Research Article

Reinhard Farwig and Ryo Kanamaru

Optimality of Serrin type extension criteria to the Navier-Stokes equations

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Abstract: We prove that a strong solution $u$ to the Navier-Stokes equations on $(0, T)$ can be extended if either $u \in L^p(0, T; U_{\infty, 1}^{\alpha, \beta})$ for $2/\theta + \alpha = 1$, $0 < \alpha < 1$ or $u \in L^2(0, T; \tilde{V}_{\infty}^{\alpha})$, where $U_{\infty, 1}^{\alpha, \beta}$ and $\tilde{V}_{\infty}^{\alpha}$ are Banach spaces that may be larger than the homogeneous Besov space $\dot{B}_p^s$. Our method is based on a bilinear estimate and a logarithmic interpolation inequality.

Keywords: Serrin type extension criterion; Navier-Stokes equations; bilinear estimate; logarithmic interpolation inequality

MSC: 35Q30 (primary), 35B65, 46E35, 76D05

1 Introduction

The motion of a viscous incompressible fluid in $\mathbb{R}^n$, $n \geq 2$, is governed by the Navier-Stokes equations:

$$\begin{cases}
\partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = 0, & x \in \mathbb{R}^n, t > 0, \\
\text{div} u = 0, & x \in \mathbb{R}^n, t > 0, \\
u|_{t=0} = u_0,
\end{cases} \tag{N-S}$$

where $u = (u_1(x, t), \cdots, u_n(x, t))$ and $\pi = \pi(x, t)$ denote the velocity vector field and the pressure of the fluid at the point $x \in \mathbb{R}^n$ and time $t > 0$, respectively, while $u_0 = u_0(x)$ is the given initial vector field for $u$.

It is known that for every $u_0 \in H^s \equiv W^{s,2}(\mathbb{R}^n)$ ($s > n/2 - 1$), there exists a unique solution $u \in C([0, T); H^s)$ to (N-S) for some $T > 0$. Such a solution is in fact smooth in $\mathbb{R}^n \times (0, T)$. See, for instance Fujita-Kato [9]. It is an important open question whether $T$ may be taken as $T = \infty$ or $T < \infty$. In this direction, Giga [10] gave a Serrin type criterion, i.e., if the solution $u$ satisfies the condition

$$\int_0^T \|u(t)\|_{L^p}^{\theta} \, dt < \infty, \quad \frac{2}{\theta} + \frac{n}{p} = 1, \, n < p \leq \infty, \tag{1.1}$$

then $u$ can be extended to the solution in the class $C([0, T'); H^s)$ for some $T' > T$. Later on, the condition (1.1) was relaxed from the $L^p$-criterion to

$$\int_0^T \|u(t)\|_{\dot{B}_p^{\alpha}}^{\theta} \, dt < \infty, \quad \frac{2}{\theta} + \alpha = 1, \, 0 \leq \alpha < 1 \tag{1.2}$$

Reinhard Farwig, Department of Mathematics, Darmstadt University of Technology, 64289 Darmstadt, Germany, E-mail: farwig@mathematik.tu-darmstadt.de

Ryo Kanamaru, Department of Pure and Applied Mathematics, School of Fundamental Science and Engineering Waseda University, Tokyo 169-8555, Japan, E-mail: ryo-kana@suou.waseda.jp

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by Kozono-Ogawa-Taniuchi [16] and Kozono-Shimada [17]. In a recent work, Nakao-Taniuchi [22] gave a new criterion, instead of (1.1) and (1.2) with \( p = \infty \) and \( \alpha = 0 \) (\( \theta = 2 \)), in a such way that

\[
\int_0^T \|\nabla u(r)\|_{V_{1/2}}^\theta \, dt < \infty. \tag{1.3}
\]

Here, \( V_\beta, \beta > 0 \), is introduced by

\[
V_\beta := \{ f \in S \mid \|f\|_{V_\beta} < \infty \}, \\
\|f\|_{V_\beta} := \sup_{N=1,2,\ldots} \|\psi_N * f\|_{L^\beta},
\]

where \( \psi \in S \) is a radially symmetric function with \( \hat{\psi}(\xi) = 1 \) in \( B(0, 1) \) and \( \hat{\psi}(\xi) = 0 \) in \( B(0, 2) \) and \( \psi_N(x) := 2^{nN} \psi(2^n x) \). This function space \( V_\beta \) is called the Vishik space and admits a continuous embedding \( L^\infty \subset V_\beta \) for each \( \beta > 0 \). The above three criteria are important from a viewpoint of scaling invariance. Indeed, it is easy to show that if \( (u, \pi) \) satisfies (N-S), then so does \( (u_\lambda, \pi_\lambda) \) for all \( \lambda > 0 \), where \( u_\lambda(x, t) := \lambda u(\lambda x, \lambda^2 t) \) and \( \pi_\lambda(x, t) := \lambda^2 \pi(\lambda x, \lambda^2 t) \). We call a Banach space \( X \) scaling invariant for the velocity \( u \) with respect to (N-S) if

\[
\|u_\lambda\|_X = \|u\|_X \quad \text{holds for all} \quad \lambda > 0.
\]

In fact, the spaces \( L^p(0, \infty; L^p) \) with \( 2/\theta + n/p = 1 \) and \( L^\theta(0, \infty; \dot{B}^0_{\infty, \infty}) \) with \( 2/\theta + \alpha = 1 \) and \( L^2(0, \infty; V_{1/2}) \) are scaling invariant for \( u \) with respect to (N-S).

On the other hand, Beale-Kato-Majda [1] and Beirão da Veiga [2] gave a criterion by means of the vorticity, \( \i.e., \) if the solution \( u \) satisfies the condition

\[
\int_0^T \|\rot u(t)\|_{L^p}^\theta \, dt < \infty, \quad \frac{2}{\theta} + \frac{n}{p} = 2, \quad \frac{n}{2} < p < \infty, \tag{1.4}
\]

then \( u \) can be extended to a solution in the class \( C([0, T'); H^\delta(\mathbb{R}^n)) \) for some \( T' > T \). Later on, the condition (1.4) was relaxed from the \( L^p \)-criterion to

\[
\int_0^T \|\rot u(t)\|^\theta_{B^0_{p, \infty}} \, dt < \infty, \quad \frac{2}{\theta} + \frac{n}{p} = 2, \quad n \leq p \leq \infty \tag{1.5}
\]

by Kozono-Ogawa-Taniuchi [16]. Moreover, Nakao-Taniuchi [21] gave a similar type of the criterion as (1.3), instead of (1.4) and (1.5) with \( p = \infty \) (\( \theta = 1 \)), in such a way that

\[
\int_0^T \|\rot u(t)\|_{V_1} \, dt < \infty.
\]

Note that \( V_\beta \) admits the following continuous embeddings in the case \( \beta = 1 \):

\[
L^\infty \subset bmo \subset B^0_{\infty, \infty} \subset V_1.
\]

Furthermore, the author [12] improved the \( B^0_{p, \infty} \)-criterion (1.5) to

\[
\int_0^T \|\rot u(t)\|^\theta_{B^0_{p, \infty}} \, dt < \infty, \quad \frac{2}{\theta} + \frac{n}{p} = 2, \quad r \leq p \leq \infty \tag{1.6}
\]

for \( L'(n < r < \infty) \) strong solutions to (N-S). Here, \( V^s_{p, q, \theta} \) is a Banach space introduced by Definition 2.1 and has a continuous embedding \( B^0_{p, \infty} \subset V^s_{p, q, \theta} \). The above criteria by means of the vorticity are also important from a viewpoint of scaling invariance. Indeed, since \( u_\lambda = \lambda^2 \rot u(\lambda x, \lambda^2 t) \), the spaces \( L^\theta(0, \infty; L^p), L^\theta(0, \infty; \dot{B}^0_{p, \infty}), L^\theta(0, \infty; V^0_{p, \infty, \theta}) \) with \( 2/\theta + n/p = 2 \) and \( L^4(0, \infty; V_1) \) are scaling invariant for the vorticity with respect to (N-S).
The aim of this paper is to improve the extension criterion (1.2) to the Navier-Stokes equations by means of Banach spaces which are larger than \( \dot{B}^{a}_{\infty, \infty} \) in the same way that the condition (1.5) was relaxed to (1.6). In fact, we prove that if the solution \( u \) to (N-S) on \((0, T)\) satisfies the condition either

\[
\int_0^T \| u(t) \|_{\dot{B}^{2/\theta + a}_{\infty, 1/\theta, \infty}}^2 \, dt < \infty, \quad \frac{2}{\theta} + a = 1, \ 0 < a < 1
\]  

(1.7)

or

\[
\int_0^T \| u(t) \|_{\dot{B}^{2/\theta + a}_{\infty, 1/\theta, \infty}}^2 \, dt < \infty,
\]  

(1.8)

then \( u \) can be extended to a solution in the class \( C([0, T'); H^s(\mathbb{R}^n)) \) for some \( T' > T \). Here, \( \dot{U}^s_{p, \beta, \alpha} \) is a Banach space introduced by Definition 2.2 and has the following continuous embeddings:

\[
\dot{B}^{a}_{\infty, \infty} \subset \dot{V}^{-a}_{\infty, \infty, \theta} \subset \dot{U}^{-a}_{\infty, 1/\theta, \infty}, \quad \frac{2}{\theta} + a = 1, \ 0 < a < 1.
\]

Hence, we see that (1.7) and (1.8) may be regarded as a weaker condition than (1.2). Moreover, note that the spaces \( L^2(0, \infty; \dot{U}^{a}_{\infty, 1/\theta, \infty}) \) with \( 2/\theta + a = 1 \) and \( L^2(0, \infty; \dot{V}^{0}_{\infty, \infty, 2}) \) are also scaling invariant for solutions \( u \) to (N-S). In order to obtain our extension principle, we need a logarithmic interpolation inequality by means of \( \dot{U}^s_{p, \beta, \alpha} \):

\[
\| f \|_{\dot{U}^{s}_{p, \beta, \alpha}} \leq C \left( 1 + \| f \|_{\dot{U}^{s}_{p, \beta, \alpha}} \log \left( 1 + \| f \|_{\dot{U}^{s}_{p, \beta, \alpha}} \right) \right).
\]

This is related to the Brezis-Gallouet-Wainger inequality given in Brezis-Gallouet [5] and Brezis-Wainger [6]. Several inequalities of Brezis-Gallouet-Wainger type were established in [1], [7], [8], [11], [12], [15], [16], [19], [20], [21], [22], [23], [24], [25]. Moreover, we prove that \( \dot{U}^s_{p, \beta, \alpha} \) is the weakest normed space that satisfies such a logarithmic interpolation inequality. Thus, roughly speaking, new conditions (1.7) and (1.8) may be regarded as optimal Serrin type criteria that guarantee \emph{a priori} estimates of \( H^s \) strong solutions to (N-S) with double exponential growth form.

The present paper is organized as follows. In the next section, we shall state our main results. In section 3 and 4, proofs of our main results are established.

## 2 Results

### 2.1 Function spaces

We first introduce some notation. Let \( S = S(\mathbb{R}^n) \) be the set of all Schwartz functions on \( \mathbb{R}^n \), and \( S' \) the set of tempered distributions. The \( L^p \)-norm on \( \mathbb{R}^n \) is denoted by \( \| \cdot \|_p \). We recall the Littlewood-Paley decomposition and use the functions \( \psi, \psi_j \in S, j \in \mathbb{Z} \), such that

\[
\hat{\psi}(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| \geq 2, \end{cases}
\]

\[
\hat{\phi}(\xi) := \hat{\psi}(\xi) - \hat{\psi}(2\xi), \quad \hat{\phi}_j(\xi) := \hat{\phi}(\xi/2^j).
\]

Let \( \mathbb{Z} := \{ f \in S; D^a \hat{f}(0) = 0 \text{ for all } a \in \mathbb{N}^n \} \) and \( \mathbb{Z}' \) denote the dual space of \( \mathbb{Z} \). We note that \( \mathbb{Z}' \) can be identified with the quotient space \( S'/\mathcal{P} \) of \( S' \) with respect to the space of polynomials, \( \mathcal{P} \). Furthermore, the homogeneous Besov space \( \dot{B}^s_{p, q} := \{ f \in \mathbb{Z}; \| f \|_{\dot{B}^s_{p, q}} < \infty \} \) is defined by the norm

\[
\| f \|_{\dot{B}^s_{p, q}} := \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| \phi_j \ast f \|_p^q \right)^{\frac{1}{q}}, & q \neq \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \| \phi_j \ast f \|_p, & q = \infty. \end{cases}
\]
See Bergh-Löfström [3, Chapter 6.3] and Triebel [26, Chapter 5] for details. Let \( C^\infty_0(\mathbb{R}^n) \) denote the set of all \( C^\infty \) functions with compact support in \( \mathbb{R}^n \) and \( C^\infty_{0,a} := \{ \phi \in (C^\infty_0(\mathbb{R}^n))^n : \text{div} \, \phi = 0 \} \). Concerning Sobolev spaces we use the notation \( H^p(\mathbb{R}^n) \) for all \( s \in \mathbb{R} \). Then \( H^p_0 \) is the closure of \( C^\infty_{0,a} \) with respect to \( H^p \)-norm. In Section 4 we will also use homogeneous Sobolev spaces \( \dot{H}^p(\mathbb{R}^n) \) and note that \( \dot{H}^p = B^s_{2,2} \) for all \( s \in \mathbb{R} \).

We now introduce Banach spaces \( \dot{V}^s_{p,q,\theta} \) and \( \dot{U}^s_{p,q,\theta} \), which are larger than the homogeneous Besov spaces \( B^s_{p,q} \). These spaces may be regarded as modified versions of spaces defined by Nakao-Taniuchi [22] and Vishik [27].

**Definition 2.1.** Let \( s \in \mathbb{R}, 1 \leq p, q, \theta \leq \infty \) and let \( \{ \phi_j \}_{j=-\infty}^\infty \) be the Littlewood-Paley decomposition. Then, 
\[
\dot{V}^s_{p,q,\theta}(\mathbb{R}^n) := \{ f \in \mathcal{Z} ; \| f \|_{\dot{V}^s_{p,q,\theta}} < \infty \}
\]

is introduced by the norm
\[
\| f \|_{\dot{V}^s_{p,q,\theta}} := \begin{cases}
\left( \sum_{|\alpha| \leq N} 2^{js\theta} \| \phi_j * f \|_p^\theta \right)^{\frac{1}{\theta}}, & \theta \neq \infty,
\sup_{N=1,2,...} N^{\frac{s}{2}} \max_{|\alpha| \leq N} 2^{j\alpha} \| \phi_j * f \|_p, & \theta = \infty.
\end{cases}
\]

**Definition 2.2.** Let \( s, \beta \in \mathbb{R}, 1 \leq p, q, \sigma \leq \infty \) and let \( \{ \phi_j \}_{j=-\infty}^\infty \) be the Littlewood-Paley decomposition. Then, 
\[
\dot{U}^s_{p,q,\theta}(\mathbb{R}^n) := \{ f \in \mathcal{Z} ; \| f \|_{\dot{U}^s_{p,q,\theta}} < \infty \}
\]

is equipped with the norm
\[
\| f \|_{\dot{U}^s_{p,q,\theta}} := \begin{cases}
\left( \sum_{|\alpha| \leq N} 2^{j\alpha} \| \phi_j * f \|_p^\sigma \right)^{\frac{1}{\sigma}}, & \sigma \neq \infty,
\sup_{N=1,2,...} N^{\beta} \max_{|\alpha| \leq N} 2^{j\alpha} \| \phi_j * f \|_p, & \sigma = \infty.
\end{cases}
\]

We see from the following proposition that \( \dot{V}^s_{p,q,\theta} \) and \( \dot{U}^s_{p,q,\theta} \) are extensions of \( B^s_{p,q} \) and \( V^s_{p,q,\theta} \), respectively.

**Proposition 2.3.**
(i) Let \( s \in \mathbb{R}, 1 \leq p, q \leq \infty \) and \( 1 \leq \theta_1 \leq \theta_2 \leq q < \theta_3 \). Then, it holds that
\[
\{ 0 \} = \dot{V}^s_{p,q,\theta_1} \subset \dot{B}^s_{p,q} = \dot{V}^s_{p,q,\theta_2} \subset \dot{V}^s_{p,q,\theta_3} \subset \dot{V}^s_{p,q,\theta_0}.
\]

(ii) Let \( s \in \mathbb{R}, 1 \leq p, \sigma \leq \infty \) and \( \beta_1 \leq p \leq \beta_2 \leq \beta_3 \). Then, it holds that
\[
\{ 0 \} = \dot{U}^s_{p,\beta_1,\sigma} \subset \dot{B}^s_{p,\beta_2} = \dot{U}^s_{p,\beta_2,\sigma} \subset \dot{U}^s_{p,\beta_3,\sigma} \subset \dot{U}^s_{p,\beta_2,\sigma} \subset \dot{U}^s_{p,\beta_3,\sigma}.
\]

(iii) Let \( s, \beta \in \mathbb{R}, 1 \leq p, q, \theta \leq \infty \), \( \tilde{\beta} = \frac{1}{\beta} - \frac{1}{p} \) and \( 1 \leq \sigma_1 \leq \sigma_2 \leq \infty \). Then, it holds that
\[
\dot{V}^s_{p,q,\theta} = \dot{U}^s_{p,\tilde{\beta},\theta} \quad \text{and} \quad \dot{U}^s_{p,\tilde{\beta},\theta} \subset \dot{U}^s_{p,\tilde{\beta},\sigma_1}.
\]

**Proof.** We easily prove \( \dot{V}^s_{p,q,\theta} \subset \dot{V}^s_{p,q,\theta_1} \) in (i) by the standard and the reverse Hölder’s inequality. The others follow from the definitions of \( B^s_{p,q} \), \( V^s_{p,q,\theta} \) and \( U^s_{p,q,\theta} \). \( \square \)

It follows by Proposition 2.3 (i) and (iii) that
\[
\dot{B}^s_{p,q,\theta} \subset \dot{V}^s_{p,q,\theta} \subset \dot{U}^s_{p,q,\theta} \quad \text{for} \ s \in \mathbb{R} \quad \text{and} \ 1 \leq \theta < \infty.
\]

We observe from the following examples that the continuous embeddings (2.1) are proper if \( s > -n \) and \( 1 \leq \theta < \infty \), which is important in terms of Theorem 2.9.
Example 2.4. (1) The continuous embedding $\bar{B}_{\infty,\infty,\theta} \subset \dot{V}^s_{\infty,\infty,\theta}$ is proper if $s > -n$ and $1 \leq \theta < \infty$. We now introduce a distribution $f \in \dot{V}^s_{\infty,\infty,\theta} \setminus \bar{B}_{\infty,\infty,\theta}$ for $s > -n$ and $1 \leq \theta < \infty$. Let $f \in \mathcal{Z}$ defined as

$$\hat{f}(\xi) := \begin{cases} k^{-2(n+s)|k^{\theta+1}|} 2^{|k^{\theta+1}| - 1} \leq |\xi| \leq 2^{|k^{\theta+1}| + 1} & (k = 1, 2, \ldots), \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, since $\hat{f} \in L^\infty$ holds, we obtain $f \in \mathcal{Z}$. We easily see that

$$\|\phi_j * f\|_\infty = \int_\mathbb{R}^{n} \hat{\phi}_j(\xi) \hat{f}(\xi) \, d\xi = \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} \hat{\phi}_j(\xi) \hat{f}(\xi) \, d\xi$$

$$= \begin{cases} 2^{-2|k^{\theta+1}|}k \|\hat{\phi}\|_1 & \text{for } j = [k^{\theta+1}] \quad (k = 1, 2, \ldots), \\ \leq 2^{-2|k^{\theta+1}|}2^n k \|\hat{\phi}\|_1 & \text{for } j = [k^{\theta+1}] \pm 1 \quad (k = 1, 2, \ldots), \\ 0 & \text{for } j \in \mathbb{Z} \setminus \bigcup_{k=1,2,\ldots} \{[k^{\theta+1}], [k^{\theta+1}] \pm 1\}. \end{cases}$$

Hence, it holds that

$$\|f\|_{\bar{B}_{\infty,\infty,\theta}} \geq \sup_{k=1,2,\ldots} 2^{2|k^{\theta+1}|} \|\phi_j * f\|_\infty = \sup_{k=1,2,\ldots} k \|\hat{\phi}\|_1 = \infty. \quad (2.2)$$

On the other hand, for any $N = 1, 2, \ldots$, there exists $k_N \in \mathbb{N}$ such that $k_N^{\theta+1} \leq N < (k_N + 1)^{\theta+1}$. Therefore, we obtain

$$\sum_{\{j \leq N\} \setminus \{k_N^{\theta+1}\}} 2^{2j\theta} \|\phi_j * f\|_\infty \leq \sum_{k=1}^{k_N^{\theta+1}} \sum_{\{j \leq N\} \setminus \{k\}} 2^{2j\theta} \|\phi_j * f\|_\infty$$

$$\leq \sum_{k=1}^{k_N^{\theta+1}} \sum_{\{j \leq N\} \setminus \{k\}} 2^{2j\theta} (2^{-2|k^{\theta+1}|}) 2^n k \|\hat{\phi}\|_1$$

$$= C \sum_{k=1}^{k_N^{\theta+1}} k^{\theta} \leq C(k_N + 1)^{\theta+1} \leq Ck_N^{\theta+1} \leq CN,$$

where $C$ is dependent only on $n$, $s$ and $\theta$. Thus, it follows that

$$\|f\|_{\dot{V}^s_{\infty,\infty,\theta}} = \sup_{N=1,2,\ldots} \left( \frac{\sum_{\{j \leq N\} \setminus \{k_N^{\theta+1}\}} 2^{2j\theta} \|\phi_j * f\|_\infty}{N^{\frac{1}{\theta}}} \right) \leq \sup_{N=1,2,\ldots} \frac{C_N^{\frac{1}{\theta}}}{N^{\frac{1}{\theta}}} < \infty. \quad (2.3)$$

From (2.2) and (2.3), we get $f \in \dot{V}^s_{\infty,\infty,\theta} \setminus \bar{B}_{\infty,\infty,\theta}$.

(2) The continuous embedding $\dot{V}^s_{\infty,\infty,\theta} = \dot{U}^s_{\infty,1/\theta,\theta} \subset \dot{U}^s_{\infty,1/\theta,\theta}$ is also proper if $s > -n$ and $1 \leq \theta < \infty$. We now introduce a distribution $g \in \dot{U}^s_{\infty,1/\theta,\theta} \setminus \dot{V}^s_{\infty,\infty,\theta}$ for $s > -n$ and $1 \leq \theta < \infty$. Let $g \in \mathcal{Z}$ defined as

$$\hat{g}(\xi) := \begin{cases} k^{\frac{1}{\theta} + \frac{2}{n+s}k^{\theta+1}} 2^{|k^{\theta+1}| - 1} \leq |\xi| \leq 2^{|k^{\theta+1}| + 1} & (k = 1, 2, \ldots), \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, since $\hat{g} \in L^\infty$ holds, we obtain $g \in \mathcal{Z}$. We easily see that

$$\|\phi_j * g\|_\infty = \int_\mathbb{R}^{n} \hat{\phi}_j(\xi) \hat{g}(\xi) \, d\xi = \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} \hat{\phi}_j(\xi) \hat{g}(\xi) \, d\xi$$

$$= \begin{cases} 2^{-2|k^{\theta+1}|}k^{\frac{2j}{\theta}} \|\hat{\phi}\|_1 & \text{for } j = [k^{\theta+1}] \quad (k = 1, 2, \ldots), \\ \leq 2^{-2|k^{\theta+1}|}2^n k^{\frac{2j}{\theta}} \|\hat{\phi}\|_1 & \text{for } j = [k^{\theta+1}] \pm 1 \quad (k = 1, 2, \ldots), \\ 0 & \text{for } j \in \mathbb{Z} \setminus \bigcup_{k=1,2,\ldots} \{[k^{\theta+1}], [k^{\theta+1}] \pm 1\}. \end{cases}$$
For any \( N = 1, 2, \ldots \), we take \( k_N \in \mathbb{N} \) such that \( k_N^{\theta+1} \leq N < (k_N + 1)^{\theta+1} \). Then, it holds that
\[
\sum_{j \leq N} 2^{js} \|\phi_j * g\|_\infty \geq \sum_{1 \leq j \leq k_N} 2^{js} [k_N^{\theta+1}]^\theta \|\phi[j-1] * g\|_\infty = C_1 \sum_{1 \leq j \leq k_N} k^{j\theta+1} \geq C_1 k_N^{\theta+2} \geq C_1 (k_N + 1)^{\theta+2} \geq C_1 N^{\theta+2} / \bar{N},
\]
where \( C_1 \) is dependent only on \( n \) and \( \theta \). Hence, we have
\[
\|f\|_{\mathcal{V}_{m,\omega,\theta}} = \sup_{N=1,2,\ldots} \left( \frac{1}{N} \sum_{j \leq N} 2^{js} \|\phi_j * f\|_\infty \right)^\frac{1}{s} \geq \sup_{N=1,2,\ldots} C_1 N^{ \frac{\theta+2}{s} \bar{N} } = \infty, \tag{2.4}
\]
On the other hand, it follows that
\[
\max_{j \leq N} 2^{js} \|\phi_j * g\|_\infty \leq \max_{1 \leq j \leq k_N} 2^{js} \|\phi_j * g\|_\infty \\
\leq \max_{1 \leq j \leq k_N} \max_{j-1 \leq \ell \leq j+1} 2^{js} 2^{-s[2^{0j}]} 2^n k^{\theta+1} \|\phi\|_1 \\
\leq C_2 \max_{1 \leq j \leq k_N} k^{\theta+1} = C_2 (k_N + 1)^{\theta+1} \leq C_2 N^{\theta+1},
\]
where \( C_2 \) is dependent only on \( n \) and \( s \). Thus, we obtain
\[
\|g\|_{\mathcal{V}_{m,\omega,\theta}} = \sup_{N=1,2,\ldots} \frac{\max_{j \leq N} 2^{js} \|\phi_j * f\|_\infty}{N^{\theta+1}} \leq \sup_{N=1,2,\ldots} \frac{C_2 N^{\theta}}{N^{\theta+1}} < \infty. \tag{2.5}
\]
From (2.4) and (2.5), we get \( g \in \mathcal{U}_{\omega,1/\theta,\omega,\theta}^s \setminus \mathcal{V}_{\omega,\omega,\omega}^s \).

### 2.2 Logarithmic interpolation inequalities and optimality

**Theorem 2.5.** (i) Let \( s_0, s_1, s_2 \in \mathbb{R} \) satisfy \( s_1 < s_0 < s_2 \), let \( 0 \leq \beta < \infty \) and \( 1 \leq p, \sigma \leq \infty \). Then there exists a positive constant \( C \) depending only on \( s_0, s_1, s_2, \) but not on \( p, \beta, \sigma \) such that
\[
\|f\|_{\mathcal{B}_{p,\beta,\sigma}^{s_0}} \leq C \left( 1 + \|f\|_{\mathcal{U}_{p,\beta,\sigma}^{s_0}} \log^\beta \left( e + \|f\|_{\mathcal{B}_{p,\beta,\sigma}^{s_1} \cap \mathcal{B}_{p,\beta,\sigma}^{s_2}} \right) \right) \tag{2.6}
\]
for all \( f \in \mathcal{B}_{p,\beta,\sigma}^{s_1} \cap \mathcal{B}_{p,\beta,\sigma}^{s_2} \).

(ii) Let \( s_0 \in \mathbb{R}, 0 \leq \beta < \infty \) and \( 1 \leq p, \sigma \leq \infty \), and let \( X \) be a normed space of distributions on \( \mathcal{Z} \). Assume that \( X \) satisfies the following conditions:

(C1) \( X \to \mathcal{Z} \);  
(C2) there exists a constant \( K_1 > 0 \) such that
\[
\|f(\cdot - y)\|_X \leq K_1 \|f\|_X \quad \text{for all} \; f \in X \text{ and all} \; y \in \mathbb{R}^n; \]
(C3) there exists a constant \( K_2 > 0 \) such that
\[
\|\rho * f\|_X \leq K_2 \|\rho\|_1 \|f\|_X \quad \text{for all} \; \rho \in \mathcal{Z} \text{ and all} \; f \in X; \]
(C4) there exist \( s_1, s_2 \in \mathbb{R} \) satisfy \( s_1 < s_0 < s_2 \) and \( K_3 > 0 \) such that
\[
\|f\|_{\mathcal{B}_{p,\beta,\sigma}^{s_0}} \leq K_3 \left( 1 + \|f\|_X \log^\beta \left( e + \|f\|_{\mathcal{B}_{p,\beta,\sigma}^{s_1} \cap \mathcal{B}_{p,\beta,\sigma}^{s_2}} \right) \right) \quad \text{for all} \; f \in X \cap \mathcal{Z}. \]
Then, \( X \to \tilde{U}^{\beta}_{p,\beta,\sigma} \) holds.

**Remark 2.6.** (1) In the first part of Theorem 2.5, the assumption \( s_1 < s_0 < s_2 \) is essential. If either of \( s_1 \) or \( s_2 \) tends to \( s_0 \), then the constant \( C \) appearing on the right hand side diverges to infinity.

(2) By Proposition 2.3 (ii), we observed that the following continuous embeddings hold for \( s_1 < s_0 < s_2 \) and \( \beta \geq 0 \):

\[
\dot{B}^{s_2}_{p,\infty} \cap \dot{B}^{s_2}_{p,\infty} \subset \dot{B}^{s_0}_{p,\sigma} \subset \dot{U}^{s_0}_{p,\beta,\sigma},
\]

Thus, (2.6) may be regarded as an interpolation inequality.

(3) From Theorem 2.5 (i), we see that \( \dot{U}^{s_0}_{p,\beta,\sigma} \) satisfies conditions (C1)-(C4). Hence, Theorem 2.5 (ii) implies that \( \dot{U}^{s_0}_{p,\beta,\sigma} \) is the weakest normed space that satisfies (C1)-(C4).

(4) By Proposition 2.3 (iii), we see that Theorem 2.5 covers the result given by the author [12]. Indeed, by setting \( \beta = \frac{1}{2} - \frac{1}{q} \), \( \sigma = \theta \) \((1 \leq q \leq \infty, 1 \leq \theta \leq q)\) in (2.6), it holds that

\[
\|f\|_{\dot{B}^{\theta}_{p,q}} \leq C \left( 1 + \|f\|_{\dot{B}^{\theta_{s_0}}_{p,q}} \log^{\frac{1}{\theta}} \left( e + \|f\|_{\dot{B}^{s_2}_{p,q} \cap \dot{B}^{s_2}_{p,\infty}} \right) \right)
\]

for all \( f \in \dot{B}^{s_2}_{p,\infty} \cap \dot{B}^{s_2}_{p,\infty} \).

### 2.3 Serrin type regularity criteria for Navier-Stokes systems

**Definition 2.7.** Let \( s > n/2 - 1 \) and let \( u_0 \in H^s_\theta \). A measurable function \( u \) on \( \mathbb{R}^n \times (0, T) \) is called a strong solution to (N-S) in the class \( CL_s(0, T) \) if

(i) \( u \in C((0, T); H^s_\theta) \cap C^1((0, T); H^s_\theta) \cap C((0, T); H^{s+2}_\theta) \);

(ii) \( u \) satisfies (N-S) with some distribution \( \pi \) such that \( \nabla \pi \in C((0, T); H^s) \).

**Remark 2.8.** For \( s > n/2 - 1 \), the existence of a strong solution to (N-S) in the class \( CL_s(0, T) \) has been proven in Fujita-Kato [9], Kato [14] and Giga [10].

Our result on extension of strong solutions now reads as follows:

**Theorem 2.9.** (i) Let \( 0 < \alpha < 1, s > n/2 - \alpha \) and let \( u_0 \in H^s_\theta \). Assume that \( u \) is a strong solution to (N-S) in the class \( CL_{s}(0, T) \). If the solution \( u \) satisfies

\[
\int_0^T \|u(t)\|^\theta_{\dot{U}^{s}_{\infty,\infty}} \, dt < \infty, \quad \frac{2}{\theta} + \alpha = 1,
\]

then \( u \) can be extended to a strong solution to (N-S) in the class \( CL_s(0, T') \) for some \( T' > T \).

(ii) Let \( s > n/2 \) and let \( u_0 \in H^s_\theta \). Assume that \( u \) is a strong solution to (N-S) in the class \( CL_{s}(0, T) \). If the solution \( u \) satisfies

\[
\int_0^T \|u(t)\|^2_{\dot{U}^{s}_{\infty,1}} \, dt < \infty,
\]

then \( u \) can be extended to a strong solution to (N-S) in the class \( CL_s(0, T') \) for some \( T' > T \).

**Remark 2.10.** (1) Let \( 0 < \alpha < 1 \). As is mentioned Example 2.4, we have proper embeddings \( \dot{B}^{s}_{\infty,\infty} \subset \dot{U}^{s}_{\infty,\infty,\theta} \subset \dot{U}^{s}_{\infty,1/\theta,\infty} \) and hence Theorem 2.9 (i) covers the extension criterion in \( \dot{B}^{s}_{\infty,\infty} \) given by Kozono-Shimada [17] for \( s > n/2 - \alpha \). Indeed, if the solution \( u \) satisfies either

\[
\int_0^T \|u(t)\|^{\theta}_{\dot{B}^{\theta}_{\infty,\infty}} \, dt < \infty, \quad \frac{2}{\theta} + \alpha = 1,
\]
or
\[
\int_0^T \|u(t)\|_{p-\delta}^{\theta} \, dt < \infty, \quad \frac{2}{\theta} + \alpha = 1,
\]
then the estimate (2.8) is easily obtained, so that the solution can be extended beyond \( t = T \).

(2) From Example 2.4, the proper embeddings \( \dot{B}_{\infty,\infty}^0 \subset \dot{V}_{\infty,\infty}^0 \subset \dot{U}_{\infty,1/2,\infty}^0 \) hold. Hence, Theorem 2.9 (ii) may be regarded as an extension of the \( \dot{B}_{\infty,\infty}^0 \)-criterion given by Kozono-Ogawa-Taniuchi [16] for \( s > n/2 \). On the other hand, it seems to be difficult to obtain the same result as in Theorem 2.9 (ii) under the condition
\[
\int_0^T \|u(t)\|_{\dot{U}_{\infty,1/2,\infty}^0}^{\theta} \, dt < \infty.
\]
This stems from inapplicability of Lemma 4.1 with \( \alpha = 0 \).

As an immediate consequence of the above Theorem 2.9, we have the following blow-up criteria of strong solutions:

**Corollary 2.11.** (i) Let \( 0 < \alpha < 1 \), \( s > n/2 - \alpha \) and let \( u_0 \in H_{\infty}^s \). Assume that \( u \) is a strong solution to (N-S) in the class \( \text{CL}_s(0, T) \). If \( T \) is maximal, i.e., \( u \) cannot be extended in the class \( \text{CL}_s(0, T') \) for any \( T' > T \), then it holds that
\[
\int_0^T \|u(t)\|_{\dot{B}_{\infty,\infty}^0}^{\theta} \, dt = \infty, \quad \frac{2}{\theta} + \alpha = 1.
\]
In particular, we have \( \lim \sup_{t \to T} \|u(t)\|_{\dot{B}_{\infty,\infty}^0} = \infty \).

(ii) Let \( s > n/2 \) and let \( u_0 \in H_{\infty}^s \). Assume that \( u \) is a strong solution to (N-S) in the class \( \text{CL}_s(0, T) \). If \( T \) is maximal, then it holds that
\[
\int_0^T \|u(t)\|_{\dot{U}_{\infty,1/2,\infty}^0}^{\theta} \, dt = \infty.
\]
In particular, \( \lim \sup_{t \to T} \|u(t)\|_{\dot{U}_{\infty,1/2,\infty}^0} = \infty \).

3 Proof of Theorem 2.5

We first prove Theorem 2.5 (i). To this aim, we use arguments given in Kozono-Ogawa-Taniuchi [16], Nakao-Taniuchi [21] and Kanamaru [12].

**Proof of Theorem 2.5 (i).** We first consider the case \( 1 \leq \sigma < \infty \). By the definition of the Besov space, we obtain
\[
\|f\|_{B_{p,q,\sigma}^s} = \left( \sum_{j \in \mathbb{Z}} 2^{js \sigma q} \|\phi_j \ast f\|_{q}^{\sigma} \right)^{\frac{1}{\sigma}}
\]
\[
\leq \sum_{j < -N} 2^{js} \|\phi_j \ast f\|_{p} + \sum_{j > N} 2^{js} \|\phi_j \ast f\|_{p} + \left( \sum_{j \in \mathbb{Z}} 2^{js \sigma q} \|\phi_j \ast f\|_{p}^{\sigma} \right)^{\frac{1}{\sigma}} \quad (3.1)
\]
\[
= S_1 + S_2 + S_3
\]
Concerning $S_1$, it holds that
\[
S_1 \leq \sum_{j < N} 2^{js_1} \| \phi_j \|_p 2^{r(s_0 - s_1)}
\leq \| f \|_{L^p_{r,s_1}} \sum_{j < N} 2^{r(s_0 - s_1)}
\leq C_1 2^{-(s_0 - s_1)N} \| f \|_{L^p_{r,s_1}},
\] (3.2)
where $C_1$ is dependent only on $s_0$ and $s_1$. For $S_2$, in the same way as (3.2), we have
\[
S_2 \leq C_2 2^{-(s_2 - s_0)N} \| f \|_{L^p_{r,s_2}},
\] (3.3)
where $C_2$ is dependent only on $s_0$ and $s_2$.

We finally estimate $S_3$. By Definition 2.2, it clearly follows that
\[
S_3 \leq N^\theta \| f \|_{L^p_{r,\beta}}.
\] (3.4)

Combining (3.2), (3.3) and (3.4) with (3.1), we obtain
\[
\| f \|_{L^p_{r,\beta}} \leq C \left( 2^{-\Delta N} \| f \|_{L^p_{r,\beta}} + N^\theta \| f \|_{L^p_{r,\beta}} \right)
\] (3.5)
for $s^* := \min(s_0 - s_1, s_2 - s_0)$ and $C = C(s_0, s_1, s_2)$. In the case $\| f \|_{L^p_{r,\beta}} \leq 1$, we take $N = 1$ in (3.5). Then it holds that
\[
\| f \|_{L^p_{r,\beta}} \leq C \left( 1 + \| f \|_{L^p_{r,\beta}} \right) \leq C \left( 1 + \| f \|_{L^p_{r,\beta}} \log^\theta(e + \| f \|_{L^p_{r,\beta}}) \right);
\]
this is the desired estimate (2.6). In the case $\| f \|_{L^p_{r,\beta}} > 1$, we take $N = 1 + \left[ \log(e + \| f \|_{L^p_{r,\beta}}) / (s^* \log 2) \right]$ in (3.5), where $[\cdot]$ denotes the Gauß symbol. Then, we get (2.6) again.

In the case $\sigma = \infty$, we obtain, instead of (3.1),
\[
\| f \|_{L^p_{r,\infty}} \leq \sup_{j < N} 2^{js_0} \| \phi_j \|_p + \sup_{j > N} 2^{js_0} \| \phi_j \|_p + \max_{j \geq N} 2^{js_0} \| \phi_j \|_p
\leq \tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3
\] (3.6)
Therefore, using the same argument as in the previous case $1 \leq \sigma < \infty$, we get (2.6).

In order to prove the second part of Theorem 2.5, we use the following Lemma.

**Lemma 3.1.** Let $\rho \in \mathcal{Z}$ and Let $X$ be a normed space. Assume that $X$ satisfies conditions (C1) and (C2) given in Theorem 2.5 (ii). Then, it holds that
\[
\rho \ast g \in L^\infty \quad \text{for all} \quad g \in X.
\] (3.7)

**Proof.** By (C1), we get that for all $\phi \in \mathcal{Z}$, there exists a constant $C = C(\phi) > 0$ such that
\[
|g(\phi)| \leq C\|g\|_X \quad \text{for all} \quad g \in X.
\] (3.8)
Assume that (3.8) does not hold. Then, there is a $\phi_0 \in \mathcal{Z}$ with the following property: for each positive integer $N$, there is a $g_N \in X$ such that
\[
|g_N(\phi_0)| > N\|g_N\|_X.
\] (3.9)
Letting $h_N := \frac{g_N}{N^{\frac{1}{2}}\|g_N\|_X} \in X$, we obtain $\|h_N\|_X = N^{-\frac{1}{2}} \to 0$ as $N \to \infty$, which implies $h_N \to 0$ in $X$. By (C1), this convergence holds in $\mathcal{Z}'$. On the other hand, by (3.9),
\[
|h_N(\phi_0)| = \frac{|g_N(\phi_0)|}{N^{\frac{1}{2}}\|g_N\|_X} > N^{\frac{1}{2}} \to \infty \quad \text{as} \quad N \to \infty,
\]
which contradicts $h_N \to 0$ in $\mathcal{Z}'$. Thus we get (3.8).
We finally prove (3.7). Note that

$$\rho * g(x) = g(\tau_x \hat{\rho}) = \tau_{-x} g(\hat{\rho}),$$

where $\tau_x f(y) = f(y - x)$ and $\hat{f}(y) = f(-y)$. Hence, from (3.8) and (C2), we obtain

$$|\rho * g(x)| \leq C(\rho) \|\tau_{-x} g\|_X \leq C(\rho, K_1) \|g\|_X$$

for all $x \in \mathbb{R}^n$, which means (3.7).

We are now in position to prove the second part of Theorem 2.5 and follow arguments given by Nakao-Taniuchi [21] and the author [12].

**Proof of Theorem 2.5 (ii).** Substituting $f = \frac{h}{\varepsilon \|h\|_p \|g\|_p}$ into the inequality given in (C4), we obtain

$$\|h\|_{B^0_{p,\alpha}} \leq K_3 \left( \varepsilon \|h\|_p \|g\|_p + \|h\|_X \log \left( e + \frac{1}{\varepsilon} \right) \right)$$

(3.10)

for all $h \in X \cap \mathcal{Z}$ and all $\varepsilon > 0$. Let $g \in X$ and $\Phi_N := \sum_{|j| \leq N} \Phi_j$ for $N = 1, 2, \ldots$. By Lemma 3.1, $\Phi_N * g \in L^\infty$. Hence, we have $\Phi_N * g \in \mathcal{Z}$. On the other hand, it holds from (C3) that

$$\|\Phi_N * g\|_X \leq 2K_2 \|\psi_1\|_X \leq 2K_2 \|\psi_1\|_X$$

(3.11)

where $\psi_j(x) := 2^{|j|} \psi(2^{|j|} x)$. Thus, we also get $\Phi_N * g \in X$. Substituting $h = \Phi_N * g (\in X \cap \mathcal{Z})$ into (3.10), we obtain

$$\|\Phi_N * g\|_{B^0_{p,\alpha}} \leq K_3 \varepsilon \|\Phi_N * g\|_X \log \left( e + \frac{1}{\varepsilon} \right).$$

(3.12)

We first consider the case $1 \leq \sigma < \infty$.

The left-hand side of (3.12) can be estimated from below as follows. Noting that $\text{supp} \hat{\Phi}_N \subset \{2^{-N-1} \leq |\xi| \leq 2^{N+1} \}$, we get

$$\|\Phi_N * g\|_{B^0_{p,\alpha}}^\sigma = \sum_{|j| \leq N+1} 2^{j \sigma} \|\phi_j * \Phi_N * g\|_p^\sigma$$

(3.13)

Concerning the second term on the right-hand side of (3.13), we obtain

$$\sum_{j=N+1}^{2^{j \sigma}} 2^{j \sigma} \|\phi_j * \Phi_N * g\|_p^\sigma \geq 2^{-|j_0| \sigma} 2^{N_0 \sigma} \sum_{j=N+1}^{2^{j \sigma}} \|\Phi_j * \Phi_N * g\|_p^\sigma$$

$$\geq 2^{-|j_0| \sigma} 2^{N_0 \sigma} 2^{-\sigma} \left( \sum_{j=N+1}^{2^{j \sigma}} \|\phi_j * \Phi_N * g\|_p^\sigma \right)$$

(3.14)

As in (3.14), similar estimates hold when replacing $N$ and $N + 1$ by $-N$ and $-N - 1$, respectively. Summarizing (3.13), (3.14) we obtain that

$$\|\Phi_N * g\|_{B^0_{p,\alpha}} \geq 2^{-(|j_0| + 1) \sigma} \left( \sum_{|j| \leq N} 2^{j \sigma} \|\phi_j * g\|_p^\sigma \right)^{\frac{1}{\sigma}}$$

(3.15)
Next, we estimate the first term on the right-hand side of (3.12). From Young’s inequality and Hölder’s inequality, it holds that
\[
\| \Phi_N \ast g \|_{B_p^{s_1,\infty}} = \sup_{|j| \leq N+1} 2^{j s_1} \| \Phi_j \ast \Phi_N \ast g \|_p \\
\leq \sup_{|j| \leq N+1} 2^{j s_1} \| \Phi_j \|_1 \| \Phi_N \ast g \|_p \\
 \leq C_1 2^{(|s_1|+1)N} \sum_{|j| \leq N} 2^{-j s_0} \| \Phi_j \ast g \|_p \\
 \leq C_1 2^{(|s_1|+|s_2|+1)N} \left( \sum_{|j| \leq N} 2^{j s_0} \| \Phi_j \ast g \|_p^q \right)^{\frac{1}{q}} \\
 \leq C_1 2^{(|s_1|+|s_2|+1)N} \left( \sum_{|j| \leq N} 2^{j s_0} \| \Phi_j \ast g \|_p^q \right)^{\frac{1}{q}},
\]
where \( C_1 \) depends only on \( n \) and \( s_1 \). In the same way as (3.16), we have
\[
\| \Phi_N \ast g \|_{B_p^{s_2,\infty}} \leq C_2 2^{(|s_0|+|s_2|+1)N} \left( \sum_{|j| \leq N} 2^{j s_0} \| \Phi_j \ast g \|_p^q \right)^{\frac{1}{q}}, \tag{3.17}
\]
where \( C_2 \) depends only on \( n \) and \( s_2 \). In the end, from (3.16) and (3.17), we get that
\[
\| \Phi_N \ast g \|_{B_p^{s_1,\infty} \cap B_p^{s_2,\infty}} \leq C_3 2^{s N} \left( \sum_{|j| \leq N} 2^{j s_0} \| \Phi_j \ast g \|_p^q \right)^{\frac{1}{q}}, \tag{3.18}
\]
for \( s^* := |s_0| + \max(|s_1|, |s_2|) + 1 \) and \( C_3 = C_3(n, s_1, s_2) \).

Thus, combining (3.11), (3.15) and (3.18) with (3.12), we obtain
\[
\left( \sum_{|j| \leq N} 2^{j s_0} \| \Phi_j \ast g \|_p^q \right)^{\frac{1}{q}} \leq C 2^{s^* N} \left( \sum_{|j| \leq N} 2^{j s_0} \| \Phi_j \ast g \|_p^q \right)^{\frac{1}{q}} + C \| g \|_X \log^\beta \left( e + \frac{1}{\varepsilon} \right)
\]
for all \( N = 1, 2, \cdots \), all \( \varepsilon > 0 \) and \( C = C(n, s_0, s_1, s_2, K_2, K_3) \). Taking \( \varepsilon = \frac{1}{2 e 2^{p q}} \), from the above inequality, we get
\[
\left( \sum_{|j| \leq N} 2^{j s_0} \| \Phi_j \ast g \|_p^q \right)^{\frac{1}{q}} \leq C N^\beta \| g \|_X \text{ for all } N = 1, 2, \cdots .
\]
This implies
\[
\| g \|_{B_p^{s_0,\infty}} \leq C \| g \|_X \text{ for all } g \in X,
\]
\text{i.e., the embedding } X \to U_p^{s_0,\infty}.

In the case \( \sigma = \infty \), we obtain, instead of (3.13),
\[
\| \Phi_N \ast g \|_{\dot{B}_{p,\infty}^{s_0}} = \max \left( \max_{|j| \leq N-1} 2^{j s_0} \| \Phi_j \ast g \|_p, \max_{j=0}^{N-1} 2^{j s_0} \| \Phi_j \ast \Phi_N \ast g \|_p, \max_{j=0}^{N-1} 2^{j s_0} \| \Phi_j \ast \Phi_N \ast \Phi_N \ast g \|_p \right).
\]
Therefore, by using the same argument as in the case \( 1 \leq \sigma < \infty \), we get
\[
\| g \|_{\dot{B}_{p,\infty}^{s_0}} \leq C \| g \|_X \text{ for all } g \in X.
\]
This proves Theorem 2.5 (ii).
4 Proof of Theorem 2.9

In order to prove Theorem 2.9, we need bilinear estimates which are related to Leibniz’ rule. Therefore, we first recall the following two lemmata.

Lemma 4.1 ([13], Proposition 2.2). Let $1 \leq p, q \leq \infty$, $s_0 > 0$, $\alpha > 0$ and $\beta > 0$. Moreover, assume that $1 \leq p_1, p_2, p_3, p_4 \leq \infty$ satisfy $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. Then, there exists a constant $C(n, s_0, \alpha, \beta) > 0$ such that

$$\|f \cdot g\|_{B_{p_1, q}^{s_0}} \leq C \left( \|f\|_{B_{p_1}^{s_0}} \|g\|_{B_{p_2}^{s_0}} + \|f\|_{B_{p_3}^{s_0}} \|g\|_{B_{p_4}^{s_0}} \right)$$

(4.1)

for all $f \in B_{p_1, q}^{s_0} \cap B_{p_3, s_0}$ and $g \in B_{p_2, q}^{s_0} \cap B_{p_4, q}^*.$

Lemma 4.2 ([18], Lemma 1). Let $1 < p < \infty$ and Let $a, \beta \in \mathbb{N}^n$. Then, there exists a constant $C(n, p, a, \beta) > 0$ such that

$$\|\alpha^a f \cdot \beta^\beta g\|_p \leq C \left( \|f\|_{BMO} \|(-\Delta)^{\frac{|a|}{2}} g\|_p + \|(-\Delta)^{\frac{|a|}{2}} f\|_p \|g\|_{BMO} \right)$$

(4.2)

for all $f, g \in BMO \cap W^{a, \beta} |p|.$

We are now in a position to prove Theorem 2.9 and follow arguments given by Kozono-Ogawa-Taniuchi [16], Kozono-Shimada [17], Kozono-Taniuchi [18] and the author [12].

Proof of Theorem 2.9. (i) It is well-known that the local existence time $T_*$ of the strong solution to (N-S) can be estimated from below as

$$T_* \geq \frac{C(n, s)}{\|u_0\|_{H^{n+1}}}$$

see e.g. [10] and [14]. Hence by the standard argument of continuation of local solutions, it suffices to establish the following a priori estimate:

$$\sup_{\varepsilon_0 < t < T} \|u(t)\|_{H^{n+1}} \leq C \left( \|u(\varepsilon_0)\|_{H^{n+1}}, \int_{\varepsilon_0}^T \|u(t)\|_{H^{n+1}}^{\beta_1} \|u(t)\|_{H^{n+1}}^{\beta_2} \right)$$

(4.3)

for some $\varepsilon_0 \in (0, T)$, where $[\cdot]$ denotes the Gauß symbol.

Applying $\partial^k$ with $|k| = 0, 1, \cdots, [s] + 1$ to (N-S), we have

$$\partial_t v_k - \Delta v_k + \nabla q_k = F_k,$$

(4.4)

where $v_k := \partial^k u$, $q_k := \partial^k \pi$ and $F_k := -\partial^k (u \cdot \nabla u) - \partial^k \nabla \cdot u \circ u$. Taking the inner product in $L^2$ between (4.4) and $2v_k$, and then integrating the resulting identity on the time interval $(\varepsilon_0, t)$, we obtain

$$\|v_k(t)\|^2 + \int_{\varepsilon_0}^t \|\nabla v_k\|^2 \, dr \leq \|v_k(\varepsilon_0)\|^2 + 2 \int_{\varepsilon_0}^t \|F_k, v_k\| \, dr, \quad \varepsilon_0 < t < T,$$

(4.5)

where

$$\|F_k, v_k\| = \|(\Delta)^{\frac{|k|}{2}} \nabla \cdot u \circ u, (\Delta)^{\frac{|k|}{2}} v_k\| \leq C \|u \circ u\|_{B_{\frac{1}{2}, 1}^{1+|k|}} \|v_k\|_{H^s}.$$ 

By the bilinear estimate Lemma 4.1 (4.1) with $p = q = 2, p_1 = p_2 = 2, p_3 = p_4 = \infty, s_0 = 1 + |k| - \alpha, \beta = \alpha$, it follows that

$$\|u \circ u\|_{B_{\frac{1}{2}, 1}^{1+|k|}} \leq C \|u\|_{B_{\frac{1}{2}, \infty}^s} \|u\|_{B_{\frac{1}{2}, \infty}^s}.$$

Together with an interpolation inequality applied to $\|v_k\|_{H^s}$ we conclude from Young’s inequality that

$$\|F_k, v_k\| \leq C \|u\|_{B_{\frac{1}{2}, \infty}^s} \|u\|_{H^{n+1}} \|v_k\|_{H^s}^{p-1} \|v_k\|_{H^s}^{1-a} \leq C \|u\|_{B_{\frac{1}{2}, \infty}^s} \|\nabla v_k\|_{L^2}^{1+a} \|v_k\|_{L^2}^{1-a} \leq C \|u\|_{B_{\frac{1}{2}, \infty}^s}^{\frac{1}{2}} \|v_k\|^2 + \frac{1}{2} \|\nabla v_k\|^2,$$
where $\theta = \frac{2}{1-a}$. $C$ depends on $n, s, a$. Inserting (4.6) to the right-hand side of (4.5), summing for $|k| = 0, 1, \cdots, |s| + 1$, and absorbing the terms $\|\nabla v_k\|_2^2$ from the right-hand side by the left-hand side, we obtain
\[
\|u(t)\|_{H^{0,1}}^2 \leq \|u(\epsilon_0)\|_{H^{0,1}}^2 + C \int_{\epsilon_0}^t \|u(t)\|_{L^{\infty,0}}^2 \|u(t)\|_{H^{0,1}}^2 \, dr,
\]
for all $\epsilon_0 \leq t < T$. By using Gronwall’s inequality, we get
\[
\|u(t)\|_{H^{0,1}} \leq \|u(\epsilon_0)\|_{H^{0,1}} \exp \left( C \int_{\epsilon_0}^t \|u(t)\|_{L^{\infty,0}}^2 \, dr \right),
\]
for (4.7).

Now, applying the logarithmic interpolation inequality (2.6) with $s_0 = -a$, $s_1 = -n/2(s-1)$, $s_2 = s - n/2(-a)$, $\beta = 1/\theta$, $p = \sigma = \infty$ to $f = \|u(t)\|_{H^{0,1}}$, it follows that
\[
\|u(t)\|_{L^{\infty,0}} \leq C \left( 1 + \|u(t)\|_{L^{\infty,0}} \log^\frac{1}{\theta} \|e + \|u(t)\|_{H^{0,1}}\|_{L^{\infty,0}} \right).
\]
By the embeddings $\dot{B}_{2,\infty}^{s_0} \subset \dot{B}_{2,\infty}^{s_0 - \frac{n}{2}} \subset \dot{B}_{2,\infty}^{s_0}$ and $H^s \subset B_{2,\infty}^s = L^2 \cap \dot{B}^s_{2,\infty} \subset \dot{B}_{2,\infty}^{s_0} \cap \dot{B}_{2,\infty}^{s_0}$, we have
\[
\|u(t)\|_{L^{\infty,0}} \leq C \|u(t)\|_{H^s} \leq C \|u(t)\|_{H^s} \leq C \|u(t)\|_{H^s}.
\]
Hence, by (4.7), (4.8) and (4.9), it holds that
\[
\|u(t)\|_{H^{0,1}} \leq \|u(\epsilon_0)\|_{H^{0,1}} \exp \left( C \int_{\epsilon_0}^t \left( 1 + \|u(t)\|_{L^{\infty,0}} \log(e + \|u(t)\|_{H^{0,1}}) \right) \, dr \right),
\]
where $C = C(n, s, a)$. Therefore, with $g(t) \equiv \log(e + \|u(t)\|_{H^{0,1}})$, we obtain
\[
g(t) \leq g(\epsilon_0) + C \int_{\epsilon_0}^t \left( 1 + \|u(t)\|_{L^{\infty,0}} \right) g(\tau) \, d\tau.
\]
Then Gronwall’s inequality implies that
\[
g(t) \leq g(\epsilon_0) \exp \left( C \int_{\epsilon_0}^t \left( 1 + \|u(t)\|_{L^{\infty,0}} \right) \, d\tau \right),
\]
for all $\epsilon_0 \leq t < T$. Thus, we get the estimate (4.3) in the form
\[
\sup_{\epsilon_0 \leq t \leq T} \|u(t)\|_{H^{0,1}} \leq \left( e + \|u(\epsilon_0)\|_{H^{0,1}} \right) \exp \left( C T + C \int_{\epsilon_0}^T \|u(t)\|_{L^{\infty,0}} \, dr \right).
\]

(ii) By the same argument as in the above proof, it suffices to establish the following a priori estimate:
\[
\sup_{\epsilon_0 \leq t \leq T} \|u(t)\|_{H^{0,1}} \leq C \left( n, s, T, \|u(\epsilon_0)\|_{H^{0,1}} \int_{\epsilon_0}^T \|u(t)\|_{L^{\infty,0}}^2 \, dr \right)
\]
for some $\epsilon_0 \in (0, T)$. Applying $\partial^k$ with $|k| = 0, 1, \cdots, |s| + 1$ to (N-S), we have
\[
\partial_t v_k - \Delta v_k + u \cdot \nabla v_k + \nabla q_k = G_k,
\]
where $v_k := \partial^k u$, $q_k := \partial^k \pi$ and $G_k := -\sum_{l \leq k, |l| = |k| - 1} \left( \frac{k}{l} \right) \partial^{k-l} u \cdot \nabla (\partial^l u)$. Testing (4.11) with $v_k$ and integrating the resulting identity on the time interval $(\epsilon_0, t)$, we obtain
\[
\|v_k(t)\|_2^2 + 2 \int_{\epsilon_0}^t \|\nabla v_k\|_2^2 \, dr \leq \|v_k(\epsilon_0)\|_2^2 + 2 \int_{\epsilon_0}^t \|G_k, v_k\|_2 \, dr, \quad \epsilon_0 \leq t < T.
\]
Now the bilinear estimate (4.2) with $p = 2$, $|\alpha| = |k| - |l|$, $|\beta| = |l| + 1$, implies that
\[
\|G_k\|_2 \leq C\|u\|_{BMO} (-\Delta)^{|k|+1} u_2. \tag{4.13}
\]
From (4.13) and Young’s inequality we conclude that
\[
|G_k, v_k| \leq \|G_k\|_2 \|v_k\|_2 \leq C\|u\|_{BMO} (-\Delta)^{|k|+1} u_2 \|v_k\|_2 \\
\leq C\|u\|_{BMO}^2 \|v_k\|_2 + \frac{1}{2} \|\nabla v_k\|_2^2,
\]
with $C = C(n, s)$. Inserting (4.14) to the right-hand side of (4.12) and summing for $|k| = 0, 1, \cdots, |s| + 1$, we obtain that
\[
\|u(t)\|_{H^{|s|+1}}^2 \leq \|u(\varepsilon_0)\|_{H^{|s|+1}}^2 + C \int_{\varepsilon_0}^{t} \|u(\tau)\|_{BMO}^2 \|u(\tau)\|_{H^{|s|+1}}^2 d\tau,
\]
for all $\varepsilon_0 \leq t < T$. By using Gronwall’s inequality and then the continuous embedding $B_{\varepsilon_0,2} \subset BMO$, we get
\[
\|u(t)\|_{H^{|s|+1}} \leq \|u(\varepsilon_0)\|_{H^{|s|+1}} \exp \left( C \int_{\varepsilon_0}^{t} \|u(\tau)\|_{BMO}^2 d\tau \right) \\
\leq \|u(\varepsilon_0)\|_{H^{|s|+1}} \exp \left( C \int_{\varepsilon_0}^{t} \|u(\tau)\|_{B_{\varepsilon_0,2}}^2 d\tau \right)
\]
(4.15)
Now, by applying the logarithmic interpolation inequality (2.6) with $s_1 = -n/2 < s_0 = 0 < s_2 = s - n/2$, $\beta = 1/2$, $p = \infty$ and $\sigma = 2$ to $f = u(\tau)$, it follows that
\[
\|u(\tau)\|_{B_{\varepsilon_0,2}} \leq C \left( 1 + \|u(\tau)\|_{B_{\varepsilon_0,\infty}} \log^\beta \left( e + \|u(\tau)\|_{B_{\varepsilon_0,\infty}} \right) \right).
\]
(4.16)
Here, we note that $\tilde{U}_{\varepsilon_0,1/2,2} \equiv \psi_{\varepsilon_0,\infty,2}$ holds due to Proposition 2.3 (iii). Hence, combining (4.15), (4.16) and (4.9), it holds that
\[
\|u(t)\|_{H^{|s|+1}} \leq \|u(\varepsilon_0)\|_{H^{|s|+1}} \exp \left( C \int_{\varepsilon_0}^{t} \left( 1 + \|u(\tau)\|_{\tilde{U}_{\varepsilon_0,\infty,2}} \log(e + \|u(\tau)\|_{H^{|s|+1}}) \right) d\tau \right),
\]
where $C = C(n, s)$. Therefore, letting $g(t) \equiv \log(e + \|u(t)\|_{H^{|s|+1}})$, we obtain
\[
g(t) \leq g(\varepsilon_0) + C \int_{\varepsilon_0}^{t} \left( 1 + \|u(\tau)\|_{\tilde{U}_{\varepsilon_0,\infty,2}} g(\tau) \right) d\tau,
\]
which by Gronwall’s inequality implies that
\[
g(t) \leq g(\varepsilon_0) \exp \left( C \int_{\varepsilon_0}^{t} \left( 1 + \|u(\tau)\|_{\tilde{U}_{\varepsilon_0,\infty,2}} \right) d\tau \right)
\]
for all $\varepsilon_0 \leq t < T$. Thus, we get the estimate
\[
\sup_{\varepsilon_0 \leq t < T} \|u(t)\|_{H^{|s|+1}} \leq \left( e + \|u(\varepsilon_0)\|_{H^{|s|+1}} \right) \exp \left( C T + C \int_{\varepsilon_0}^{t} \|u(\tau)\|_{\tilde{U}_{\varepsilon_0,\infty,2}} d\tau \right),
\]
which is the desired estimate (4.10).

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References


