Research Article

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Nehari-type ground state solutions for a Choquard equation with doubly critical exponents

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Abstract: This paper deals with the following Choquard equation with a local nonlinear perturbation:

$$\begin{cases} -\Delta u + u = \left(I_\alpha * |u|^{\frac{\alpha}{2}+1}\right) |u|^{\frac{\alpha}{2}-1} u + f(u), & x \in \mathbb{R}^2; \\ u \in H^1(\mathbb{R}^2), \end{cases}$$

where $\alpha \in (0, 2)$, $I_\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the Riesz potential and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ is of critical exponential growth in the sense of Trudinger-Moser. The exponent $\frac{\alpha}{2} + 1$ is critical with respect to the Hardy-Littlewood-Sobolev inequality. We obtain the existence of a nontrivial solution or a Nehari-type ground state solution for the above equation in the doubly critical case, i.e. the appearance of both the lower critical exponent $\frac{\alpha}{2} + 1$ and the critical exponential growth of $f(u)$.

Keywords: Choquard equation; Lower critical exponent; Nehari-type ground state solution; Critical exponential growth; Trudinger-Moser

MSC: 35J20, 35J62, 35Q55

1 Introduction

In the past few years, the following Choquard equation:

$$\begin{cases} -\Delta u + u = (I_\alpha * |u|^q) |u|^{q-1} u, & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \quad (1.1)$$

has attracted considerable attention, where $N \geq 1$, $\alpha \in (0, N)$, $2 < q < 2^*$ and $I_\alpha : \mathbb{R}^N \rightarrow \mathbb{R}$ is the Riesz potential. Physical motivation of (1.1) comes from the case that $N = 3$, $\alpha = 2$ and $q = 2$. In this case, Eq.(1.1) is called the Choquard-Pekar equation [21, 31], Hartree equation [19] or Schrödinger-Newton equation [27, 39], depending on its physical backgrounds and derivations. The existence of a ground state in this case was studied in [21, 22, 28] via variational arguments. In a pioneering work, Lieb [21] first obtained the existence and uniqueness of positive solutions to (1.1) with $N = 3$, $\alpha = 2$ and $q = 2$. Later, Lions [22, 23] got the existence and multiplicity results of normalized solution on the same topic. Moroz and Van Schaftingen [28] proved that (1.1) has a ground state solution if

$$\frac{N-2}{N+\alpha} < \frac{1}{q} < \frac{N}{N+\alpha};$$

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and it has no nontrivial solution when either $q \leq \frac{\alpha}{N} + 1$ or $q \geq \frac{N+\alpha}{N-2}$. The endpoints of the above interval are critical exponents. The upper critical exponent $\frac{N+\alpha}{N-2}$ plays a similar role as the Sobolev critical exponent in the local semilinear equations [9, 40]. The lower critical exponent $\frac{\alpha}{N} + 1$ comes from the Hardy-Littlewood-Sobolev inequality. So far, there are a variety of interesting results concerning the existence of nontrivial solutions for more general Choquard equation with upper critical growth, see for example, see [6, 7, 17, 24] and the references cited therein. However, to the best of our knowledge, it seems that the only available works regarding the existence of nontrivial solutions for Choquard equation with lower critical exponent and a local nonlinear perturbation are the papers [35, 38, 41]. In details, Van Schaftingen and Xia [35] proved that the following Choquard equation:

$$\begin{cases} -\Delta u + u = \left(I_\alpha \star |u|^{\frac{\alpha}{N}+1}\right) |u|^{\frac{\alpha}{N}-1} u + f(u), & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N) \end{cases} \quad (1.2)$$

admits a ground state solution if there exists $\Lambda > 0$ such that f satisfies the following three assumptions:

(PG) $f \in C(\mathbb{R}, \mathbb{R})$, $f(t) = o(|t|)$ as $t \rightarrow 0$ and $f(t) = o(|t|^{2N/(N-2)})$ as $|t| \rightarrow \infty$.

(AR) there exists $\mu > 2$ such that $0 < \mu F(t) \leq f(t)t$ for all $t \neq 0$, where $F(t) = \int_0^t f(s)ds$;

(ZS) $\liminf_{|t| \rightarrow 0} \frac{F(t)}{|t|^{\frac{\alpha}{N}+2}} > \Lambda$.

When $N \geq 3$, by using the Pohožaev identity argument, Wang and Liao [41] obtained the same conclusion only under (PG) and (ZS). When $f(u) = \lambda|u|^{p-2}u$ with $\lambda > 0$ and $2 < p < 2^*$, Tang, Wei and Chen [38] obtained the existence of a ground state solution to (1.2) for every $p \in (2, 2^*)$. For more existence results on (1.1) or related results, we refer to [1, 4, 5, 8, 12–15, 18, 25, 26, 29, 30, 32–34, 42].

In above-mentioned works [35, 38, 41], it was only considered the case when $f(u)$ has polynomial growth. When $N = 2$, the corresponding Sobolev embedding yields $H^1(\mathbb{R}^2) \subset L^s(\mathbb{R}^2)$ for all $s \in [2, +\infty)$, but $H^1(\mathbb{R}^2) \not\subset L^\infty(\mathbb{R}^2)$. In this case, the Pohožaev-Trudinger-Moser inequality in $\mathbb{R}^2$ can be seen as a substitute of the Sobolev inequality, which was first established by Cao in [11], see also [2, 10], and reads as follows.

Lemma 1.1. i) If $\beta > 0$ and $u \in H^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} (e^{\beta u^2} - 1) dx < \infty;$$

ii) if $u \in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2^2 \leq 1$, $\|u\|_2 \leq M < \infty$, and $\beta < 4\pi$, then there exists a constant $\mathcal{C}(M, \beta)$, which depends only on M and β , such that

$$\int_{\mathbb{R}^2} (e^{\beta u^2} - 1) dx \leq \mathcal{C}(M, \beta).$$

Based on Lemma 1.1, we say $f(t)$ has subcritical exponential growth at $t = \pm\infty$ if it verifies

(F1) $f \in C(\mathbb{R}, \mathbb{R})$ and

$$\lim_{|t| \rightarrow \infty} \frac{|f(t)|}{e^{\beta t^2}} = 0, \quad \text{for all } \beta > 0; \quad (1.3)$$

and $f(t)$ has critical exponential growth at $t = \pm\infty$ if it verifies

(F1') $f \in C(\mathbb{R}, \mathbb{R})$ and there exists $\beta_0 > 0$ such that

$$\lim_{|t| \rightarrow \infty} \frac{|f(t)|}{e^{\beta t^2}} = 0, \quad \text{for all } \beta > \beta_0 \quad (1.4)$$

and

$$\lim_{|t| \rightarrow \infty} \frac{|f(t)|}{e^{\beta t^2}} = +\infty, \quad \text{for all } \beta < \beta_0. \quad (1.5)$$

This notion of criticality was introduced by Adimurthi and Yadava [3], see also de Figueiredo, Miyagaki and Ruf [16], for the study of the planar Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x, u), & x \in \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases} \quad (1.6)$$

which is the maximal growth that allows to treat the problem variationally in $H^1(\mathbb{R}^2)$.

Inspired by [16, 35, 38, 41], in the present paper, we consider the following planar Choquard equation:

$$\begin{cases} -\Delta u + u = \left(I_\alpha \star |u|^{\frac{\alpha}{2}+1}\right) |u|^{\frac{\alpha}{2}-1} u + f(u), & x \in \mathbb{R}^2; \\ u \in H^1(\mathbb{R}^2), \end{cases} \quad (1.7)$$

where $\alpha \in (0, 2)$, f satisfies (F1) or (F1'), and $I_\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma\left(\frac{2-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) 2^\alpha \pi |x|^{2-\alpha}}, \quad x \in \mathbb{R}^2 \setminus \{0\}.$$

In (1.7), as what mentioned before, $\frac{\alpha}{2} + 1$ is the lower critical exponent coming from the Hardy-Littlewood-Sobolev inequality. Naturally, we are interested in whether (1.7) admits a nontrivial solution or a ground state solution when f has critical exponential growth in the sense of Trudinger-Moser. The double difficulties due to both the lower critical exponent $\frac{\alpha}{2} + 2$ and the critical exponential growth of f enforce the implementation of new ideas and tricks.

To state our theorems, we need to introduce some notations and assumptions.

By the Hardy-Littlewood-Sobolev inequality, one has

$$\mathcal{S} \left[\int_{\mathbb{R}^2} \left(I_\alpha \star |u|^{\frac{\alpha}{2}+1}\right) |u|^{\frac{\alpha}{2}-1} u^2 dx \right]^{\frac{2}{2+\alpha}} \leq \int_{\mathbb{R}^2} u^2 dx. \quad (1.8)$$

In view of [20, Theorem 4.3], the sharp constant $\mathcal{S}$ is achieved by a function $u \in H^1(\mathbb{R}^2)$ if and only if for every $x \in \mathbb{R}^2$,

$$u(x) = A(y^2 + |x - a|^2)^{-1} \quad (1.9)$$

for $a \in \mathbb{R}^2$, $A > 0$ and $y > 0$. In what follows, we let $A = A_0$ and A_0 is determined by

$$A_0^{\alpha+2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{dx dy}{(1 + |x|^2)^{\frac{\alpha}{2}+1} |x - y|^{2-\alpha} (1 + |y|^2)^{\frac{\alpha}{2}+1}} = \frac{2^\alpha \pi \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(1 - \frac{\alpha}{2}\right)}. \quad (1.10)$$

Set $T_0 = \left[\frac{(\alpha+2)(3\mathcal{S}+2\pi A_0^2)}{6} \right]^{1/\alpha}$. Let t_0 be the unique positive root in the interval $(0, \mathcal{S}^{\frac{1}{\alpha}})$ of the following equation

$$\frac{2\pi(p-2)A_0^2}{3} \left(\frac{4}{p^p}\right)^{\frac{1}{p-2}} t^2 = \frac{\alpha \mathcal{S}^{\frac{2}{\alpha}+1} - (\alpha+2)\mathcal{S}t^2 + 2t^{\alpha+2}}{\alpha+2}, \quad (1.11)$$

and let s_0 be the unique positive root in the interval $(0, \mathcal{S}^{\frac{1}{\alpha}})$ of the following equation

$$4 \left(\frac{\pi}{3p}\right)^{\frac{p}{2}} (p-2)^{\frac{p}{2}-1} A_0^p t^p = \left[\frac{\alpha}{\alpha+2} \left(\mathcal{S}^{\frac{2}{\alpha}+1} - \frac{4\pi}{\beta_0}\right)_+ + \frac{2\pi A_0^2}{3} t^2 \right] \left[\frac{4\pi\alpha - (\alpha+2)\beta_0 \mathcal{S}t^2 + 2\beta_0 t^{\alpha+2}}{2(\alpha+2)\beta_0} \right]^{\frac{p}{2}-1}, \quad (1.12)$$

where $a_+ = \max\{a, 0\}$.

In addition (F1) and (F1'), we also assume f satisfies the following conditions:

(F2) $f(t) = o(|t|)$ as $t \rightarrow 0$;

(F3) $\liminf_{|t| \rightarrow 0} \frac{F(t)}{|t|^\alpha} > A_0^{-2} \mathcal{S}^{-\frac{2}{\alpha}}$;

(F4) there exist $p > 4$ and $\lambda > \frac{p-1}{3(A_0 t_0)^{p-2}}$ such that

$$F(t) \geq \lambda |t|^p, \quad \forall 0 \leq t \leq A_0 T_0;$$

(F5) there exist $p > 2$ and $\lambda > \lambda_0$ such that

$$F(t) \geq \lambda |t|^p, \quad \forall 0 \leq t \leq A_0 T_0,$$

where

$$\lambda_0 = \frac{(p-1) \left[3\alpha \left(\beta_0 \mathcal{S}^{\frac{2}{\alpha}+1} - 4\pi\right)_+ + 2\pi(\alpha+2)\beta_0(A_0 s_0)^2 \right]}{6\pi(\alpha+2)\beta_0(A_0 s_0)^p};$$

(WN) $\frac{f(t)}{|t|}$ is non-decreasing on $(-\infty, 0) \cup (0, \infty)$.

In view of (1.8) and Lemma 1.1, under (F1) (or (F1')) and (F2), the energy functional $\mathcal{J} : H^1(\mathbb{R}^2) \rightarrow \mathbb{R}$ associated with (1.7)

$$\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) dx - \frac{1}{2+\alpha} \int_{\mathbb{R}^2} (I_\alpha * |u|^{\frac{\alpha}{2}+1}) |u|^{\frac{\alpha}{2}+1} dx - \int_{\mathbb{R}^2} F(u) dx \quad (1.13)$$

is continuously differentiable, and

$$\langle \mathcal{J}'(u), v \rangle = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + uv) dx - \int_{\mathbb{R}^2} (I_\alpha * |u|^{\frac{\alpha}{2}+1}) |u|^{\frac{\alpha}{2}-1} uv dx - \int_{\mathbb{R}^2} f(u)v dx, \quad \forall u, v \in H^1(\mathbb{R}^2), \quad (1.14)$$

moreover its critical points correspond to the weak solutions of (1.7). As usual, a solution is called a ground state solution if its energy is minimal among all nontrivial solutions.

Our main results are as follows.

Theorem 1.2. Assume that (F1), (F2), (F3) (or (F4)) and (AR) hold. Then (1.7) has a solution $\bar{u} \in H^1(\mathbb{R}^2) \setminus \{0\}$.

Theorem 1.3. Assume that (F1), (F2), (F3) (or (F4)) and (WN) hold. Then (1.7) has a solution $\bar{u} \in \mathcal{N}$ such that

$$\mathcal{J}(\bar{u}) = \inf_{\mathcal{N}} \mathcal{J} = \inf_{u \in H^1(\mathbb{R}^2) \setminus \{0\}} \max_{t>0} \mathcal{J}(tu) > 0,$$

where

$$\mathcal{N} := \left\{ u \in H^1(\mathbb{R}^2) \setminus \{0\} : \langle \mathcal{J}'(u), u \rangle = 0 \right\} \quad (1.15)$$

is the Nehari manifold of $\mathcal{J}$.

Theorem 1.4. Assume that (F1'), (F2) and (AR) with $\mu = \alpha + 2$ hold. Further suppose that one of the following conditions:

- (i) $\beta_0 \leq 4\pi S^{-\frac{2}{\alpha}-1}$ and (F3) (or (F4)) hold;
- (ii) $\beta_0 > 4\pi S^{-\frac{2}{\alpha}-1}$ and (F5) holds.

Then (1.7) has a solution $\bar{u} \in H^1(\mathbb{R}^2) \setminus \{0\}$.

Theorem 1.5. Assume that (F1'), (F2) and (WN) hold. Further suppose that one of the following conditions:

- (i) $\beta_0 \leq 4\pi S^{-\frac{2}{\alpha}-1}$ and (F3) (or (F4)) hold;
- (ii) $\beta_0 > 4\pi S^{-\frac{2}{\alpha}-1}$ and (F5) holds.

Then (1.7) has a solution $\bar{u} \in \mathcal{N}$ such that

$$\mathcal{J}(\bar{u}) = \inf_{\mathcal{N}} \mathcal{J} = \inf_{u \in H^1(\mathbb{R}^2) \setminus \{0\}} \max_{t>0} \mathcal{J}(tu) > 0.$$

The paper is organized as follows. In Section 2, we give some useful lemmas. We give the proofs of Theorems 1.2 and 1.3 in Sections 3 and prove Theorems 1.4 and 1.5 in Section 4.

Throughout the paper we also make use of the following notations:

- $H^1(\mathbb{R}^2)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + uv) dx, \quad \|u\| = (u, u)^{1/2}, \quad \forall u, v \in H^1(\mathbb{R}^2).$$

- $L^s(\mathbb{R}^2)$ ($1 \leq s < \infty$) denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_{\mathbb{R}^2} |u|^s dx \right)^{1/s}$.
- $C_1, C_2, \dots$ denote positive constants possibly different in different places.

2 Some useful lemmas

By a simple calculation, we can verify the following lemma.

Lemma 2.1. Assume that (F1) (or (F1')), (F2) and (WN) hold. Then the following two inequalities hold:

$$\frac{1-t^2}{2}f(\tau)\tau - F(\tau) + F(t\tau) \geq 0, \quad \forall t \in [0, +\infty) \quad (2.1)$$

and

$$\alpha - (\alpha + 2)t^2 + 2t^{\alpha+2} > 0, \quad \forall t \in [0, 1) \cup (1, +\infty). \quad (2.2)$$

Inspired by [37], we establish a key functional inequality as follows.

Lemma 2.2. Assume that (F1) (or (F1')), (F2) and (WN) hold. Then for any $u \in H^1(\mathbb{R}^2)$ and $t \geq 0$, one has

$$\mathcal{J}(u) \geq \mathcal{J}(tu) + \frac{1-t^2}{2} \langle \mathcal{J}'(u), u \rangle. \quad (2.3)$$

Proof. Note that

$$\mathcal{J}(tu) = \frac{t^2}{2} \|u\|^2 - \frac{t^{\alpha+2}}{\alpha+2} \int_{\mathbb{R}^2} \left(I_\alpha \star |u|^{\frac{\alpha}{2}+1} \right) |u|^{\frac{\alpha}{2}+1} dx - \int_{\mathbb{R}^2} F(tu) dx. \quad (2.4)$$

Thus, by (1.13), (1.14), (2.1), (2.2) and (2.4), one has

$$\begin{aligned} \mathcal{J}(u) - \mathcal{J}(tu) &= \frac{1-t^2}{2} \|u\|^2 - \frac{1-t^{\alpha+2}}{\alpha+2} \int_{\mathbb{R}^2} \left(I_\alpha \star |u|^{\frac{\alpha}{2}+1} \right) |u|^{\frac{\alpha}{2}+1} dx - \int_{\mathbb{R}^2} [F(u) - F(tu)] dx \\ &= \frac{1-t^2}{2} \left\{ \|u\|^2 - \int_{\mathbb{R}^2} \left(I_\alpha \star |u|^{\frac{\alpha}{2}+1} \right) |u|^{\frac{\alpha}{2}+1} dx - \int_{\mathbb{R}^2} f(u)u dx \right\} \\ &\quad + \int_{\mathbb{R}^2} \left[\frac{1-t^2}{2} f(u)u - F(u) + F(tu) \right] dx \\ &\quad + \frac{\alpha - (\alpha+2)t^2 + 2t^{\alpha+2}}{2(\alpha+2)} \int_{\mathbb{R}^2} \left(I_\alpha \star |u|^{\frac{\alpha}{2}+1} \right) |u|^{\frac{\alpha}{2}+1} dx \\ &\geq \frac{1-t^2}{2} \langle \mathcal{J}'(u), u \rangle. \end{aligned}$$

This shows that (2.3) holds. $\square$

From Lemma 2.2, we have the following corollary.

Corollary 2.3. Assume that (F1) (or (F1')), (F2) and (WN) hold. Then for $u \in \mathcal{N}$,

$$\mathcal{J}(u) = \max_{t>0} \mathcal{J}(tu). \quad (2.5)$$

Lemma 2.4. Assume that (F1) (or (F1')) and (F2) hold. Then there exists $\rho_0 > 0$ such that

$$\kappa := \{\mathcal{J}(u) : u \in H^1(\mathbb{R}^2), \|u\| = \rho_0\} > 0. \quad (2.6)$$

Proof. By (F1) (or (F1')) and (F2), one has for some constants $\alpha > 0$ and $C_1 > 0$

$$|F(t)| \leq \frac{1}{4}t^2 + C_1 \left(e^{\alpha t^2} - 1 \right) t^3, \quad \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}. \quad (2.7)$$

In view of Lemma 1.1 ii), we have

$$\int_{\mathbb{R}^2} \left(e^{2\alpha u^2} - 1 \right) dx = \int_{\mathbb{R}^2} \left(e^{2\alpha \|u\|^2 (u/\|u\|)^2} - 1 \right) dx \leq \mathcal{C}(1, 2\pi), \quad \forall \|u\| \leq \sqrt{\pi/\alpha}. \quad (2.8)$$

From (2.7) and (2.8), we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} F(u) dx &\leq \frac{1}{4} \|u\|_2^2 + C_1 \int_{\mathbb{R}^2} (e^{au^2} - 1) |u|^3 dx \\ &\leq \frac{1}{4} \|u\|_2^2 + C_1 \left[\int_{\mathbb{R}^2} (e^{2au^2} - 1) dx \right]^{1/2} \|u\|_6^3 \\ &\leq \frac{1}{4} \|u\|^2 + C_2 \|u\|^3, \quad \forall \|u\| \leq \sqrt{\pi/\alpha}. \end{aligned} \quad (2.9)$$

Hence, it follows from (1.8), (1.13) and (2.9) that

$$\begin{aligned} \mathcal{J}(u) &= \frac{1}{2} \|u\|^2 - \frac{1}{\alpha+2} \int_{\mathbb{R}^2} (I_\alpha \star |u|^{\frac{\alpha}{2}+1}) |u|^{\frac{\alpha}{2}+1} dx - \int_{\mathbb{R}^2} F(u) dx \\ &\geq \frac{1}{4} \|u\|^2 - C_2 \|u\|^3 - C_3 \|u\|^{\alpha+2}, \quad \forall \|u\| \leq \sqrt{\pi/\alpha}. \end{aligned}$$

Therefore, there exists $0 < \rho_0 < \sqrt{\pi/\alpha}$ such that (2.6) holds. $\square$

By Lemma 2.4 and the classical mountain pass theorem [40], we can prove the following lemma by a standard argument.

Lemma 2.5. Assume that (F1) (or (F1')) and (F2) hold. If $F(t) \geq 0$, then there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ satisfying

$$\mathcal{J}(u_n) \rightarrow c_0, \quad \|\mathcal{J}'(u_n)\|(1 + \|u_n\|) \rightarrow 0. \quad (2.10)$$

where

$$c_0 = \inf_{y \in \Gamma} \max_{t \in [0,1]} \mathcal{J}(y(t)) \quad (2.11)$$

and

$$\Gamma = \{y \in \mathcal{C}([0,1], H^1(\mathbb{R}^2)) : y(0) = 0, \mathcal{J}(y(1)) < 0\}. \quad (2.12)$$

Lemma 2.6. Assume that (F1) (or (F1')), (F2) and (WN) hold. Then for any $u \in H^1(\mathbb{R}^2) \setminus \{0\}$, there exists a $t_u > 0$ such that $t_u u \in \mathcal{N}$.

Proof. Let $u \in H^1(\mathbb{R}^2) \setminus \{0\}$ be fixed and define a function $\zeta(t) := \mathcal{J}(tu)$ on $[0, \infty)$. Clearly, by (1.14) and (2.4), we have

$$\begin{aligned} \zeta'(t) = 0 &\Leftrightarrow t^2 \|u\|^2 - t^{\alpha+2} \int_{\mathbb{R}^2} (I_\alpha \star |u|^{\frac{\alpha}{2}+1}) |u|^{\frac{\alpha}{2}+1} dx - \int_{\mathbb{R}^2} f(tu) t u dx = 0 \\ &\Leftrightarrow \langle \mathcal{J}'(tu), tu \rangle = 0 \Leftrightarrow tu \in \mathcal{N}. \end{aligned} \quad (2.13)$$

By Lemma 2.4, one has $\zeta(0) = 0$ and $\zeta(t) > 0$ for $t > 0$ small and $\zeta(t) < 0$ for t large. Therefore $\max_{t \in (0, \infty)} \zeta(t)$ is achieved at some $t_u > 0$ so that $\zeta'(t_u) = 0$ and $t_u u \in \mathcal{N}$. $\square$

From Corollary 2.3 and Lemma 2.6, we have $\mathcal{N} \neq \emptyset$ and the following minimax characterization.

Lemma 2.7. Assume that (F1) (or (F1')), (F2) and (WN) hold. Then $\inf_{u \in \mathcal{N}} \mathcal{J}(u) := m = \inf_{u \in H^1(\mathbb{R}^2) \setminus \{0\}} \max_{t > 0} \mathcal{J}(tu)$.

By Corollary 2.3 and Lemma 2.6, we have the following lemma.

Lemma 2.8. Assume that (F1) (or (F1')), (F2) and (WN) hold. Then

$$m \geq \kappa = \{\mathcal{J}(u) : u \in H^1(\mathbb{R}^2), \|u\| = \rho_0\} > 0,$$

where κ and ρ_0 are given in Lemma 2.4.

Next, we apply the non-Nehari manifold method introduced in [36] to prove the following lemma which is key to obtain a Nehari-type ground state solution.

Lemma 2.9. *Assume that (F1) (or (F1')), (F2) and (WN) hold. Then there exist a constant $c_* \in [\kappa, m]$ and a sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ satisfying*

$$\mathcal{J}(u_n) \rightarrow c_*, \quad \|\mathcal{J}'(u_n)\|(1 + \|u_n\|) \rightarrow 0. \quad (2.14)$$

Proof. Choose $v_k \in \mathcal{N}$ such that

$$m \leq \mathcal{J}(v_k) < m + \frac{1}{k}, \quad k \in \mathbb{N}. \quad (2.15)$$

From (WN) and (2.4), we have

$$\begin{aligned} \mathcal{J}(tv_k) &= \frac{t^2}{2} \|v_k\|^2 - \frac{t^{\alpha+2}}{\alpha+2} \int_{\mathbb{R}^2} \left(I_\alpha * |v_k|^{\frac{\alpha}{2}+1} \right) |v_k|^{\frac{\alpha}{2}+1} dx - \int_{\mathbb{R}^2} F(tv_k) dx \\ &\leq \frac{t^2}{2} \|v_k\|^2 - \frac{t^{\alpha+2}}{\alpha+2} \int_{\mathbb{R}^2} \left(I_\alpha * |v_k|^{\frac{\alpha}{2}+1} \right) |v_k|^{\frac{\alpha}{2}+1} dx. \end{aligned} \quad (2.16)$$

It follows that there exists $t_k > 0$ such that $\mathcal{J}(t_k v_k) < 0$ for all $k \in \mathbb{N}$. Hence, in view of Lemma 2.5, there exist a constant $c_k \in [\kappa, \sup_{t \in [0, t_k]} \mathcal{J}(tv_k)]$ and a sequence $\{u_{k,n}\}_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^2)$ satisfying

$$\mathcal{J}(u_{k,n}) \rightarrow c_k, \quad \|\mathcal{J}'(u_{k,n})\|(1 + \|u_{k,n}\|) \rightarrow 0, \quad k \in \mathbb{N}. \quad (2.17)$$

By virtue of Corollary 2.4, one can get that

$$\mathcal{J}(v_k) \geq \mathcal{J}(tv_k), \quad \forall t \geq 0. \quad (2.18)$$

Hence, by (2.15), (2.17) and (2.18), one has

$$\mathcal{J}(u_{k,n}) \rightarrow c_k < m + \frac{1}{k}, \quad \|\mathcal{J}'(u_{k,n})\|(1 + \|u_{k,n}\|) \rightarrow 0, \quad k \in \mathbb{N}.$$

Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\mathcal{J}(u_{k,n_k}) < m + \frac{1}{k}, \quad \|\mathcal{J}'(u_{k,n_k})\|(1 + \|u_{k,n_k}\|) < \frac{1}{k}, \quad k \in \mathbb{N}.$$

Let $u_k = u_{k,n_k}$, $k \in \mathbb{N}$. Then, going if necessary to a subsequence, we have

$$\mathcal{J}(u_n) \rightarrow c_* \in [\kappa, m], \quad \|\mathcal{J}'(u_n)\|(1 + \|u_n\|) \rightarrow 0.$$

□

Next, we give an estimate on the energy level m , which is essential in ensuring compactness.

Lemma 2.10. *Assume that (F1) (or (F1')), (F2) and (F3) hold, moreover $F(t) \geq 0$ for all $t \in \mathbb{R}$. Then $\max\{c_0, m\} < m_* := \frac{\alpha}{2(\alpha+2)} S^{\frac{2}{\alpha+1}}$.*

Proof. We set $U(x) = A_0(1 + |x|^2)^{-1}$, where A_0 satisfies (1.10). By a simple calculation, we have

$$\|U\|_4^4 = \int_{\mathbb{R}^2} |U|^4 dx = 2\pi A_0^4 \int_0^{+\infty} \frac{r}{(1+r^2)^4} dr = \frac{\pi A_0^4}{3} \quad (2.19)$$

and

$$\|\nabla U\|_2^2 = \int_{\mathbb{R}^2} |\nabla U|^2 dx = 8\pi A_0^2 \int_0^{+\infty} \frac{r^3}{(1+r^2)^4} dr = \frac{2\pi A_0^2}{3}. \quad (2.20)$$

By (F3), we can choose $\Lambda_0 > 0$ such that

$$\liminf_{|t| \rightarrow 0} \frac{F(t)}{|t|^4} \geq \Lambda_0 > A_0^{-2} S^{-\frac{2}{\alpha}}. \quad (2.21)$$

Let $t_* = S^{\frac{1}{\alpha}}$. Then we can choose $\epsilon > 0$ such that

$$\frac{(\Lambda_0 - \epsilon)(t_* - \epsilon)^2}{1 + \epsilon} \geq \frac{\|\nabla U\|_2^2}{2\|U\|_4^4}. \quad (2.22)$$

We set

$$g(t) = \frac{S}{2} t^2 - \frac{t^{\alpha+2}}{\alpha+2}. \quad (2.23)$$

By (2.21), we can choose $\epsilon \in (0, 1)$ such that

$$\frac{F(\epsilon t U(x))}{|\epsilon t U(x)|^4} \geq \Lambda_0 - \epsilon, \quad \forall x \in \mathbb{R}^2, \quad 0 \leq t \leq T_0 \quad (2.24)$$

and

$$\frac{1}{2} \epsilon^2 T_0^2 \|\nabla U\|_2^2 < m_* - g(t_* + \epsilon), \quad \frac{1}{2} \epsilon^2 t_*^2 \|\nabla U\|_2^2 < m_* - g(t_* - \epsilon), \quad (2.25)$$

where

$$T_0 = \left[\frac{(\alpha+2)(3S+2\pi A_0^2)}{6} \right]^{\frac{1}{\alpha}} = \left[\frac{(\alpha+2)(S+\|\nabla U\|_2^2)}{2} \right]^{\frac{1}{\alpha}}. \quad (2.26)$$

Now we define a function $\varphi_\epsilon(t)$ as follows:

$$\varphi_\epsilon(t) = \frac{\epsilon^2 \|\nabla U\|_2^2}{2} t^2 - \epsilon^{-2} \int_{\mathbb{R}^2} F(\epsilon t U) dx. \quad (2.27)$$

It is easy to see that $g(t) < g(t_*) = \frac{\alpha}{2(\alpha+2)} S^{\frac{2}{\alpha}+1} = m_*$ for $t \in [0, t_*) \cup (t_*, \infty)$. We set $U_\epsilon(x) = \epsilon U(\epsilon x)$. Then it follows from the definition of S that

$$\|U_\epsilon\|_2^2 = S \quad \text{and} \quad \int_{\mathbb{R}^2} \left(I_\alpha * |U_\epsilon|^{\frac{\alpha}{2}+1} \right) |U_\epsilon|^{\frac{\alpha}{2}+1} dx = 1 \quad (2.28)$$

and

$$\|\nabla U_\epsilon\|_2^2 = \epsilon^2 \|\nabla U\|_2^2. \quad (2.29)$$

From (2.4), (2.23), (2.27), (2.28) and (2.29), we have

$$\begin{aligned} \mathcal{J}(t U_\epsilon) &= \frac{1}{2} \left(\|\nabla U_\epsilon\|_2^2 + \|U_\epsilon\|_2^2 \right) t^2 - \frac{t^{\alpha+2}}{\alpha+2} \int_{\mathbb{R}^2} \left(I_\alpha * |U_\epsilon|^{\frac{\alpha}{2}+1} \right) |U_\epsilon|^{\frac{\alpha}{2}+1} dx \\ &\quad - \int_{\mathbb{R}^2} F(t U_\epsilon) dx \\ &= \frac{S}{2} t^2 - \frac{t^{\alpha+2}}{\alpha+2} + \frac{\epsilon^2 \|\nabla U\|_2^2}{2} t^2 - \epsilon^{-2} \int_{\mathbb{R}^2} F(t \epsilon U) dx \\ &= g(t) + \varphi_\epsilon(t). \end{aligned} \quad (2.30)$$

There are four possible subcases:

Subcase i) $t \geq T_0$. Then it follows from (2.23), (2.26), (2.27) and (2.30) that

$$\max_{t \geq T_0} \mathcal{J}(t U_\epsilon) = \max_{t \geq T_0} [g(t) + \varphi_\epsilon(t)] \leq 0 < m_*.$$

Subcase ii) $t_* + \epsilon \leq t \leq T_0$. Then it follows from (2.23), (2.25), (2.27) and (2.30) that

$$\max_{t_* + \epsilon \leq t \leq T_0} \mathcal{J}(t U_\epsilon) = \max_{t_* + \epsilon \leq t \leq T_0} [g(t) + \varphi_\epsilon(t)]$$

$$\leq g(t_* + \epsilon) + \frac{1}{2}\epsilon^2 T_0^2 \|\nabla U\|_2^2 < m_*.$$

Subcase iii) $t_* - \epsilon \leq t \leq t_* + \epsilon$. Then from (2.22), (2.24) and (2.27), one has

$$\begin{aligned} \varphi_\epsilon(t) &= \frac{\epsilon^2 \|\nabla U\|_2^2}{2} t^2 - \epsilon^{-2} \int_{\mathbb{R}^2} F(\epsilon t U) dx \\ &\leq \frac{\epsilon^2 \|\nabla U\|_2^2}{2} t^2 - (\Lambda_0 - \epsilon) \epsilon^2 \|U\|_4^4 t^4 \\ &\leq -\frac{\epsilon \epsilon^2 \|\nabla U\|_2^2}{2} (t_* - \epsilon)^2. \end{aligned} \quad (2.31)$$

Hence, it follows from (2.23), (2.27), (2.30) and (2.31) that

$$\begin{aligned} \max_{t_* - \epsilon \leq t \leq t_* + \epsilon} \mathcal{J}(tU_\epsilon) &= \max_{t_* - \epsilon \leq t \leq t_* + \epsilon} [g(t) + \varphi_\epsilon(t)] \\ &\leq g(t_*) + \max_{t_* - \epsilon \leq t \leq t_* + \epsilon} \varphi_\epsilon(t) \\ &\leq m_* - \frac{1}{2}\epsilon \epsilon^2 (t_* - \epsilon)^2 \|\nabla U\|_2^2 < m_*. \end{aligned}$$

Subcase iv) $0 \leq t \leq t_* - \epsilon$. Then it follows from (2.23), (2.25), (2.27) and (2.30) that

$$\begin{aligned} \max_{0 \leq t \leq t_* - \epsilon} \mathcal{J}(tU_\epsilon) &= \max_{0 \leq t \leq t_* - \epsilon} [g(t) + \varphi_\epsilon(t)] \\ &\leq g(t_* - \epsilon) + \frac{1}{2}\epsilon^2 t_*^2 \|\nabla U\|_2^2 < m_*. \end{aligned}$$

The above four subcases and Lemmas 2.5 and 2.7 show that

$$\max\{c_0, m\} \leq \max_{t \geq 0} \mathcal{J}(tU_\epsilon) < m_*.$$

□

Lemma 2.11. Assume that (F1) (or (F1')), (F2) and (F4) hold, moreover $F(t) \geq 0$ for all $t \in \mathbb{R}$. Then $\max\{c_0, m\} < m_* = \frac{\alpha}{2(\alpha+2)} S^{\frac{2}{\alpha}+1}$.

Proof. As in the proof of Lemma 2.10, we set $U(x) = A_0(1 + |x|^2)^{-1}$. By a simple calculation, we have

$$\begin{aligned} \|U\|_p^p &= \int_{\mathbb{R}^2} |U|^p dx = 2\pi A_0^p \int_0^{+\infty} \frac{r}{(1+r^2)^p} dr \\ &= \pi A_0^p \int_0^{+\infty} \frac{dt}{(1+t)^p} = \frac{\pi A_0^p}{p-1}. \end{aligned} \quad (2.32)$$

Now we define a function $\varphi(t)$ as follows:

$$\varphi(t) = \frac{\|\nabla U\|_2^2}{2} t^2 - \int_{\mathbb{R}^2} F(tU) dx. \quad (2.33)$$

It follows from the definition of S that

$$\|U\|_2^2 = S \text{ and } \int_{\mathbb{R}^2} \left(I_\alpha * |U|^{\frac{\alpha}{2}+1} \right) |U|^{\frac{\alpha}{2}+1} dx = 1. \quad (2.34)$$

From (2.4), (2.23), (2.33) and (2.34), we have

$$\mathcal{J}(tU) = \frac{1}{2} \left(\|\nabla U\|_2^2 + \|U\|_2^2 \right) t^2 - \frac{t^{\alpha+2}}{\alpha+2} \int_{\mathbb{R}^2} \left(I_\alpha * |U|^{\frac{\alpha}{2}+1} \right) |U|^{\frac{\alpha}{2}+1} dx$$

$$\begin{aligned}
& - \int_{\mathbb{R}^2} F(tU) dx \\
& = \frac{s}{2} t^2 - \frac{t^{\alpha+2}}{\alpha+2} + \frac{\|\nabla U\|_2^2}{2} t^2 - \int_{\mathbb{R}^2} F(tU) dx \\
& = g(t) + \varphi(t).
\end{aligned} \tag{2.35}$$

By (F4), (1.11), (2.20) and (2.32), we have

$$\lambda > \frac{\|\nabla U\|_2^2}{2\|U\|_p^p t_0^{p-2}} \tag{2.36}$$

and

$$\lambda > \left(\frac{p-2}{2p}\right)^{\frac{p-2}{2}} \frac{\|\nabla U\|_2^p}{[m_* - g(t_0)]^{(p-2)/2} p \|U\|_p^p}. \tag{2.37}$$

From (F4) and (2.33), one has

$$\varphi(t) \leq \frac{\|\nabla U\|_2^2}{2} t^2 - \lambda \|U\|_p^p t^p, \quad \forall 0 \leq t \leq T_0. \tag{2.38}$$

There are three possible subcases:

Subcase i) $t \geq T_0$. Then it follows from (2.23), (2.33) and (2.35) that

$$\max_{t \geq T_0} \mathcal{J}(tU) = \max_{t \geq T_0} [g(t) + \varphi(t)] \leq 0 < m_*.$$

Subcase ii) $t_0 \leq t \leq T_0$. Then it follows from (2.23), (2.35), (2.36) and (2.38) that

$$\begin{aligned}
\max_{t_0 \leq t \leq T_0} \mathcal{J}(tU) &= \max_{t_0 \leq t \leq T_0} [g(t) + \varphi(t)] \\
&\leq g(t_*) + \max_{t_0 \leq t \leq T_0} \varphi(t) \\
&\leq m_* + \frac{1}{2} t_0^2 \left[\|\nabla U\|_2^2 - 2\lambda \|U\|_p^p t_0^{p-2} \right] < m_*.
\end{aligned}$$

Subcase iii) $0 \leq t \leq t_0$. Then it follows from (2.23), (2.35), (2.37) and (2.38) that

$$\begin{aligned}
\max_{0 \leq t \leq t_0} \mathcal{J}(tU) &= \max_{0 \leq t \leq t_0} [g(t) + \varphi(t)] \\
&\leq g(t_0) + \frac{(p-2)\|\nabla U\|_2^{2p/(p-2)}}{2p\lambda\|U\|_p^{2p/(p-2)}} < m_*.
\end{aligned}$$

The above three subcases and Lemmas 2.5 and 2.7 show that

$$\max\{c_0, m\} \leq \max_{t \geq 0} \mathcal{J}(tU_\varepsilon) < m_*.$$

□

Lemma 2.12. Assume that (F1'), (F2) and (F5) hold, moreover $F(t) \geq 0$ for all $t \in \mathbb{R}$. Then $\max\{c_0, m\} < \omega_0 := \frac{2\pi\alpha}{(\alpha+2)\beta_0}$.

Proof. We set $U(x)$, $g(t)$ and $\varphi(t)$ are the same as in the proof of Lemma 2.11. Then (2.20), (2.26), (2.32) and (2.35) hold. By (F5), (1.11), (2.20) and (2.32), we have

$$\lambda > \frac{2(m_* - \omega_0)_+ + \|\nabla U\|_2^2 s_0^2}{2\|U\|_p^p s_0^p} \tag{2.39}$$

and

$$\lambda > \left(\frac{p-2}{2p}\right)^{\frac{p-2}{2}} \frac{\|\nabla U\|_2^p}{[\omega_0 - g(s_0)]^{(p-2)/2} p \|U\|_p^p}. \tag{2.40}$$

From (F5) and (2.33), one has

$$\varphi(t) \leq \frac{\|\nabla U\|_2^2}{2} t^2 - \lambda \|U\|_p^p t^p, \quad \forall 0 \leq t \leq T_0. \quad (2.41)$$

There are three possible subcases:

Subcase i) $t \geq T_0$. Then it follows from (2.23), (2.26), (2.33) and (2.35) that

$$\max_{t \geq T_0} \mathcal{J}(tU) = \max_{t \geq T_0} [g(t) + \varphi(t)] \leq 0 < \omega_0.$$

Subcase ii) $s_0 \leq t \leq T_0$. Then it follows from (2.23), (2.35), (2.39) and (2.41) that

$$\begin{aligned} \max_{s_0 \leq t \leq T_0} \mathcal{J}(tU) &= \max_{s_0 \leq t \leq T_0} [g(t) + \varphi(t)] \\ &\leq g(t_*) + \max_{s_0 \leq t \leq T_0} \varphi(t) \\ &\leq m_* + \frac{1}{2} s_0^2 \left[\|\nabla U\|_2^2 - 2\lambda \|U\|_p^p s_0^{p-2} \right] < \omega_0. \end{aligned}$$

Subcase iii) $0 \leq t \leq s_0$. Then it follows from (2.23), (2.35), (2.40) and (2.41) that

$$\begin{aligned} \max_{0 \leq t \leq s_0} \mathcal{J}(tU) &= \max_{0 \leq t \leq s_0} [g(t) + \varphi(t)] \\ &\leq g(s_0) + \frac{(p-2)\|\nabla U\|_2^{2p/(p-2)}}{2p(p\lambda\|U\|_p^p)^{2/(p-2)}} < \omega_0. \end{aligned}$$

The above three subcases and Lemmas 2.5 and 2.7 show that

$$\max\{c_0, m\} \leq \max_{t \geq 0} \mathcal{J}(tU_\varepsilon) < \omega_0. \quad \square$$

3 Sub-critical case

Lemma 3.1. Assume that (F1), (F2) and (AR) hold. Then any sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ satisfying (2.10) is bounded in $H^1(\mathbb{R}^2)$.

The proof of Lemma 3.1 is standard, so we omit it.

Lemma 3.2. Assume that (F1), (F2), (F3) (or (F4)) and (WN) hold. Then any sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ satisfying (2.14) is bounded in $H^1(\mathbb{R}^2)$.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \rightarrow \infty$. Let $v_n = u_n/\|u_n\|$. Then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^2)$, $v_n \rightarrow v$ in $L_{\text{loc}}^s(\mathbb{R}^2)$, $2 \leq s < \infty$, $v_n \rightarrow v$ a.e. on $\mathbb{R}^2$. If

$$\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B_1(y)} v_n^2 dx = 0,$$

then by Lions' concentration compactness principle [40, Lemma 1.21], $v_n \rightarrow 0$ in $L^s(\mathbb{R}^2)$ for $2 < s < \infty$. Set $\beta \in (0, 1/\delta t_*^2)$, where δ is determined by (1.8) and t_* is given in the proof of Lemma 2.10. In view of Lemmas 2.10 and 2.11, we have $m < m_*$. By (F1) and (F2), there exists $C_1 > 0$ such that

$$|F(t)| \leq \frac{m_* - m}{4t_*^2 \delta} t^2 + C_1 |t| \left(e^{\beta t^2} - 1 \right), \quad \forall t \in \mathbb{R}. \quad (3.1)$$

Then (3.1) and Lemma 1.1 ii) lead to

$$\int_{\mathbb{R}^2} F(t_* \sqrt{\delta} v_n) dx \leq \frac{m_* - m}{4} \|v_n\|_2^2 + C_1 t_* \sqrt{\delta} \int_{\mathbb{R}^2} \left(e^{\beta \delta t_*^2 v_n^2} - 1 \right) |v_n| dx$$

$$\begin{aligned}
&\leq \frac{m^* - m}{4} + C_1 t_* \sqrt{S} \left[\int_{\mathbb{R}^2} \left(e^{\beta S t_*^2 v_n^2} - 1 \right)^{3/2} dx \right]^{2/3} \|v_n\|_3 \\
&\leq \frac{m^* - m}{4} + C_1 t_* \sqrt{S} \left[\int_{\mathbb{R}^2} \left(e^{3\beta S t_*^2 v_n^2/2} - 1 \right) dx \right]^{2/3} \|v_n\|_3 \\
&\leq \frac{m^* - m}{4} + o(1).
\end{aligned} \tag{3.2}$$

Let $t_n = t_* \sqrt{S} / \|u_n\|$. Hence, from (1.8), (2.3), (2.4), (2.14) and (3.2), we derive

$$\begin{aligned}
c^* + o(1) &= \mathcal{J}(u_n) \geq \mathcal{J}(t_n u_n) + \frac{1 - t_n^2}{2} \langle \mathcal{J}'(u_n), u_n \rangle \\
&= \frac{t_n^2}{2} \|u_n\|^2 - \frac{t_n^{\alpha+2}}{\alpha+2} \int_{\mathbb{R}^2} \left(I_\alpha * |u_n|^{\frac{\alpha}{2}+1} \right) |u_n|^{\frac{\alpha}{2}+1} dx - \int_{\mathbb{R}^2} F(t_n u_n) dx + o(1) \\
&= \frac{S}{2} t_*^2 - \frac{S^{\frac{\alpha}{2}+1} t_*^{\alpha+2}}{\alpha+2} \int_{\mathbb{R}^2} \left(I_\alpha * |v_n|^{\frac{\alpha}{2}+1} \right) |v_n|^{\frac{\alpha}{2}+1} dx - \int_{\mathbb{R}^2} F(t_* \sqrt{S} v_n) dx + o(1) \\
&\geq \frac{S}{2} t_*^2 - \frac{t_*^{\alpha+2}}{\alpha+2} \|v_n\|_2^{\alpha+2} - \frac{m^* - m}{4} + o(1) \\
&\geq \frac{S}{2} t_*^2 - \frac{t_*^{\alpha+2}}{\alpha+2} - \frac{m^* - m}{4} + o(1) \\
&= m^* - \frac{m^* - m}{4} + o(1),
\end{aligned}$$

which is a contradiction due the fact that $c^* \leq m < m^*$. This shows that $\delta > 0$. The rest of the proof is standard, so we omit it. $\square$

Lemma 3.3. Assume that (F1) (or (F1')) and (F2) hold. Let $v_n \rightharpoonup \bar{v}$ in $H^1(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} f(v_n) v_n dx \leq K_0 \tag{3.3}$$

for some constant $K_0 > 0$. Then for every $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} f(v_n) \phi dx = \int_{\mathbb{R}^2} f(\bar{v}) \phi dx. \tag{3.4}$$

Proof. Let $\Omega = \text{supp } \phi$. For any given $\varepsilon > 0$, we have

$$\int_{|v_n| \geq K_0 \|\phi\|_\infty \varepsilon^{-1}} |f(v_n) \phi| dx \leq \frac{\varepsilon}{K_0} \int_{|v_n| \geq K_0 \|\phi\|_\infty \varepsilon^{-1}} f(v_n) v_n dx < \varepsilon. \tag{3.5}$$

Since $f(\bar{v}) \phi \in L^1(\Omega)$, it follows that there exists $\delta > 0$ such that

$$\int_A |f(\bar{v}) \phi| dx < \varepsilon \text{ if } \text{meas}(A) \leq \delta \tag{3.6}$$

for all measurable set $A \subset \Omega$. Next using the fact that $\bar{v} \in L^1(\Omega)$ we find $M_1 > 0$ such that

$$\text{meas}(\{x \in \Omega : |\bar{v}(x)| \geq M_1\}) \leq \delta. \tag{3.7}$$

Let $M_\varepsilon = \max\{M_1, K_0 \|\phi\|_\infty \varepsilon^{-1}\}$. Then we have

$$\int_{|v_n| \geq M_\varepsilon} |f(v_n) \phi| dx < \varepsilon, \quad \int_{|\bar{v}| \geq M_\varepsilon} |f(x, \bar{v}) \phi| dx < \varepsilon. \tag{3.8}$$

Since $|f(v_n)\phi|\chi_{|v_n|\leq M_\varepsilon} \rightarrow |f(\bar{v})\phi|\chi_{|\bar{v}|\leq M_\varepsilon}$ a.e. in Ω , moreover,

$$|f(v_n)\phi|\chi_{|v_n|\leq M_\varepsilon} \leq \|\phi\|_\infty \max_{|t|\leq M_\varepsilon} |f(t)| < \infty, \quad \forall x \in \Omega.$$

Then Lebesgue dominated convergence theorem leads to

$$\int_{|v_n|\leq M_\varepsilon} |f(v_n)\phi| dx \rightarrow \int_{|\bar{v}|\leq M_\varepsilon} |f(\bar{v})\phi| dx. \quad (3.9)$$

It follows from (3.8) and (3.9) that (3.4) holds due to the arbitrariness of $\varepsilon > 0$. $\square$

Proof of Theorem 1.2. By Lemmas 2.5, 2.10, 2.11 and 3.1, there exists a bounded sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ satisfying (2.10) with $c_0 \in [\kappa, m^*)$. Choose $C_2 > 0$ such that $\|u_n\| \leq C_2$. If

$$\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B_1(y)} u_n^2 dx = 0,$$

then by Lions' concentration compactness principle [40, Lemma 1.21], one has $u_n \rightarrow 0$ in $L^s(\mathbb{R}^2)$ for $2 < s < \infty$. Let $\beta \in (0, 1/C_2^2)$. By virtue of (F1) and (F2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon |t| + C_\varepsilon (e^{\beta t^2} - 1), \quad \forall t \in \mathbb{R}. \quad (3.10)$$

From (3.10) and Lemma 1.1 ii), similarly to (3.2), we can prove that

$$\begin{aligned} \int_{\mathbb{R}^2} f(u_n) u_n dx &\leq \varepsilon \|u_n\|_2^2 + C_\varepsilon \int_{\mathbb{R}^2} (e^{\beta u_n^2} - 1) |u_n| dx \\ &\leq C_2^2 \varepsilon + o(1). \end{aligned} \quad (3.11)$$

Due to the arbitrariness of $\varepsilon > 0$, we obtain from (3.11)

$$\int_{\mathbb{R}^2} f(u_n) u_n dx = o(1). \quad (3.12)$$

Similarly, we have

$$\int_{\mathbb{R}^2} F(u_n) dx = o(1). \quad (3.13)$$

From (1.13), (1.14), (2.10), (3.12) and (3.13), one can get

$$\|u_n\|^2 = \int_{\mathbb{R}^2} \left(I_\alpha \star |u_n|^{\frac{\alpha}{2}+1} \right) |u_n|^{\frac{\alpha}{2}+1} dx + o(1) \quad (3.14)$$

and

$$\begin{aligned} c_0 + o(1) &= \mathcal{J}(u_n) \\ &= \frac{1}{2} \|u_n\|^2 - \frac{1}{\alpha+2} \int_{\mathbb{R}^2} \left(I_\alpha \star |u_n|^{\frac{\alpha}{2}+1} \right) |u_n|^{\frac{\alpha}{2}+1} dx + o(1) \\ &= \frac{\alpha}{2(\alpha+2)} \int_{\mathbb{R}^2} \left(I_\alpha \star |u_n|^{\frac{\alpha}{2}+1} \right) |u_n|^{\frac{\alpha}{2}+1} dx + o(1). \end{aligned} \quad (3.15)$$

It follows from (1.8), (3.14) and (3.15) that

$$c_0 + o(1) = \frac{\alpha}{2(\alpha+2)} \int_{\mathbb{R}^2} \left(I_\alpha \star |u_n|^{\frac{\alpha}{2}+1} \right) |u_n|^{\frac{\alpha}{2}+1} dx + o(1)$$

$$\begin{aligned}
&\geq \frac{\alpha}{2(\alpha+2)} \|u_n\|_2^2 + o(1) \\
&\geq \frac{\alpha S}{2(\alpha+2)} \left[\int_{\mathbb{R}^2} \left(I_\alpha \star |u_n|^{\frac{\alpha}{2}+1} \right) |u_n|^{\frac{\alpha}{2}+1} dx \right]^{\frac{2}{\alpha+2}} + o(1) \\
&= \frac{\alpha S}{2(\alpha+2)} \left[\frac{2c_0(\alpha+2)}{\alpha} \right]^{\frac{2}{\alpha+2}} + o(1),
\end{aligned} \tag{3.16}$$

which leads to

$$c_0 \geq \frac{\alpha}{2(\alpha+2)} S^{\frac{2}{\alpha}+1} = m^*. \tag{3.17}$$

This contradicts with the fact that $c_0 < m^*$. Hence, $\delta > 0$, and so there exists a sequence $\{y_n\} \subset \mathbb{R}^2$ such that $\int_{B_1(y_n)} |u_n|^2 dx > \delta/2 > 0$. Let $\hat{u}_n(x) = u_n(x + y_n)$. Then we have $\|\hat{u}_n\| = \|u_n\|$ and

$$\mathcal{J}(\hat{u}_n) \rightarrow c_0, \quad \|\mathcal{J}'(\hat{u}_n)\|(1 + \|\hat{u}_n\|) \rightarrow 0, \quad \int_{B_1(0)} |\hat{u}_n|^2 dx > \frac{\delta}{2}. \tag{3.18}$$

Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^2) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \hat{u}_n \rightharpoonup \hat{u}, & \text{in } H^1(\mathbb{R}^2); \\ \hat{u}_n \rightarrow \hat{u}, & \text{in } L_{\text{loc}}^s(\mathbb{R}^2), \forall s \in [1, \infty); \\ \hat{u}_n \rightarrow \hat{u}, & \text{a.e. on } \mathbb{R}^2. \end{cases} \tag{3.19}$$

From (1.14), (3.18) and (3.19), we have

$$C_2^2 \geq \|\hat{u}_n\|^2 = \int_{\mathbb{R}^2} \left(I_\alpha \star |\hat{u}_n|^{\frac{\alpha}{2}+1} \right) |\hat{u}_n|^{\frac{\alpha}{2}+1} dx + \int_{\mathbb{R}^2} f(\hat{u}_n) \hat{u}_n dx + o(1) \tag{3.20}$$

Therefore, (1.14), (3.20) and Lemma 3.3 yield for every $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$,

$$\langle \mathcal{J}'(\hat{u}), \phi \rangle = \lim_{n \rightarrow \infty} \langle \mathcal{J}'(\hat{u}_n), \phi \rangle = 0.$$

Hence $\mathcal{J}'(\hat{u}) = 0$. This completes the proof. $\square$

Proof of Theorem 1.3. By Lemmas 2.9, 2.10, 2.11 and 3.2, there exists a bounded sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ satisfying (2.14) with $c^* \in [\kappa, m^*)$. Similarly to the proof of Theorem 1.2, we can prove that there exists a sequence $\{\hat{u}_n\}$ and $\hat{u} \in H^1(\mathbb{R}^2) \setminus \{0\}$ such that

$$\begin{cases} \hat{u}_n \rightharpoonup \hat{u}, & \text{in } H^1(\mathbb{R}^2); \\ \hat{u}_n \rightarrow \hat{u}, & \text{in } L_{\text{loc}}^s(\mathbb{R}^2), \forall s \in [1, \infty); \\ \hat{u}_n \rightarrow \hat{u}, & \text{a.e. on } \mathbb{R}^2 \end{cases} \tag{3.21}$$

and

$$\mathcal{J}(\hat{u}_n) \rightarrow c^*, \quad \|\mathcal{J}'(\hat{u}_n)\|(1 + \|\hat{u}_n\|) \rightarrow 0, \quad \mathcal{J}'(\hat{u}) = 0. \tag{3.22}$$

It follows that $\mathcal{J}(\hat{u}) \geq m$. From (WN), (1.13), (1.14), (3.21), (3.22) and Fatou' Lemma, we have

$$\begin{aligned}
m &\geq c^* \\
&= \lim_{n \rightarrow \infty} \left[\mathcal{J}(\hat{u}_n) - \frac{1}{2} \langle \mathcal{J}'(\hat{u}_n), \hat{u}_n \rangle \right] \\
&= \lim_{n \rightarrow \infty} \left[\frac{\alpha}{2(\alpha+2)} \int_{\mathbb{R}^2} \left(I_\alpha \star |\hat{u}_n|^{\frac{\alpha}{2}+1} \right) |\hat{u}_n|^{\frac{\alpha}{2}+1} dx + \frac{1}{2} \int_{\mathbb{R}^2} [f(\hat{u}_n) \hat{u}_n - 2F(\hat{u}_n)] dx \right] \\
&\geq \frac{\alpha}{2(\alpha+2)} \int_{\mathbb{R}^2} \left(I_\alpha \star |\hat{u}|^{\frac{\alpha}{2}+1} \right) |\hat{u}|^{\frac{\alpha}{2}+1} dx + \frac{1}{2} \int_{\mathbb{R}^2} [f(\hat{u}) \hat{u} - 2F(\hat{u})] dx
\end{aligned}$$

$$\begin{aligned}
&= \mathcal{J}(\hat{u}) - \frac{1}{2} \langle \mathcal{J}'(\hat{u}), \hat{u} \rangle \\
&= \mathcal{J}(\hat{u}) \geq m.
\end{aligned}$$

This shows that $\mathcal{J}(\hat{u}) = m$, which, together with Lemma 2.7, completes the proof. $\square$

4 Critical case

Lemma 4.1. Assume that (F1'), (F2) and (AR) with $\mu = \alpha + 2$ hold. Then any sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ satisfying (2.10) is bounded in $H^1(\mathbb{R}^2)$.

The proof of Lemma 4.1 is standard, so we omit it.

Lemma 4.2. Suppose all assumptions in Theorem 1.5 hold. Then any sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ satisfying (2.14) is bounded in $H^1(\mathbb{R}^2)$.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \rightarrow \infty$. Let $v_n = u_n / \|u_n\|$. Then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightarrow v$ in $H^1(\mathbb{R}^2)$, $v_n \rightarrow v$ in $L^s_{\text{loc}}(\mathbb{R}^2)$, $2 \leq s < \infty$, $v_n \rightarrow v$ a.e. on $\mathbb{R}^2$. If

$$\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B_1(y)} v_n^2 dx = 0,$$

then by Lions' concentration compactness principle [40, Lemma 1.21], $v_n \rightarrow 0$ in $L^s(\mathbb{R}^2)$ for $2 < s < \infty$. There are two cases to distinguish:

Case 1). $\beta_0 \leq 4\pi S^{-\frac{2}{\alpha}-1}$ and (F3) (or (F4)) hold. In view of Lemmas 2.10 and 2.11, we have $c_* \leq m < m_*$. Let $c_* := (1 - 3\epsilon)m_*$. By (F1') and (F2), there exists $C_1 > 0$ such that

$$|F(t)| \leq \frac{\epsilon m_*}{t_*^2(1 - \epsilon)S} t^2 + C_1 |t| \left(e^{(1+\epsilon)\beta_0 t^2} - 1 \right), \quad \forall t \in \mathbb{R}. \quad (4.1)$$

Let $q \in (1, 2)$ such that $q(1 - \epsilon^2) < 1$ and let $q' = q/(q - 1)$. Then (4.1) and Lemma 1.1 ii) lead to

$$\begin{aligned}
\int_{\mathbb{R}^2} F(t_* \sqrt{(1 - \epsilon)S} v_n) dx &\leq \epsilon m_* \|v_n\|_2^2 + C_1 t_* \sqrt{S} \int_{\mathbb{R}^2} \left(e^{(1-\epsilon^2)\beta_0 S t_*^2 v_n^2} - 1 \right) |v_n| dx \\
&\leq \epsilon m_* + C_1 t_* \sqrt{S} \left[\int_{\mathbb{R}^2} \left(e^{(1-\epsilon^2)\beta_0 S t_*^2 v_n^2} - 1 \right)^q dx \right]^{1/q} \|v_n\|_{q'} \\
&\leq \epsilon m_* + C_1 t_* \sqrt{S} \left[\int_{\mathbb{R}^2} \left(e^{q(1-\epsilon^2)\beta_0 S t_*^2 v_n^2} - 1 \right) dx \right]^{1/q} \|v_n\|_{q'} \\
&\leq \epsilon m_* + C_1 t_* \sqrt{S} \left[\int_{\mathbb{R}^2} \left(e^{4\pi q(1-\epsilon^2)v_n^2} - 1 \right) dx \right]^{1/q} \|v_n\|_{q'} \\
&\leq \epsilon m_* + o(1).
\end{aligned} \quad (4.2)$$

Let $t_n = t_* \sqrt{(1 - \epsilon)S} / \|u_n\|$. Hence, from (1.8), (2.3), (2.4), (2.14) and (4.2), we derive

$$\begin{aligned}
c_* + o(1) &= \mathcal{J}(u_n) \geq \mathcal{J}(t_n u_n) + \frac{1 - t_n^2}{2} \langle \mathcal{J}'(u_n), u_n \rangle \\
&= \frac{t_n^2}{2} \|u_n\|^2 - \frac{t_n^{\alpha+2}}{\alpha+2} \int_{\mathbb{R}^2} \left(I_\alpha * |u_n|^{\frac{\alpha}{2}+1} \right) |u_n|^{\frac{\alpha}{2}+1} dx
\end{aligned}$$

$$\begin{aligned}
& - \int_{\mathbb{R}^2} F(t_n u_n) dx + o(1) \\
& = \frac{(1-\epsilon)\mathcal{S}}{2} t_*^2 - \frac{[(1-\epsilon)\mathcal{S}]^{\frac{\alpha}{2}+1} t_*^{\alpha+2}}{\alpha+2} \int_{\mathbb{R}^2} \left(I_\alpha * |v_n|^{\frac{\alpha}{2}+1} \right) |v_n|^{\frac{\alpha}{2}+1} dx \\
& \quad - \int_{\mathbb{R}^2} F(t_* \sqrt{(1-\epsilon)\mathcal{S}} v_n) dx + o(1) \\
& \geq \frac{(1-\epsilon)\mathcal{S}}{2} t_*^2 - \frac{(1-\epsilon)^{\frac{\alpha}{2}+1}}{\alpha+2} \|v_n\|_2^{\alpha+2} t_*^{\alpha+2} - \epsilon m_* + o(1) \\
& \geq (1-\epsilon) \left(\frac{\mathcal{S}}{2} t_*^2 - \frac{t_*^{\alpha+2}}{\alpha+2} \right) - \epsilon m_* + o(1) \\
& = (1-\epsilon)m_* - \epsilon m_* + o(1),
\end{aligned}$$

which is a contradiction.

Case 2). $\beta_0 > 4\pi\mathcal{S}^{-\frac{2}{\alpha}-1}$ and (F5) holds. In view of Lemma 2.12, we deduce that $c_* \leq m < \omega_0 < m_*$. Let $c_* := (1-3\epsilon)\omega_0$ and $t_{**} = \sqrt{\frac{4\pi}{\beta_0\mathcal{S}}}$. It is easy to verify $t_{**} \in (0, t_*)$ and

$$\frac{\mathcal{S}}{2} t_{**}^2 - \frac{1}{\alpha+2} t_{**}^{\alpha+2} > \frac{2\pi\alpha}{(\alpha+2)\beta_0} = \omega_0. \quad (4.3)$$

By (F1') and (F2), there exists $C_2 > 0$ such that

$$|F(t)| \leq \frac{\epsilon\omega_0}{t_{**}^2(1-\epsilon)\mathcal{S}} t^2 + C_2 |t| \left(e^{(1+\epsilon)\beta_0 t^2} - 1 \right), \quad \forall t \in \mathbb{R}. \quad (4.4)$$

Let $q \in (1, 2)$ such that $q(1-\epsilon^2) < 1$ and let $q' = q/(q-1)$. Then (4.4) and Lemma 1.1 ii) lead to

$$\begin{aligned}
\int_{\mathbb{R}^2} F(t_{**} \sqrt{(1-\epsilon)\mathcal{S}} v_n) dx & \leq \epsilon\omega_0 \|v_n\|_2^2 + C_2 t_{**} \sqrt{\mathcal{S}} \int_{\mathbb{R}^2} \left(e^{(1-\epsilon^2)\beta_0 \mathcal{S} t_{**}^2 v_n^2} - 1 \right) |v_n| dx \\
& \leq \epsilon\omega_0 + C_2 t_{**} \sqrt{\mathcal{S}} \left[\int_{\mathbb{R}^2} \left(e^{(1-\epsilon^2)\beta_0 \mathcal{S} t_{**}^2 v_n^2} - 1 \right)^q dx \right]^{1/q} \|v_n\|_{q'} \\
& \leq \epsilon\omega_0 + C_2 t_{**} \sqrt{\mathcal{S}} \left[\int_{\mathbb{R}^2} \left(e^{q(1-\epsilon^2)\beta_0 \mathcal{S} t_{**}^2 v_n^2} - 1 \right) dx \right]^{1/q} \|v_n\|_{q'} \\
& = \epsilon\omega_0 + C_2 t_{**} \sqrt{\mathcal{S}} \left[\int_{\mathbb{R}^2} \left(e^{4\pi q(1-\epsilon^2)v_n^2} - 1 \right) dx \right]^{1/q} \|v_n\|_{q'} \\
& \leq \epsilon\omega_0 + o(1).
\end{aligned} \quad (4.5)$$

Let $t_n = t_{**} \sqrt{(1-\epsilon)\mathcal{S}} / \|u_n\|$. Hence, from (1.8), (2.3), (2.4), (2.14), (4.3) and (4.5), we derive

$$\begin{aligned}
c_* + o(1) & = \mathcal{J}(u_n) \geq \mathcal{J}(t_n u_n) + \frac{1-t_n^2}{2} \langle \mathcal{J}'(u_n), u_n \rangle \\
& = \frac{t_n^2}{2} \|u_n\|^2 - \frac{t_n^{\alpha+2}}{\alpha+2} \int_{\mathbb{R}^2} \left(I_\alpha * |u_n|^{\frac{\alpha}{2}+1} \right) |u_n|^{\frac{\alpha}{2}+1} dx \\
& \quad - \int_{\mathbb{R}^2} F(t_n u_n) dx + o(1) \\
& = \frac{(1-\epsilon)\mathcal{S}}{2} t_{**}^2 - \frac{[(1-\epsilon)\mathcal{S}]^{\frac{\alpha}{2}+1} t_{**}^{\alpha+2}}{\alpha+2} \int_{\mathbb{R}^2} \left(I_\alpha * |v_n|^{\frac{\alpha}{2}+1} \right) |v_n|^{\frac{\alpha}{2}+1} dx
\end{aligned}$$

$$\begin{aligned}
& - \int_{\mathbb{R}^2} F(t_{**} \sqrt{(1-\epsilon)S} v_n) dx + o(1) \\
& \geq \frac{(1-\epsilon)S}{2} t_{**}^2 - \frac{(1-\epsilon)^{\frac{\alpha}{2}+1}}{\alpha+2} \|v_n\|_2^{\alpha+2} t_{**}^{\alpha+2} - \epsilon \omega_0 + o(1) \\
& \geq (1-\epsilon) \left(\frac{S}{2} t_{**}^2 - \frac{t_{**}^{\alpha+2}}{\alpha+2} \right) - \epsilon \omega_0 + o(1) \\
& > (1-\epsilon) \omega_0 - \epsilon \omega_0 + o(1),
\end{aligned}$$

which is a contradiction.

Both Case 1) and Case 2) shows that $\delta > 0$. The rest of the proof is standard, so we omit it. $\square$

Proof of Theorem 1.4. By Lemmas 2.5, there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ satisfying (2.10). Since $\mu = \alpha + 2$. Then it follows from (AR), (1.13), (1.14) and (2.10) that

$$\begin{aligned}
c_0 + o(1) &= \mathcal{J}(u_n) - \frac{1}{\alpha+2} \langle \mathcal{J}'(u_n), u_n \rangle \\
&= \frac{\alpha}{2(\alpha+2)} \|u_n\|^2 + \frac{1}{\alpha+2} \int_{\mathbb{R}^2} [f(u_n)u_n - (\alpha+2)F(u_n)] dx \\
&\geq \frac{\alpha}{2(\alpha+2)} \|u_n\|^2.
\end{aligned} \tag{4.6}$$

Hence, it follows from (4.6) that

$$\|u_n\|^2 \leq \frac{2(\alpha+2)c_0}{\alpha} + o(1). \tag{4.7}$$

There are two cases to distinguish:

Case 1). $\beta_0 \leq 4\pi S^{-\frac{2}{\alpha}-1}$ and (F3) (or (F4)) hold. In view of Lemmas 2.10 and 2.11, we have

$$\frac{2(\alpha+2)c_0}{\alpha} < S^{\frac{2}{\alpha}+1} \leq \frac{4\pi}{\beta_0}. \tag{4.8}$$

Case 2). $\beta_0 > 4\pi S^{-\frac{2}{\alpha}-1}$ and (F5) holds. In view of Lemma 2.12, we can also deduce

$$\frac{2(\alpha+2)c_0}{\alpha} < \frac{4\pi}{\beta_0}. \tag{4.9}$$

The above two cases show that there exist $\epsilon > 0$ and $q \in (1, 2)$ such that

$$\frac{2(\alpha+2)c_0}{\alpha} \leq \frac{4\pi}{\beta_0} (1-3\epsilon) \tag{4.10}$$

and

$$\frac{(1+\epsilon)(1-3\epsilon)q}{1-\epsilon} < 1. \tag{4.11}$$

From (4.7), (4.10) and (4.11), we have

$$q\beta_0(1+\epsilon)\|u_n\|^2 \leq 1-\epsilon + o(1). \tag{4.12}$$

If

$$\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B_1(y)} |u_n|^2 dx = 0,$$

then by Lions' concentration compactness principle [40, Lemma 1.21], one has $u_n \rightarrow 0$ in $L^s(\mathbb{R}^2)$ for $2 < s < \infty$. By virtue of (F1') and (F2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon \left(e^{\beta_0(1+\epsilon)t^2} - 1 \right), \quad \forall t \in \mathbb{R}. \tag{4.13}$$

Let $q' = q/(q-1)$. It follows from (4.12), (4.13) and Lemma 1.1 ii) that

$$\begin{aligned} \int_{\mathbb{R}^2} f(u_n) u_n dx &\leq \varepsilon \|u_n\|_2^2 + C_\varepsilon \int_{\mathbb{R}^2} \left(e^{\beta_0(1+\varepsilon)u_n^2} - 1 \right) |u_n| dx \\ &\leq C_3 \varepsilon + C_\varepsilon \left[\int_{\mathbb{R}^2} \left(e^{\beta_0(1+\varepsilon)u_n^2} - 1 \right)^q dx \right]^{1/q} \|u_n\|_{q'} \\ &\leq C_3 \varepsilon + C_\varepsilon \left[\int_{\mathbb{R}^2} \left(e^{q\beta_0(1+\varepsilon)u_n^2} - 1 \right) dx \right]^{1/q} \|u_n\|_{q'} \\ &\leq C_3 \varepsilon + o(1). \end{aligned} \quad (4.14)$$

Due to the arbitrariness of $\varepsilon > 0$, we obtain from (4.14)

$$\int_{\mathbb{R}^2} f(u_n) u_n dx = o(1). \quad (4.15)$$

Similarly, we have

$$\int_{\mathbb{R}^2} F(u_n) dx = o(1). \quad (4.16)$$

The rest proof is the same as one of Theorem 1.2, so we omit it. $\square$

Proof of Theorem 1.5. By Lemmas 2.9 and 4.2, there exists a bounded sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ satisfying (2.14). It follows from (WN), (1.13) and (2.14) that $2c_* \leq \liminf_{n \rightarrow \infty} \|u_n\|^2 \leq \limsup_{n \rightarrow \infty} \|u_n\|^2 < \infty$. If

$$\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B_1(y)} |u_n|^2 dx = 0,$$

then by Lions' concentration compactness principle [40, Lemma 1.21], one has $u_n \rightarrow 0$ in $L^s(\mathbb{R}^2)$ for $2 < s < \infty$. It follows that $v_n := u_n/\|u_n\| \rightarrow 0$ in $L^s(\mathbb{R}^2)$ for $2 < s < \infty$. Similarly to the proof of Lemma 4.2, we can show that $\delta > 0$ by distinguishing two cases :

Case 1). $\beta_0 \leq 4\pi S^{-\frac{2}{a}-1}$ and (F3) (or (F4)) hold.

Case 2). $\beta_0 > 4\pi S^{-\frac{2}{a}-1}$ and (F5) holds.

Hence, there exists a sequence $\{y_n\} \subset \mathbb{R}^2$ such that $\int_{B_1(y_n)} |u_n|^2 dx > \delta/2 > 0$. Let $\hat{u}_n(x) = u_n(x + y_n)$. Then we have $\|\hat{u}_n\| = \|u_n\|$ and

$$\mathcal{J}(\hat{u}_n) \rightarrow c_*, \quad \|\mathcal{J}'(\hat{u}_n)\|(1 + \|\hat{u}_n\|) \rightarrow 0, \quad \int_{B_1(0)} |\hat{u}_n|^2 dx > \frac{\delta}{2}. \quad (4.17)$$

Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^2) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \hat{u}_n \rightharpoonup \hat{u}, & \text{in } H^1(\mathbb{R}^2); \\ \hat{u}_n \rightarrow \hat{u}, & \text{in } L_{\text{loc}}^s(\mathbb{R}^2), \forall s \in [1, 2^*]; \\ \hat{u}_n \rightarrow \hat{u}, & \text{a.e. on } \mathbb{R}^2. \end{cases} \quad (4.18)$$

Similarly to the proof of Theorem 1.3, we can prove that $\mathcal{J}'(\hat{u}) = 0$ and $\mathcal{J}(\hat{u}) = m$. $\square$

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