Research article

Józef Banaś and Weronika Woś*

Solvability of an infinite system of integral equations on the real half-axis

<https://doi.org/10.1515/anona-2020-0114>

Received March 7, 2020; accepted April 16, 2020.

Abstract: The aim of the paper is to investigate the solvability of an infinite system of nonlinear integral equations on the real half-axis. The considerations will be located in the space of function sequences which are bounded at every point of the half-axis. The main tool used in the investigations is the technique associated with measures of noncompactness in the space of functions defined, continuous and bounded on the real half-axis with values in the space l_∞ consisting of real bounded sequences endowed with the standard supremum norm. The essential role in our considerations is played by the fact that we will use a measure of noncompactness constructed on the basis of a measure of noncompactness in the mentioned sequence space l_∞ . An example illustrating our result will be included.

Keywords: Space of functions continuous and bounded on the half-axis; Sequence space; Measure of noncompactness; Fixed point theorem of Darbo type; Infinite system of integral equations

MSC 2010: Primary 47H08, Secondary 45G1

1 Introduction

The paper is dedicated to the study of the existence of solutions of an infinite system of nonlinear integral equations on the real half-axis $\mathbb{R}_+ = [0, \infty)$. More precisely, we will look for solutions of the mentioned infinite system in the space $BC(\mathbb{R}_+, l_\infty)$ consisting of functions defined, continuous and bounded on the interval $\mathbb{R}_+$ with values in the classical sequence space l_∞ which is equipped with the standard supremum norm. Thus, any solution of our infinite system of integral equations will be treated as a sequence of functions $(x_n(t))$ defined on $\mathbb{R}_+$ and such that for any fixed $t \in \mathbb{R}_+$ the sequence $(x_n(t))$ is an element of the space l_∞ . Further details concerning the properties of the solutions in question will be formulated in the sequel of our paper.

The present paper is a continuation of papers [1, 2], where we constructed measures of noncompactness needed in our study. Particularly, in paper [2] we constructed measures of noncompactness in the Banach space $BC(\mathbb{R}_+, E)$ containing functions defined, continuous and bounded on the interval $\mathbb{R}_+$ and taking values in a given Banach space E . The construction of those measures depends strongly on a given measure of noncompactness in the space E . Additionally, in the mentioned paper [2] we applied one of the constructed measures of noncompactness to the study of the solvability of an infinite system of integral equations in the space $BC(\mathbb{R}_+, l_\infty)$.

In this paper we are going to study a more general infinite system of nonlinear integral equations in the same Banach space $BC(\mathbb{R}_+, l_\infty)$ but with the use of another measure of noncompactness. Such an approach allows us to obtain an existence result concerning the mentioned infinite system, but under weaker assump-

Józef Banaś, Department of Nonlinear Analysis, Rzeszów University of Technology, al. Powstańców Warszawy 8, 35-959 Rzeszów, Poland, E-mail: jbanas@prz.edu.pl

*Corresponding Author: Weronika Woś, Faculty of Mechanics and Technology, Rzeszów University of Technology - branch in Stalowa Wola, ul. Kwiatkowskiego 4, 37-450 Stalowa Wola, Poland, E-mail: weronikawos@prz.edu.pl

tions than the existence result obtained in [2]. Thus, our result creates a generalization of the existence result contained in paper [2]. It is also worthwhile mentioning that in paper [1] it was also studied the solvability of an infinite system of integral equations in the Banach space $BC(\mathbb{R}_+, E)$ but we assumed that E is a Banach space with a regular measure of noncompactness being equivalent to the so-called Hausdorff measure of noncompactness. Let us point out that in some Banach spaces such measures of noncompactness are not known [3].

Finally, let us mention that the theory of infinite systems of integral equations has recently been intensively developed and up to now there have appeared a lot of papers concerning those infinite systems [4–9]. That theory is closely related to the theory of infinite systems of differential equations (cf. [4, 7, 10] and references therein).

However, in the papers published up to now the authors investigated mainly infinite systems of integral equations in the Banach space $C([0, T], E)$ consisting of functions defined and continuous on the bounded interval $[0, T]$ with values in a Banach space E . The generalization to the space $BC(\mathbb{R}_+, E)$ is rather a quite new branch of the theory of infinite systems of integral equations (cf. [1, 2]).

2 Notations, definitions and auxiliary facts

In this section we establish the notations used in the paper and we provide definitions creating the basis of our study conducted further on. Apart from this we give some facts concerning the theory of measures of noncompactness being the basic tool utilized in our considerations.

In the paper we will denote by $\mathbb{R}$ the set of real numbers while $\mathbb{N}$ stands for the set of natural numbers. We will put $\mathbb{R}_+ = [0, \infty)$. Further assume that E is a Banach space with the norm $\|\cdot\|_E$ and the zero element θ . We will denote by $B(x, r)$ the closed ball centered at x and with radius r . We write B_r to denote the ball $B(\theta, r)$. If X, Y are subsets of the Banach space E and $\lambda \in \mathbb{R}$ then the standard algebraic operations on sets will be denoted by $X + Y$ and λX . Moreover, the symbol $\bar{X}$ denotes the closure of the set X while $\text{Conv} X$ stands for the closed convex hull of the set X .

Next, denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

In our considerations we will accept the following definition of the concept of a measure of noncompactness (cf. [3, 4]).

Definition 2.1. A function $\mu : \mathfrak{M}_E \rightarrow \mathbb{R}_+$ will be called a measure of noncompactness in the space E if it satisfies the following conditions:

- (i) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
- (ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
- (iii) $\mu(\bar{X}) = \mu(X)$.
- (iv) $\mu(\text{Conv } X) = \mu(X)$.
- (v) $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
- (vi) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and if $\lim_{n \rightarrow \infty} \mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ from axiom (i) will be called the kernel of the measure of noncompactness μ . Let us also observe that the set X_∞ defined in axiom (vi) is an element of the family $\ker \mu$. In fact, it follows easily from the inclusion $X_\infty \subset X_n$ for $n = 1, 2, \dots$ and axiom (ii) which implies the inequality $\mu(X_\infty) \leq \mu(X_n)$ for any $n \in \mathbb{N}$. Hence we have that $\mu(X_\infty) = 0$ and consequently, $X_\infty \in \ker \mu$. This simple conclusion is very crucial in applications.

In what follows assume that μ is a measure of noncompactness in the space E . The measure μ is called sublinear [3] if it satisfies the following additional conditions

- (vii) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
 (viii) $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$.

If the measure of noncompactness μ satisfies the condition

$$(ix) \mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$$

then we say that it has the maximum property. Moreover, if $\ker \mu = \mathfrak{N}_E$ we say that μ is full. If μ is a sublinear and full measure of noncompactness with the maximum property, then μ is said to be regular.

Let us recall that from the historical point of view the first measure of noncompactness was defined in 1930 by K. Kuratowski [11], but the most important and useful measure of noncompactness is the so-called Hausdorff measure of noncompactness which was defined in [12, 13] by the formula

$$\chi(X) = \inf \{ \varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in } E \},$$

where $X \in \mathfrak{M}_E$. The importance of the Hausdorff measure χ is caused by the fact that in some Banach spaces like $C([a, b])$, c_0 and l_p we can give formulas expressing the measure χ in connection with the structure of the mentioned Banach spaces. Let us mention that χ is a regular measure of noncompactness [3, 14, 15].

However, in a lot of Banach spaces we are not in a position to give formulas for the Hausdorff measure of noncompactness χ . Even more, we are not able to provide formulas for full measures of noncompactness [3, 4]. Therefore, in such a situation we restrict ourselves to measures of noncompactness in the sense of Definition 2.1, which are not full (cf. also [3, 4, 14]). In the present paper we will also consider a measure of noncompactness of such a type.

Namely, assume that E is a given Banach space with the norm $\|\cdot\|_E$ and μ is a measure of noncompactness in the space E . Consider the Banach space $BC(\mathbb{R}_+, E)$ consisting of all functions $x : \mathbb{R}_+ \rightarrow E$ which are continuous and bounded on the interval $\mathbb{R}_+$. The space $BC(\mathbb{R}_+, E)$ will be equipped with the standard supremum norm

$$\|x\|_\infty = \sup \{ \|x(t)\|_E : t \in \mathbb{R}_+ \}.$$

Further, take an arbitrary nonempty and bounded subset X of the space $BC(\mathbb{R}_+, E)$. Fix $x \in X$ and $\varepsilon > 0$. We define the modulus of the uniform continuity of the function x (cf. [2]) by putting

$$\omega^\infty(x, \varepsilon) = \sup \{ \|x(t) - x(s)\|_E : t, s \in \mathbb{R}_+, |t - s| \leq \varepsilon \}.$$

Obviously, $\lim_{\varepsilon \rightarrow 0} \omega^\infty(x, \varepsilon) = 0$ if and only if the function x is uniformly continuous on the interval $\mathbb{R}_+$.

Further, let us define the following quantities

$$\begin{aligned} \omega^\infty(X, \varepsilon) &= \sup \{ \omega^\infty(x, \varepsilon) : x \in X \}, \\ \omega_0^\infty(X) &= \lim_{\varepsilon \rightarrow 0} \omega^\infty(X, \varepsilon). \end{aligned} \quad (2.1)$$

Next, let us consider the function $\bar{\mu}_\infty$ defined on the family $\mathfrak{M}_{BC(\mathbb{R}_+, E)}$ in the following way

$$\bar{\mu}_\infty(X) = \lim_{T \rightarrow \infty} \bar{\mu}_T(X), \quad (2.2)$$

where $\bar{\mu}_T(X)$ is defined by the formula

$$\bar{\mu}_T(X) = \sup \{ \mu(X(t)) : t \in [0, T] \}$$

for any fixed $T > 0$.

Finally, for a given $T > 0$ let us put

$$a_T(X) = \sup_{x \in X} \left\{ \sup \{ \|x(t)\|_E : t \geq T \} \right\}$$

and

$$a_\infty(X) = \lim_{T \rightarrow \infty} a_T(X). \quad (2.3)$$

Now, linking quantities defined by (2.1), (2.2) and (2.3), we can consider the following function μ_a defined on the family $\mathfrak{M}_{BC(\mathbb{R}_+, E)}$:

$$\mu_a(X) = \omega_0^\infty(X) + \bar{\mu}_\infty(X) + a_\infty(X). \quad (2.4)$$

It can be shown that the function μ_a is a measure of noncompactness in the space $BC(\mathbb{R}_+, E)$ (cf. [2]). The kernel $\ker \mu_a$ of the measure μ_a consists of all bounded subsets X of the space $BC(\mathbb{R}_+, E)$ such that functions from X are uniformly continuous and equicontinuous on $\mathbb{R}_+$ (equivalently we can say that functions from X are equiuniformly continuous on $\mathbb{R}_+$) and tend to zero at infinity with the same rate. Apart from this, all cross-sections $X(t) = \{x(t) : x \in X\}$ of the set X belong to the kernel $\ker \mu$ of the measure of noncompactness μ in the Banach space E (cf. [2]). The measure μ_a is not full and has the maximum property. If the measure μ is sublinear in E then the measure μ_a defined by (2.4) is also sublinear [2].

Let us notice that in the similar way as above we may define other measures of noncompactness in the space $BC(\mathbb{R}_+, E)$ (see [2]). We will not provide the definitions of those measures since we will use only the measure of noncompactness μ_a further on.

In what follows, taking into account our further purposes, we will consider as the Banach space E , the sequence space l_∞ endowed with the standard supremum norm.

Thus, let us consider the Banach space $BC(\mathbb{R}_+, l_\infty)$ consisting of functions $x : \mathbb{R}_+ \rightarrow l_\infty$ which are continuous and bounded on $\mathbb{R}_+$. Observe, that if $x \in BC(\mathbb{R}_+, l_\infty)$, then we can write this function in the form

$$x(t) = (x_n(t)) = (x_1(t), x_2(t), \dots)$$

for any $t \in \mathbb{R}_+$, where the sequence $(x_n(t))$ is an element of the space l_∞ for any fixed t . The norm of the function $x = x(t) = (x_n(t))$ is defined by the equality

$$\|x\|_\infty = \sup \{ \|x(t)\|_{l_\infty} : t \in \mathbb{R}_+ \} = \sup_{t \in \mathbb{R}_+} \left\{ \sup \{ |x_n(t)| : n = 1, 2, \dots \} \right\}.$$

In our further considerations the space $BC(\mathbb{R}_+, l_\infty)$ will be denoted by BC_∞ .

Now, we provide the formula expressing the measure of noncompactness μ_a defined by (2.4) in the space BC_∞ , provided the measure of noncompactness μ in the sequence space l_∞ is defined by the formula [3]

$$\mu^1(X) = \lim_{n \rightarrow \infty} \left\{ \sup_{x=(x_i) \in X} \left\{ \sup \{ |x_k| : k \geq n \} \right\} \right\}$$

for $X \in \mathfrak{M}_{l_\infty}$. In such a case the component $\bar{\mu}_\infty$ defined by (2.2) will be denoted by $\bar{\mu}_\infty^1$.

Thus, our measure of noncompactness μ_a defined by (2.4) will be now denoted by μ_a^1 and is defined as a particular case of (2.4) by the formula

$$\mu_a^1(X) = \omega_0^\infty(X) + \bar{\mu}_\infty^1(X) + a_\infty(X), \quad (2.5)$$

where the components on the right hand side of formula (2.5) are defined in the following way:

$$\omega_0^\infty(X) = \lim_{\varepsilon \rightarrow 0} \left\{ \sup_{x \in X} \left\{ \sup \left\{ \sup_{n \in \mathbb{N}} |x_n(t) - x_n(s)| : t, s \in \mathbb{R}_+, |t - s| \leq \varepsilon \right\} \right\} \right\}, \quad (2.6)$$

$$\bar{\mu}_\infty^1(X) = \lim_{T \rightarrow \infty} \left\{ \sup_{t \in [0, T]} \left\{ \lim_{n \rightarrow \infty} \left\{ \sup_{x \in X} \left\{ \sup \{ |x_k(t)| : k \geq n \} \right\} \right\} \right\} \right\}, \quad (2.7)$$

$$a_\infty(X) = \lim_{T \rightarrow \infty} \left\{ \sup_{x \in X} \left\{ \sup \left\{ \sup_{n \in \mathbb{N}} |x_n(t)| : t \geq T \right\} \right\} \right\}. \quad (2.8)$$

In what follows we give an other formula expressing the quantity $\bar{\mu}_\infty$ defined by (2.2). To this end we prove the following lemma.

Lemma 2.2. *The following equality is satisfied*

$$\bar{\mu}_\infty(X) = \sup \{ \mu(X(t)) : t \in \mathbb{R}_+ \},$$

where $\bar{\mu}_\infty$ is defined by formula (2.2).

Proof. Obviously, for any fixed $T > 0$ we have

$$\sup \{ \mu(X(t)) : t \in [0, T] \} \leq \sup \{ \mu(X(t)) : t \in \mathbb{R}_+ \}.$$

Hence, we get

$$\bar{\mu}_\infty(X) = \lim_{T \rightarrow \infty} \left\{ \sup \{ \mu(X(t)) : t \in [0, T] \} \right\} \leq \sup \{ \mu(X(t)) : t \in \mathbb{R}_+ \}. \quad (2.9)$$

To prove the converse inequality, let us denote

$$\delta = \sup \{ \mu(X(t)) : t \in \mathbb{R}_+ \}.$$

Further, fix an arbitrary number $\varepsilon > 0$. Then we can find a number $t_0 \in \mathbb{R}_+$ such that

$$\delta - \varepsilon \leq \mu(X(t_0)).$$

Hence, for $T \geq t_0$ we obtain

$$\delta - \varepsilon \leq \sup \{ \mu(X(t)) : t \in [0, T] \}. \quad (2.10)$$

Since the function $T \rightarrow \sup \{ \mu(X(t)) : t \in [0, T] \}$ is nondecreasing, we get

$$\sup \{ \mu(X(t)) : t \in [0, T] \} \leq \lim_{T \rightarrow \infty} \left\{ \sup \{ \mu(X(t)) : t \in [0, T] \} \right\}. \quad (2.11)$$

Combining (2.10) and (2.11), we have

$$\delta - \varepsilon \leq \lim_{T \rightarrow \infty} \left\{ \sup \{ \mu(X(t)) : t \in [0, T] \} \right\}.$$

Consequently, in view of the arbitrariness of the number ε , we derive the following inequality

$$\delta \leq \lim_{T \rightarrow \infty} \left\{ \sup \{ \mu(X(t)) : t \in [0, T] \} \right\} = \bar{\mu}_\infty(X). \quad (2.12)$$

Finally, linking (2.9) and (2.12) we obtain the desired equality. $\square$

Now, let us notice that taking into account Lemma 2.2 and formula (2.7) expressing the quantity $\bar{\mu}_\infty$ in the case of the space BC_∞ , we obtain the following corollary.

Corollary 2.3. *The quantity (2.7) can be expressed by the following formula*

$$\bar{\mu}_\infty^1(X) = \sup_{t \geq 0} \left\{ \lim_{n \rightarrow \infty} \left\{ \sup_{x \in X} \left\{ \sup \{ |x_k(t)| : k \geq n \} \right\} \right\} \right\}.$$

At the end of this section we recall a useful fixed point theorem of Darbo type [3, 16].

To this end let us assume that E is a Banach space and μ is a measure of noncompactness (in the sense of Definition 2.1) in the space E .

Theorem 2.4. *Assume that Ω is a nonempty, bounded, closed and convex subset of a Banach space E and $Q : \Omega \rightarrow \Omega$ is a continuous operator such that there exists a constant $k \in [0, 1)$ for which $\mu(QX) \leq k\mu(X)$ for an arbitrary nonempty subset X of Ω . Then there exists at least one fixed point of the operator Q in the set Ω .*

Remark 2.5. *It can be shown that the set $\text{Fix } Q$ of all fixed points of the operator Q belongs to the family $\ker \mu$ [3].*

This simple observation is very essential in characterization of solutions of considered operator equations which are proved with help of Theorem 2.4.

3 Solvability of an infinite system of quadratic integral equations on the real half-axis

The aim of this section is to investigate the infinite system of the quadratic integral equations of Volterra-Hammerstein type having the form

$$x_n(t) = a_n(t) + f_n(t, x_1(t), x_2(t), \dots) \int_0^t k_n(t, s) g_n(s, x_1(s), x_2(s), \dots) ds, \quad (3.1)$$

where $t \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

Our goal is to prove the existence of solutions of the infinite system of integral equations (3.1).

Considerations conducted further on will be situated in the Banach space $BC_\infty = BC(\mathbb{R}_+, l_\infty)$ described in details previously. Moreover, in our study we are going to use the measure of noncompactness $\mu_a^1(X)$ defined by formula (2.5).

Now, we formulate the assumptions under which the infinite system (3.1) will be considered.

- (i) The sequence $(a_n(t))$ is an element of the space BC_∞ such that $\lim_{t \rightarrow \infty} a_n(t) = 0$ uniformly with respect to $n \in \mathbb{N}$ i.e., the following condition is satisfied

$$\forall \varepsilon > 0 \exists T > 0 \forall t \geq T \forall n \in \mathbb{N} |a_n(t)| \leq \varepsilon.$$

Moreover, $\lim_{n \rightarrow \infty} a_n(t) = 0$ for any $t \in \mathbb{R}_+$.

- (ii) The functions $k_n(t, s) = k_n : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ are continuous on the set $\mathbb{R}_+^2$ ($n = 1, 2, \dots$). Moreover, the functions $t \rightarrow k_n(t, s)$ are equicontinuous on the set $\mathbb{R}_+$ uniformly with respect to $s \in \mathbb{R}_+$ i.e., the following condition is satisfied

$$\forall \varepsilon > 0 \exists \delta > 0 \forall n \in \mathbb{N} \forall s \in \mathbb{R}_+ \forall t_1, t_2 \in \mathbb{R}_+ \left[|t_2 - t_1| \leq \delta \Rightarrow |k_n(t_2, s) - k_n(t_1, s)| \leq \varepsilon \right].$$

- (iii) There exists a constant $K_1 > 0$ such that

$$\int_0^t |k_n(t, s)| ds \leq K_1$$

for any $t \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

- (iv) The sequence $(k_n(t, s))$ is equibounded on $\mathbb{R}_+^2$ i.e., there exists a constant $K_2 > 0$ such that $|k_n(t, s)| \leq K_2$ for $t, s \in \mathbb{R}_+$ and $n = 1, 2, \dots$.
- (v) The function f_n is defined on the set $\mathbb{R}_+ \times \mathbb{R}^\infty$ and takes real values for $n = 1, 2, \dots$. Moreover, the function $t \rightarrow f_n(t, x_1, x_2, \dots)$ is uniformly continuous on $\mathbb{R}_+$ uniformly with respect to $x = (x_i) \in l_\infty$ and uniformly with respect to $n \in \mathbb{N}$ i.e., the following condition is satisfied

$$\begin{aligned} & \forall \varepsilon > 0 \exists \delta > 0 \forall (x_i) \in l_\infty \forall n \in \mathbb{N} \forall t, s \in \mathbb{R}_+ \left[|t - s| \leq \delta \right. \\ & \quad \left. \Rightarrow |f_n(t, x_1, x_2, \dots) - f_n(s, x_1, x_2, \dots)| \leq \varepsilon \right]. \end{aligned}$$

- (vi) There exists a function $l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that l is nondecreasing on $\mathbb{R}_+$, continuous at 0 and there exists a sequence of functions $(\bar{f}_n)$ being an element of the space BC_∞ , taking nonnegative values and such that $\lim_{t \rightarrow \infty} \bar{f}_n(t) = 0$ uniformly with respect to $n \in \mathbb{N}$ (cf. assumption (i)) and $\lim_{n \rightarrow \infty} \bar{f}_n(t) = 0$ for any $t \in \mathbb{R}_+$. Moreover, for any $r > 0$ the following inequality is satisfied

$$|f_n(t, x_1, x_2, \dots)| \leq \bar{f}_n(t) + l(r) \sup \{ |x_i| : i \geq n \}$$

for each $x = (x_i) \in l_\infty$ such that $\|x\|_{l_\infty} \leq r$, for every $t \in \mathbb{R}_+$ and for $n = 1, 2, \dots$.

Observe that in view of assumption (vi) we can define the finite constant $\bar{F} = \sup \{\bar{f}_n(t) : t \in \mathbb{R}_+, n = 1, 2, \dots\}$.

Now, we can formulate the other assumptions concerning the infinite system (3.1).

(vii) There exists a function $m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that m is nondecreasing on $\mathbb{R}_+$, continuous at 0 and the following inequality is satisfied

$$|f_n(t, x_1, x_2, \dots) - f_n(t, y_1, y_2, \dots)| \leq m(r)\|x - y\|_{l_\infty}$$

for any $r > 0$, for $x = (x_i)$, $y = (y_i) \in l_\infty$ such that $\|x\|_{l_\infty} \leq r$, $\|y\|_{l_\infty} \leq r$ and for all $t \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

(viii) The function g_n is defined on the set $\mathbb{R}_+ \times \mathbb{R}^\infty$ and takes real values for $n = 1, 2, \dots$. Moreover, the operator g defined on the set $\mathbb{R}_+ \times l_\infty$ by the formula

$$(gx)(t) = (g_n(t, x)) = (g_1(t, x), g_2(t, x), \dots)$$

transforms the set $\mathbb{R}_+ \times l_\infty$ into l_∞ and is such that the family of functions $\{(gx)(t)\}_{t \in \mathbb{R}_+}$ is equicontinuous on the space l_∞ i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|(gy)(t) - (gx)(t)\|_{l_\infty} \leq \varepsilon$$

for any $t \in \mathbb{R}_+$ and for all $x, y \in l_\infty$ such that $\|x - y\|_{l_\infty} \leq \delta$.

(ix) The operator g defined in assumption (viii) is bounded on the set $\mathbb{R}_+ \times l_\infty$. More precisely, there exists a positive constant $\bar{G}$ such that $\|(gx)(t)\|_{l_\infty} \leq \bar{G}$ for any $x \in l_\infty$ and $t \in \mathbb{R}_+$.

(x) There exists a positive solution r_0 of the inequality

$$A + \bar{F} \bar{G} K_1 + \bar{G} K_1 r l(r) \leq r$$

such that $\bar{G} K_1 \max \{l(r_0), m(r_0)\} < 1$, where the constants $\bar{F}$, $\bar{G}$, K_1 were defined above and the constant A is defined in the following way

$$A = \sup \{|a_n(t)| : t \in \mathbb{R}_+, n = 1, 2, \dots\}. \quad (3.2)$$

Before formulating our main result we indicate some consequences of assumption (i).

Lemma 3.1. *Let the function $x(t) = (x_n(t))$ be an element of the space BC_∞ . Then the sequence (x_n) is equibounded and locally equicontinuous on $\mathbb{R}_+$.*

The proof can be conducted analogously as the proof of Lemma 4.1 in paper [1] and is therefore omitted.

Lemma 3.2. *Let the sequence $(a_n(t))$ be an element of the space BC_∞ such that $\lim_{t \rightarrow \infty} a_n(t) = 0$ uniformly with respect to $n \in \mathbb{N}$ (cf. assumption (i)). Then the sequence (a_n) is equibounded and equicontinuous on $\mathbb{R}_+$.*

Proof. The equiboundedness of the sequence (a_n) on $\mathbb{R}_+$ follows from Lemma 3.1. To prove the equicontinuity of (a_n) on $\mathbb{R}_+$ let us fix $\varepsilon > 0$. Keeping in mind the remaining part of the assumption in our lemma we can find a number $T > 0$ such that $|a_n(t)| \leq \varepsilon/4$ for $t \geq T$ and $n = 1, 2, \dots$. On the other hand, in view of Lemma 3.1 we deduce that the sequence (a_n) is locally equicontinuous on $\mathbb{R}_+$. Thus, we can find a number $\delta > 0$ such that $|a_n(t_2) - a_n(t_1)| \leq \varepsilon/2$ for $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| \leq \delta$ and for any $n = 1, 2, \dots$. Now, let us take arbitrary $t_1, t_2 \in \mathbb{R}_+$ such that $|t_2 - t_1| \leq \delta$. Without loss of generality we can assume that $t_1 < t_2$. If $t_1, t_2 \in [0, T]$, then in view of the above established fact we have that $|a_n(t_2) - a_n(t_1)| \leq \varepsilon/2$ for $n = 1, 2, \dots$.

If $t_1, t_2 \geq T$, then we obtain

$$|a_n(t_2) - a_n(t_1)| \leq |a_n(t_2)| + |a_n(t_1)| \leq \frac{\varepsilon}{2}$$

for any $n = 1, 2, \dots$.

Assume now that $t_1 < T \leq t_2$. Then, fixing arbitrarily $n \in \mathbb{N}$ and taking into account the above derived facts we get

$$|a_n(t_2) - a_n(t_1)| \leq |a_n(t_2) - a_n(T)| + |a_n(T) - a_n(t_1)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that the sequence (a_n) is equicontinuous on $\mathbb{R}_+$. $\square$

In what follows let us notice that as an immediate consequence of Lemma 3.1 we obtain the following corollary.

Corollary 3.3. *The constant A defined by equality (3.2) is finite.*

Now, we are in a position to formulate an existence theorem concerning the infinite system of integral equations (3.1).

Theorem 3.4. *Under assumptions (i) – (x) the infinite system (3.1) has at least one solution $x(t) = (x_n(t))$ in the space $BC_\infty = BC(\mathbb{R}_+, l_\infty)$.*

Proof. At the beginning we define three operators F, V, Q on the space BC_∞ in the following way:

$$\begin{aligned}(Fx)(t) &= ((F_n x)(t)) = (f_n(t, x(t))) = (f_n(t, x_1(t), x_2(t), \dots)), \\ (Vx)(t) &= ((V_n x)(t)) = \left(\int_0^t k_n(t, s) g_n(s, x_1(s), x_2(s), \dots) ds \right), \\ (Qx)(t) &= ((Q_n x)(t)) = (a_n(t) + (F_n x)(t)(V_n x)(t)).\end{aligned}$$

At first we show that the operator F transforms the space BC_∞ into itself.

To this end fix the function $x = x(t) = (x_n(t)) \in BC_\infty$. Then, in view of assumption (vi) we have the following estimate

$$|(F_n x)(t)| = |f_n(t, x_1(t), x_2(t), \dots)| \leq \bar{f}_n(t) + l(\|x(t)\|_{l_\infty}) \sup \{|x_i(t)| : i \geq n\} \quad (3.3)$$

for $t \in \mathbb{R}_+$ and for $n = 1, 2, \dots$, where the functions $\bar{f}_n$ and $l = l(r)$ were specified in assumption (vi). Hence, on the basis of (3.3) we deduce that

$$\|Fx\|_{BC_\infty} \leq \bar{F} + l(\|x\|_{BC_\infty}) \|x\|_{BC_\infty} \quad (3.4)$$

for any $x \in BC_\infty$. This shows that the function Fx is bounded on $\mathbb{R}_+$.

In order to prove the continuity of the function Fx on the interval $\mathbb{R}_+$, let us fix $\varepsilon > 0$. Then, from assumption (v) we obtain that there exists $\delta > 0$ such that for $t, s \in \mathbb{R}_+$ and $|t - s| \leq \delta$ the following inequality is satisfied

$$|f_n(t, x_1, x_2, \dots) - f_n(s, x_1, x_2, \dots)| \leq \varepsilon$$

for any $x = (x_i) \in l_\infty$. This implies that

$$\|(Fx)(t) - (Fx)(s)\|_{l_\infty} \leq \varepsilon$$

provided $t, s \in \mathbb{R}_+$ are such that $|t - s| \leq \delta$. But this means that the function Fx is continuous (even uniformly continuous) on $\mathbb{R}_+$. Finally we conclude that the operator F transforms the space BC_∞ into itself.

Now, we are going to show that the operator V acts from the space BC_∞ into itself.

Thus, similarly as above, fix a function $x = x(t) = (x_n(t)) \in BC_\infty$. Then, for arbitrarily fixed numbers $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$, in view of assumptions (iii) and (ix), we get

$$\begin{aligned}|(V_n x)(t)| &\leq \int_0^t |k_n(t, s)| |g_n(s, x_1(s), x_2(s), \dots)| ds \\ &\leq \int_0^t |k_n(t, s)| \bar{G} ds \leq \bar{G} \int_0^t |k_n(t, s)| ds \leq \bar{G} K_1.\end{aligned} \quad (3.5)$$

Particularly, the above estimate yields that the function Vx is bounded on the interval $\mathbb{R}_+$.

Next, fix $\varepsilon > 0$ and choose a number $\delta > 0$ according to assumption (ii). Then, for arbitrarily fixed numbers $t_1, t_2 \in \mathbb{R}_+$ such that $|t_2 - t_1| \leq \delta$, on the basis of assumptions (ii) and (ix) (assuming additionally that $t_1 < t_2$), we obtain

$$\begin{aligned} |(V_n x)(t_2) - (V_n x)(t_1)| &\leq \left| \int_0^{t_2} k_n(t_2, s) g_n(s, x_1(s), x_2(s), \dots) ds - \int_0^{t_2} k_n(t_1, s) g_n(s, x_1(s), x_2(s), \dots) ds \right| \\ &\quad + \left| \int_0^{t_2} k_n(t_1, s) g_n(s, x_1(s), x_2(s), \dots) ds - \int_0^{t_1} k_n(t_1, s) g_n(s, x_1(s), x_2(s), \dots) ds \right| \\ &\leq \int_0^{t_2} |k_n(t_2, s) - k_n(t_1, s)| |g_n(s, x_1(s), x_2(s), \dots)| ds \\ &\quad + \int_{t_1}^{t_2} |k_n(t_1, s)| |g_n(s, x_1(s), x_2(s), \dots)| ds \\ &\leq \int_0^{t_2} \omega_k(\delta) |g_n(s, x_1(s), x_2(s), \dots)| ds + \int_{t_1}^{t_2} K_2 |g_n(s, x_1(s), x_2(s), \dots)| ds, \end{aligned} \quad (3.6)$$

where K_2 is a constant appearing in assumption (iv) and $\omega_k(\delta)$ denotes a common modulus of continuity of the sequence of functions $t \rightarrow k_n(t, s)$ on the interval $\mathbb{R}_+$ (according to assumption (ii)). Obviously we have that $\omega_k(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Further on, taking into account estimate (3.6) and assumption (ix), we obtain

$$|(V_n x)(t_2) - (V_n x)(t_1)| \leq \overline{G} \omega_k(\delta) + K_2 \overline{G} \delta. \quad (3.7)$$

Hence we deduce that the function Vx is continuous on the interval $\mathbb{R}_+$. Linking boundedness of the function Vx with its continuity on $\mathbb{R}_+$ we conclude that the operator V transforms the space BC_∞ into itself.

Now, keeping in mind the fact that the space $BC_\infty = BC(\mathbb{R}_+, l_\infty)$ forms a Banach algebra with respect to the coordinatewise multiplication of function sequences and taking into account the definition of the operator Q as well as assumption (i), we infer that for an arbitrarily fixed function $x = x(t) \in BC_\infty$ the function $(Qx)(t) = ((Q_n x)(t)) = (a_n(t) + (F_n x)(t)(V_n x)(t))$ acts from the interval $\mathbb{R}_+$ into the space l_∞ . Indeed, in view of the fact that $((F_n x)(t)) \in l_\infty$ for any $t \in \mathbb{R}_+$ and in the light of estimate (3.5) we get

$$|(Q_n x)(t)| \leq |a_n(t)| + \overline{G} K_1 |(F_n x)(t)|.$$

Hence, applying (3.3) we deduce that $(Qx)(t) = ((Q_n x)(t)) \in l_\infty$ for each $t \in \mathbb{R}_+$.

Next, let us notice that the continuity of the function Qx on $\mathbb{R}_+$ is a simple consequence of the fact that both the function Fx and the function Vx are continuous on $\mathbb{R}_+$. Similarly we can derive that the function Qx is bounded on the interval $\mathbb{R}_+$. Indeed, it is only sufficient to make use assumption (i) and Lemma 3.1.

Finally, let us observe that combining all the above established properties of the function Qx we conclude that the operator Q transforms the space BC_∞ into itself.

Further, let us note that based on estimates (3.4) and (3.5), for arbitrarily fixed $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$ we get

$$\begin{aligned} |(Q_n x)(t)| &\leq |a_n(t)| + |(F_n x)(t)| |(V_n x)(t)| \\ &\leq A + \left[\overline{F} + l(\|x\|_{BC_\infty}) \|x\|_{BC_\infty} \right] \overline{G} K_1 \\ &\leq A + \overline{F} \overline{G} K_1 + \overline{G} K_1 l(\|x\|_{BC_\infty}) \|x\|_{BC_\infty}. \end{aligned}$$

From the above estimate and assumption (x) we infer that there exists a number $r_0 > 0$ such that the operator Q transforms the ball B_{r_0} into itself.

In what follows we are going to show that the operator Q is continuous on the ball B_{r_0} . Keeping in mind the representation of the operator Q given at the beginning of our proof we see it is sufficient to prove the continuity of the operators F and V , separately.

To this end let us fix $\varepsilon > 0$ and $x \in B_{r_0}$. Next, choose an arbitrary point $y \in B_{r_0}$ such that $\|x - y\|_{BC_\infty} \leq \varepsilon$. Then, for each fixed $t \in \mathbb{R}_+$, in virtue of assumption (vii) we get

$$\|(Fy)(t) - (Fx)(t)\|_{l_\infty} \leq m(r_0)\|x - y\|_{l_\infty} \leq \varepsilon m(r_0).$$

Particularly this shows that the operator F is continuous at every point of the ball B_{r_0} .

To prove the continuity of the operator V on the ball B_{r_0} let us define the function $\delta = \delta(\varepsilon)$ by putting

$$\delta(\varepsilon) = \sup \{ |g_n(t, y) - g_n(t, x)| : x, y \in l_\infty, \|y - x\|_{l_\infty} \leq \varepsilon, t \in \mathbb{R}_+, n \in \mathbb{N} \}.$$

Obviously, in view of assumption (viii) we have that $\delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Next, taking $x, y \in B_{r_0}$ such that $\|y - x\|_{BC_\infty} \leq \varepsilon$ and $t \in \mathbb{R}_+$, for arbitrary $n \in \mathbb{N}$ we obtain

$$\begin{aligned} |(V_n y)(t) - (V_n x)(t)| &\leq \int_0^t |k_n(t, s)| |g_n(s, y_1(s), y_2(s), \dots) - g_n(s, x_1(s), x_2(s), \dots)| ds \\ &\leq \int_0^t |k_n(t, s)| \delta(\varepsilon) ds \leq K_1 \delta(\varepsilon). \end{aligned}$$

This yields to the estimate

$$\|Vx - Vy\|_{BC_\infty} \leq K_1 \delta(\varepsilon).$$

Thus we see that the operator V is continuous on the ball B_{r_0} .

In the sequel let us fix an arbitrary number $\varepsilon > 0$. Next, choose a number $\delta > 0$ according to assumption (v). Further fix a nonempty subset X of the ball B_{r_0} and take an arbitrary function $x \in X$ and $n \in \mathbb{N}$. Then, for arbitrary $t, s \in \mathbb{R}_+$ such that $|t - s| \leq \delta$, in virtue of assumption (v) we get

$$\begin{aligned} |(F_n x)(t) - (F_n x)(s)| &= |f_n(t, x_1(t), x_2(t), \dots) - f_n(s, x_1(s), x_2(s), \dots)| \\ &\leq |f_n(t, x_1(t), x_2(t), \dots) - f_n(s, x_1(t), x_2(t), \dots)| \\ &\quad + |f_n(s, x_1(t), x_2(t), \dots) - f_n(s, x_1(s), x_2(s), \dots)| \\ &\leq \varepsilon + m(r_0) \sup \{ \|x(t) - x(s)\|_{l_\infty} : t, s \in \mathbb{R}_+, |t - s| \leq \delta \} \\ &\leq \varepsilon + m(r_0) \omega^\infty(x, \delta). \end{aligned}$$

Now, on the basis of the above estimate we obtain

$$\omega^\infty(Fx, \varepsilon) \leq \varepsilon + m(r_0) \omega^\infty(x, \delta). \quad (3.8)$$

Further, let us fix a number $\varepsilon > 0$ and choose $t_1, t_2 \in \mathbb{R}_+$ such that $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we may assume that $t_1 < t_2$. Then, in view of estimate (3.7) we derive the following inequality

$$|(V_n x)(t_2) - (V_n x)(t_1)| \leq \bar{G} \omega_k(\varepsilon) + \bar{G} K_2 \varepsilon.$$

This yields the estimate

$$\omega^\infty(Vx, \varepsilon) \leq \bar{G} \omega_k(\varepsilon) + \bar{G} K_2 \varepsilon. \quad (3.9)$$

Now, taking into account the representation of the operator Q , for an arbitrary function $x \in X$ and for arbitrary numbers $t, s \in \mathbb{R}_+$, we obtain

$$\|(Qx)(t) - (Qx)(s)\|_{l_\infty} \leq \|a(t) - a(s)\|_{l_\infty} + \|(Vx)(t)\|_{l_\infty} \|(Fx)(t) - (Fx)(s)\|_{l_\infty} + \|(Fx)(s)\|_{l_\infty} \|(Vx)(t) - (Vx)(s)\|_{l_\infty},$$

where $a(t) = (a_n(t))$.

Further, fix $\varepsilon > 0$ and assume that $|t - s| \leq \varepsilon$. Then, from the above inequality and estimates (3.8), (3.9), (3.4), and (3.5), we get

$$\omega^\infty(Qx, \varepsilon) \leq \omega^\infty(a, \varepsilon) + \bar{G} K_1 \omega^\infty(Fx, \varepsilon) + (\bar{F} + r_0 l(r_0)) (\bar{G} \omega_k(\varepsilon) + \bar{G} K_2 \varepsilon)$$

$$\leq \omega^\infty(a, \varepsilon) + \overline{G}K_1 m(r_0) \omega^\infty(x, \varepsilon) + \overline{G}K_1 \varepsilon + (\overline{F} + r_0 l(r_0)) (\overline{G} \omega_k(\varepsilon) + \overline{G}K_2 \varepsilon).$$

Now, in view of Lemma 3.2 we infer that $\omega^\infty(a, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Next, keeping in mind that $\omega_k(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, from the above obtained estimate we derive the following inequality

$$\omega^\infty(QX) \leq \overline{G}K_1 m(r_0) \omega_0^\infty(X). \quad (3.10)$$

In what follows we will consider the second term of the measure of noncompactness μ_a^1 (cf. formula (2.5)) which is denoted by $\overline{\mu}_\infty^1$ and is defined by formula (2.7).

To this end fix a nonempty subset X of the ball B_{r_0} and choose an arbitrary function $x = x(t) \in X$. Further, take a natural number n and $T > 0$. Then, for any fixed $t \in [0, T]$, in view of the representation of the operator Q and estimates (3.3) and (3.5), we obtain

$$\begin{aligned} |(Q_n x)(t)| &\leq |a_n(t)| + |f_n(t, x_1(t), x_2(t), \dots)| \int_0^t |k_n(t, s)| |g_n(s, x_1(s), x_2(s), \dots)| ds \\ &\leq |a_n(t)| + \left[\overline{f}_n(t) + l(\|x(t)\|_{l_\infty}) \sup \{ |x_i(t)| : i \geq n \} \right] \overline{G}K_1. \end{aligned}$$

Now, taking supremum over all $x \in X$, from the above estimate we get

$$\sup_{x \in X} |(Q_n x)(t)| \leq |a_n(t)| + \overline{G}K_1 \left[\overline{f}_n(t) + l(r_0) \sup_{x \in X} \left\{ \sup \{ |x_i(t)| : i \geq n \} \right\} \right].$$

Hence, taking into account assumption (i) and (vi), we derive the following inequality

$$\lim_{n \rightarrow \infty} \left\{ \sup_{x \in X} |(Q_n x)(t)| \right\} \leq \overline{G}K_1 l(r_0) \left\{ \lim_{n \rightarrow \infty} \left\{ \sup_{x \in X} \left\{ \sup \{ |x_i(t)| : i \geq n \} \right\} \right\} \right\}.$$

Finally, taking supremum over $t \in [0, T]$ on both sides of the above inequality and next, passing with $T \rightarrow \infty$, in virtue of formula (2.7) we get

$$\overline{\mu}_\infty^1(QX) \leq \overline{G}K_1 l(r_0) \overline{\mu}_\infty^1(X). \quad (3.11)$$

In order to estimate the last term a_∞ of the measure of noncompactness μ_a^1 (cf. formula (2.5)) expressed by formula (2.8), let us take a nonempty subset X ($X \subset B_{r_0}$) and choose a function $x \in X$. Further, fix arbitrarily $T > 0$. Then, taking $t \geq T$ and keeping in mind the previously obtained inequalities (3.3) and (3.5), we obtain

$$\begin{aligned} \sup \{ |(Q_n x)(t)| : n \in \mathbb{N} \} &\leq \sup \{ |a_n(t)| : n \in \mathbb{N} \} \\ &\quad + \sup \left\{ |f_n(t, x_1(t), x_2(t), \dots)| \int_0^t |k_n(t, s)| |g_n(s, x_1(s), x_2(s), \dots)| ds : n \in \mathbb{N} \right\} \\ &\leq \sup \{ |a_n(t)| : n \in \mathbb{N} \} + \sup \left[\overline{f}_n(t) + l(r_0) \sup \{ |x_i(t)| : i \geq n \} : n \in \mathbb{N} \right] \overline{G}K_1 \\ &\leq \sup \{ |a_n(t)| : n \in \mathbb{N} \} + l(r_0) \overline{G}K_1 \sup \left[\sup \{ |x_i(t)| : i \geq n \} : n \in \mathbb{N} \right] \\ &\quad + \overline{G}K_1 \sup \{ \overline{f}_n(t) : n \in \mathbb{N} \}. \end{aligned}$$

Now, taking in the above estimate supremum over $t \geq T$ and next, over $x \in X$, we get

$$\begin{aligned} \sup_{x \in X} \left\{ \sup_{t \geq T} \left\{ \sup \{ |(Q_n x)(t)| : n \in \mathbb{N} \} \right\} \right\} &\leq \sup_{t \geq T} \left\{ \sup \{ |a_n(t)| : n \in \mathbb{N} \} \right\} \\ &\quad + l(r_0) \overline{G}K_1 \left\{ \sup_{x \in X} \left\{ \sup_{t \geq T} \left\{ \sup \{ |x_n(t)| : n \in \mathbb{N} \} \right\} \right\} \right\} \\ &\quad + \overline{G}K_1 \left\{ \sup_{t \geq T} \left\{ \sup \{ \overline{f}_n(t) : n \in \mathbb{N} \} \right\} \right\}. \end{aligned}$$

Further, passing with $T \rightarrow \infty$ and taking into account assumptions (i) and (vi), we derive the following estimate

$$a_\infty(QX) \leq l(r_0) \overline{G} K_1 a_\infty(X). \quad (3.12)$$

Finally, combining estimates (3.10)-(3.12) and keeping in mind formula (2.5), we obtain the following inequality for an arbitrary nonempty subset X of the ball B_{r_0} :

$$\mu_a^1(QX) \leq \overline{G} K_1 \max \{l(r_0), m(r_0)\} \mu_a^1(X).$$

Hence, in view of the fact that the operator Q is a continuous self-mapping of the ball B_{r_0} , assumption (x) and Theorem 2.4 we conclude that the infinite system of Volterra-Hammerstein integral equations (3.1) has at least one solution $x = x(t)$ in the space $BC_\infty = BC(\mathbb{R}_+, l_\infty)$ which belongs to the ball B_{r_0} and is uniformly continuous on the interval $\mathbb{R}_+$.

The proof is complete. $\square$

4 An example

Now, we are going to provide an example illustrating the existence result contained in Theorem 3.4.

To this end, let us consider the following infinite system of nonlinear Volterra-Hammerstein integral equations having the form:

$$x_n(t) = \frac{\alpha t}{1 + n^2 + t^2} + \left(\frac{\beta}{n^2 + t^2} + \frac{yx_n(t)}{1 + x_1^2(t)} + \frac{yx_{n+1}(t)}{n + x_2^2(t)} \right) \int_0^t \frac{s}{1 + n(s^2 + t^2)} \arctan \left(\frac{x_1(s) + x_n(s)}{n + s^2} \right) ds, \quad (4.1)$$

for $n = 1, 2, \dots$ and $t \in \mathbb{R}_+$. Moreover, we assume that α, β, γ appearing in the above system are positive constants.

Notice that infinite system (4.1) is a particular case of system (3.1) if we put

$$a_n(t) = \frac{\alpha t}{1 + n^2 + t^2}, \quad (4.2)$$

$$f_n(t, x_1, x_2, \dots) = \frac{\beta}{n^2 + t^2} + \frac{yx_n}{1 + x_1^2} + \frac{yx_{n+1}}{n + x_2^2}, \quad (4.3)$$

$$k_n(t, s) = \frac{s}{1 + n(s^2 + t^2)}, \quad (4.4)$$

$$g_n(t, x_1, x_2, \dots) = \arctan \left(\frac{x_1 + x_n}{n + t^2} \right) \quad (4.5)$$

for $n = 1, 2, \dots$ and $t, s \in \mathbb{R}_+$.

We are going to show that the infinite system of integral equations (4.1) has a solution in the Banach space $BC_\infty = BC(\mathbb{R}_+, l_\infty)$. To this end we will apply Theorem 3.4.

Thus, we show that functions defined by (4.2)-(4.5) satisfy assumptions (i)-(x) of Theorem 3.4.

At the beginning let us observe that the function $a_n(t)$ defined by (4.2) is an element of the space BC_∞ for $n = 1, 2, \dots$. In view of the inequality $|a_n(t)| = a_n(t) \leq \frac{\alpha t}{1+t^2}$ we derive that $\lim_{t \rightarrow \infty} a_n(t) = 0$ uniformly with respect to $n \in \mathbb{N}$. Moreover, $\lim_{n \rightarrow \infty} a_n(t) = 0$ for any $t \in \mathbb{R}_+$. This shows that the sequence $(a_n(t))$ satisfies assumption (i). Apart from this we have that $A = \alpha/2\sqrt{2}$, where the constant A is defined by (3.2).

Further, let us notice that the function $k_n(t, s)$ defined by (4.4) ($n = 1, 2, \dots$) is continuous on $\mathbb{R}_+^2$. Additionally, using standard tools of differential calculus it is easily seen that

$$|k_n(t_2, s) - k_n(t_1, s)| \leq \frac{1}{n} |t_2 - t_1|$$

for $n = 1, 2, \dots$ and for $t_1, t_2 \in \mathbb{R}_+$. This means that the sequence of functions $(k_n(\cdot, s))$ is equicontinuous on $\mathbb{R}_+$ uniformly with respect to $s \in \mathbb{R}_+$.

Summing up, we see that there is satisfied assumption (ii).

Next, let us observe that for each $n \in \mathbb{N}$ and for arbitrary $t, s \in \mathbb{R}_+$ we have the following estimate

$$|k_n(t, s)| \leq \frac{s}{1 + ns^2} \leq \frac{s}{1 + s^2} \leq \frac{1}{2}.$$

Hence it follows that the sequence $(k_n(t, s))$ is equibounded on $\mathbb{R}_+$ with the constant $K_2 = \frac{1}{2}$. This shows that there is satisfied assumption (iv).

On the other hand we obtain

$$\int_0^t |k_n(t, s)| ds = \int_0^t \frac{s}{1 + n(s^2 + t^2)} ds = \frac{1}{2} \ln \left(\frac{1 + 2nt^2}{1 + nt^2} \right) \leq \frac{1}{2n} \ln 2 \leq \frac{1}{2} \ln 2.$$

This proves that the function sequence $(k_n(t, s))$ satisfies assumption (iii) with the constant $K_1 = \frac{1}{2} \ln 2$.

Now, let us take into account the function $t \rightarrow f_n(t, x_1, x_2, \dots)$ defined by formula (4.3) for $n = 1, 2, \dots$. Fix arbitrary $t_1, t_2 \in \mathbb{R}_+$ and $x = (x_n) \in l_\infty$. Then, we get

$$\begin{aligned} |f_n(t_2, x_1, x_2, \dots) - f_n(t_1, x_1, x_2, \dots)| &\leq \beta \left| \frac{1}{n^2 + t_2^2} - \frac{1}{n^2 + t_1^2} \right| \\ &\leq \beta \frac{|t_2 - t_1|(t_1 + t_2)}{(n^2 + t_1^2)(n^2 + t_2^2)} = \beta |t_2 - t_1| \left[\frac{t_1}{(n^2 + t_1^2)(n^2 + t_2^2)} + \frac{t_2}{(n^2 + t_1^2)(n^2 + t_2^2)} \right] \\ &\leq \beta |t_2 - t_1| \left[\frac{1}{2} \frac{1}{n^2 + t_2^2} + \frac{1}{2} \frac{1}{n^2 + t_1^2} \right] \leq \beta |t_2 - t_1| \end{aligned}$$

for any $n = 1, 2, \dots$. This shows that the functions f_n ($n = 1, 2, \dots$) satisfy assumption (v).

In order to verify assumption (vi) let us fix a number $r > 0$ and choose $x = (x_i) \in l_\infty$ such that $\|x\|_{l_\infty} \leq r$. Then, for arbitrarily fixed $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, we obtain

$$\begin{aligned} |f_n(t, x_1, x_2, \dots)| &\leq \frac{\beta}{n^2 + t^2} + y \left[\frac{|x_n|}{1 + x_1^2} + \frac{|x_{n+1}|}{n + x_2^2} \right] \\ &\leq \frac{\beta}{n^2 + t^2} + y(|x_n| + |x_{n+1}|) \leq \frac{\beta}{n^2 + t^2} + 2y \sup \{ |x_i| : i \geq n \}. \end{aligned}$$

This shows that the inequality from assumption (vi) is satisfied with the following functions

$$\begin{aligned} \bar{f}_n(t) &= \frac{\beta}{n^2 + t^2}, \\ l(r) &= 2y \end{aligned}$$

for $n = 1, 2, \dots$. Since $\bar{f}_n(t) \leq \beta/(1 + t^2)$ we infer that $\lim_{t \rightarrow \infty} \bar{f}_n(t) = 0$ uniformly with respect to $n \in \mathbb{N}$. Apart from this we have that $\lim_{n \rightarrow \infty} \bar{f}_n(t) = 0$ for any $t \in \mathbb{R}_+$.

Summing up we see that assumption (vi) is satisfied. Moreover, let us notice that

$$\bar{F} = \sup \{ \bar{f}_n(t) : t \in \mathbb{R}_+, n = 1, 2, \dots \} = \beta.$$

Next, let us fix a number $r > 0$ and take arbitrary $x = (x_i), y = (y_i) \in l_\infty$ such that $\|x\|_{l_\infty} \leq r, \|y\|_{l_\infty} \leq r$. Then, for a fixed $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, we get

$$\begin{aligned} |f_n(t, x_1, x_2, \dots) - f_n(t, y_1, y_2, \dots)| &\leq y \left| \frac{x_n}{1 + x_1^2} - \frac{y_n}{1 + y_1^2} \right| + y \left| \frac{x_{n+1}}{n + x_2^2} - \frac{y_{n+1}}{n + y_2^2} \right| \\ &\leq y \frac{|x_n + x_n y_1^2 - y_n - y_n x_1^2|}{(1 + x_1^2)(1 + y_1^2)} + y \frac{|nx_{n+1} + x_{n+1} y_2^2 - ny_{n+1} - x_2^2 y_{n+1}|}{(n + x_2^2)(n + y_2^2)} \\ &\leq y |x_n - y_n| + y \frac{|(x_n y_1^2 - y_n y_1^2) + (y_n y_1^2 - y_n x_1^2)|}{(1 + x_1^2)(1 + y_1^2)} \\ &\quad + y n \frac{|x_{n+1} - y_{n+1}|}{(n + x_2^2)(n + y_2^2)} + y \frac{|(x_{n+1} y_2^2 - y_{n+1} y_2^2) + (y_{n+1} y_2^2 - y_{n+1} x_2^2)|}{(n + x_2^2)(n + y_2^2)} \end{aligned}$$

$$\begin{aligned}
&\leq y|x_n - y_n| + y \left[\frac{y_1^2|x_n - y_n|}{(1+x_1^2)(1+y_1^2)} + \frac{|y_n||y_1^2 - x_1^2|}{(1+x_1^2)(1+y_1^2)} \right] \\
&\quad + y|x_{n+1} - y_{n+1}| + \frac{yy_2^2|x_{n+1} - y_{n+1}|}{(n+x_2^2)(n+y_2^2)} + y|y_{n+1}| \frac{(|x_2| + |y_2|)|y_2 - x_2|}{(n+x_2^2)(n+y_2^2)} \\
&\leq 2y|x_n - y_n| + yr \left(\frac{|y_1|}{(1+x_1^2)(1+y_1^2)} + \frac{|x_1|}{(1+x_1^2)(1+y_1^2)} \right) |x_1 - y_1| \\
&\quad + 2y|x_{n+1} - y_{n+1}| + yr|x_2 - y_2| \left(\frac{|x_2|}{(n+x_2^2)(n+y_2^2)} + \frac{|y_2|}{(n+x_2^2)(n+y_2^2)} \right) \\
&\leq 2y|x_n - y_n| + yr|x_1 - y_1| + 2y|x_{n+1} - y_{n+1}| + yr|x_2 - y_2| \\
&\leq (4y + 2ry)\|x - y\|_{l_\infty} = 2y(2+r)\|x - y\|_{l_\infty}.
\end{aligned}$$

Thus we see that assumption (vii) is satisfied with the function $m(r) = 2y(2+r)$.

In the next step of our proof we are going to verify assumption (viii). To this end fix arbitrarily $n \in \mathbb{N}$ and consider the function $g_n(t, x) = g_n(t, x_1, x_2, \dots)$ defined by formula (4.5) i.e.,

$$g_n(t, x_1, x_2, \dots) = \arctan \left(\frac{x_1 + x_n}{n + t^2} \right).$$

Then, from the estimate

$$|g_n(t, x_1, x_2, \dots)| \leq \frac{|x_1| + |x_n|}{n + t^2} \leq \frac{|x_1| + |x_n|}{n}$$

we deduce that the operator g defined in assumption (viii) by the equality

$$(gx)(t) = (g_n(t, x)) = (g_1(t, x), g_2(t, x), \dots)$$

transforms the set $\mathbb{R}_+ \times l_\infty$ into l_∞ .

Further on, fix $t \in \mathbb{R}_+$ and take $x = (x_i)$, $y = (y_i) \in l_\infty$. Then we have

$$|g_n(t, x) - g_n(t, y)| \leq \left| \frac{x_1 + x_n}{n + t^2} - \frac{y_1 + y_n}{n + t^2} \right| \leq \frac{|x_1 - y_1|}{n + t^2} + \frac{|x_n - y_n|}{n + t^2} \leq \frac{|x_1 - y_1|}{n} + \frac{|x_n - y_n|}{n}.$$

This allows us to derive the following estimate:

$$\begin{aligned}
\|(gx)(t) - (gy)(t)\|_{l_\infty} &= \sup \{ |g_n(t, x) - g_n(t, y)| : n \in \mathbb{N} \} \\
&\leq \sup \left\{ \frac{|x_1 - y_1|}{n} + \frac{|x_n - y_n|}{n} : n \in \mathbb{N} \right\} \\
&\leq 2 \sup \left\{ \frac{|x_n - y_n|}{n} : n \in \mathbb{N} \right\} \leq 2\|x - y\|_{l_\infty}.
\end{aligned}$$

From the above estimate we infer that the operator g satisfies assumption (viii).

Moreover, it is easily seen that for an arbitrary $x \in l_\infty$ and $t \in \mathbb{R}_+$ we get

$$\|(gx)(t)\|_{l_\infty} = \sup \{ |g_n(t, x)| : n \in \mathbb{N} \} \leq \frac{\pi}{2}.$$

This means that the operator g satisfies assumption (ix) with the constant $\bar{G} = \pi/2$.

Finally, let us consider the first inequality from assumption (x). Obviously, in our case that inequality has the form

$$\frac{\alpha}{2\sqrt{2}} + \frac{\pi}{4} \ln 2 (\beta + 2yr) < r. \quad (4.6)$$

On the other hand, taking the second inequality required in assumption (x), we get

$$y \frac{\pi}{2} \ln 2 (2 + r_0) < 1. \quad (4.7)$$

It is easy to check that choosing $y < \frac{1}{\pi \ln 2}$ and taking $r_0 > \frac{\alpha}{\sqrt{2}} + \frac{\beta}{2y}$, we can easily verify that both inequalities (4.6) and (4.7) are satisfied.

Thus, in the light of Theorem 3.4 we infer that infinite system of nonlinear integral equations (4.1) has at least one solution belonging to the ball B_{r_0} in the space $BC(\mathbb{R}_+, l_\infty)$.

References

- [1] J. Banaś and A. Chlebowicz, On solutions of an infinite system of nonlinear integral equations on the real half-axis, *Banach Journal of Mathematical Analysis* **13** (2019), no. 4, 944–968.
- [2] J. Banaś, A. Chlebowicz and W. Woś, On measures of noncompactness in the space of functions defined on the half-axis with values in a Banach space, *Journal of Mathematical Analysis and Applications* **489**(2020), 124187
- [3] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, vol. 60, Marcel Dekker Inc., New York, 1980.
- [4] J. Banaś and M. Mursaleen, *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*, Springer, New Delhi, 2014.
- [5] J. Banaś, M. Mursaleen and S.M.H. Rizvi, Existence of solutions to a boundary-value problem for an infinite system of differential equations, *Electronic Journal of Differential Equations* **2017** (2017), no. 262, 1–12.
- [6] J. Banaś and B. Rzepka, On solutions of infinite systems of integral equations of Hammerstein type, *Journal of Nonlinear and Convex Analysis* **18** (2017), no. 2, 261–278.
- [7] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.
- [8] M. Mursaleen and S.M.H. Rizvi, Solvability of infinite systems of second order differential equations in c_0 and l_1 by Meir-Keeler condensing operators, *Proceedings of the American Mathematical Society* **144** (2016), no. 10, 4279–4289.
- [9] B. Rzepka and K. Sadarangani, On solutions of an infinite system of singular integral equations, *Mathematical and Computer Modelling* **45** (2007), no. 9-10, 1265–1271.
- [10] K. Deimling, *Ordinary Differential Equations in Banach Spaces*, Lecture Notes in Mathematics 596, Springer, Berlin, 1977.
- [11] K. Kuratowski, Sur les espaces complets, *Fundamenta Mathematicae* **15** (1930), no. 1, 301–309.
- [12] L.S. Goldenšteĭn, I.T. Gohberg and A.S. Markus, Investigations of some properties of bounded linear operators with their q -norms, *Učen. Zap. Kishinevsk. Univ.* **29** (1957), 29–36.
- [13] L.S. Goldenšteĭn and A.S. Markus, On a measure of noncompactness of bounded sets and linear operators, *Studies in Algebra and Mathematical Analysis* **1** (1965), 45–54.
- [14] R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina and B.N. Sadovskii, *Measure of Noncompactness and Condensing Operators. Operator Theory: Advanced and Applications vol. 55*, Birkhäuser Verlag, Basel, 1992.
- [15] J.M. Ayerbe Toledano, T. Dominguez Benavides and G. Lopez Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Birkhäuser Verlag, Basel, 1997.
- [16] G. Darbo, Punti uniti in trasformazioni a codominio non compatto, *Rendiconti del Seminario Matematico della Università di Padova* **24** (1955), 84–92.