Research article

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Blow-up criteria and instability of normalized standing waves for the fractional Schrödinger-Choquard equation

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Abstract: In this paper, we study blow-up criteria and instability of normalized standing waves for the fractional Schrödinger-Choquard equation

\[ i\partial_t \psi - (-\Delta)^s \psi + (I_\alpha * |\psi|^p)|\psi|^{p-2}\psi = 0. \]

By using localized virial estimates, we firstly establish general blow-up criteria for non-radial solutions in both \(L^2\)-critical and \(L^2\)-supercritical cases. Then, we show existence of normalized standing waves by using the profile decomposition theory in \(H^s\). Combining these results, we study the strong instability of normalized standing waves. Our obtained results greatly improve earlier results.

Keywords: Fractional Schrödinger-Choquard equation; Blow-up criteria; Strong instability; Normalized standing waves

MSC: 35Q55; 35J10

1 Introduction

Over the past decade, there has been a great deal of interest in studying the fractional Schrödinger equation (NLS)

\[ i\partial_t \psi = (-\Delta)^s \psi + f(\psi), \tag{1.1} \]

where \(0 < s < 1\) and \(f(\psi)\) is the nonlinearity. The fractional differential operator \((-\Delta)^s\) is defined by \((-\Delta)^s \psi = \mathcal{F}^{-1}[|\xi|^{2s}\mathcal{F}(\psi)]\), where \(\mathcal{F}\) and \(\mathcal{F}^{-1}\) are the Fourier transform and inverse Fourier transform, respectively. The fractional NLS (1.1) was first deduced by Laskin in [29, 30] by extending the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. The fractional NLS also arises in the description of Bonson stars as well as in water wave dynamics (see e.g. [22]) and in the continuum limit of discrete models with long-range interactions (see e.g. [28]).

In this paper, we consider blow-up criteria and instability of normalized standing waves for the fractional nonlinear Schrödinger-Choquard equation

\[
\begin{align*}
    i\partial_t \psi - (-\Delta)^s \psi + (I_\alpha * |u|^p)|\psi|^{p-2}\psi &= 0, & (t, x) \in [0, T^*) \times \mathbb{R}^N, \\
    \psi(0, x) &= \psi_0(x),
\end{align*}
\tag{1.2}
\]

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where \( \psi : [0, T^*) \times \mathbb{R}^N \to \mathbb{C} \) is the complex valued function, \( N \geq 1, \psi_0 \in H^s, 0 < s < 1, 0 < T^* \leq \infty, 1 + \frac{a}{N} < p < \frac{N+a}{N-2s}, I_a : \mathbb{R}^N \to \mathbb{R} \) is the Riesz potential defined by

\[
I_a(x) = \frac{A(a)}{|x|^{N-a}}, \quad A(a) := \frac{\Gamma(\frac{N-a}{2})}{\Gamma(\frac{N}{2})} m^{N/2} 2^a
\]

with \( a \in (0, N) \) and \( \Gamma \) is the Gamma function.

Equation (1.2) enjoys the scaling invariance. That is, if \( \psi \) is a solution of (1.2) with initial data \( \psi_0 \), then

\[
\psi_\mu(t, x) := \mu^{\frac{2-N}{2}} \psi(\mu t, \mu x) \quad \text{for all} \quad \mu > 0
\]

is also a solution of (1.2) with initial data \( \mu^{\frac{2-N}{2}} \psi_0(\mu x) \). In particular, \( \|\psi_\mu(t)\|_{H^n} = \|\psi(t)\|_{H^n} \), where

\[
s_c := \frac{N}{2} - \frac{a+2s}{2p-2}.
\]

Thus, \( s_c \) is referred as the critical Sobolev exponent of (1.2). If the initial data \( \psi_0 \in H^s \), then equation (1.2) enjoys mass and energy conservation laws:

\[
\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \quad E(\psi(t)) = E(\psi_0),
\]

where the energy \( E \) is defined by

\[
E(\psi(t)) = \frac{1}{2} \|\psi(t)\|_{H^s}^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_a * |\psi(t)|^p)(x) |\psi(t, x)|^p \, dx.
\]  

(1.4)

Before entering our main results, we firstly recall some known blow-up results for NLS. For the classical NLS, i.e., \( s = 1 \), when initial data \( \psi_0 \in \Sigma := \{ \psi_0 \in H^1 \text{ and } x\psi_0 \in L^2 \} \), the following Variance-Virial Law holds

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |x|^2 |\psi(t, x)|^2 \, dx = 2\text{Im} \int_{\mathbb{R}^N} \psi(t, x) x \cdot \nabla \psi(t, x) \, dx.
\]  

(1.5)

By using (1.5) and the virial identity, ones can prove existence of blow-up solutions for the classical NLS with negative energy \( E(\psi_0) < 0 \), see [7]. However, since identity (1.5) fails for \( s < 1 \), which readily checks by dimensional analysis, this argument cannot work. Rather, a possible generalization of the variance for the fractional NLS is given by the nonnegative quantity

\[
\gamma(s)[\psi(t)] := \int_{\mathbb{R}^N} \psi(t, x) x \cdot (-\Delta)^{1-s} \psi(t, x) \, dx = \| x(-\Delta)^{\frac{1-s}{2}} \psi(t) \|_{L^2}^2.
\]  

(1.6)

Let \( \psi(t) \) be a sufficiently regular and spatially localized solution of equation \( i\partial_t \psi = (-\Delta)^s \psi \), it follows that

\[
\frac{1}{2} \frac{d}{dt} \gamma(s)[\psi(t)] := 2\text{Im} \int_{\mathbb{R}^N} \tilde{\psi}(t, x) x \cdot \nabla \tilde{\psi}(t, x) \, dx.
\]

(1.7)

This method has been successfully applied to prove the existence of radial blow-up solutions of (1.1) with focusing Hartree-type nonlinearities, i.e., \( f(\psi) = -(|x|^r * |\psi|^2) \psi \) with \( r \geq 1 \), see [8, 9, 48]. But this method can not work due to the nontrivial error terms which seem very hard to control for the local nonlinearities \( f(\psi) = -|\psi|^p \psi \), see [6]. In [6], Boulenger, Himmelsbach and Lenzmann applied the Balakrishman’s formula

\[
(-\Delta)^s = \frac{\sin \pi s}{\pi} \int_0^\infty m^{s-1} \frac{-\Delta}{-\Delta + m} \, dm,
\]

(1.8)

and obtained the differential inequality

\[
\frac{d}{dt} \left( \text{Im} \int_{\mathbb{R}^N} \psi(t) \nabla \varphi_R \cdot \nabla \psi(t) \, dx \right) \leq 4pN E(\psi_0) - 2\delta \| (-\Delta)^{\frac{s}{2}} \psi(t) \|_{L^2}^2 + c_R(1 + \| (-\Delta)^{\frac{s}{2}} \psi(t) \|_{L^1}^{p(s+1)}),
\]

where \( \varphi_R \) is a standard mollifier.
where $\delta = pN - 2s$. Based on this key estimate, they proved the existence of radial blow-up solutions by applying a standard comparison ODE argument.

For the fractional Schrödinger-Choquard equation (1.2), Saanouni in [39] proved the existence of radial blow-up solutions by using the method in [6]. In this paper, we will further study the existence of blow-up solutions of (1.2) for non-radial initial data by using the idea of Du, Wu and Zhang in [13]. The main difficulty is the appearance of the fractional order Laplacian $(-\Delta)^s$. When $s = 1$, the time derivative of the virial action can be easily obtained, that is

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \varphi(x)|\psi(t, x)|^2 \, dx = 2\text{Im} \int_{\mathbb{R}^N} \bar{\psi}(t, x) \nabla \varphi(x) \cdot \nabla \psi(t, x) \, dx.$$  

(1.9)

Using this identity, Du, Wu and Zhang in [13] derived an $L^2$-estimate in the exterior ball. Combining this $L^2$-estimate and the virial estimates, they established blow-up criteria for the classical NLS. When $s \in (\frac{1}{2}, 1)$, the identity (1.9) does not hold. However, by exploiting the ideas in [6, 12] and using the Balakrishnan’s formula (1.8), we can obtain the time derivative of the virial action, see Lemma 2.9. Thus, we can establish the blow-up criteria for (1.2).

**Theorem 1.1.** Let $N \geq 1$, $s \in (\frac{1}{2}, 1)$, $1 + \frac{2sN}{N-2s} \leq p < \frac{N+s}{N-2s}$, $\psi_0 \in H^s$ and $\psi \in C([0, T^*], H^s)$ be the corresponding solution of (1.2). Furthermore, we suppose either $E(\psi_0) < 0$, or, if $E(\psi_0) \geq 0$ and

$$\begin{cases}
E(\psi_0)^{\frac{s}{2}} \|\psi_0\|_{L^2}^{2(s-s_c)} < E(\psi)^{\frac{s}{2}} \|\psi\|_{L^2}^{2(s-s_c)}, \\
\|(-\Delta)^{s/2} \psi_0\|_{L^2}^{2(s-s_c)} > \|(-\Delta)^{s/2} \psi\|_{L^2}^{2(s-s_c)},
\end{cases}$$

(1.10)

where $s_c$ is defined by (1.3) and $u$ is a ground state of the following elliptic equation

$$(-\Delta)^s u + u - (I_\alpha \ast |u|^p)|u|^{p-2} u = 0.$$  

(1.11)

Then one of the following statements holds true:

- $\psi(t)$ blows up in finite time, i.e. $T^* < +\infty$;
- $\psi(t)$ blows up infinite time and there exists a time sequence $(t_n)_{n=1}$ such that $t_n \to +\infty$ and

$$\lim_{n \to \infty} \|(-\Delta)^{s/2} \psi(t_n)\|_{L^2} = \infty.$$

**Remark 1.** The uniqueness of ground state solutions to (1.11) is still unknown. However, it follows from the optimal constant in (2.2) that all ground states have the same $L^1$-norm. Moreover, we see from Pohozaev’s identities (2.3) that all ground states have the same $H^s$-norm and energy. Therefore, for different ground states, the quantities $E(\psi)^{\frac{s}{2}} \|\psi\|_{L^2}^{2(s-s_c)}$ and $\|(-\Delta)^{s/2} \psi\|_{L^2}^{2(s-s_c)}$ are same. These imply that the assumption (1.10) is reasonable.

**Remark 2.** When $p = 2$, similar blow-up criteria for (1.2) with radial solutions have been established in [8, 9, 25, 33, 37, 40, 41, 47, 48]. Here, we remove the assumption of radial solutions and extend these results to more general Choquard-type nonlinearity.

Based on blow-up criteria (1.10), we study the strong instability of normalized standing waves of (1.2). Firstly, we introduce some notations. Equation (1.2) enjoys a class of special solutions, which are called standing waves, namely solutions of the form $e^{i\omega t} u_\omega$, where $\omega \in \mathbb{R}$ is a frequency and $u_\omega \in H^s$ is a nontrivial solution to the elliptic equation

$$(-\Delta)^s u_\omega + \omega u_\omega - (I_\alpha \ast |u_\omega|^p)|u_\omega|^{p-2} u_\omega = 0.$$  

(1.12)

At this moment, our intention is reduced to study (1.12). To do this, there exist two substantially different choices in terms of the frequency $\omega$. One is to fix the frequency $\omega \in \mathbb{R}$. In this situation, every solution to (1.12) corresponds to a critical point of the action functional $S_\omega(u)$ on $H^s$, where

$$S_\omega(u) := \frac{1}{2} \|u\|_{H^s}^2 + \frac{\omega}{2} \|u\|_{L^2}^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha \ast |u|^p)(x)|u(x)|^p \, dx.$$  

(1.13)
Alternatively, it is interesting to study solutions of (1.12) having prescribed $L^2$-norm. That is, for any given $c > 0$, ones study solutions of (1.12) satisfying the $L^2$-norm constraint
\begin{equation}
S(c) = \{ u \in H^s : \| u \|^2_{L^2} = c \}, \quad c > 0. \tag{1.14}
\end{equation}
Physically, such solutions are called normalized solutions of (1.12), which formally corresponds to critical points of the energy functional $E(u)$ restricted on $S(c)$, where $E(u)$ is defined by (1.4). In particular, in this situation, the frequency $\omega \in \mathbb{R}$ is an unknown part, which corresponds to the associated Lagrange multiplier. Recently, these questions have received more attention, see [1–5, 26, 32, 42, 45].

In the $L^2$-subcritical case, i.e., $1 + \frac{a}{N} < p < 1 + \frac{2s+a}{N}$, the energy $E(u)$ is bounded from below on $S(c)$. Feng and Zhang in [21] studied existence of normalized ground states to (1.12) by using the profile decomposition theory in $H^s$. On the contrary, in the $L^2$-supercritical case, the energy $E(u)$ restricted on $S(c)$ becomes unbounded from below for any $c > 0$. For this reason, it is unlikely to obtain a solution to (1.12)-(1.14) by developing a global minimizing problem. Motivated by minimizing method on Pohozaev manifold, we try to construct a submanifold of $S(c)$, on which $E(u)$ is bounded from below and coercive, and then we look for minimizers of $E(u)$ on such a submanifold. Precisely, we introduce the following minimizing problem
\begin{equation}
m(c) := \inf_{u \in V(c)} E(u), \tag{1.15}
\end{equation}
where the constraint $V(c)$ is defined by
\begin{equation}
V(c) := \{ u \in S(c) : K(u) = 0 \}, \tag{1.16}
\end{equation}
and the functional $K(u)$ is defined by
\begin{equation}
K(u) := \partial_\lambda S_\omega(u^1)|_{\lambda=1} = s\| u \|^2_{H^s} - \frac{\theta}{2p} \int_{\mathbb{R}^N} (I_a * |u|^p)(x)|u(x)|^p \, dx, \tag{1.17}
\end{equation}
where
\begin{equation}
\theta = Np - N - a, \quad u^1(x) := \lambda^{N/2} u(\lambda x). \tag{1.18}
\end{equation}
Indeed, the identity $K(u) = 0$ is the Pohozaev identity related to (1.12). The constraint $V(c)$ is the so-called Pohozaev manifold related to (1.12)-(1.14). In the following theorem, we can prove the existence of minimizers of (1.15).

**Theorem 1.2.** Let $1 + \frac{a+2s}{N} < p < \frac{N+a}{N-2s}$ and $c > 0$. Then there exists $u_c \in V(c)$ such that $E(u_c) = m(c)$.

**Remark.** This theorem can be proved by using the method in [18]. Here, we will use the profile decomposition of bounded sequences in $H^s$ to prove this theorem. The profile decomposition theory has been extensively applied to study existence of normalized standing waves in the $L^2$-subcritical case, see, e.g., [20, 21, 49]. Here, we successfully apply it to study existence of normalized standing waves in the $L^2$-supercritical case. Therefore, our approach is of particular interest.

Next, we denote the set of minimizers of $E$ on $V(c)$ as
\begin{equation}
M_c := \{ u \in V(c) : E(u) = \inf_{v \in V(c)} E(v) \}. \tag{1.19}
\end{equation}
In the following theorem, we can show any minimizer to (1.15) is a ground state to (1.12)-(1.14).

**Theorem 1.3.** Let $1 + \frac{2s+a}{N} < p < \frac{N+a}{N-2s}$. Then for any $u_c \in M_c$, there exists $\omega_c > 0$ such that $(u_c, \omega_c) \in H^s \times \mathbb{R}$ is a weak solution to problem (1.12). Furthermore, $u_c$ is a ground state solution to problem (1.12) with $\omega = \omega_c$.

Finally, we consider the strong instability of normalized standing waves. The usual strategy to study the strong instability of standing waves for the classical NLS ($s=1$) is to use the variational characterization of the
ground states as minimizers of the action functional and obtain the key estimate $K(\psi(t)) \leq 2(S_\omega(\psi_0) - S_\omega(u_\omega))$. Then, it follows from the virial identity that

$$\frac{d^2}{dt^2} \|x\psi(t)\|^2_{L^2} = 8K(\psi(t)) \leq 16(S_\omega(\psi_0) - S_\omega(u_\omega)) < 0,$$

where $K(\psi(t))$ is defined by (1.17) with $s = 1$. This implies that the solution $\psi(t)$ of (1.1) with $s = 1$ blows up in finite time. Thus, one can prove the strong instability of ground state standing waves, see [7, 11, 16, 17, 23, 24, 31, 34–36, 38, 43, 44].

Here, we only need to use the blow-up criterion (1.10) to study the strong instability of normalized standing waves.

**Theorem 1.4.** Let $N \geq 1$, $s \in (\frac{1}{2}, 1)$, $1 + \frac{2s+a}{N} < p < \frac{N+a}{N-2s}$, $c > 0$. Then for any $u_c \in \mathcal{M}_c$, the standing wave $\psi(t, x) = e^{i\omega t}u_c(x)$ is strongly unstable in the following sense: there exists $\{\psi_{0,n}\} \subset H^s$ such that $\psi_{0,n} \rightarrow u_c$ in $H^s$ as $n \rightarrow \infty$ and the corresponding solution $\psi_n$ of (1.2) with initial data $\psi_{0,n}$ blows up in finite or infinite time for any $n \geq 1$.

**Remark 1.** In previous results, in order to construct blow-up solutions around the ground state solution, one need to assume that the ground state solution $u_\omega$ is radial or $u_\omega \in \Sigma := \{v \in H^s \text{ and } xv \in L^2\}$. Here, we remove these assumptions, so our result greatly improve some previous results.

**Remark 2.** When $p = 2$ and $N - a = 2s$, i.e., in the $L^2$-critical case, Zhang and Zhu in [46] proved the strong instability of radial ground state standing waves of (1.2). Here, we remove this radial assumption and extend this result to the $L^2$-supercritical case and more general Choquard-type nonlinearity.

This paper is organized as follows: in Section 2, we will recall and prove some lemmas such as the local well-posedness theory of (1.2), a sharp Gagliardo-Nirenberg type inequality and the localized virial estimate. Then, it follows from the virial identity that $K(\psi(t)) \leq 16(S_\omega(\psi_0) - S_\omega(u_\omega)) < 0$, where $K(\psi(t))$ is the so-called Gagliardo semi-norm of $\psi$. In this paper, we use the following notations. For any $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N); \int_{\mathbb{R}^N} \left(1 + |\xi|^{2s}\right) |\hat{u}(\xi)|^2 d\xi < \infty \right\},$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \|u\|_{H^s(\mathbb{R}^N)},$$

where up to a multiplicative constant

$$\|u\|_{H^s(\mathbb{R}^N)} = \left\{ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy \right\}^{\frac{1}{2}}$$

is the so-called Gagliardo semi-norm of $u$. In this paper, we often use the abbreviations $L^r = L^r(\mathbb{R}^N)$, $H^s = H^s(\mathbb{R}^N)$. For $J \subset \mathbb{R}$ and $q, r \in [1, \infty]$, we define the mixed norm

$$\|u\|_{L^q(J; L^r)} := \left( \int_J \left( \int_{\mathbb{R}^N} |u(t, x)|^r dx \right)^{\frac{q}{r}} \right)^{\frac{1}{q}},$$

with the usual modification when either $q$ or $r$ are infinity. In the case $q = r$, we shall use $L^q(J \times \mathbb{R}^N)$ instead of $L^q(J, L^r)$. 

2 Preliminaries

In this section, we recall some preliminary results that will be used later. Firstly, we recall the local well-posedness for the Cauchy problem (1.2). Hong and Sire in [27] first studied the local well-posedness of the fractional NLS in $H^s$ by using Strichartz’s estimates and the contraction mapping argument. Since Strichartz’s estimates for non-radial data have a loss of derivatives, a weak local well-posedness holds in the energy space compared to the classical nonlinear Schrödinger equation, see [10, 27] for more details. One can remove the loss of derivatives in Strichartz’s estimates by considering radially symmetric data. However, it needs a restriction on the validity of $s$, namely $\frac{N}{2N-1} \leq s < 1$.

**Proposition 2.1.** [15, Proposition 2.3][Non-radial $H^s$ LWP] Let $s \in (0, 1) \setminus \{1/2\}$, $2 \leq p < \frac{N+\alpha}{N-2\alpha}$, and $\max\{0, N-4s\} < \alpha < N$ be such that

$$s > \begin{cases} \frac{2}{2} - \frac{2s}{\max\{2p-2, 4\}} & \text{if } N = 1, \\ \frac{N}{2} - \frac{s}{p} & \text{if } N \geq 2. \end{cases}$$

(2.1)

Then for all $\psi_0 \in H^s$, there exist $T^* \in (0, +\infty)$ and a unique solution $\psi \in C([0, T^*), H^s) \cap L^q_t(0, T^*), L^\infty)$, for some $q > \max\{2p-2, 4\}$ when $N = 1$ and some $q > 2p-2$ when $N \geq 2$. Moreover, the following properties hold:

- If $T^* < +\infty$, then $\|\psi(t)\|_{H^s} \to \infty$ as $t \uparrow T^*$.
- The solution enjoys conservation of mass and energy, i.e., $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}$ and $E(u(t)) = E(\psi_0)$ for all $t \in [0, T^*)$, where $E(\psi(t))$ defined by (1.4).

**Remark.** When $1 + \frac{a}{N} < p < \frac{N+\alpha}{N-2\alpha}$, it follows from the Hardy-Littlewood-Sobolev inequality that $\int_{\mathbb{R}^N} (I_a * |\psi|^p)(x) \psi(x)^p \, dx$ is well-defined for $\psi \in H^s$. Therefore, we guess that these results also hold for $1 +\frac{a}{N} < p < 2$. However, we cannot prove these results since the nonlinearity $(I_a * |\psi|^p) |\psi|^{p-2}$ is singular when $1 +\frac{a}{N} < p < 2$, see [19].

**Proposition 2.2.** [15, Proposition 2.3][Radial $H^s$ LWP] Let $N \geq 2$, $\frac{N}{2N-1} \leq s < 1$, $2 \leq p < \frac{N+\alpha}{N-2\alpha}$, and $\max\{0, N-4s\} < \alpha < N$. Then for any $\psi_0 \in H^s$ radial, there exist $T^* \in (0, +\infty)$ and a unique solution $\psi \in C([0, T^*), H^s)$ to (1.2). Moreover, the following properties hold:

- $\psi \in L^q_{loc}([0, T), W^{a,b})$ for any fractional admissible pair $(a, b)$.
- If $T^* < +\infty$, then $\|\psi(t)\|_{H^s} \to \infty$ as $t \uparrow T^*$.
- The solution enjoys conservation of mass and energy, i.e., $M(\psi(t)) = M(\psi_0)$ and $E(\psi(t)) = E(\psi_0)$ for all $t \in [0, T^*)$.

Next, we recall a sharp Gagliardo-Nirenberg type inequality established in [21].

**Lemma 2.3.** [21, Theorem 2.3] Let $0 < s < 1$ and $1 + \frac{a}{N} < p < \frac{N+\alpha}{N-2\alpha}$. Then, for all $u \in H^s$,

$$\int_{\mathbb{R}^N} (I_a * |u|^p)|u|^p \, dx \leq C_{opt} \|(-\Delta)^{s/2} u\|_{L^{2s/2}} \|u\|_{L^2}^{2-2p},$$

(2.2)

where the optimal constant $C_{opt}$ is given by

$$C_{opt} = \frac{2sp}{2sp-Np+N+a} \left( \frac{2sp-Np+N+a}{Np-N-a} \right)^{\frac{Np-N-a}{2s}} \|Q\|_{L^2}^{2-2p},$$

where $Q$ is the ground state of the elliptic equation (1.11). In particular, in the $L^2$-critical case, i.e., $p = 1 + \frac{2s+a}{N}$,

$$C_{opt} = p \|Q\|_{L^2}^{2-2p}.$$

Moreover, the following Pohozaev’s identities hold true:

$$\|Q\|_{H^s}^2 = \frac{Np-N-a}{2sp-Np+N+a} \int_{\mathbb{R}^N} (I_a * |Q|^p)(x) Q(x)^p \, dx = \frac{Np-N-a}{2sp-Np+N+a} \|Q\|_{L^2}^2,$$

(2.3)
Next, we recall the profile decomposition of bounded sequences in $H^s$, which has been established in [48].

**Lemma 2.4.** Let $N \geq 3, 0 < s < 1$ and $1 + \frac{a}{N} < p < \frac{N+a}{N-2s}$. If $\{u_n\}_{n=1}^\infty$ is a bounded sequence in $H^s$, then there exist a subsequence of $\{u_n\}_{n=1}^\infty$ (still denoted by $\{u_n\}_{n=1}^\infty$), a family $\{x_n^j\}_{j=1}^\infty$ of sequences in $\mathbb{R}^N$ and a sequence $\{U^j\}_{j=1}^\infty$ in $H^s$ such that

(i) for every $k \neq j$, $|x_n^j - x_n^k| \to +\infty$ as $n \to \infty$;
(ii) for every $l \geq 1$ and every $x \in \mathbb{R}^N$, we have

$$u_n(x) = \sum_{j=1}^l U^j(x - x_n^j) + r_n^l, \quad (2.4)$$

with $\lim \sup_{n \to \infty} \|r_n^l\|_{L^q} \to 0$ as $l \to \infty$ for every $q \in (2, \frac{2N}{N-2s})$. Moreover,

$$\|u_n\|_{L^2}^2 = \sum_{j=1}^l \|U^j\|_{L^2}^2 + \|r_n^l\|_{L^2}^2 + o(1), \quad (2.5)$$

$$\|(-\Delta)^{s/2} u_n\|_{L^2}^2 = \sum_{j=1}^l \|(-\Delta)^{s/2} U^j\|_{L^2}^2 + \|(-\Delta)^{s/2} r_n^l\|_{L^2}^2 + o(1), \quad (2.6)$$

$$\int_{\mathbb{R}^N} I_a \left[ \sum_{j=1}^l |U^j(\cdot - x_n^j)|^p \right] \sum_{j=1}^l |U^j(\cdot - x_n^j)|^p \, dx$$

$$= \sum_{j=1}^l \int_{\mathbb{R}^N} I_a \left[ |U^j(\cdot - x_n^j)|^p \right] |U^j(\cdot - x_n^j)|^p \, dx + o(1), \quad (2.7)$$

where $o(1) = o_n(1) \to 0$ as $n \to \infty$.

Finally, we recall and prove some virial estimates related to (1.2) which is the main ingredient in the proof of Theorem 1.1.

**Lemma 2.5 ([6]).** Let $N \geq 1$ and suppose $\varphi : \mathbb{R}^N \to \mathbb{R}$ is such that $\nabla \varphi \in W^{1,\infty}(\mathbb{R}^N)$. Then, for all $u \in H^\frac{1}{2}$, it holds that

$$\int_{\mathbb{R}^N} \nabla \varphi(x) \cdot \nabla u(x) \, dx \leq C \| \nabla \varphi \|_{W^{1,\infty}} \left( \|u\|_{H^\frac{1}{2}}^2 + \|u\|_{L^2} \|u\|_{H^\frac{1}{2}} \right),$$

for some constant $C > 0$ that depends only on $N$.

In order to study localized virial estimates for (1.2), we need to introduce the auxiliary function

$$u_m(x) := c_s \frac{1}{-\Delta + m} u(x) = c_s s^{-1} \left( \frac{\tilde{u}(\xi)}{\xi^2 + m} \right), \quad m > 0, \quad (2.8)$$

where

$$c_s := \sqrt{\frac{\sin \pi s}{\pi}}.$$

**Lemma 2.6 ([6]).** Let $N \geq 1, s \in (0, 1)$ and suppose $\varphi : \mathbb{R}^N \to \mathbb{R}$ with $\Delta \varphi \in W^{2,\infty}(\mathbb{R}^N)$. Then, for all $u \in L^2$, it holds that

$$\int_0^\infty m^s \int_{\mathbb{R}^N} |(\Delta^2 \varphi)| u_m |^2 \, dx \, dm \leq C \|\Delta^2 \varphi\|_{L^\infty}^s \|\Delta \varphi\|_{L^\infty}^{1-s} \|u\|_{L^2}^2,$$

for some constant $C > 0$ that depends only on $s$ and $N$. 
We refer the reader to [6, Appendix A] for the proof of Lemma 2.5 and Lemma 2.6. Using the fact
\[
\frac{\sin ns}{n} \int_0^\infty \frac{m^s}{\langle \xi \rangle^2 + m^2} \, dm = s|\xi|^{2s-2},
\]
the Plancherel’s and Fubini’s theorems imply
\[
\int_0^\infty m^s \int_\mathbb{R}^N \langle \nabla u_m \rangle^2 \, dx \, dm = \int_\mathbb{R}^N \left( \frac{\sin ns}{n} \int_0^\infty \frac{m^s \, dm}{\langle \xi \rangle^2 + m^2} \right) |\xi|^2 |\tilde{u}(\xi)|^2 \, d\xi
\]
\[
= \int_\mathbb{R}^N (s|\xi|^{2s-2}) |\tilde{u}(\xi)|^2 \, d\xi = s\langle -\Delta \rangle^s u_{L^2},
\]
for any \( u \in \dot{H}^s \).

**Lemma 2.7.** [12, Lemma 4.2] Let \( N \geq 1, s \in (1/2, 1) \) and \( \varphi : \mathbb{R}^N \to \mathbb{R} \) be such that \( \nabla \varphi \in W^{1,\infty} \). Then for any \( u \in L^2 \), it holds that
\[
\int_0^\infty \left( \int_{\mathbb{R}^N} m^s (\Delta \varphi) |u_m|^2 \, dx \right) \, dm \leq C \|\Delta \varphi\|_{L^{2s-1}}^2 \|\nabla \varphi\|_{L^{2s}}^2 \|u\|_{L^2}^2,
\]
for some constant \( C > 0 \) that depends only on \( s \) and \( N \).

By the same argument as in Lemma 2.7 and using in addition Lemma 2.5, we obtain the following estimate.

**Lemma 2.8.** Let \( N \geq 1, s \in (1/2, 1) \) and \( \varphi : \mathbb{R}^N \to \mathbb{R} \) be such that \( \nabla \varphi \in W^{1,\infty} \). Then for any \( u \in H^{1/2} \), it holds that
\[
\int_0^\infty \left( \int_{\mathbb{R}^N} m^s (\nabla \varphi \cdot \nabla u_m) \, dx \right) \, dm \leq C \|\nabla \varphi\|_{W^{1,\infty}}^2 \|u\|_{H^{1/2}}^2,
\]
for some constant \( C > 0 \) depending only on \( N \).

Let \( N \geq 1, 1/2 < s < 1 \) and \( \varphi : \mathbb{R}^N \to \mathbb{R} \) be such that \( \varphi \in W^{2,\infty} \). Assume that \( \psi \in C([0, T^*), H^s) \) is a solution to (1.2). We define the localized virial action of \( \psi \) associated to \( \varphi \) by
\[
\mathcal{V}_\varphi[\psi(t)] := \int_{\mathbb{R}^N} \varphi(x) |\psi(t, x)|^2 \, dx.
\]

**Lemma 2.9.** [12, Lemma 4.5] [Virial identity] Let \( N \geq 1, s \in (1/2, 1) \) and \( \varphi : \mathbb{R}^N \to \mathbb{R} \) be such that \( \varphi \in W^{2,\infty} \). Assume that \( \psi \in C([0, T^*), H^s) \) is a solution to (1.2). Then for any \( t \in [0, T^*) \), it holds that
\[
\frac{d}{dt} \mathcal{V}_\varphi[\psi(t)] = -i \int_{\mathbb{R}^N} \langle \Delta \varphi \rangle |\psi_m(t)|^2 \, dx \, dm - 2i \int_{\mathbb{R}^N} \overline{\psi_m(t)} \nabla \varphi \cdot \nabla \psi_m(t) \, dx \, dm,
\]
where \( \psi_m(t) = c_s (-\Delta + m)^{-1} \psi(t) \).

A direct consequence of Lemmas 2.7, Lemma 2.8 and 2.9 is the following estimate.

**Corollary 2.10.** Let \( N \geq 1, s \in (1/2, 1) \) and \( \varphi : \mathbb{R}^N \to \mathbb{R} \) be such that \( \varphi \in W^{2,\infty} \). Assume that \( \psi \in C([0, T^*), H^s) \) is a solution to (1.2). Then for any \( t \in [0, T^*) \),
\[
\left| \frac{d}{dt} \mathcal{V}_\varphi[\psi(t)] \right| \leq C \|\nabla \varphi\|_{W^{1,\infty}} \|\psi(t)\|_{H^s}^2,
\]
for some constant \( C > 0 \) depending only on \( s \) and \( N \).
We next define the localized Morawetz action of \( \psi \) associated to \( \varphi \) by
\[
\mathcal{M}_\varphi[\psi(t)] := 2 \operatorname{Im} \int_{\mathbb{R}^N} \bar{\psi}(t, x) \nabla \varphi(x) \cdot \nabla \psi(t, x) dx.
\] (2.10)

By Lemma 2.5, we obtain the bound
\[
| \mathcal{M}_\varphi[\psi(t)] | \leq C (\| \nabla \varphi \|_{L^\infty}, \| \Delta \varphi \|_{L^\infty}, \| \psi(t) \|_{H^1}^2).
\]

Hence the quantity \( \mathcal{M}_\varphi[\psi(t)] \) is well-defined, since \( \psi(t) \in H^s \) with some \( s > \frac{1}{2} \) by assumption.

By a similar argument as that in [6, Lemma 2.1], we have the following time evolution of \( \mathcal{M}_\varphi[\psi(t)] \).

**Lemma 2.11** (Morawetz identity). Let \( N \geq 1, s \in (1/2, 1) \) and \( \varphi : \mathbb{R}^N \rightarrow \mathbb{R} \) be such that \( \nabla \varphi \in W^{3,\infty} \). Assume that \( \psi \in C([0, T^*], H^s) \) is a solution to (1.2). Then for any \( t \in [0, T^*) \), it holds that
\[
\frac{d}{dt} \mathcal{M}_\varphi[\psi(t)] = \int_{\mathbb{R}^N} \left\{ 4 \partial_k \psi_\varphi(t)(\partial_k \varphi) \partial_1 \psi_m(t) - (\Delta^2 \varphi)(\psi_m(t))^2 \right\} dx + \frac{2p-4}{p} \int_{\mathbb{R}^N} \Delta \varphi(I_\alpha \ast |\psi(t)|^p)|\psi(t)|^p dx
\]
\[
- \frac{2N-2a}{p} A(\alpha) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\psi(t, x)|^p|\psi(t, y)|^p(x-y) \cdot (\nabla \varphi(x) - \nabla \varphi(y))}{|x-y|^{N-a+2}} dx dy,
\] (2.11)

where \( \psi_m(t) = \psi_m(t, x) \) is defined in (2.8).

**Proof.** It follows from an integration by parts that
\[
\langle \psi(t), [(I_\alpha \ast |\psi(t)|^p)|\psi(t)|^{p-2}, i\Gamma \varphi] \psi(t) \rangle = -\langle \psi(t), [(I_\alpha \ast |\psi(t)|^p)|\psi(t)|^p, \nabla \varphi \cdot \nabla + \nabla \cdot \nabla \varphi] \psi(t) \rangle
\]
\[
= 2 \int_{\mathbb{R}^N} \nabla \varphi \cdot \nabla (I_\alpha \ast |\psi(t)|^p)|\psi(t)|^p dx + 2 \int_{\mathbb{R}^N} (I_\alpha \ast |\psi(t)|^p)|\psi(t)|^2 \nabla \varphi \cdot \nabla (|\psi(t)|^{p-2}) dx
\]
\[
= -\frac{2p-4}{p} \int_{\mathbb{R}^N} \Delta \varphi(I_\alpha \ast |\psi(t)|^p)|\psi(t)|^p dx
\]
\[
- \frac{2N-2a}{p} A(\alpha) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\psi(t, x)|^p|\psi(t, y)|^p(x-y) \cdot (\nabla \varphi(x) - \nabla \varphi(y))}{|x-y|^{N-a+2}} dx dy.
\]

Therefore, following the method used in [6], we prove Lemma 2.11. \( \square \)

### 3 Blow-up criteria

In this section, we will prove Theorem 1.1. To this end, we will establish the following blow-up criterion for (1.2).

**Lemma 3.1.** Let \( N \geq 1, s \in (1/2, 1) \) and \( 1 + \frac{2sN}{N-a} \leq p < \frac{N+a}{N-2s} \). Assume that \( \psi_0 \in H^s \) and \( \psi \in C([0, T^*], H^s) \) is the corresponding solution of (1.2). If there exists \( \delta > 0 \) such that
\[
\sup_{t \in [0, T^*)} K(\psi(t)) \leq -\delta < 0.
\] (3.1)

then one of the following statements holds true:

- \( \psi(t) \) blows up in finite time, i.e. \( T^* < +\infty \);
• \( \psi(t) \) blows up infinite time and there exists a time sequence \( (t_n)_{n \geq 1} \) such that \( t_n \to +\infty \) and

\[
\lim_{n \to \infty} \|(-\Delta)^{\frac{3}{2}} \psi(t_n)\|_{L^2} = \infty.
\]

**Proof.** If \( T^* < +\infty \), then the proof is completed. If \( T^* = +\infty \), then we show (3.2). Assume by contradiction that the solution \( \psi(t) \) exists globally and there exists \( C_0 > 0 \) such that

\[
C_0 := \sup_{t \in [0, +\infty)} \|(-\Delta)^{\frac{3}{2}} \psi(t)\|_{L^2} < \infty.
\]

Combining this and the conservation of mass, we have

\[
C_1 := \sup_{t \in [0, +\infty)} \|\psi(t)\|_{H^1} < \infty.
\]

Next, we introduce a smooth function \( \theta : [0, \infty) \to [0, 1] \) and satisfy

\[
\theta(r) = \begin{cases} 
0 & \text{if } 0 \leq r \leq 1/2, \\
1 & \text{if } r \geq 1.
\end{cases}
\]

For \( R > 1 \), we define the radial function

\[
\phi_R(x) = \phi_R(r) := \theta(r/R), \quad r = |x|.
\]

After some simple calculations, we can obtain

\[
\nabla \phi_R(x) = \frac{x}{rR} \theta'(r/R), \quad \Delta \phi_R(x) = \frac{1}{R^2} \theta''(r/R) + \left( \frac{N-1}{rR} \right) \theta'(r/R).
\]

These imply

\[
\|\nabla \phi_R\|_{W^{1,\infty}} \sim \|\nabla \phi_R\|_{L^\infty} + \|\Delta \phi_R\|_{L^\infty} \lesssim R^{-1}.
\]

Thus, we can define the localized virial function

\[
\mathcal{V}_{\phi_R}[\psi(t)] := \int_{\mathbb{R}^N} \phi_R(x)|\psi(t, x)|^2 \, dx.
\]

It easily follows that

\[
\mathcal{V}_{\phi_R}[\psi(t)] = \mathcal{V}_{\phi_R}[\psi_0] + \int_0^t \frac{d}{dt} \mathcal{V}_{\phi_R}[\psi(t)] \, dt \\
\leq \mathcal{V}_{\phi_R}[\psi_0] + \left( \sup_{t \in [0, t]} \left| \frac{d}{dt} \mathcal{V}_{\phi_R}[\psi(t)] \right| \right) t.
\]

Combining Corollary 2.10, (3.4) and (3.5), we can obtain

\[
\sup_{t \in [0, t]} \left| \frac{d}{dt} \mathcal{V}_{\phi_R}[\psi(t)] \right| \lesssim \|\nabla \phi_R\|_{W^{1,\infty}} \sup_{t \in [0, t]} \|\psi(t)\|_{H^1}^2 \lesssim C_1^2 R^{-1},
\]

for some constant \( C > 0 \) independent of \( R \) and \( C_1 \). We consequently obtain

\[
\mathcal{V}_{\phi_R}[\psi(t)] \leq \mathcal{V}_{\phi_R}[\psi_0] + CC_1^2 R^{-1} t,
\]

for all \( t \geq 0 \). We infer from the definition of \( \theta \) that

\[
\mathcal{V}_{\phi_R}[\psi_0] = \int_{\mathbb{R}^N} \phi_R(x)|\psi_0(x)|^2 \, dx \leq \int_{|x| > R/2} |\psi_0(x)|^2 \, dx \to 0,
\]
as \( R \to \infty \). This implies that \( \nabla \varphi_k \varphi_0 = o_R(1) \). In addition, it follows that

\[
\int_{|x| = R} |\psi(t, x)|^2 \, dx \leq \nabla \varphi_k \varphi(t).
\]

Collecting the above estimates, we can obtain the following control about the \( L^2 \)-norm of the solution \( \psi(t) \) outside a large ball.

**Lemma 3.2.** Let \( \eta > 0, R > 1 \) and \( C_1 \) be as in (3.4). Then there exists a constant \( C > 0 \) independent of \( R \) and \( C_1 \) such that for any \( t \in [0, T_0] \) with \( T_0 := \frac{\eta R}{CC_1} \),

\[
\int_{|x| > R} |\psi(t, x)|^2 \, dx \leq \eta + o_R(1).
\]

Next, we introduce a radial function \( \varphi(x) = \varphi(r) \) which satisfies

\[
\varphi(r) = \begin{cases} 
\frac{r^2}{2} & \text{for } r \leq 1, \\
\text{const.} & \text{for } r \geq 10,
\end{cases}
\]

and \( \varphi''(r) \leq 1 \) for \( r \geq 0 \). For any \( R > 0 \), we define the rescaled function \( \varphi_R : \mathbb{R}^N \to \mathbb{R} \) by

\[
\varphi_R(x) := R^2 \varphi \left( \frac{x}{R} \right).
\]

It easily follows that

\[
1 - \varphi_R''(r) \geq 0, \quad 1 - \frac{\varphi_R'(r)}{r} \geq 0, \quad N - \Delta \varphi_R(x) \geq 0,
\]

for all \( r \geq 0 \) and all \( x \in \mathbb{R}^N \). It is easy to see that

\[
\| \nabla^k \varphi_R \|_{L^\infty} \lesssim R^{2-k}, \quad k = 0, \ldots, 4,
\]

and

\[
\text{supp}(\nabla^k \varphi_R) \subset \begin{cases} 
\{|x| \leq 10R\} & \text{for } k = 1, 2, \\
\{R \leq |x| \leq 10R\} & \text{for } k = 3, 4.
\end{cases}
\]

Applying Lemma 2.11, we can obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^N} m^s \left\{ 4 \Delta \varphi_R \varphi(t) \partial_t \varphi(t) \varphi_m(t) - (\Delta^2 \varphi_R) \varphi(t) \right\} \, dx \, dm 
- \frac{2p - 4}{p} \int_{\mathbb{R}^N} \Delta \varphi_R (I_n \varphi(t)^p \varphi(t) ) \, dx 
- \frac{2N - 2}{p} A(a) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(t, x)|^p |\psi(t, y)|^p (x - y) \cdot (\nabla \varphi_R (x) - \nabla \varphi_R (y))}{|x - y|^{N-a+2}} \, dx \, dy
\]

where \( \psi_m(t) = \psi_m(t, x) \) is defined in (2.8). Due to \( \text{supp}(\Delta^2 \varphi_R) \subset \{|x| \geq R\} \), we infer from Lemma 2.6 that

\[
\int_0^\infty m^s \left( \Delta^2 \varphi_R \right) \varphi(t) \, dx \, dm 
\lesssim \| \Delta^2 \varphi_R \|_{L^\infty} \| \Delta \varphi_R \|_{L^\infty} \| \varphi(t) \|_{L^2(\{x \geq R\})}^2 
\lesssim R^{-2s} \| \varphi(t) \|_{L^2(\{x \geq R\})}^2.
\]

Since \( \varphi_R \) is a radial function, applying

\[
\partial_{jj}^2 \left( \frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3} \right) \partial_r + \frac{x_j x_k}{r^2} \partial_r^2,
\]
we can obtain
\[
\int_0^\infty m^s \int_{\mathbb{R}^N} \partial_k \psi_m(t)(\partial_k^2 \varphi_R) \partial_t \psi_m(t) \, dx \, dm = \int_0^\infty m^s \int_{\mathbb{R}^N} \frac{\varphi_R'}{r} |\nabla \psi_m(t)|^2 \, dx \, dm \\
+ \int_0^\infty m^s \int_{\mathbb{R}^N} \left( \frac{\varphi_R''}{r^2} - \frac{\varphi_R'}{r^3} \right) |x \cdot \nabla \psi_m(t)|^2 \, dx \, dm.
\]

It follows from (2.9) that
\[
\int_0^\infty m^s \int_{\mathbb{R}^N} \frac{\varphi_R'}{r} |\nabla \psi_m(t)|^2 \, dx \, dm = s\|(-\Delta)^{s/2} \psi(t)\|_{L^2}^2 + \int_0^\infty m^s \int_{\mathbb{R}^N} \left( \frac{\varphi_R'}{r} - 1 \right) |\nabla \psi_m(t)|^2 \, dx \, dm.
\]

Since \(\varphi_R'' \leq 1\), we deduce from Cauchy-Schwarz inequality that
\[
\int_0^\infty m^s \int_{\mathbb{R}^N} \varphi_R' |\nabla \psi_m(t)|^2 \, dx \, dm + \int_0^\infty m^s \int_{\mathbb{R}^N} \left( \varphi_R' - \varphi_R'' \right) \frac{|x \cdot \nabla \psi_m(t)|^2}{r^2} \, dx \, dm \leq 0.
\]

Thus, we can obtain
\[
4 \int_0^\infty m^s \int_{\mathbb{R}^N} \partial_k \psi_m(t)(\partial_k^2 \varphi_R) \partial_t \psi_m(t) \, dx \, dm \leq 4s\|(-\Delta)^{s/2} \psi(t)\|_{L^2}^2.
\] (3.9)

Next, we write
\[
- \frac{2p-4}{p} \int_{\mathbb{R}^N} \Delta \varphi_R(Ia \ast |\psi(t)|^p) |\psi(t)|^p \, dx = - \frac{(2p-4)N}{p} \int_{\mathbb{R}^N} (Ia \ast |\psi(t)|^p) |\psi(t)|^p \, dx \\
+ \frac{2p-4}{p} \int_{\mathbb{R}^N} (N - \Delta \varphi_R)(Ia \ast |\psi(t)|^p) |\psi(t)|^p \, dx.
\]

By the Hardy-Littlewood-Sobolev inequality and the conservation of mass, we can estimate as follows
\[
\int_{\mathbb{R}^N} (N - \Delta \varphi_R)(Ia \ast |\psi(t)|^p) |\psi(t)|^p \, dx \lesssim \int_{|x| \leq R} (Ia \ast |\psi(t)|^p) |\psi(t)|^p \, dx
\lesssim \|Ia \ast |\psi(t)|^p\|_{L^{2p/(p+1)}(|x| \leq R)} \|\psi(t)|^p\|_{L^{2p/(p+1)}(|x| \leq R)}
\lesssim \|\psi(t)|^p\|_{L^{2p/(p+1)}(|x| \leq R)}
\lesssim \|\psi(t)|^p\|_{L^{2p/(p+1)}(|x| \leq R)}
\lesssim \|\psi(t)|^p\|_{L^{2p/(p+1)}(|x| \leq R)}.
\]

We consequently obtain
\[
- \frac{2p-4}{p} \int_{\mathbb{R}^N} \Delta \varphi_R(Ia \ast |\psi(t)|^p) |\psi(t)|^p \, dx \leq - \frac{(2p-4)N}{p} \int_{\mathbb{R}^N} (Ia \ast |\psi(t)|^p) |\psi(t)|^p \, dx \\
+ C C_1^{\frac{Np-N-a}{a}} \|\psi(t)|^p\|_{L^{2p/(p+1)}(|x| \leq R)}.
\] (3.10)

Denote the last term in (3.7) by \(\mathcal{I}\). We can obtain
\[
\mathcal{I} = - \frac{2N-2a}{p} A(a) \int_{\mathbb{R}^N} (Ia \ast |\psi(t)|^p) |\psi(t)|^p \, dx
\]
\[
+ \frac{2N - 2\alpha}{p} A(\alpha) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left[ |x - y|^2 - (x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y)) \right] \frac{|\psi(t, x)|^p |\psi(t, y)|^p}{|x - y|^{N - \alpha + 2}} \, dx \, dy.
\]

Note that
\[
supp((x - y)^2 - (x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y))) \subset \{|x| \leq R\} \cup \{|y| \geq R\}.
\]

In the region \(|x| \geq R\), it follows that
\[
||x - y|^2 - (x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y))|| \lesssim |x - y|^2.
\]

This implies that
\[
\left| A(\alpha) \int_{|x| \geq R} \int_{\mathbb{R}^N} \left[ |x - y|^2 - (x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y)) \right] \frac{|\psi(t, x)|^p |\psi(t, y)|^p}{|x - y|^{N - \alpha + 2}} \, dx \, dy \right| \lesssim \int_{|x| \geq R} (I_a * |\psi(t)|^p)|\psi(t)|^p \, dx.
\]

We have a similar control in the region \(|y| \geq R\). By a similar argument as above, we can obtain
\[
\mathcal{J} \leq -\frac{2N - 2\alpha}{p} \int_{\mathbb{R}^N} (I_a * |\psi(t)|^p)|\psi(t)|^p \, dx + C_1^{\frac{N - \alpha}{2}} \|\psi(t)\|_{L^2(|x| \geq R)}^2 + C_1^{\frac{2N - 2\alpha}{2}} \|\psi(t)\|_{L^2(|x| \geq R)}^2.
\]  

Combining (3.8) – (3.11), we obtain
\[
\frac{d}{dt} \mathcal{M}_{\psi_R} [\psi(t)] \leq 4s ||(-\Delta)^{\frac{s}{2}} \psi(t)||_{L^2}^2 - \frac{2N - 2\alpha}{p} \int_{\mathbb{R}^N} (I_a * |\psi(t)|^p)|\psi(t)|^p \, dx + C R^{-2s} \|\psi(t)\|_{L^2(|x| \geq R)}^2 + C C_1^{\frac{2N - 2\alpha}{2}} \|\psi(t)\|_{L^2(|x| \geq R)}^2.
\]  

Applying Lemma 3.2, for any \(\eta > 0\) and any \(R > 1\), there exists \(C > 0\) independent of \(R\) and \(C_1\) such that for any \(t \in [0, T_0]\) with \(T_0 = \frac{\eta R}{CC_1}\),
\[
\frac{d}{dt} \mathcal{M}_{\psi_R} [\psi(t)] \leq 4K(\psi(t)) + CR^{-2s}(\eta + o_R(1))^2 + C C_1^{\frac{2N - 2\alpha}{2}} (\eta + o_R(1)) \frac{N - \alpha + (N - 2)s}{\alpha} \lesssim -4\delta + CR^{-2s}(\eta^2 + o_R(1)) + C C_1^{\frac{2N - 2\alpha}{2}} (\eta \frac{N - \alpha + (N - 2)s}{\alpha} + o_R(1))
\]

We first choose \(\eta > 0\) small enough so that
\[
CC_1^{\frac{2N - 2\alpha}{2}} \eta \frac{N - \alpha + (N - 2)s}{\alpha} \leq 2\delta.
\]

We next choose \(R > 1\) large enough so that
\[
\frac{d}{dt} \mathcal{M}_{\psi_R} [\psi(t)] \leq -\delta < 0,
\]  

for any \(t \in [0, T_0]\) with \(T_0 = \frac{\eta R}{CC_1}\). Note that \(\eta > 0\) is fixed, so we can choose \(R > 1\) large enough so that \(T_0\) is as large as we want. By (3.13), it follows that
\[
\mathcal{M}_{\psi_R} [\psi(t)] \leq -ct,
\]

for all \(t \in [t_0, T_0]\) with some sufficiently large \(t_0 \in [0, T_0]\). The constant \(c > 0\) depends only on \(\delta\). On the other hand, we deduce from Lemma 2.5 and the conservation of mass that for any \(t \in [0, +\infty)\),
\[
|\mathcal{M}_{\psi_R} [\psi(t)]| \lesssim C(\varphi_R) \left( \|\psi(t)\|_{H^{\frac{1}{2}}}^2 + \|\psi(t)\|_{L^2} \|\psi(t)\|_{H^{\frac{1}{2}}} \right) \lesssim C(\varphi_R) \left( \|\psi(t)\|_{H^{\frac{1}{2}}}^2 + \|\psi(t)\|_{L^2}^2 \right).
\]
By interpolating between $L^2$ and $\dot{H}^s$, we get for any $t \in [t_0, T_0]$,

$$ct \leq -M_{\varphi_R}[\psi(t)] = |M_{\varphi_R}[\psi(t)]| \lesssim C(\varphi_R) \left(\|(-\Delta)^{s/2}\psi(t)\|_{L^2}^{\frac{1}{s}} + 1\right).$$

This implies that

$$\|(-\Delta)^{s/2}\psi(t)\|_{L^2} \geq Ct^s,$$  \hspace{1cm} (3.14)

for all $t \in [t_1, T_0]$ with some sufficiently large $t_1 \in [t_0, T_0]$. Taking $t$ close to $T_0 = \frac{\eta R}{CC_s}$, we see that $\|(-\Delta)^{s/2}\psi(t)\|_{L^2} \to \infty$ as $R \to \infty$. Taking $R > 1$ sufficiently large, we have a contradiction with (3.4). The proof is complete.

\[\square\]

Applying Lemma 3.1, we can prove Theorem 1.1.

**Proof of Theorem 1.1.** We need only to check that (3.1) follows. In the $L^2$-critical case, i.e., $s_c = 0$, we infer from (1.10) that $\|\psi_0\|_{L^2} < \|u\|_{L^2}$ and $\|\psi_0\|_{L^2} > \|u\|_{L^2}$, which is an contradiction. Thus, when $s_c = 0$, we have $E(\psi_0) < 0$. Applying the conservation of energy and $1 + \frac{2sp}{\theta} \leq p < \frac{N+2}{N-2}$, it follows that

$$K(\psi(t)) = s\|\psi(t)\|_{H^s}^2 - \frac{\theta}{2p} \int_{\mathbb{R}^N} (I_a \ast |\psi(t)|^p)(x)\psi(t)(x) dx$$

$$= 2sE(\psi(t)) + \frac{2s-\theta}{2p} \int_{\mathbb{R}^N} (I_a \ast |\psi(t)|^p)(x)\psi(t)(x) dx \leq 2sE(\psi_0),$$

for all $t \in [0, T^*)$. Hence, (3.1) follows with $\delta = -2sE(\psi_0)$.

Next, we consider the case $E(\psi_0) > 0$. We deduce from the assumption (1.10) that

$$E(\psi_0)\|\psi_0\|_{L^2}^{2\sigma} < E(u)\|u\|_{L^2}^{2\sigma},$$

$$\|(-\Delta)^{s/2}\psi_0\|_{L^2}^{2\sigma} > \|(-\Delta)^{s/2}u\|_{L^2}^{2\sigma},$$ \hspace{1cm} (3.15)

where

$$\sigma := \frac{s-s_c}{s_c} = \frac{2sp-\theta}{\theta-2s}.$$

Notice that the sharp constant in Gagliardo-Nirenberg inequality (2.2) is

$$C_{\text{opt}} = \frac{\int_{\mathbb{R}^N} (I_a \ast |u|^p)(x)|u(x)|^p dx}{\|u\|_{H^s}^2\|u\|_{L^2}^{2\sigma}}.$$ \hspace{1cm} (3.16)

By (2.3), we can rewrite $C_{\text{opt}}$ as

$$C_{\text{opt}} = \frac{2sp}{\theta} \left(\frac{1}{\|u\|_{H^s}^2\|u\|_{L^2}^{2\sigma}}\right)^{rac{2\sigma}{p}}.$$ \hspace{1cm} (3.17)

By a direct calculation, we also have

$$E(u)\|u\|_{L^2}^{2\sigma} = \frac{\theta-2s}{2\theta}\left(\|u\|_{H^s}^2\|u\|_{L^2}^{2\sigma}\right)^2.$$ \hspace{1cm} (3.18)

Multiplying both sides of $E(\psi(t))$ by $\|\psi(t)\|_{L^2}^{2\sigma}$ and use the sharp Gagliardo-Nirenberg inequality (2.2), we obtain

$$E(\psi(t))\|\psi(t)\|_{L^2}^{2\sigma} = \frac{1}{2} \|\psi(t)\|_{H^s}^2\|\psi(t)\|_{L^2}^{2\sigma} - \frac{1}{2p} \int_{\mathbb{R}^N} (I_a \ast |\psi(t)|^p)(x)|\psi(t)(x)|^p dx \|u(t)\|_{L^2}^{2\sigma}$$

$$\geq \frac{1}{2}\left(\|\psi(t)\|_{H^s}^2\|\psi(t)\|_{L^2}^{2\sigma}\right)^2 - \frac{C_{\text{opt}}}{2p}\left(\|\psi(t)\|_{H^s}^2\|\psi(t)\|_{L^2}^{2\sigma}\right)^{\frac{\theta-2s}{\theta}}.$$
where \( f(x) := \frac{1}{2}x^2 - \frac{C_{opt}}{2p}x^\frac{p}{2} \). It easily follows that \( f \) is increasing on \((0, x_0)\) and decreasing on \((x_0, \infty)\), where
\[
x_0 = \left( \frac{2sp}{C_{opt} \theta} \right)^{\frac{1}{p}} = \|u\|_{H^s} \|u\|_{L^2}^p,
\]
where the last equality follows from (3.17). It follows from (3.17) and (3.18) that
\[
f(\|u\|_{H^s}, \|u\|_{L^2}^p) = E(u) = \|u\|_{H^s}^2.
\]
Thus the conservation of mass and energy together with the first condition in (1.10) imply
\[
f(\|\psi(t)\|_{H^s}, \|\psi(t)\|_{L^2}^p) \leq E(\psi(t)) \|\psi(t)\|_{L^2}^2 = E(\psi_0) \|\psi_0\|_{L^2}^2
\]
\[
< E(u) \|\psi_0\|_{L^2}^2 = f(\|u\|_{H^s}, \|u\|_{L^2}^p),
\]
for all \( t \in [0, T^\ast) \). Using the second condition (1.10), the continuity argument shows that
\[
\|\psi(t)\|_{H^s} \|\psi(t)\|_{L^2}^p > \|u\|_{H^s} \|u\|_{L^2}^p
\]
for any \( t \in [0, T^\ast) \). On the other hand, since \( E(\psi_0) \|\psi_0\|_{L^2}^2 < E(u) \|\psi_0\|_{L^2}^2 \), we pick \( \eta > 0 \) small enough so that
\[
E(\psi_0) \|\psi_0\|_{L^2}^2 \leq (1 - \eta) E(u) \|\psi_0\|_{L^2}^2.
\]
Thus, by the conservation of energy, (3.18) and (3.19), we have
\[
K(\psi(t)) \|\psi(t)\|_{L^2}^2 = \theta E(\psi(t)) \|\psi(t)\|_{L^2}^2 - \frac{\theta - 2s}{2} \|\psi(t)\|_{H^s} \|\psi(t)\|_{L^2}^p
\]
\[
= \theta E(\psi_0) \|\psi_0\|_{L^2}^2 - \frac{\theta - 2s}{2} (\|\psi(t)\|_{H^s} \|\psi(t)\|_{L^2}^p)^2
\]
\[
\leq \eta(1 - \eta) E(u) \|\psi_0\|_{L^2}^2 - \frac{\theta - 2s}{2} (\|\psi_{0}\|_{H^s} \|\psi_0\|_{L^2}^p)^2
\]
\[
= - \eta \theta E(u) \|\psi_0\|_{L^2}^2,
\]
for all \( t \in [0, T^\ast) \). This implies (3.1) with \( \delta = \eta \theta E(u) \|\psi_0\|_{L^2}^2 \). Thus, the solution \( \psi(t) \) of (1.2) blows up in finite or infinite time. This completes the proof.

### 4 Existence and instability of normalized standing waves

In this section, we will prove the existence and instability of normalized standing waves of (1.2). Firstly, we prove Theorem 1.2.

**Proof of Theorem 1.2.** We first show \( m(c) > 0 \). By \( K(v) = 0 \) and the inequality (2.2), we have
\[
s \|v\|_{H^s}^2 = \frac{\theta}{2p} \int \langle I_a * |v|^3 \rangle |v|^p dx \leq C \|v\|_{H^s}^2 \|v\|_{L^2}^{2p-4} \leq C \|v\|_{H^s}^2,
\]
where \( \theta = Np - N - \alpha \), which implies that there exists \( C_1 > 0 \) such that \( \|v\|_{H^s} \geq C_1 > 0 \). Thus, it follows from \( K(v) = 0 \) that
\[
E(v) = \frac{1}{2s} K(v) + \frac{\theta - 2s}{4sp} \int \langle I_a * |v|^3 \rangle |v|^p dx = \frac{\theta - 2s}{2\theta} \|v\|_{H^s}^2 \geq \frac{\theta - 2s}{2\theta} C_1.
\]
(4.1)
Taking the infimum over \( v \in V(c) \), we have \( m(c) > 0 \).

Next, let \( \{v_n\} \subseteq V(c) \) be a minimizing sequence of (1.15), i.e., \( K(v_n) = 0 \), \( \|v_n\|_{L^2}^2 = c \) and \( E(v_n) \to m(c) \) as \( n \to \infty \). Thus, it follows from (4.1) that
\[
\|v_n\|_{H^s}^2 = \frac{2\theta}{\theta - 2s} E(v_n) \to \frac{2\theta m(c)}{\theta - 2s},
\]
for all \( t \in [0, T^\ast) \).
which implies that \( \{v_n\} \) is bounded in \( H^s \).

On the other hand, we see from (2.7) that
\[
\{K\}
\quad
\text{We claim that}
\quad
\text{and}
\quad
\|v\|_{H^s}^2,
\quad
(4.2)
\]
and (2.5)-(2.7) hold. Moreover, we deduce from (2.5)-(2.6) that
\[
0 = K(v_n) = s\|v_n\|_{H^s}^2 - \frac{\theta}{2p} \int_{\mathbb{R}^N} (I_\alpha \ast |v_n|^p) |v_n|^p \, dx
\]
\[
= s \sum_{j=1}^I \|U_j^I\|_{H^s}^2 + s\|v_n^I\|_{H^s}^2 - \frac{\theta}{2p} \int_{\mathbb{R}^N} (I_\alpha \ast |v_n|^p) |v_n|^p \, dx + o_n(1)
\]
\[
= \sum_{j=1}^I K(U_j^I) + \frac{\theta}{2p} \sum_{j=1}^I \int_{\mathbb{R}^N} (I_\alpha \ast |U_j^I|^p) |U_j^I|^p \, dx + s\|v_n^I\|_{H^s}^2
\]
\[
- \frac{\theta}{2p} \int_{\mathbb{R}^N} (I_\alpha \ast |v_n|^p) |v_n|^p \, dx + o_n(1),
\quad
(4.3)
\]
where \( o_n(1) \to 0 \) as \( n \to \infty \). Since \( K(v_n) = 0 \),
\[
\int_{\mathbb{R}^N} (I_\alpha \ast |v_n|^p) |v_n|^p \, dx = \frac{4sp}{\theta - 2s} E(v_n) \to \frac{4spm(c)}{\theta - 2s}, \quad \text{as } n \to \infty
\]
and \( s\|v_n^I\|_{H^s}^2 \geq 0 \) for all \( n \geq 1 \), we infer that
\[
\sum_{j=1}^I K(U_j^I) + \frac{\theta}{2p} \sum_{j=1}^I \int_{\mathbb{R}^N} (I_\alpha \ast |U_j^I|^p) |U_j^I|^p \, dx - \frac{4spm(c)}{\theta - 2s} \leq 0,
\quad
(4.4)
\]
or equivalently,
\[
s \sum_{j=1}^I \|U_j^I\|_{H^s}^2 = \frac{2sm(c)\theta}{\theta - 2s} \leq 0.
\quad
(4.5)
\]
On the other hand, we see from (2.7) that
\[
\frac{4spm(c)}{\theta - 2s} = \lim_{n \to \infty} \int_{\mathbb{R}^N} (I_\alpha \ast |v_n|^p) |v_n|^p \, dx = \sum_{j=1}^I \int_{\mathbb{R}^N} (I_\alpha \ast |U_j^I|^p) |U_j^I|^p \, dx.
\quad
(4.6)
\]
Combining (4.4)-(4.6), we obtain
\[
\sum_{j=1}^I K(U_j^I) \leq 0, \quad \sum_{j=1}^I \|U_j^I\|_{H^s}^2 \leq \frac{2m(c)\theta}{\theta - 2s}.
\quad
(4.7)
\]
We claim that \( K(U_j^I) = 0 \) for all \( j \geq 1 \). Indeed, suppose that there exists \( j_0 \geq 1 \) such that \( K(U_j^I) < 0 \). Notice that
\[
K(\lambda U_j^I) = \lambda^2 s\|U_j^I\|_{H^s}^2 - \frac{\theta}{2p} \lambda^2 \int_{\mathbb{R}^N} (I_\alpha \ast |U_j^I|^p) |U_j^I|^p \, dx > 0,
\quad
(4.8)
\]
for sufficiently small \( \lambda > 0 \). There exists \( \lambda_0 \in (0, 1) \) such that \( K(\lambda_0 U_j^I) = 0 \). Let \( V = \lambda_0 U_j^I \), then we have
\[
K(V) = 0, \quad \int_{\mathbb{R}^N} |V(x)|^2 \, dx < c.
\quad
(4.9)
\]
let $V_\mu = \mu^{\frac{2s}{N+2s}} V(\mu x)$,

$$K(V_\mu) = \mu^{\frac{2s}{N+2s} - N} K(V) = 0.$$  \hspace{1cm} (4.10)

Since $p > \frac{N+2s}{N}$ and

$$\int_{\mathbb{R}^N} |V_\mu(x)|^2 dx = \mu^{\frac{2s}{N+2s}} \int_{\mathbb{R}^N} |V(x)|^2 dx = c.$$  \hspace{1cm} (4.11)

there exists $\mu_0 \in (0, 1)$ such that $\|V_\mu\|_{L^2}^2 = c$. We consequently estimate as follows:

$$m(c) \leq E(V_{\mu_0}) = \frac{\theta - 2s}{4sp} \int_{\mathbb{R}^N} (I_a * |V_{\mu_0}|^p) |V_{\mu_0}|^p dx$$

$$= \frac{\theta - 2s}{4sp} \mu_0^{\frac{2s}{N+2s} - N} \int_{\mathbb{R}^N} (I_a * |U_0|^p) |U_0|^p dx$$

$$< \frac{\theta - 2s}{4sp} \int_{\mathbb{R}^N} (I_a * |U_0|^p) |U_0|^p dx$$

$$\leq \frac{\theta - 2s}{4sp} \frac{4spm(c)}{\theta - 2s} = m(c),$$  \hspace{1cm} (4.12)

which is a contraction. Finally, we claim that there exists only one term $U^j \neq 0$. Indeed, if there exist two terms $U^{j_1} \neq 0$ and $U^{j_2} \neq 0$, it follows from (4.7) that $K(U^{j_1}) = 0, K(U^{j_2}) = 0$ and

$$\int_{\mathbb{R}^N} (I_a * |U^{j_1}|^p) |U^{j_1}|^p dx < \frac{4spm(c)}{\theta - 2s}$$

and

$$\int_{\mathbb{R}^N} (I_a * |U^{j_2}|^p) |U^{j_2}|^p dx < \frac{4spm(c)}{\theta - 2s}.$$  \hspace{1cm} (4.13)

Next, we set

$$U^{j_1}_\mu = \mu^{\frac{2s}{N+2s}} U^{j_1}(\mu x), \quad U^{j_2}_\mu = \mu^{\frac{2s}{N+2s}} U^{j_2}(\mu x).$$  \hspace{1cm} (4.14)

It follows from that $K(U^{j_1}_\mu) = K(U^{j_1}) = 0$, and $K(U^{j_2}_\mu) = K(U^{j_2}) = 0$ for all $\mu > 0$. By $\int_{\mathbb{R}^N} |U^{j_1}|^2 dx < c$ and $\int_{\mathbb{R}^N} |U^{j_2}|^2 dx < c$, we obtain that there exist $\mu_1, \mu_2 \in (0, 1)$ such that

$$\int_{\mathbb{R}^N} |U^{j_1}_{\mu_1}|^2 dx = c, \quad \int_{\mathbb{R}^N} |U^{j_2}_{\mu_2}|^2 dx = c.$$  \hspace{1cm} (4.15)

Thus, we can estimate as follows:

$$m(c) \leq E(U^{j_1}_{\mu_1}) = \frac{\theta - 2s}{4sp} \int_{\mathbb{R}^N} (I_a * |U^{j_1}_{\mu_1}|^p) |U^{j_1}_{\mu_1}|^p dx$$

$$= \frac{\theta - 2s}{4sp} \mu_1^{\frac{2s}{N+2s} - N} \int_{\mathbb{R}^N} (I_a * |U_0|^p) |U_0|^p dx$$

$$\leq \frac{\theta - 2s}{4sp} \frac{4spm(c)}{\theta - 2s} = m(c),$$

which is a contradiction. Therefore, there exists only one term $U_0^{j_0} \neq 0$ in the decomposition (4.2) and $K(U^{j_0}) = 0$, which together with (2.5) implies that the infimum of the variational problem (1.15) is attained at $U^{j_0}$. This completes the proof.

**Proof of Theorem 1.3.** With Theorem 1.2 in hand, one can prove Theorem 1.3 by a similar argument as Theorem 1.3 in [18]. So we omit it.

**Proof of Theorem 1.4.** Let $u_\epsilon \in \mathcal{M}_\epsilon$, a direct computation shows

$$E(u_\epsilon^j) = \frac{1}{2} \Lambda^{2s} \|u_\epsilon|_{H^s} - \frac{\lambda^q}{2p} \int_{\mathbb{R}^N} (I_a * |u_c|^p) |u_c(x)|^p dx,$$
and
\[ \partial_t E(u_c^\lambda) = s \lambda^{2s-1} \|u_c\|_p^2 - \frac{\theta \lambda^{\theta-1}}{2p} \int_{\mathbb{R}^n} (I_A * |u_c|^p)(x)|u_c(x)|^p \, dx = \frac{K(u_c^\lambda)}{\lambda}. \]

It is easy to see that the equation \( \partial_t E(u_c^\lambda) = 0 \) has a unique non-zero solution
\[ \left( \frac{2sp\|u_c\|_p^2}{\theta \int_{\mathbb{R}^n} (I_A * |u_c|^p)(x)|u_c(x)|^p \, dx} \right)^{\frac{1}{2p}} = 1. \]

The last inequality comes from the fact that \( K(u_c) = 0 \), which follows from Pohozaev’s identities (2.3). We thus obtain
\[ \begin{cases} 
\partial_t E(u_c^\lambda) > 0 & \text{if } \lambda \in (0, 1), \\
\partial_t E(u_c^\lambda) < 0 & \text{if } \lambda \in (1, \infty).
\end{cases} \]

This implies that for any \( \lambda > 1 \)
\[ E(u_c^\lambda) < E(u_c). \quad (4.16) \]

Let \( \lambda_n > 1 \) such that \( \lim_{n \to \infty} \lambda_n = 1 \). We take the initial data
\[ \psi_{0,n}(x) = u_c^{\lambda_n}(x) = \lambda_n^\frac{n}{s} u_c(\lambda_n x). \]

By Brezis-Lieb’s lemma, we have \( \psi_{0,n} \to u_c \) in \( H^s \) as \( n \to \infty \). We deduce from (4.16) that
\[ E(\psi_{0,n}) < E(u_c), \]
and
\[ \|\langle-D\rangle^{s/2} \psi_{0,n}\|_2^2 = \lambda_n^s \|\langle-D\rangle^{s/2} u_c\|_2^2 > \|\langle-D\rangle^{s/2} u_c\|_2^2. \]

On the other hand, let \( u_c(x) = \omega_{\lambda_n^2} \omega_{\lambda_n^2} u_c(\lambda_n^2 x) \) in (1.12), then \( u \) satisfies equation (1.11). In particular, by some basic calculations, we have
\[ E(u_c)^{s_c} \|u_c\|_{L^2}^{2(s_c-1)} = E(u)^{s_c} \|u\|_{L^2}^{2(s_c-1)}, \quad (4.17) \]
and
\[ \|\langle-D\rangle^2 u_c\|_{L^2}^{s_c} \|u_c\|_{L^2}^{2s_c} = \|\langle-D\rangle^2 u\|_{L^2}^{s_c} \|u\|_{L^2}^{2s_c}. \quad (4.18) \]

Thus, by (4.17), (4.18) and \( \|\psi_{0,n}\|_{L^2} = \|u_c\|_{L^2}, \) we have
\[ E(\psi_{0,n})^{s_c} \|\psi_{0,n}\|_{L^2}^{2(s_c-1)} < E(u_c)^{s_c} \|u_c\|_{L^2}^{2(s_c-1)} = E(u)^{s_c} \|u\|_{L^2}^{2(s_c-1)}, \]
and
\[ \|\langle-D\rangle^{s/2} \psi_{0,n}\|_{L^2}^{s_c} \|\psi_{0,n}\|_{L^2}^{2s_c} > \|\langle-D\rangle^{s/2} u_c\|_{L^2}^{s_c} \|u_c\|_{L^2}^{2s_c} = \|\langle-D\rangle^{s/2} u\|_{L^2}^{s_c} \|u\|_{L^2}^{2s_c}, \]
where \( s_c = \frac{N}{2} - \frac{a+\alpha}{2p-2} \). Applying Theorem 1.1, the solution \( \psi_n \) of (1.2) and initial data \( \psi_{0,n} \) blows up in finite or infinite time. This completes the proof.

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