that the following result holds. Moreover, this result will be used in the proof of Theorem 1.3 and Theorem 1.4.

Lemma 4.1. *Under the assumptions of Theorem 1.1 or Theorem 1.2, the following two conclusions hold*

- (1) $\alpha := \inf\{\|z\| : z \in \mathcal{N}\} > 0$;
- (2) $c_0 := \inf\{\Phi(z) : z \in \mathcal{N}\} > 0$.

Proof. Conclusion (1). Let $\{z_n\} \subset \mathcal{N}$ such that $\|z_n\| \rightarrow \alpha$. Observe that

$$0 = \langle \Phi'(z_n), z_n^+ - z_n^- \rangle = \|z_n\|^2 - \int_{\mathbb{R}^N} H_z(x, z_n)(z_n^+ - z_n^-),$$

jointly with (2.1) and (2.2), which implies that

$$\|z_n\|^2 \leq \epsilon \pi_2^{-2} \|z_n\|^2 + c_\epsilon \pi_p^{-p} \|z_n\|^p,$$

where $p \in (2, 2^*)$. Taking $\epsilon = \frac{1}{2} \pi_2^2$, then

$$\|z_n\|^2 \leq \frac{1}{2} \|z_n\|^2 + c_{10} \|z_n\|^p,$$

and

$$\alpha + o(1) = \|z_n\| \geq (2c_{10})^{-1/(p-2)} > 0.$$

This shows that conclusion (1) holds.

Conclusion (2). First, we note that $\mathcal{N} \subset \mathcal{M}$, then $c_0 = \inf_{\mathcal{N}} \Phi \geq \inf_{\mathcal{M}} \Phi = m_0 > 0$. Hence, conclusion (2) holds under the assumptions of Theorem 1.1.

Next, we show that conclusion (2) holds under the assumptions of Theorem 1.2. Indeed, suppose to the contrary that there exists a sequence $\{z_n\} \subset \mathcal{N}$ such that $\Phi(z_n) \rightarrow 0$ and $\Phi'(z_n) = 0$. Clearly, $\{z_n\}$ is a $(C)_0$ -sequence of Φ . Moreover, by some arguments as in [33], $\{z_n\}$ is bounded in E . Together with conclusion (1), we know that $\alpha \leq \|z_n\| \leq C$ for some $C > 0$. Observe that

$$o(1) = \Phi(z_n) - \frac{1}{2} \langle \Phi'(z_n), z_n \rangle = \int_{\mathbb{R}^N} \mathcal{H}(x, z_n). \quad (4.9)$$

Let $w_n = z_n / \|z_n\|$, then $\|w_n\| = 1$. Set

$$\Xi_n := \left\{ x \in \mathbb{R}^N : \frac{|H_z(x, z_n)|}{|z_n|} \leq a - \delta_0 \right\}.$$

According to Lemma 2.2, there holds $\alpha|w_n|_2^2 \leq \|w_n\|^2$. Using this fact, we get

$$\int_{\Xi_n} \frac{|H_z(x, z_n)|}{|z_n|} |w_n| |w_n^+ - w_n^-| \leq (\alpha - \delta_0) |w_n|_2^2 \leq 1 - \frac{\delta_0}{\alpha}. \quad (4.10)$$

On the other hand, using the fact $\alpha \leq \|z_n\| \leq C$, (f_5) , (4.9) and Hölder inequality we obtain

$$\begin{aligned} & \frac{1}{\|z_n\|^{1-\sigma}} \int_{\mathbb{R}^N \setminus \Xi_n} \frac{|H_z(x, z_n)|}{|z_n|^\sigma} |w_n|^\sigma |w_n^+ - w_n^-| \\ & \leq \frac{1}{\|z_n\|^{1-\sigma}} \left(\int_{\mathbb{R}^N \setminus \Xi_n} \left(\frac{|H_z(x, z_n)|}{|z_n|^\sigma} \right)^{\frac{2^*}{2^*-1-\sigma}} \right)^{\frac{2^*-1-\sigma}{2^*}} |w_n|_2^{\sigma^*} |w_n^+ - w_n^-|_{2^*} \\ & \leq \frac{C_{11}}{\|z_n\|^{1-\sigma}} \left(\int_{\mathbb{R}^N \setminus \Xi_n} \mathcal{H}(x, z_n) \right)^{\frac{2^*-1-\sigma}{2^*}} = o(1). \end{aligned} \quad (4.11)$$

Therefore, it follows from (4.10) and (4.11) that

$$\begin{aligned} 1 &= \frac{\|z_n\|^2 - \langle \Phi'(z_n), z_n^+ - z_n^- \rangle}{\|z_n\|^2} = \left(\int_{\Xi_n} + \int_{\mathbb{R}^N \setminus \Xi_n} \right) \frac{H_z(x, z_n)(z_n^+ - z_n^-)}{\|z_n\|^2} \\ &\leq \int_{\Xi_n} \frac{|H_z(x, z_n)|}{|z_n|} |w_n| |w_n^+ - w_n^-| + \frac{1}{\|z_n\|^{1-\sigma}} \int_{\mathbb{R}^N \setminus \Xi_n} \frac{|H_z(x, z_n)|}{|z_n|^\sigma} |w_n|^\sigma |w_n^+ - w_n^-| \\ &\leq 1 - \frac{\delta_0}{\alpha} + o(1), \end{aligned}$$

which is a contradiction since $\delta_0 \in (0, \alpha)$. This shows that conclusion (2) holds. $\square$

Proof of Theorem 1.2. According to [33, Theorem 1.1], we have $\mathcal{N} \neq \emptyset$. Let $\{z_n\} \subset \mathcal{N}$ such that $\Phi(z_n) \rightarrow c_0$. Similarly to the proof [33, Lemma 2.3], we can prove that $\{z_n\}$ is bounded in E . Moreover, by Lions' concentration compactness principle, $\{z_n\}$ is nonvanishing. Hence, by some similar arguments as Lemma 3.9, we deduce that there exists $\tilde{z} \in E$ such that $\Phi'(\tilde{z}) = 0$. Moreover, Lemma 4.1 shows that $\Phi(\tilde{z}) = c_0 > 0$. $\square$

5 The proofs of Theorems 1.3 and 1.4

In this section, we are devoted to looking for infinitely many geometrically distinct solutions for system (1.1), and give the proofs of Theorem 1.3 and Theorem 1.4. To this end, we need some notations. For $d \geq e > -\infty$ we put

$$\Phi^d := \{z \in E : \Phi(z) \leq d\}, \quad \Phi_e := \{z \in E : \Phi(z) \geq e\}, \quad \Phi_e^d := \Phi_e \cap \Phi^d,$$

$$\mathcal{N} := \{z \in E \setminus \{0\} : \Phi'(z) = 0\}, \quad \mathcal{N}_c := \{z \in \mathcal{N} : \Phi(z) = c\}.$$

Using some standard arguments in [6, 26], we can obtain the following results without proof.

Lemma 5.1. Assume that (f_1) and (f_2) hold. If $z_n \rightharpoonup z$ in E , then, passing to a subsequence,

$$\lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} (H(x, z_n) - H(x, z) - H(x, z_n - z)) \right) = 0,$$

$$\lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} (H_z(x, z_n) - H_z(x, z) - H_z(x, z_n - z)) \varphi \right) = 0$$

uniformly in $\varphi \in E$.

Lemma 5.2. Assume that (f_1) and (f_2) hold. If $z_n \rightarrow z$ in E , then

$$\Phi(z_n) = \Phi(z) + \Phi(z_n - z) + o(1),$$

$$\Phi'(z_n) = \Phi'(z) + \Phi'(z_n - z) + o(1).$$

In the following we discuss further the $(C)_c$ -sequence. Let $[l]$ denote the integer part of $l \in \mathbb{R}$. Combining Lemma 4.1, Lemma 5.2 and some standard arguments, we have the following lemma (see Coti-Zelati and Rabinowitz [5]).

Lemma 5.3. Under the assumptions of Theorem 1.1 or Theorem 1.2, let $\{z_n\} \subset E$ be a $(C)_c$ -sequence of Φ . Then either

(i) $z_n \rightarrow 0$ (and hence $c = 0$); or

(ii) $c \geq c_0$ and there exist a positive integer $l \leq [\frac{c}{c_0}]$, $z_1, \dots, z_l \in \mathcal{N}$ and sequences $\{a_n^i\} \subset \mathbb{Z}^N$, $i = 1, 2, \dots, l$, such that, after extraction of a subsequence of $\{z_n\}$,

$$\left\| z_n - \sum_{i=1}^l a_n^i \star z_i \right\| \rightarrow 0 \text{ and } \sum_{i=1}^l \Phi(z_i) = c$$

and for $i \neq k$,

$$|a_n^i - a_n^k| \rightarrow \infty.$$

As in [5, 24], we choose a subset $\mathcal{F}$ of $\mathcal{N}$ such that $\mathcal{F} = -\mathcal{F}$ and each orbit $\mathcal{O}(w) \subset \mathcal{N}$ has a unique representative in $\mathcal{F}$. In order to prove Theorems 1.3 and 1.4, it suffices to show that the set $\mathcal{F}$ is infinite. Arguing by contradiction, we assume that

$$\mathcal{F} \text{ is a finite set.} \quad (5.1)$$

For any $c \geq c_0$, as in [5], let

$$\mathcal{F}_c := \left\{ \sum_{i=1}^j (a_i \star z_i) : 1 \leq j \leq \left[\frac{c}{c_0} \right], a_i \in \mathbb{Z}^N, z_i \in \mathcal{F} \right\}.$$

Following some arguments in [5] and [24], we have

Lemma 5.4. Let $c \geq c_0$. Then $y_c := \inf \{\|z_1 - z_2\| : z_1, z_2 \in \mathcal{F}_c, z_1 \neq z_2\} > 0$.

The following lemma plays an important role in expressing the discreteness property of the Cerami sequence.

Lemma 5.5. (Discreteness of Cerami sequences) Let $c \geq c_0$. If $\{z_n^1\}, \{z_n^2\} \subset \Phi_{c_0}^c$ are two Cerami sequences for Φ , then either $\|z_n^1 - z_n^2\| \rightarrow 0$ or $\limsup_{n \rightarrow \infty} \|z_n^1 - z_n^2\| \geq y_c$.

Proof. Since $\{z_n^1\}$ and $\{z_n^2\}$ are Cerami sequences, by some arguments used in the proofs of Theorems 1.1 and 1.2, we know that $\{z_n^1\}$ and $\{z_n^2\}$ are bounded. In view of Lemma 5.3, there exist two sequence $\{w_n^1\}, \{w_n^2\} \subset \mathcal{F}_c$ such that

$$\|z_n^i - w_n^i\| \rightarrow 0, \quad i = 1, 2. \quad (5.2)$$

On the other hand, from Lemma 5.4, we have $\|w_n^1 - w_n^2\| \rightarrow 0$ or $\limsup_{n \rightarrow \infty} \|w_n^1 - w_n^2\| \geq y_c$. Therefore, the conclusion holds from (5.2). $\square$

Since Φ is even, it is well known that Φ admits an odd pseudo-gradient vector field, i.e., there exists an odd locally Lipschitz continuous map $K : E \setminus (\mathcal{N} \cup \{0\}) \rightarrow E$ such that

$$\begin{cases} \|K(z)\| \leq 2\|\Phi'(z)\|, \\ \langle \Phi'(z), K(z) \rangle \geq \|\Phi'(z)\|^2. \end{cases} \quad (5.3)$$

Set

$$g(z) = \frac{(1 + \|z\|)K(z)}{\|\Phi'(z)\|}, \quad z \in E \setminus (\mathcal{N} \cup \{0\}). \quad (5.4)$$

For each $z \in E \setminus (\mathcal{N} \cup \{0\})$, now we consider the Cauchy problem

$$\begin{cases} \frac{d}{dt}\eta(t, z) = -g(\eta(t, z)), \\ \eta(0, z) = z. \end{cases} \quad (5.5)$$

The basic existence-uniqueness theorem for ordinary differential equations implies that, for each $z \in E \setminus (\mathcal{N} \cup \{0\})$, (5.5) has a unique solution $\eta(t, z)$ defined on $[0, \infty)$, and $\eta(t, z)$ is odd with respect to z .

Lemma 5.6. *Let $c > c_0$, $b \in (0, (c - c_0)/2)$ and $z \in E \setminus (\mathcal{N} \cup \{0\})$ be such that $c - b \leq \Phi(\eta(t, z)) \leq c + b$ for all $t \geq 0$. Then the limit $\lim_{t \rightarrow \infty} \eta(t, z)$ exists and is a critical point of Φ .*

Proof. We employ a similar argument in [26], for the completeness, we give the details. Observe that, from (5.3)-(5.5), we have

$$\begin{aligned} \frac{d}{dt}\Phi(\eta(t, z)) &= -\langle \Phi'(\eta(t, z)), g(\eta(t, z)) \rangle \\ &= -\frac{1 + \|\eta(t, z)\|}{\|\Phi'(\eta(t, z))\|} \langle \Phi'(\eta(t, z)), W(\eta(t, z)) \rangle \\ &\leq -\frac{1 + \|\eta(t, z)\|}{\|\Phi'(\eta(t, z))\|} \|\Phi'(\eta(t, z))\|^2 \\ &= -(1 + \|\eta(t, z)\|) \|\Phi'(\eta(t, z))\| \\ &\leq 0. \end{aligned}$$

This shows that $\Phi(\eta(t, z))$ is decreasing on $t \in [0, \infty)$, and so $\lim_{t \rightarrow \infty} \Phi(\eta(t, z))$ exists. To prove that $\lim_{t \rightarrow \infty} \eta(t, z)$ exists, it clearly suffices to show that

$$\text{for every } \varepsilon > 0, \text{ there exists } t_\varepsilon > 0 \text{ such that } \|\eta(t_\varepsilon, z) - \eta(t, z)\| < \varepsilon, \quad \forall t \geq t_\varepsilon. \quad (5.6)$$

We suppose by contradiction that (5.6) is false. Then there exist $0 < \varepsilon_0 < \frac{1}{2}y_{c+b}$ and a sequence $\{t_n\} \subset [0, \infty)$ with $t_n \rightarrow \infty$ and $\|\eta(t_{n+1}, z) - \eta(t_n, z)\| = \varepsilon_0$. Choose the smallest $t_n^1 \in (t_n, t_{n+1})$ and $s_n^1 \in [t_n, t_n^1]$ such that

$$\|\eta(t_n^1, z) - \eta(t_n, z)\| = \frac{\varepsilon_0}{3} \text{ and } \|\Phi'(\eta(s_n^1, z))\| = \min_{t \in [t_n, t_n^1]} \|\Phi'(\eta(t, z))\|. \quad (5.7)$$

Then it follows from (5.3)-(5.5) and (5.7) that

$$\begin{aligned}
 \frac{\varepsilon_0}{3} &= \|\eta(t_n^1, z) - \eta(t_n, z)\| \leq \int_{t_n}^{t_n^1} \|g(\eta(t, z))\| dt \\
 &= \int_{t_n}^{t_n^1} \frac{(1 + \|\eta(t, z)\|) \|K(\eta(t, z))\|}{\|\Phi'(\eta(t, z))\|} dt \\
 &\leq 2 \int_{t_n}^{t_n^1} \frac{(1 + \|\eta(t, z)\|) \|\Phi'(\eta(t, z))\|}{\|\Phi'(\eta(t, z))\|} dt \\
 &\leq \frac{2}{\|\Phi'(\eta(s_n^1, z))\|} \int_{t_n}^{t_n^1} \frac{(1 + \|\eta(t, z)\|) \|\Phi'(\eta(t, z))\|^2}{\|\Phi'(\eta(t, z))\|} dt \\
 &\leq \frac{2}{\|\Phi'(\eta(s_n^1, z))\|} \int_{t_n}^{t_n^1} \langle \Phi'(\eta(t, z)), g(\eta(t, z)) \rangle dt \\
 &= \frac{2}{\|\Phi'(\eta(s_n^1, z))\|} \left(\Phi(\eta(t_n, z)) - \Phi(\eta(t_n^1, z)) \right).
 \end{aligned}$$

Since $\lim_{t \rightarrow \infty} \Phi(\eta(t, z))$ exists, then $\Phi(\eta(t_n, z)) - \Phi(\eta(t_n^1, z)) \rightarrow 0$, and $\Phi'(\eta(s_n^1, z)) \rightarrow 0$ from the above inequality. Similarly, we find a largest $t_n^2 \in (t_n, t_{n+1})$ and $s_n^2 \in [t_n^2, t_{n+1}]$ such that $\|\eta(t_{n+1}, z) - \eta(t_n^2, z)\| = \frac{\varepsilon_0}{3}$ and $\Phi'(\eta(s_n^2, z)) \rightarrow 0$. Let $z_n^1 := \eta(s_n^1, z)$ and $z_n^2 := \eta(s_n^2, z)$. Since $\|\eta(s_n^1, z)\|$ and $\|\eta(s_n^2, z)\|$ are bounded, then $\{z_n^1\}$ and $\{z_n^2\}$ are two Cerami sequences for Φ such that $\frac{\varepsilon_0}{3} \leq \|z_n^1 - z_n^2\| \leq 2\varepsilon_0 < \gamma_{c+b}$. This however contradicts Lemma 5.5, therefore (5.6) is true. So $\lim_{t \rightarrow \infty} \eta(t, z)$ exists, and obviously it must be a critical point of Φ . $\square$

Lemma 5.7. Let $c > c_0$. If $\mathcal{N}_c = \emptyset$, then there exists $\varepsilon > 0$ such that $\lim_{t \rightarrow \infty} \Phi(\eta(t, z)) < c - \varepsilon$ for $z \in \Phi^{c+\varepsilon}$.

Proof. It follows from (5.1) and the translation invariance of Φ that $\Phi(\mathcal{N}) := \{\Phi(z) : z \in \mathcal{N}\}$ is a finite set. Hence, there exists $\varepsilon \in (0, (c - c_0)/2)$ such that $\Phi_{c-\varepsilon}^{c+\varepsilon} \cap \mathcal{N} = \emptyset$. We divide into two cases to finish the proof. For $z \in \Phi^{c+\varepsilon} \setminus \Phi_{c-\varepsilon}^{c+\varepsilon}$, then $\Phi(z) < c - \varepsilon$. By the monotonicity of $\Phi(\eta(t, \cdot))$, we have $\Phi(\eta(t, z)) \leq \Phi(z) < c - \varepsilon$ for all $t \geq 0$. Obviously, the desired conclusion holds. For $z \in \Phi_{c-\varepsilon}^{c+\varepsilon}$, if $\lim_{t \rightarrow \infty} \Phi(\eta(t, z)) \geq c - \varepsilon$, then there exists $t_0 > 0$ such that $\Phi(\eta(t, z)) \geq c - \varepsilon$ for all $t \geq t_0$. Therefore, by Lemma 5.6 we know that $\lim_{t \rightarrow \infty} \Phi(\eta(t, z)) := z_\infty \in \Phi_{c-\varepsilon}^{c+\varepsilon} \cap \mathcal{N}$, this contradicts with $\Phi_{c-\varepsilon}^{c+\varepsilon} \cap \mathcal{N} = \emptyset$. So, the desired conclusion holds. The proof is completed. $\square$

In the following, for a subset $P \subset E$ and $\delta > 0$, we define $U_\delta(P) := \{w \in E : \text{dist}(w, P) < \delta\}$. Using the deformation arguments from [28], we have the following result.

Lemma 5.8. Let $c \geq c_0$. Then for every $\delta \in (0, \gamma_c/4)$, there is $\varepsilon > 0$ and an odd and continuous map $\phi : \Phi^{c+\varepsilon} \setminus U_\delta(\mathcal{F}_c) \rightarrow \Phi^{c-\varepsilon}$.

Proof. We fix $\delta \in (0, \gamma_c/4)$ and let $S = E \setminus U_\delta(\mathcal{F}_c)$. We first show that there exists $\varepsilon > 0$ such that

$$(1 + \|z\|) \|\Phi'(z)\| \geq 8\varepsilon \text{ for all } z \in \Phi_{c-2\varepsilon}^{c+2\varepsilon} \cap S_{2\varepsilon}, \quad (5.8)$$

where $S_{2\varepsilon} := \{z \in E : \|z - w\| \leq 2\varepsilon, w \in S\}$. Suppose by contradiction that there exist sequences of numbers $\varepsilon_n > 0$ and functions $z_n \in S_{2\varepsilon_n}$ such that $\varepsilon_n \rightarrow 0$ and

$$(1 + \|z_n\|) \|\Phi'(z_n)\| < 8\varepsilon_n \text{ and } c - 2\varepsilon_n \leq \Phi(z_n) \leq c + 2\varepsilon_n,$$

then $\{z_n\}$ is a $(C)_c$ -sequence. By Lemma 5.3, up to a subsequence, there exists $\{w_n\} \subset \mathcal{F}_c$ such that $\|z_n - w_n\| \rightarrow 0$. Moreover, there exists $u_n \in S$ such that $\|z_n - u_n\| \leq 2\varepsilon_n$ for $n \in \mathbb{N}$. Since $w_n \in \mathcal{F}_c$ and $u_n \in S$, from the above facts, we deduce that

$$0 < \delta \leq \|u_n - w_n\| \leq \|z_n - u_n\| + \|z_n - w_n\| \leq 2\varepsilon_n + o(1) = o(1).$$

This contradiction shows that (5.8) holds. Since Φ is even, using the deformation lemma from [28], there exists an odd and continuous function $\eta : [0, 1] \times E \rightarrow E$ such that

- (i) $\eta(t, z) = z$ if $t = 0$ or $z \notin \Phi_{c-\varepsilon}^{c+\varepsilon} \cap S_{2\varepsilon}$;
- (ii) $\eta(1, \Phi^{c+\varepsilon} \cap S) \subset \Phi^{c-\varepsilon}$;
- (iii) $t \mapsto \Phi(\eta(t, z))$ is nonincreasing for all $z \in E$.

Let $\phi(z) = \eta(1, z)$, hence it follows from the above conclusions that ϕ has the asserted properties. $\square$

Let $\mathcal{A} \subset E \setminus \{0\}$ be a closed and symmetric subset (i.e., $\mathcal{A} = -\mathcal{A} = \bar{\mathcal{A}}$) and let $y(\mathcal{A})$ denote the usual Krasnoselskii genus of $\mathcal{A}$ (see [22]).

Lemma 5.9. *Let $c \geq c_0$. Then for every $\delta \in (0, y_c/4)$, $y(\overline{U_\delta(\mathcal{F}_c)}) = 1$.*

Proof. Since $\mathcal{F}$ is a finite set and symmetric, then we can assume that $\mathcal{F}_c = \{\pm z_n : n \in \mathbb{N}\}$. Plainly, $\mathcal{F}_c$ is also finite and symmetric. Moreover, by [22, Example 7.2], we know $y(\mathcal{F}_c) = 1$. Additionally, since $\delta \in (0, y_c/4)$, then $\overline{U_\delta(\mathcal{F}_c)}$ is a closed and symmetric set. Therefore, it follows from the continuity property of the genus that $y(\overline{U_\delta(\mathcal{F}_c)}) = 1$. $\square$

Proof of Theorems 1.3 and 1.4. For $j \in \mathbb{N}$, we consider the family Σ_j of all closed and symmetric subsets $\mathcal{A} \subset E \setminus \{0\}$ with $y(\mathcal{A}) \geq j$. Moreover, we consider the nondecreasing sequence of Lusternik-Schnirelman values for Φ defined by

$$c_k := \inf\{c \geq c_0 : y(\Phi^c) \geq k\}, \quad k \in \mathbb{N}.$$

We claim

$$\mathcal{N}_{c_k} \neq \emptyset \text{ and } c_k < c_{k+1} \text{ for all } k \in \mathbb{N}. \quad (5.9)$$

Suppose by contradiction that $\mathcal{N}_{c_k} = \emptyset$ for some $k \in \mathbb{N}$. Then $c_k > c_0$. According to Lemma 5.7, there exists $\varepsilon > 0$ such that $\lim_{t \rightarrow \infty} \Phi(\eta(t, z)) < c_k - \varepsilon$ for $z \in \Phi^{c_k+\varepsilon}$. Let $t_z := \inf\{t \geq 0 : \Phi(\eta(t, z)) \leq c_k - \varepsilon\}$. Since Φ is even and $\eta(t, z)$ is odd with respect to z , it implies that $t_z = t_{-z}$. Define a map

$$\zeta : \Phi^{c_k+\varepsilon} \rightarrow \Phi^{c_k-\varepsilon}, \quad \zeta(z) = \eta(t_z, z).$$

Then ζ is odd and continuous. Hence, by the mapping property of the genus, we have $y(\Phi^{c_k+\varepsilon}) \leq y(\Phi^{c_k-\varepsilon})$. Obviously, this contradicts with the definition of c_k .

Next, we show $c_k < c_{k+1}$ for all $k \in \mathbb{N}$. Indeed, for each k , from Lemma 5.8, it follows that there exist $\delta_k \in (0, y_{c_k}/4)$, $\varepsilon_k > 0$ and an odd and continuous map $\phi : \Phi^{c_k+\varepsilon_k} \setminus U_{\delta_k}(\mathcal{F}_{c_k}) \rightarrow \Phi^{c_k-\varepsilon_k}$. Again by the mapping property of the genus, we have $y(\Phi^{c_k+\varepsilon_k} \setminus U_{\delta_k}(\mathcal{F}_{c_k})) \leq y(\Phi^{c_k-\varepsilon_k}) \leq k-1$. Moreover, by the subadditivity of the genus and Lemma 5.9, we obtain

$$k \leq y(\Phi^{c_k+\varepsilon_k}) \leq y(\overline{U_{\delta_k}(\mathcal{F}_{c_k})}) + k - 1 = k,$$

which implies that $y(\Phi^{c_k+\varepsilon_k}) = k$. On the other hand, note that $c_{k+1} \geq c_k$. If $c_{k+1} = c_k$, then $y(\Phi^{c_k+\varepsilon_k}) \geq k+1$, a contradiction. Hence, $c_{k+1} > c_k$.

Finally, it follows from (5.9) that there is an infinite sequence $\{z_k\}$ of pairs of geometrically distinct critical points of Φ with $\Phi(z_k) = c_k$, contrary to (5.1). The proof is finished.

Acknowledgement: This work was supported by the NNSF (No. 11601145, 11701173), by the Project of China Postdoctoral Science Foundation (2019M652790), by the Natural Science Foundation of Hunan Province (2018JJ2198, 2019JJ40142), and by the Scientific Research Project of Hunan Province Education Department (18B342, 19C1049).

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