Research Article

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Compact Sobolev-Slobodeckij embeddings and positive solutions to fractional Laplacian equations

<https://doi.org/10.1515/anona-2020-0133>

Received May 6, 2020; accepted April 7, 2021.

Abstract: In this work, we study the existence of a positive solution to an elliptic equation involving the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$, for $n \geq 2$, such as

$$(-\Delta)^s u + E(x)u + V(x)u^{q-1} = K(x)f(u) + u^{2_s^*-1}. \quad (0.1)$$

Here, $s \in (0, 1)$, $q \in [2, 2_s^*)$ with $2_s^* := \frac{2n}{n-2s}$ being the fractional critical Sobolev exponent, $E(x), K(x), V(x) > 0 : \mathbb{R}^n \rightarrow \mathbb{R}$ are measurable functions which satisfy joint “vanishing at infinity” conditions in a measure-theoretic sense, and $f(u)$ is a continuous function on $\mathbb{R}$ of quasi-critical, super- q -linear growth with $f(u) \geq 0$ if $u \geq 0$. Besides, we study the existence of multiple positive solutions to an elliptic equation in $\mathbb{R}^n$ such as

$$(-\Delta)^s u + E(x)u + V(x)u^{q-1} = \lambda K(x)u^{r-1}, \quad (0.2)$$

where $2 < r < q < \infty$ (both possibly (super-)critical), $E(x), K(x), V(x) > 0 : \mathbb{R}^n \rightarrow \mathbb{R}$ are measurable functions satisfying joint integrability conditions, and $\lambda > 0$ is a parameter. To study (0.1)-(0.2), we first describe a family of general fractional Sobolev-Slobodeckij spaces $M^{s,q,p}(\mathbb{R}^n)$ as well as their associated compact embedding results.

Keywords: Sobolev-Slobodeckij spaces, compact embedding, fractional Laplacian, positive solutions

MSC: Primary 46E35, 35R11. Secondary 35J20, 35P05

Dedicated to In loving memory of my father Jiaying Han (June 5, 1952–April 9, 2019).

1 Introduction

In this paper, we are interested in seeking a positive solution to

$$(-\Delta)^s u + E(x)u + V(x)u^{q-1} = K(x)f(u) + u^{2_s^*-1} \quad (1.1)$$

in $\mathbb{R}^n$ when $n \geq 2$. Here, and henceforth, $(-\Delta)^s$ denotes the fractional Laplace operator that, up to a normalization constant and for $0 < s < 1$, is defined to be

$$(-\Delta)^s u(x) := -\frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \quad (1.2)$$

$2 \leq q < 2_s^*$ with $2_s^* := \frac{2n}{n-2s}$ the fractional critical Sobolev index, while $E(x), K(x), V(x) > 0 : \mathbb{R}^n \rightarrow \mathbb{R}$ and $f(u) : \mathbb{R} \rightarrow \mathbb{R}$ are functions satisfying the conditions below.

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(EKV1) $K(x) \in L^\infty(\mathbb{R}^n)$, $\inf_{\mathbb{B}_R} V(x) \geq V_{\mathbb{B}_R} > 0$ for each $R > 0$, while for all $c > 0$ and some $\vartheta \in (q, 2_s^*)$, $\mathfrak{L}\left(\left\{x \in \mathbb{R}^n : K(x)E^{-\left(\frac{2_s^*-q}{2_s^*-2}-\tau\frac{2_s^*-q}{2_s^*-2}\right)}(x)V^{-\tau}(x) \geq c\right\}\right) < \infty$ with $\tau \in [0, \frac{2_s^*-q}{2_s^*-q}]$.

(f1) $f(u)$ is continuous and satisfies $\lim_{u \rightarrow 0^+} \frac{f(u)}{u^{q-1}} = \lim_{u \rightarrow \infty} \frac{f(u)}{u^{2_s^*-1}} = 0$.

(f2) $f(u) \geq A_\vartheta u^{\vartheta-1} \geq 0$ when $u \geq 0$, where $A_\vartheta > \max\left\{2, 2^{\frac{\vartheta}{2}}\left(\frac{s\vartheta}{n(\vartheta-2)\int_{\mathbb{R}^n} K V_0^\vartheta dx} S^{\frac{n}{2s}}\right)^{\frac{2-\vartheta}{2}}\right\} > 0$ is a constant, $v_0 > 0$ is a mountain pass solution to

$$(-\Delta)^s u + E(x)u + V(x)u^{q-1} = K(x)u^{\vartheta-1}, \quad (1.3)$$

and S represents the best fractional critical Sobolev embedding constant.

(f3) There is some $\theta \in (q, 2_s^*)$ such that, for $F(u) := \int_0^u f(v) dv$ with $u, v \geq 0$,

$$K(x)(uf(u) - \theta F(u)) \geq -\eta_1 K_1(x)u^{r_1} - \eta_2 K_2(x)u^{r_2}, \quad (1.4)$$

where $r_1 \in [1, 2_s^*)$ and $r_2 \in [1, 2)$, $K_1(x), K_2(x) > 0 : \mathbb{R}^n \rightarrow \mathbb{R}$ are functions satisfying $K_1^{\frac{2_s^*}{2_s^*-r_1}}(x) \in L^1(\mathbb{R}^n)$ and $K_2^{\frac{2_s^*+\kappa(2_s^*-q)}{2_s^*-r_2}}(x)V^{-\kappa}(x) \in L^1(\mathbb{R}^n)$ with $\kappa \in [0, \frac{r_2}{q-r_2}]$, while $0 \leq \eta_1 \leq \left(\frac{2 \cdot 2_s^* r_1 s \theta}{3n(2_s^*-r_1)} S^{\frac{n}{2s}}\right)^{\frac{2_s^*-r_1}{2_s^*}} \cdot \frac{(2_s^*-\theta)^{\frac{r_1}{2_s^*}}}{6r_1 C_{K_1}}$ and $0 \leq \eta_2 \leq \min\left\{\left(\frac{4r_2 s \theta}{3n(2-r_2)} S^{\frac{n}{2s}}\right)^{\frac{2-r_2}{2}} \cdot \frac{(\theta-2)^{\frac{r_2}{2}}}{6r_2 C_{K_2}}, \left(\frac{q r_2 s \theta}{9n(q-r_2)} S^{\frac{n}{2s}}\right)^{\frac{q-r_2}{q}} \cdot \frac{(\theta-q)^{\frac{r_2}{q}}}{r_2 C_{K_2}}\right\}$ are some constants with $C_{K_1} := \left(\int_{\mathbb{R}^n} K_1^{\frac{2_s^*}{2_s^*-r_1}}(x) dx\right)^{\frac{2_s^*-r_1}{2_s^*}} > 0$ and $C_{K_2} > 0$ a constant that depends on some embedding constants (see Section 5 for details).

Note the hypothesis $A_\vartheta > 2$ in condition **(f2)** guarantees that

$$0 \leq \sup_{t \geq 0} \left\{ \frac{t^2}{2} + \frac{t^q}{q} - A_\vartheta \frac{t^\vartheta}{\vartheta} \right\} = \sup_{0 \leq t \leq 1} \left\{ \frac{t^2}{2} + \frac{t^q}{q} - A_\vartheta \frac{t^\vartheta}{\vartheta} \right\} \leq \sup_{t \geq 0} \left\{ t^2 - A_\vartheta \frac{t^\vartheta}{\vartheta} \right\}. \quad (1.5)$$

It may be helpful to demonstrate $f(u)$ using an example. For $\varepsilon > 0$ small, set

$$f(u) := \begin{cases} c_1(q+\varepsilon)u^{q+\varepsilon-1} + c_2 \frac{2_s^* u^{2_s^*-1} \ln(1+u) + 2_s^* u^{2_s^*} \ln(1+u) - u^{2_s^*}}{(1+u) \ln^2(1+u)} \geq 0 & \text{if } u \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

for appropriate constants $c_1, c_2 > 0$ to observe $F(u) = \begin{cases} c_1 u^{q+\varepsilon} + c_2 \frac{u^{2_s^*}}{\ln(1+u)} & \text{if } u \geq 0 \\ 0 & \text{otherwise} \end{cases}$. So, one derives

$$uf(u) - \theta F(u) = -c_1(\theta - q - \varepsilon)u^{q+\varepsilon} + c_2(2_s^* - \theta) \frac{u^{2_s^*}}{\ln(1+u)} - c_2 \frac{u^{2_s^*+1}}{(1+u) \ln^2(1+u)}$$

for $u \geq 0$. When u is small, one has $\ln(1+u) \doteq u$ and thus

$$uf(u) - \theta F(u) \geq -c_1(\theta - q - \varepsilon)u^{q+\varepsilon} + c_2(2_s^* - \theta - 1)u^{2_s^*-1} \geq -\eta_2 u^{r_2}$$

for some small $\eta_2 > 0$ if $2_s^* - 1 \geq q + \varepsilon$, since $q \geq 2 > r_2 \geq 1$; when u is large, one has

$$uf(u) - \theta F(u) \geq -c_1(\theta - q - \varepsilon)u^{q+\varepsilon} + c_2(2_s^* - \theta - 1)u^{2_s^*-\varepsilon} \geq -\eta_1 u^{r_1}$$

for some small $\eta_1 > 0$ if $2_s^* - q \geq \max\{1, 2\varepsilon\}$, provided that $q + \varepsilon < r_1 < 2_s^*$ in the case where $2_s^* - \theta - 1 \geq 0$ and $2_s^* - \varepsilon < r_1 < 2_s^*$ in the case where $2_s^* - \theta - 1 < 0$. As a consequence, our condition (1.4) is fulfilled upon the introduction of $K_1(x), K_2(x) \geq K(x) > 0$.

Since the seminal work by Brézis and Nirenberg [17], there has been a lot of research on the existence of positive solutions to elliptic problems involving the critical Sobolev exponent; see also the renowned papers of Ambrosetti, García-Azorero and Peral [5], and Benci and Cerami [12]. Problem (1.1) is motivated by the recent paper of do Ó, Miyagaki and Squassina [26], but our approach applies the insightful idea from García-Azorero and Peral [27] using the profound concentration-compactness principle of Lions [35]; see Palatucci

and Pisante [39], and Bonder, Saintier and Silva [14] in the nonlocal setting. The authors in [26] used the ingenious approach of Caffarelli and Silvestre [18] which transfers a fractional Laplacian problem on $\mathbb{R}^n$ to a locally possibly degenerate Laplacian problem on $\mathbb{R}^n \times \mathbb{R}^+$ via the s -harmonic extension; the relations of $K(x)$, $V(x)$ in [26] depend essentially on those of Alves and Souto [2], but ours are different. Finally, our condition (1.4) utilizes the embedding results and we have

$$uf(u) - \theta F(u) \geq -\eta_1 K_1(x) K^{-1}(x) u^{r_1} - \eta_2 K_2(x) K^{-1}(x) u^{r_2};$$

noting that the right-hand side term is nonpositive, (1.4) is indeed weaker than the well-known condition of Ambrosetti-Rabinowitz that demands $uf(u) \geq \theta F(u) > 0$.

Now, our first main result can be formulated as follows.

Theorem 1.1. *Suppose $n \geq 2$, $0 < s < 1$, $2 \leq q < 2_s^*$, $E(x), K(x), V(x) > 0 : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfy condition (EKV1), and $f(u) : \mathbb{R} \rightarrow \mathbb{R}$ satisfies conditions (f1)-(f3). Then, equation (1.1) has a positive solution.*

Below, we proceed to discuss the existence of multiple positive solutions to

$$(-\Delta)^s u + E(x)u + V(x)u^{q-1} = \lambda K(x)u^{r-1}, \quad (1.6)$$

where $2 < r < q < \infty$ both possibly being (super-)critical, $E(x), K(x), V(x) > 0 : \mathbb{R}^n \rightarrow \mathbb{R}$ are functions satisfying the subsequent conditions, and $\lambda > 0$ is a parameter.

(EKV2-1) $2 < r < \min\{2_s^*, q\}$, $K(x) \in L_{loc}^\infty(\mathbb{R}^n)$, $\inf_{\mathbb{B}_R} V(x) \geq V_{\mathbb{B}_R} > 0$ for each $R > 0$, while for some $\beta \in [0, \infty)$

and $\kappa \in \left(\frac{\beta(r-2)}{q-r} + \frac{r-2}{q-r} \cdot \frac{n}{2s}, \frac{\beta(r-2)}{q-r} + \frac{r}{q-r}\right]$, $K^{\psi_{2_s^*}(\beta, \kappa)}(x) E^{-\beta}(x) V^{-\kappa}(x) \in L^1(\mathbb{R}^n)$ with $\psi_{2_s^*}(\beta, \kappa) := \frac{2_s^* + \beta(2_s^* - 2) + \kappa(2_s^* - q)}{2_s^* - r}$.

(EKV2-2) $2_s^* \leq r < q$, $K(x) \in L_{loc}^\infty(\mathbb{R}^n)$, $\inf_{\mathbb{B}_R} V(x) \geq V_{\mathbb{B}_R} > 0$ for each $R > 0$, while for some $\beta \in [0, \infty)$ and

$\kappa \in \left[\frac{\beta(r-2)}{q-r} + \frac{r}{q-r}, \frac{\beta(r-2)}{q-r} + \frac{r-2}{q-r} \cdot \frac{n}{2s}\right)$, $K^{\psi_{2_s^*}(\beta, \kappa)}(x) E^{-\beta}(x) V^{-\kappa}(x) \in L^1(\mathbb{R}^n)$.

(EK) $2 < r < \min\{2_s^*, q\}$, $\inf_{\mathbb{B}_R} E(x) \geq E_{\mathbb{B}_R} > 0$ for each $R > 0$, $K(x) \in L^\infty(\Omega)$ for every subset $\Omega \subsetneq \mathbb{R}^n$ with

$\mathcal{L}(\Omega) < \infty$, and $\mathcal{L}\left(\left\{x \in \mathbb{R}^n : K(x) E^{-\frac{2_s^* - r}{2_s^* - 2}}(x) \geq c\right\}\right) < \infty$ for all $c > 0$.

Since the celebrated paper of Ambrosetti, Brézis and Cerami [3] on the concave and convex model problem, there has been a lot of studies on this topic. In 1996, Alama and Tarantello [1] considered an elliptic problem of different type that is the origin of all the research on equation (1.6). It seems to the author that Chabrowski [20] was the first person extending a result in [1] to $\mathbb{R}^n$; ever since, other important results appeared such as the renowned papers of Pucci and Rădulescu [42, 44] where for the first time the multiplicity results were discussed; Autuori and Pucci [8] later extended this research to the nonlocal setting with the same equation as (1.6) here (see also the closely related paper [7]). In this sense, our study continues their work, as we further utilize the compact embedding results for the functions $E(x)$, $K(x)$, $V(x)$, not merely those for $K(x)$ and/or $V(x)$. See [31] for more details and references.

Of course, the authors in [7, 8] used Hardy's potential for $E(x)$; here, we shall fully employ $E(x)$ by inscribing it into the compact embedding results with $K(x)$, $V(x)$, which include the one where $\beta = 0$, $\kappa = \frac{r}{q-r}$ and $\psi_{2_s^*}(0, \frac{r}{q-r}) = \frac{q}{q-r}$ in [7, 8] as a special case. Other two closely related important work are from Pucci and Zhang [43], and Rădulescu, Xiang and Zhang [45], where a weaker condition corresponding to $\beta = 0$, $\kappa = \frac{r-2}{q-r} \cdot \frac{n}{2s}$ and $\psi_{2_s^*}(0, \frac{r-2}{q-r} \cdot \frac{n}{2s}) = \frac{q-2}{q-r} \cdot \frac{n}{2s}$ was used. Notice both conditions come from those in [1]. In addition, our condition (EK) here is broader than any of those appeared in the associated earlier work.

Now, our second main result can be formulated as follows.

Theorem 1.2. *Assume that $n \geq 2$, $0 < s < 1$, $2 < r < q < \infty$ both possibly being (super-)critical, and $E(x), K(x), V(x) > 0 : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfy conditions (EKV2-1) or (EKV2-2). Then, there is a $\lambda_1 \geq 0$ such that equation (1.6) has at least a positive solution for all $\lambda > \lambda_1$. In addition to condition (EKV2-1), when condition (EK) is true, then $\lambda_1 > 0$ and equation (1.6) has at least a positive solution if and only if $\lambda \geq \lambda_1$; moreover, there exists a $\lambda_2 (\geq \lambda_1 > 0)$ such that equation (1.6) has at least two distinct positive solutions for all $\lambda > \lambda_2$.*

Note in problems (1.1) and (1.6), we are interested in the situation where $E(x)$, $K(x)$, $V(x)$ vanish at infinity, as initiated by Ambrosetti, Felli and Malchiodi in their celebrated work [4].

The paper is organized as follows: Section 2 analyzes a family of general fractional Sobolev-Slobodeckij spaces $M^{s;q,p}(\mathbb{R}^n)$, Section 3 describes a version of the fractional Moser-Trudinger inequality when $sp = n$, Section 4 discusses some compact embedding results that are particularly useful for our studies here of equations (1.1) and (1.2), while Sections 5 and 6 are devoted to the detailed proofs of Theorems 1.1 and 1.2 respectively.

An excellent reference for the approaches involved in this paper is the recent monograph of Papageorgiou, Rădulescu and Repovš [40] for nonlinear analysis in large.

It is worthwhile to remark finally that our conditions fully utilize the (compact) embedding results both in the upper-triangle case and the lower-triangle case (see Section 4 for definition) in each individual problem, which seems new even in the Laplacian setting.

2 Function space analysis

Recently, nonlocal equations and the associated fractional Sobolev function spaces $W^{s,p}(\mathbb{R}^n)$ attracted great attentions due to their impressive applications to real-world problems in various disciplines as briefed, for example, in the introduction from Di Nezza, Palatucci and Valdinoci [23]; see also Demengel and Demengel [22, Chapter 4], or Molica Bisci, Rădulescu and Servadei [37, Chapters 1-2] for some concise, self-contained treatment of this matter.

In his classical monograph [36, Section 5.1.1], Maz'ya defined a class of general Sobolev function spaces $W_{p,q}^1(\mathbb{R}^n)$; using the symbol $M^{q,p}(\mathbb{R}^n)$, the author analyzed some relations among those spaces and their compact embedding results in [28–31] hoping to provide a complement to Lemma I.1 of Lions [34] in the function space setting.

In the desire of furnishing a bit more insight regarding the similarity between $W^{1,p}(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n)$ as seen in [22, 23, 37], Sections 2-4 here are devoted to the description of a family of general fractional Sobolev-Slobodeckij spaces $M^{s;q,p}(\mathbb{R}^n)$ when $n \geq 2$, their relations and their compact embedding results that are akin to those observed for $M^{q,p}(\mathbb{R}^n)$.

In the sequel, we always assume $n \geq 2$, $1 \leq p, q \leq \infty$ and $0 < s < 1$.

Denote by Ω a bounded domain in $\mathbb{R}^n$ having a compact, Lipschitz boundary $\partial\Omega$, or $\Omega = \mathbb{R}^n$. Define $M^{s;q,p}(\Omega)$ to be the Banach space as the completion of the set $C^1(\Omega)$ if Ω is bounded, or as that of the set $C_c^1(\mathbb{R}^n)$ for $\Omega = \mathbb{R}^n$, with respect to the norm

$$\|u\|_{M^{s;q,p}(\Omega)} := [u]_{s,p,\Omega} + \|u\|_{q,\Omega}, \quad (2.1)$$

where $[u]_{s,p,\Omega}^p := \int_{\Omega} \int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dx dy$ and $\|u\|_{q,\Omega}^q := \int_{\Omega} |u|^q dx$.

Note $W^{s,p}(\Omega) = M^{s;p,p}(\Omega)$, and $M^{s;q,\infty}(\Omega) = C^{0,s}(\Omega)$ independent of q .

From now on, we shall write both continuous embedding of function spaces and convergence of functions by “ $\rightarrow$ ”, compact embedding of function spaces by “ $\hookrightarrow$ ”, and weak convergence of functions by “ $\rightharpoonup$ ”. Other notations will be clarified when appropriate.

Below, we will embark on our detailed analysis for the spaces $M^{s;q,p}(\Omega)$.

2.1 Ω is a bounded domain having a compact, Lipschitz boundary $\partial\Omega$.

In this case, it is trivial to see $M^{s;q,p}(\Omega) \rightarrow L^1(\Omega)$; so, via the fractional Poincaré's inequality of Bellido and Mora-Corral [11, Lemma 3.1], we have $M^{s;q,p}(\Omega) \rightarrow W^{s,p}(\Omega)$ through Minkowski's inequality. This, together with [22, Corollary 4.53], leads to the conclusions as follows.

(1) When $sp < n$, we have, with $p_s^* := \frac{np}{n-sp}$,

$$\begin{aligned} M^{s;q,p}(\Omega) &= W^{s,p}(\Omega) \quad \text{for } 1 \leq q \leq p_s^*, \\ M^{s;q_1,p}(\Omega) &\rightarrow M^{s;q_2,p}(\Omega) \quad \text{for } p_s^* \leq q_2 \leq q_1 \leq \infty. \end{aligned} \quad (2.2)$$

Also, there exists a constant $C_{p,q} > 0$, depending on n, p, q, s, Ω , such that

$$\|u\|_{W^{s,p}(\Omega)} \leq C_{p,q} \|u\|_{M^{s;q,p}(\Omega)}, \quad \forall u \in M^{s;q,p}(\Omega) \text{ and } q \in [1, \infty]. \quad (2.3)$$

(2) When $sp = n$, we have

$$M^{s;\infty,p}(\Omega) \rightarrow M^{s;q,p}(\Omega) = W^{s,p}(\Omega) \quad \text{for } 1 \leq q < \infty, \quad (2.4)$$

and $M^{s;q,p}(\Omega)$ in general cannot be a proper subspace of $L^\infty(\Omega)$ unless $q = \infty$. The fractional Moser-Trudinger inequality of Parini and Ruf [41] says

$$\sup_{[u]_{s,p,\mathbb{R}^n} \leq 1} \int_{\Omega} e^{\alpha|u|^{\frac{n}{n-s}}} dx \leq C_{\Omega}, \quad \forall u \in \tilde{W}_0^{s,p}(\Omega) \text{ and } \alpha \in [0, \alpha_*]. \quad (2.5)$$

Here, $\alpha_*, C_{\Omega} > 0$ are constants of n, p, s, Ω , and $\tilde{W}_0^{s,p}(\Omega)$, as a proper subspace of $W^{s,p}(\Omega)$, is the completion of $C_c^1(\Omega)$ with respect to the Gagliardo semi-norm $[u]_{s,p,\mathbb{R}^n}$ which is equivalent to (2.1) on $\mathbb{R}^n$ for $C_c^1(\Omega)$ by Brasco, Lindgren and Parini [16, Remark 2.5].

(3) When $sp > n$, we have, with $\gamma_s^* := \frac{sp-n}{p}$,

$$M^{s;q,p}(\Omega) = W^{s,p}(\Omega) \rightarrow C^{0,\gamma}(\Omega) \quad \text{for } 0 \leq \gamma \leq \gamma_s^* \text{ and } 1 \leq q \leq \infty. \quad (2.6)$$

Next, by [22, Theorem 4.54] or [23, Theorem 7.1], one easily verifies the result below.

Proposition 2.1. Assume $n \geq 2, 1 \leq p, q \leq \infty$ and $0 < s < 1$. When $sp < n$, the embedding $M^{s;q,p}(\Omega) \rightarrow L^r(\Omega)$ is continuous if $1 \leq r \leq \max\{p_s^*, q\}$ and compact if $1 \leq r < \max\{p_s^*, q\}$. When $sp \geq n$, the embedding $M^{s;q,p}(\Omega) \hookrightarrow L^r(\Omega)$ is compact if $1 \leq r < \infty$.

Proof. Viewing (2.2)-(2.4) and (2.6), we only need to consider the case $sp < n$ following almost identically the proofs of [22, Theorem 4.54] or [23, Theorem 7.1]. Actually, noticing that Ω is an extension domain, for any cube $Q(\supsetneq \Omega)$ and each $u \in M^{s;q,p}(\Omega)$, denote the extension of u to $W^{s,p}(\mathbb{R}^n)$ by $\tilde{u}$; then, for some constants $C'_{\Omega}, C'_{p,q} > 0$ depending on n, p, q, s, Ω ,

$$\|\tilde{u}\|_{W^{s,p}(Q)} \leq \|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C'_{\Omega} \|u\|_{W^{s,p}(\Omega)} \leq C'_{p,q} \|u\|_{M^{s;q,p}(\Omega)}$$

holds. So, our proof is finished using [22, Theorem 4.54] or [23, Theorem 7.1]. $\square$

Finally, recall for $sp > n$, in view of (2.6), one actually has

$$M^{s;q,p}(\Omega) \hookrightarrow C^{0,\gamma}(\Omega) \quad \text{for } 0 \leq \gamma < \gamma_s^* \text{ and } 1 \leq q \leq \infty.$$

On the other hand, from our preceding analysis and the remark at page 197 of [22], we have an equivalent definition of the space $M^{s;q,p}(\Omega)$, when $p, q \neq \infty$, such as

$$M^{s;q,p}(\Omega) := \{u : \Omega \rightarrow \mathbb{R} \text{ measurable with } [u]_{s,p,\Omega} + \|u\|_{q,\Omega} < \infty\}.$$

2.2 $\Omega = \mathbb{R}^n$.

Recall $M^{s;q,p}(\mathbb{R}^n)$ denotes the Banach space as the completion of the set $C_c^1(\mathbb{R}^n)$ in terms of the norm (2.1). Now, following Lieb and Loss [33, Sections 3.2 and 4.3], a function $u \in L_{loc}^1(\mathbb{R}^n)$ is said to *vanish at infinity*

provided $\mathcal{L}(\{x \in \mathbb{R}^n : |u(x)| \geq c\}) < \infty$ for all positive constants $c > 0$, where $\mathcal{L}$ represents the Lebesgue measure.

(1) When $sp < n$, denote by $D^{s,p}(\mathbb{R}^n)$ the space of functions $u \in L^1_{loc}(\mathbb{R}^n)$ where u vanish at infinity and $[u]_{s,p,\mathbb{R}^n} < \infty$. Then, a careful check for [23, Lemma 6.3 and Theorem 6.5] implies that $u \in L^{p^*}_s(\mathbb{R}^n)$, and the fractional Sobolev inequality reads as follows

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{\frac{n-sp}{n}} \leq C_1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy, \quad \forall u \in D^{s,p}(\mathbb{R}^n). \quad (2.7)$$

Here, $C_1 > 0$ is an absolute constant depending on n, p, s for $1 \leq p < \infty$.

By virtue of Dipierro and Valdinoci [24] with $a = 0$, $D^{s,p}(\mathbb{R}^n)$ can be equivalently defined to be the completion of $C_c^1(\mathbb{R}^n)$ through $[u]_{s,p,\mathbb{R}^n}$; so, $D^{s,p}(\mathbb{R}^n) = M^{s,p^*,p}(\mathbb{R}^n)$ and one recognizes an equivalent definition of the space $M^{s;q,p}(\mathbb{R}^n)$, for $q \neq \infty$, such as

$$M^{s;q,p}(\mathbb{R}^n) := \{u : \mathbb{R}^n \rightarrow \mathbb{R} \text{ measurable with } [u]_{s,p,\mathbb{R}^n} + \|u\|_{q,\mathbb{R}^n} < \infty\}.$$

From Chebyshev's inequality and the standard interpolation inequality, one has

$$M^{s;q_1,p}(\mathbb{R}^n) \rightarrow M^{s;q_2,p}(\mathbb{R}^n) \rightarrow D^{s,p}(\mathbb{R}^n) \text{ for } \begin{cases} \text{either } 1 \leq q_1 \leq q_2 \leq p_s^* \\ \text{or } p_s^* \leq q_2 \leq q_1 < \infty \end{cases}. \quad (2.8)$$

If one would like to include $q = \infty$, then u needs to be compactly supported.

The best fractional critical Sobolev embedding constant S_p is defined to be

$$S_p := \inf_{u \in D^{s,p}(\mathbb{R}^n) \setminus \{0\}} \frac{[u]_{s,p,\mathbb{R}^n}^p}{\|u\|_{p_s^*,\mathbb{R}^n}^p} \geq C_1^{-1} > 0. \quad (2.9)$$

As common practice, we write $D^s(\mathbb{R}^n) := D^{s,2}(\mathbb{R}^n)$ and $S := S_2$ in the sequel.

(2) When $sp = n$, the fractional Gagliardo-Nirenberg inequality of Nguyen and Squassina [38, Lemma 2.1] says

$$\|u\|_{r,\mathbb{R}^n} \leq C_2 \|u\|_{q,\mathbb{R}^n}^\zeta [u]_{s,p,\mathbb{R}^n}^{1-\zeta}, \quad \forall u \in M^{s;q,p}(\mathbb{R}^n). \quad (2.10)$$

Here, we only write a tailored version for our purpose with $1 \leq q \leq r < \infty$ and $\zeta := \frac{q}{r} \in (0, 1]$, and $C_2 > 0$ is an absolute constant depending on n, p, q, r, s . Thus, we have

$$M^{s;q_1,p}(\mathbb{R}^n) \rightarrow M^{s;q_2,p}(\mathbb{R}^n) \text{ for } 1 \leq q_1 \leq q_2 < \infty. \quad (2.11)$$

Notice here $q = \infty$ does not contribute to this continuous embedding (2.11).

(3) When $sp > n$, a careful check for Theorem 8.2 in [23], especially their estimates (8.3) and (8.9), suggests that $M^{s;q,p}(\mathbb{R}^n) \rightarrow C^{0,\gamma_s^*}(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)$. Therefore, one has

$$M^{s;q_1,p}(\mathbb{R}^n) \rightarrow M^{s;q_2,p}(\mathbb{R}^n) \rightarrow C^{0,\gamma_s^*}(\mathbb{R}^n) \text{ for } 1 \leq q_1 \leq q_2 \leq \infty. \quad (2.12)$$

Note $M^{s;q,\infty}(\mathbb{R}^n) \rightarrow C^{0,s}(\mathbb{R}^n)$ which is consistent with the observation at page 565 of [23], and $M^{s;\infty,p}(\mathbb{R}^n)$ is the largest possible function space for fixed p, s in (2.12).

Summarizing all the preceding analysis yields the result below.

Proposition 2.2. Assume $n \geq 2$, $1 \leq p, q < \infty$ and $0 < s < 1$. When $sp < n$, the embedding $M^{s;q,p}(\mathbb{R}^n) \rightarrow L^r(\mathbb{R}^n)$ is continuous if $\min\{p_s^*, q\} \leq r \leq \max\{p_s^*, q\}$. When $sp = n$ or $sp > n$, the embedding $M^{s;q,p}(\mathbb{R}^n) \rightarrow L^r(\mathbb{R}^n)$ is continuous if $q \leq r < \infty$ or $q \leq r \leq \infty$.

3 A fractional Moser-Trudinger inequality

In this section, we utilize [41] to provide an extension of the inequality (2.5) to $\mathbb{R}^n$ for functions in $M^{s,q,p}(\mathbb{R}^n)$ when $sp = n$. The proof presented here follows essentially from do Ó [25, Lemma 1] and Ruf [46, Proposition 2.1]. Consult also Li and Ruf [32], Han [30, Section 2.2], and several other important contributions in $\mathbb{R}^n$ that cannot be exhaustively listed here.

Below, we review some fundamental facts about Schwarz symmetrization based on Berestycki and Lions [13, Appendix III], and Lieb and Loss [33, Section 3.3]. Assume $f(x)$ is a measurable function vanishing at infinity in the measure-theoretic sense of Lieb and Loss, and let $f^*(x) \geq 0$ be its Schwarz symmetrization or spherical rearrangement through $\mathcal{L}(\{x \in \mathbb{R}^n : f^*(x) \geq c\}) = \mathcal{L}(\{x \in \mathbb{R}^n : |f(x)| \geq c\})$ for all positive constants $c > 0$. Then, f^* is unique, radial, decreasing in $|x|$, and lower semi-continuous (and therefore, measurable). Also, we have

$$\int_{\mathbb{R}^n} \Phi(f^*) \, dx = \int_{\mathbb{R}^n} \Phi(|f|) \, dx \quad (3.1)$$

for all continuous functions Φ with $\Phi(|f|)$ integrable; so, $\|f^*\|_{q,\mathbb{R}^n} = \|f\|_{q,\mathbb{R}^n}$ if $f \in L^q(\mathbb{R}^n)$ for $1 \leq q \leq \infty$. We recall the important Theorem 3 of Beckner [10] and observe

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f^*(x) - f^*(y)|^p}{|x - y|^{n+sp}} \, dx dy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx dy, \quad (3.2)$$

while moreover the Berestycki-Lions' radial lemma [13, Lemma A.IV] leads to

$$f^*(x) \leq |x|^{-\frac{n}{q}} \left(\frac{n}{\omega_{n-1}} \right)^{\frac{1}{q}} \|f^*\|_{q,\mathbb{R}^n}, \quad \forall f^* \in L^q(\mathbb{R}^n), \quad (3.3)$$

with $x \neq 0$, where ω_{n-1} represents the surface area of the unit sphere in $\mathbb{R}^n$.

It is notable that neither in [13, Appendix III] or [34, Lemma I.1], nor in [33, Section 3.3] or [10, Theorem 3] was the condition f being integrable needed.

Now, we characterize an inequality of fractional Moser-Trudinger type in the function space $M^{s,q,p}(\mathbb{R}^n)$ using its full norm, instead of the Gagliardo semi-norm $[u]_{s,p,\mathbb{R}^n}$.

Theorem 3.1. *Let $n \geq 2$, $1 \leq p, q < \infty$, $0 < s < 1$, $sp = n$, $0 \leq \alpha < \alpha_*$, and $u \in M^{s,q,p}(\mathbb{R}^n)$. Then, we have*

$$\sup_{\|u\|_{M^{s,q,p}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \Psi_{\alpha,q,s}(|u|) \, dx \leq C(\alpha, q, s). \quad (3.4)$$

Here, $C(\alpha, q, s) > 0$ is an absolute constant that depends on α, n, p, q, s , and for $v \geq 0$ and the least positive integer l_0 with $\frac{l_0 n}{n-s} \geq q \geq 1$, we set $\Psi_{\alpha,q,s}(v) := \sum_{l=l_0}^{\infty} \frac{\alpha^l}{l!} v^{\frac{ln}{n-s}}$.

Proof. To save notation, without loss of generality, take $u \geq 0$. Recalling (3.1) and (3.2), and noticing that $\Psi_{\alpha,q,s}(u^*) \geq 0$ for u^* , we may simply consider $u = u^*$ and decompose

$$\int_{\mathbb{R}^n} \Psi_{\alpha,q,s}(u) \, dx = \int_{\mathbb{B}_{R_0}} \Psi_{\alpha,q,s}(u) \, dx + \int_{\mathbb{B}_{R_0}^c} \Psi_{\alpha,q,s}(u) \, dx.$$

Here, and hereafter, $\mathbb{B}_R \subsetneq \mathbb{R}^n$ is the ball of radius R centered at the origin and $\mathbb{B}_R^c := \mathbb{R}^n \setminus \mathbb{B}_R$, while $R_0 > 0$ is a sufficiently large absolute constant to be determined later.

To estimate the integral over $\mathbb{B}_{R_0}$, write $v(x) := \max\{u(x) - u_0, 0\}$ for $u_0 := u(R_0) > 0$, a constant. Then, $v \equiv 0$ on $\mathbb{B}_{R_0}^c$ and $v \in \tilde{W}_0^{s,p}(\mathbb{B}_{R_0})$. In fact, as in [16, Lemma 2.4], for the ball $\mathbb{B}_{R_0}(x_0) \subsetneq \mathbb{B}_{R_0}^c$ of radius R_0 centered at a point x_0 with $|x_0| = 4R_0$, one has

$$v^p(x) = \frac{|v(x) - v(y)|^p}{|x - y|^{2n}} |x - y|^{2n} \text{ for } x \in \mathbb{B}_{R_0} \text{ and } y \in \mathbb{B}_{R_0}(x_0),$$

which further yields

$$\frac{\omega_{n-1} R_0^n}{n} v^p(x) \leq (\text{diam}(\mathbb{B}_{R_0}(x_0) \cup \mathbb{B}_{R_0}))^{2n} \int_{\mathbb{B}_{R_0}(x_0)} \frac{|v(x) - v(y)|^p}{|x - y|^{2n}} dy.$$

Thus, it follows that

$$\begin{aligned} \frac{\omega_{n-1}}{n(36R_0)^n} \int_{\mathbb{B}_{R_0}} v^p(x) dx &\leq \int_{\mathbb{B}_{R_0}} \int_{\mathbb{B}_{R_0}(x_0)} \frac{|v(x) - v(y)|^p}{|x - y|^{2n}} dy dx \\ &\leq \int_{\mathbb{B}_{R_0}} \int_{\mathbb{B}_{R_0}} \frac{|v(x) - v(y)|^p}{|x - y|^{2n}} dx dy + 2 \int_{\mathbb{B}_{R_0}} \int_{\mathbb{B}_{R_0}^c} \frac{|v(x) - v(y)|^p}{|x - y|^{2n}} dx dy \\ &= \int_{\mathbb{B}_{R_0}} \int_{\mathbb{B}_{R_0}} \frac{|u(x) - u(y)|^p}{|x - y|^{2n}} dx dy + 2 \int_{\mathbb{B}_{R_0}} \int_{\mathbb{B}_{R_0}^c} \frac{|u(x) - u_0|^p}{|x - y|^{2n}} dx dy; \end{aligned}$$

in view of $u(y) \leq u_0$ on $\mathbb{B}_{R_0}^c$, we have

$$\frac{\omega_{n-1}}{n(36R_0)^n} \|v\|_{p, \mathbb{B}_{R_0}}^p \leq [v]_{s,p, \mathbb{R}^n}^p \leq [u]_{s,p, \mathbb{R}^n}^p \leq 1.$$

Recall [25, Estimate (5)] says $(1+t)^{\frac{n}{n-s}} \leq t^{\frac{n}{n-s}} + C_3 t^{\frac{s}{n-s}} + 1$ for all $t \in (0, \infty)$ with $C_3 > 0$ an absolute constant. Actually, for

$$f(t) := \frac{(1+t)^{\frac{n}{n-s}} - t^{\frac{n}{n-s}} - 1}{t^{\frac{s}{n-s}}}$$

continuous on $(0, \infty)$, one observes $\lim_{t \rightarrow 0^+} f(t) = 0$ as $n \geq 2 > 2s$ and $\lim_{t \rightarrow \infty} f(t) = \frac{n}{n-s}$ through $\tilde{t} := \frac{1}{t}$. Hence, for $t = \frac{v(x)}{u_0}$, it follows that

$$u^{\frac{n}{n-s}} \leq (v + u_0)^{\frac{n}{n-s}} \leq v^{\frac{n}{n-s}} + C_3 v^{\frac{s}{n-s}} u_0 + u_0^{\frac{n}{n-s}}.$$

Apply Young's inequality to $C_3 v^{\frac{s}{n-s}} u_0$ to get, for an absolute constant $C'_3 > 0$,

$$u^{\frac{n}{n-s}} \leq v^{\frac{n}{n-s}} (1 + u_0^{\frac{n}{s}}) + C'_3 + u_0^{\frac{n}{n-s}}. \quad (3.5)$$

Set $\tilde{u} := v(1 + u_0^{\frac{n}{s}})^{\frac{n-s}{n}} \in \widetilde{W}_0^{s,p}(\mathbb{B}_{R_0})$ and recall $[v]_{s,p, \mathbb{R}^n}^p \leq [u]_{s,p, \mathbb{R}^n}^p$ to see

$$[\tilde{u}]_{s,p, \mathbb{R}^n}^p \leq (1 + u_0^{\frac{n}{s}})^{\frac{n-s}{s}} [u]_{s,p, \mathbb{R}^n}^p \leq (1 + u_0^{\frac{n}{s}})^{\frac{n-s}{s}} (1 - \|u\|_{q, \mathbb{R}^n})^{\frac{n}{s}}.$$

From (3.3), one has $u_0^{\frac{n}{s}} \leq \sigma \|u\|_{q, \mathbb{R}^n}^{\frac{n}{s}}$ for $\sigma := (\frac{n}{\omega_{n-1} R_0^n})^{\frac{n}{qs}}$. Define

$$g(t) := (1 + \sigma t^{\frac{n}{s}})^{\frac{n-s}{s}} (1 - t)^{\frac{n}{s}} - 1$$

with $g(0) = 0$ and $g(1) = -1$, and note

$$g'(t) = \frac{n}{s} (1 + \sigma t^{\frac{n}{s}})^{\frac{n-s}{s}} (1 - t)^{\frac{n-s}{s}} \left(\frac{n-s}{s} \sigma t^{\frac{n-s}{s}} - \frac{n}{s} \sigma t^{\frac{n}{s}} - 1 \right).$$

Write

$$h(t) := \frac{n-s}{s} \sigma t^{\frac{n-s}{s}} - \frac{n}{s} \sigma t^{\frac{n}{s}} - 1$$

with $h'(t_0) = 0$ at $t_0 := (\frac{n-s}{n})^{\frac{2}{s}}$. Thus, when R_0 is so large that $\sigma \leq (\frac{n-s}{n})^{\frac{2n-s}{s}}$, one deduces $h(t) \leq 0$ and hence $g'(t) \leq 0$ on $[0, 1]$. That is, $g(t) \leq 0$ for $0 \leq t \leq 1$, which clearly implies that $[\tilde{u}]_{s,p, \mathbb{R}^n} \leq 1$ provided $R_0^n \geq \frac{n}{\omega_{n-1}} (\frac{n-s}{n})^{\frac{q(2n-s)}{n}}$.

Thus, one may combine the preceding analysis with (2.5) and (3.5) to derive

$$\int_{\mathbb{B}_{R_0}} \Psi_{\alpha,q,s}(u) dx \leq \int_{\mathbb{B}_{R_0}} e^{au^{\frac{n}{n-s}}} dx \leq e^{aC'_3 + au_0^{\frac{n}{n-s}}} \int_{\mathbb{B}_{R_0}} e^{a\tilde{u}^{\frac{n}{n-s}}} dx \leq C_1(\alpha, q, s).$$

Here, $C_1(\alpha, q, s) > 0$ is an absolute constant depending on α, n, p, q, s .

To estimate the integral over $\mathbb{B}_{R_0}^c$, we use (3.3) with $\|u\|_{M^{s;q,p}(\mathbb{R}^n)} \leq 1$ to see

$$\begin{aligned} \int_{\mathbb{B}_{R_0}^c} u^{\frac{ln}{n-s}} dx &\leq \omega_{n-1} \left(\frac{n}{\omega_{n-1}} \right)^{\frac{ln}{q(n-s)}} \int_{R_0}^{\infty} t^{-\frac{ln^2}{q(n-s)} + n-1} dt \\ &\leq R_0^n \omega_{n-1} \left(\frac{n}{\omega_{n-1} R_0^n} \right)^{\frac{ln}{q(n-s)}} \frac{q(n-s)}{ln^2 - nq(n-s)} = R_0^n \omega_{n-1} \sigma^{\frac{ls}{n-s}} \frac{q(n-s)}{ln^2 - nq(n-s)}. \end{aligned}$$

As $\frac{l_0 n}{n-s} \geq q$, we apply Proposition 2.2 to the first term of $\Psi_{\alpha,q,s}(u)$ and observe

$$\int_{\mathbb{B}_{R_0}^c} \Psi_{\alpha,q,s}(u) dx \leq \frac{\alpha^{l_0}}{l_0!} \int_{\mathbb{B}_{R_0}^c} u^{\frac{l_0 n}{n-s}} dx + \frac{q(n-s)}{n^2} R_0^n \omega_{n-1} e^{\alpha \sigma^{\frac{s}{n-s}}} \leq C_2(\alpha, q, s).$$

Here, $C_2(\alpha, q, s) > 0$ is an absolute constant depending on α, n, p, q, s .

Therefore, (3.4) follows with $C(\alpha, q, s) := C_1(\alpha, q, s) + C_2(\alpha, q, s) > 0$. $\square$

It is worth to remark that Parini and Ruf [41] found an upper bound

$$\alpha_{s,n}^* := n \left(\frac{2\omega_{n-1}^2 \Gamma(p+1)}{n!} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n+2k)^p k!} \right)^{\frac{s}{n-s}}$$

for α^* utilizing renowned functions in analytic number theory with sophisticated calculations. Following Proposition 5.2 of [41], one observes a similar result as follows.

Proposition 3.2. Assume that $n \geq 2$, $1 \leq p, q < \infty$, $0 < s < 1$, $sp = n$, and $\alpha_{s,n}^* < \alpha < \infty$. Then, we have

$$\sup_{\|u\|_{M^{s;q,p}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \Psi_{\alpha,q,s}(|u|) dx = \infty. \quad (3.6)$$

Proof. Define the nonlocal/modified Moser function u_ε by $|\ln \varepsilon|^{1-\frac{s}{n}}$ if $|x| \leq \varepsilon$, $|\ln |x|| \cdot |\ln \varepsilon|^{-\frac{s}{n}}$ if $\varepsilon \leq |x| \leq 1$, and 0 if $|x| \geq 1$. Then, $\lim_{\varepsilon \rightarrow 0^+} [u_\varepsilon]_{s,p,\mathbb{R}^n}^p = \left(\frac{\alpha_{s,n}^*}{n} \right)^{\frac{n-s}{s}}$ as shown in Proposition 5.1 of [41] through delicate computations, and $\|u_\varepsilon\|_{q,\mathbb{R}^n}^q = O(|\ln \varepsilon|^{-\frac{sq}{n}})$ which is infinitesimal when $\varepsilon \rightarrow 0^+$. Set accordingly $v_\varepsilon := \frac{u_\varepsilon}{\|u_\varepsilon\|_{M^{s;q,p}(\mathbb{R}^n)}} \geq 0$ with $\|v_\varepsilon\|_{M^{s;q,p}(\mathbb{R}^n)} = 1$. When $\alpha > \alpha_{s,n}^*$ and ε is sufficiently small, $\alpha \|u_\varepsilon\|_{M^{s;q,p}(\mathbb{R}^n)}^{-\frac{n}{n-s}} = \alpha [u_\varepsilon]_{s,p,\mathbb{R}^n}^{-\frac{n}{n-s}} = \frac{\alpha n}{\alpha_{s,n}^*} > n$. So, one derives

$$\begin{aligned} \int_{\mathbb{R}^n} \Psi_{\alpha,q,s}(v_\varepsilon) dx &\geq \int_{\mathbb{B}_\varepsilon} \Psi_{\alpha,q,s}(v_\varepsilon) dx \\ &= \int_{\mathbb{B}_\varepsilon} e^{\alpha \|u_\varepsilon\|_{M^{s;q,p}(\mathbb{R}^n)}^{-\frac{n}{n-s}} |\ln \varepsilon|} dx - \sum_{l=0}^{l_0-1} \frac{\alpha^l}{l!} \int_{\mathbb{B}_\varepsilon} \|u_\varepsilon\|_{M^{s;q,p}(\mathbb{R}^n)}^{-\frac{ln}{n-s}} |\ln \varepsilon|^l dx \\ &\geq \frac{\omega_{n-1}}{n} \left(\varepsilon^{n-\frac{\alpha n}{\alpha_{s,n}^*}} - O(\varepsilon^n |\ln \varepsilon|^{l_0}) \right) \rightarrow \infty \text{ as } \varepsilon \rightarrow 0^+, \end{aligned}$$

which certainly implies the desired estimate (3.6) and thus finishes the proof. $\square$

Not like in the local case, it is not clear if $\alpha_{s,n}^*$ is optimal, yet, as pointed out in Remark 5.4 of [41], at least $\alpha_{s,2}^*$ coincides with the optimal exponent $\alpha_{1,2}^*$ up to the multiplicative constant appeared in the asymptotic behaviour of the Gagliardo semi-norm as $s \rightarrow 1^-$.

4 Some compact embedding results

In this section, we describe some compact embedding results from subspaces of $M^{s;q,p}(\mathbb{R}^n)$ or $D^{s,p}(\mathbb{R}^n)$ to $L_K^r(\mathbb{R}^n)$ that are useful when studying problems (1.1) and (1.6).

Some closely related results on $W^{1,p}(\mathbb{R}^n)$ may be found in Bartsch and Wang [9, Theorem 2.1], Chiappinelli [21, Theorem 1], Schneider [47, Theorem 2.3], Bonheure and Van Schaftingen [15, Theorem 4], and Han [30, Section 4.1] and [31, Propositions 2.3-2.4]; note nevertheless the results presented below seem not known in the literature, even on $W^{1,p}(\mathbb{R}^n)$.

Let $n \geq 2$, $1 \leq p, q < \infty$, $1 \leq \tilde{p} \leq p_s^*$, and $0 < s < 1$ with $sp < n$. Suppose $E(x), V(x) > 0$ with $\inf_{\mathbb{B}_R} V(x) \geq V_{\mathbb{B}_R} > 0$ for each $R > 0$ in the sequel. Denote by $M_{E,V}^{s;q,p}(\mathbb{R}^n)$ the Banach space as the completion of functions $u \in L_{loc}^1(\mathbb{R}^n)$ that vanish at infinity (measure-theoretically) with $[u]_{s,p,\mathbb{R}^n} < \infty$, or equivalently as that of $u \in C_c^1(\mathbb{R}^n)$, with respect to the norm

$$\|u\|_{M_{E,V}^{s;q,p}(\mathbb{R}^n)} := [u]_{s,p,\mathbb{R}^n} + \|u\|_{L_E^{\tilde{p}}(\mathbb{R}^n)} + \|u\|_{L_V^q(\mathbb{R}^n)} \quad (4.1)$$

for $\|u\|_{L_E^{\tilde{p}}(\mathbb{R}^n)}^{\tilde{p}} := \int_{\mathbb{R}^n} E|u|^{\tilde{p}} dx$ and $\|u\|_{L_V^q(\mathbb{R}^n)}^q := \int_{\mathbb{R}^n} V|u|^q dx$; so, $M_{E,V}^{s;q,p}(\mathbb{R}^n) \rightarrow D^{s,p}(\mathbb{R}^n)$.

Theorem 4.1. Assume that $n \geq 2$, $1 \leq p < \infty$, $0 < s < 1$ with $sp < n$, and $1 \leq \tilde{p}, q \leq r < p_s^*$. Let $E(x), K(x), V(x) > 0 : \mathbb{R}^n \rightarrow \mathbb{R}$ be Lebesgue measurable functions such that $K(x) \in L^t(\Omega)$ for some $t \in (\frac{p_s^*}{p_s^*-r}, \infty]$ on each set Ω of $\mathbb{R}^n$ having $\mathfrak{L}(\Omega) < \infty$ while $K(x)E^{-\phi_{p_s^*}(\tau)}(x)V^{-\tau}(x)$ vanishes at infinity with $\phi_{p_s^*}(\tau) := \frac{p_s^*-r}{p_s^*-\tilde{p}} - \tau \frac{p_s^*-q}{p_s^*-\tilde{p}}$ for some $\tau \in [0, \frac{p_s^*-r}{p_s^*-\tilde{p}}]$; so, $\phi_{p_s^*}(\tau) \in [0, \frac{p_s^*-r}{p_s^*-\tilde{p}}]$. Then, the embedding $M_{E,V}^{s;q,p}(\mathbb{R}^n) \hookrightarrow L_K^r(\mathbb{R}^n)$ is compact.

Proof. Without loss of generality, assume $\{u_l : l \geq 1\}$ is a sequence of functions in $M_{E,V}^{s;q,p}(\mathbb{R}^n)$ with $u_l \rightarrow 0$ when $l \rightarrow \infty$ and $\|u_l\|_{M_{E,V}^{s;q,p}(\mathbb{R}^n)}$ uniformly bounded. For each $\varepsilon > 0$, set $\mathbf{W}_\varepsilon := \{x \in \mathbb{R}^n : K(x)E^{-\phi_{p_s^*}(\tau)}(x)V^{-\tau}(x) \geq \varepsilon\}$ and $\mathbf{W}_\varepsilon^c := \mathbb{R}^n \setminus \mathbf{W}_\varepsilon$ to decompose

$$\int_{\mathbb{R}^n} K|u_l|^r dx = \int_{\mathbf{W}_\varepsilon} K|u_l|^r dx + \int_{\mathbf{W}_\varepsilon^c} K|u_l|^r dx.$$

For the integral over $\mathbf{W}_\varepsilon^c$, noting that $p_s^* = \frac{r-\tilde{p}\phi_{p_s^*}(\tau)-q\tau}{1-\phi_{p_s^*}(\tau)-\tau}$, it is easy to get

$$\begin{aligned} \int_{\mathbf{W}_\varepsilon^c} K|u_l|^r dx &= \int_{\mathbf{W}_\varepsilon^c} KE^{-\phi_{p_s^*}(\tau)} V^{-\tau} (E|u_l|^{\tilde{p}})^{\phi_{p_s^*}(\tau)} (V|u_l|^q)^\tau |u_l|^{r-\tilde{p}\phi_{p_s^*}(\tau)-q\tau} dx \\ &\leq \varepsilon \left(\int_{\mathbf{W}_\varepsilon^c} E|u_l|^{\tilde{p}} dx \right)^{\phi_{p_s^*}(\tau)} \left(\int_{\mathbf{W}_\varepsilon^c} V|u_l|^q dx \right)^\tau \left(\int_{\mathbf{W}_\varepsilon^c} |u_l|^{p_s^*} dx \right)^{1-\phi_{p_s^*}(\tau)-\tau} \\ &\leq \varepsilon C'_1 \|u_l\|_{L_E^{\tilde{p}}(\mathbb{R}^n)}^{\tilde{p}\phi_{p_s^*}(\tau)} \|u_l\|_{L_V^q(\mathbb{R}^n)}^{q\tau} [u_l]_{s,p,\mathbb{R}^n}^{r-\tilde{p}\phi_{p_s^*}(\tau)-q\tau} \leq \varepsilon C'_1 \|u_l\|_{M_{E,V}^{s;q,p}(\mathbb{R}^n)}^r \end{aligned} \quad (4.2)$$

using (2.7), with $C'_1 > 0$ an absolute constant independent of u_l for all $l \geq 1$.

For the integral over $\mathbf{W}_\varepsilon$, as $\mathfrak{L}(\mathbf{W}_\varepsilon) < \infty$ and $\inf_{\mathbb{B}_R} V(x) \geq V_{\mathbb{B}_R} > 0$, one sees from (2.3) and Proposition 2.1 that $M_{E,V}^{s;q,p}(\mathbb{B}_R) \rightarrow W^{s,p}(\mathbb{B}_R) \hookrightarrow L^{\frac{r}{r-1}}(\mathbb{B}_R)$ with $\frac{r}{r-1} \in [r, p_s^*)$, and thus

$$\begin{aligned} \int_{\mathbf{W}_\varepsilon} K|u_l|^r dx &= \int_{\mathbb{B}_R \cap \mathbf{W}_\varepsilon} K|u_l|^r dx + \int_{\mathbb{B}_R^c \cap \mathbf{W}_\varepsilon} K|u_l|^r dx \\ &\leq \left(\int_{\mathbf{W}_\varepsilon} K^t dx \right)^{\frac{1}{t}} \left\{ \left(\int_{\mathbb{B}_R} |u_l|^{\frac{r}{r-1}} dx \right)^{\frac{r-1}{r}} \right. \\ &\quad \left. + \left(\int_{\mathbb{R}^n} |u_l|^{p_s^*} dx \right)^{\frac{r}{p_s^*}} (\mathfrak{L}(\mathbb{B}_R^c \cap \mathbf{W}_\varepsilon))^{1-\frac{1}{t}-\frac{r}{p_s^*}} \right\} \rightarrow 0 \end{aligned} \quad (4.3)$$

as $R \rightarrow \infty$ and $l \rightarrow \infty$ for a subsequence $\{u_l : l \geq 1\}$ using the same notation.

So, (4.2)-(4.3) yields $\|u_l\|_{L_K^r(\mathbb{R}^n)} \rightarrow 0$ as $l \rightarrow \infty$ and thus finishes the proof. $\square$

Note the embedding $M_{E,V}^{s;q,p}(\mathbb{R}^n) \rightarrow L_K^r(\mathbb{R}^n)$ is continuous for $\frac{p_s^*}{p_s^*-r} \leq t \leq \infty$, and if $K(x) \in L^\infty(\mathbb{R}^n)$, then the integrability requirement of $K(x)$ is fulfilled automatically.

Theorem 4.2. Assume that $n \geq 2$, $1 \leq p < \infty$, $0 < s < 1$ with $sp < n$, and $1 \leq \tilde{p} \leq r < \min\{p_s^*, q\}$. Let $E(x), K(x), V(x) > 0 : \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $K(x) \in L_{loc}^t(\mathbb{R}^n)$ for some $t \in (\frac{\max\{p_s^*, q\}}{\max\{p_s^*, q\}-r}, \infty]$ and $K^{\psi_{p_s^*}(\beta, \kappa)}(x)E^{-\beta}(x)V^{-\kappa}(x) \in L^1(\mathbb{R}^n)$ with $\psi_{p_s^*}(\beta, \kappa) := \frac{p_s^*+\beta(p_s^*-\tilde{p})+\kappa(p_s^*-q)}{p_s^*-r}$ for some $\beta \in [0, \infty)$ and $\kappa \in [0, \frac{\beta(r-\tilde{p})+r}{q-r}]$. Then, the embedding $M_{E,V}^{s;q,p}(\mathbb{R}^n) \hookrightarrow L_K^r(\mathbb{R}^n)$ is compact.

Proof. Define $\mathfrak{w} := \frac{\beta(p_s^*-r)}{p_s^*+\beta(p_s^*-\tilde{p})+\kappa(p_s^*-q)}$, $\mathfrak{r} := \frac{\kappa(p_s^*-r)}{p_s^*+\beta(p_s^*-\tilde{p})+\kappa(p_s^*-q)}$, $\mathfrak{y} := \frac{r+\beta(r-\tilde{p})+\kappa(r-q)}{p_s^*+\beta(p_s^*-\tilde{p})+\kappa(p_s^*-q)}$ and $\mathfrak{z} := \frac{p_s^*-r}{p_s^*+\beta(p_s^*-\tilde{p})+\kappa(p_s^*-q)} = \psi_{p_s^*}^{-1}(\beta, \kappa)$ with $\mathfrak{w} + \mathfrak{r} + \mathfrak{y} + \mathfrak{z} = 1$. Then, for all $u \in M_{E,V}^{s;q,p}(\mathbb{R}^n)$ and on each set Ω of $\mathbb{R}^n$, one has

$$\begin{aligned} \int_{\Omega} K|u|^r dx &= \int_{\Omega} KE^{-\mathfrak{w}}V^{-\mathfrak{r}}(E|u|^{\tilde{p}})^{\mathfrak{w}}(V|u|^q)^{\mathfrak{r}}|u|^{r-\tilde{p}\mathfrak{w}-q\mathfrak{r}} dx \\ &\leq \left(\int_{\Omega} K^{\psi_{p_s^*}(\beta, \kappa)}E^{-\beta}V^{-\kappa} dx \right)^{\mathfrak{z}} \left(\int_{\Omega} E|u|^{\tilde{p}} dx \right)^{\mathfrak{w}} \left(\int_{\Omega} V|u|^q dx \right)^{\mathfrak{r}} \left(\int_{\Omega} |u|^{p_s^*} dx \right)^{\mathfrak{y}} \\ &\leq C'_1 \|K^{\psi_{p_s^*}(\beta, \kappa)}E^{-\beta}V^{-\kappa}\|_{1, \Omega}^{\psi_{p_s^*}^{-1}(\beta, \kappa)} \|u\|_{L_{E,V}^{\tilde{p}}(\Omega)}^{\tilde{p}\mathfrak{w}} \|u\|_{L_V^q(\Omega)}^{q\mathfrak{r}} [u]_{s,p,\mathbb{R}^n}^{p_s^*\mathfrak{y}} \end{aligned} \quad (4.4)$$

by Hölder's inequality and (2.7), as $\tilde{p}\mathfrak{w} + q\mathfrak{r} + p_s^*\mathfrak{y} = r$, provided $\mathfrak{w}, \mathfrak{r}, \mathfrak{y}, \mathfrak{z} \in (0, 1)$.

Note that $\beta \geq 0$ while $\beta = \infty$ furnishes a version of Theorem 4.1. As $\mathfrak{w} = \beta\mathfrak{z}$ and $\mathfrak{r} = \kappa\mathfrak{z}$, to get $\mathfrak{z} > 0$, one has $0 < \kappa < \frac{\beta(p_s^*-\tilde{p})+p_s^*}{q-p_s^*}$, while to get $\mathfrak{y} > 0$, one has either $0 < \kappa < \frac{\beta(r-\tilde{p})+r}{q-r}$ or $\kappa > \frac{\beta(p_s^*-\tilde{p})+p_s^*}{q-p_s^*}$; so, $0 < \kappa < \frac{\beta(r-\tilde{p})+r}{q-r}$. When $\beta = 0 = \mathfrak{w}$, one simply ignores $E(x)$ or treats $E(x)$ as 0 with $0^0 = 1$, while when $\kappa = 0 = \mathfrak{r}$, the same instead occurs to $V(x)$. We may set $\mathfrak{y} = 0$ to get $\kappa = \frac{\beta(r-\tilde{p})+r}{q-r}$ without having p_s^* involved, and can simultaneously take $\mathfrak{w} = \mathfrak{r} = 0$ corresponding to $M_{E,V}^{s;q,p}(\mathbb{R}^n) = D^{s,p}(\mathbb{R}^n)$ with $E(x) \equiv V(x) \equiv 0$. So, $M_{E,V}^{s;q,p}(\mathbb{R}^n) \rightarrow L_K^r(\mathbb{R}^n)$, upon replacing Ω in (4.4) by $\mathbb{R}^n$, is continuous without using $K(x) \in L_{loc}^t(\mathbb{R}^n)$.

Finally, utilizing the same $\{u_l : l \geq 1\}$ as the one in Theorem 4.1, we derive

$$\int_{\mathbb{R}^n} K|u_l|^r dx = \int_{\mathbb{B}_R} K|u_l|^r dx + \int_{\mathbb{B}_R^c} K|u_l|^r dx.$$

For the integral on $\mathbb{B}_R$, as $M_{E,V}^{s;q,p}(\mathbb{B}_R) \rightarrow M^{s;q,p}(\mathbb{B}_R) \hookrightarrow L^{\frac{rt}{t-1}}(\mathbb{B}_R)$ with $\frac{rt}{t-1} \in [r, \max\{p_s^*, q\}]$,

$$\int_{\mathbb{B}_R} K|u_l|^r dx \leq \left(\int_{\mathbb{B}_R} K^t dx \right)^{\frac{1}{t}} \left(\int_{\mathbb{B}_R} |u_l|^{\frac{rt}{t-1}} dx \right)^{\frac{t-1}{t}} \rightarrow 0 \quad (4.5)$$

when $l \rightarrow \infty$ for a subsequence $\{u_l : l \geq 1\}$ by the same index l , via $\inf_{\mathbb{B}_R} V(x) \geq V_{\mathbb{B}_R} > 0$, (2.2) and Proposition 2.1; for the integral on $\mathbb{B}_R^c$, we apply (4.4) on $\Omega = \mathbb{B}_R^c$ to deduce, as $R \rightarrow \infty$,

$$\int_{\mathbb{B}_R^c} K|u_l|^r dx \leq C'_1 \|K^{\psi_{p_s^*}(\beta, \kappa)}E^{-\beta}V^{-\kappa}\|_{1, \mathbb{B}_R^c}^{\psi_{p_s^*}^{-1}(\beta, \kappa)} \|u_l\|_{M_{E,V}^{s;q,p}(\mathbb{R}^n)}^r \rightarrow 0.$$

Hence, $u_l \rightarrow 0$ in $L_K^r(\mathbb{R}^n)$ for a subsequence relabeled using the same index l . $\square$

Theorem 4.3. Suppose $n \geq 2$, $1 \leq p < \infty$, $0 < s < 1$ with $sp < n$, and $1 \leq \tilde{p} \leq p_s^* \leq r < q$. Let $E(x), K(x), V(x) > 0 : \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $K(x) \in L_{loc}^t(\mathbb{R}^n)$ for some $t \in (\frac{q}{q-r}, \infty]$ and $K^{\psi_{p_s^*}(\beta, \kappa)}(x)E^{-\beta}(x)V^{-\kappa}(x) \in L^1(\mathbb{R}^n)$ for some $\beta \in [0, \infty)$ and $\kappa \in [\frac{\beta(r-\tilde{p})+r}{q-r}, \infty)$. Then, the embedding $M_{E,V}^{s;q,p}(\mathbb{R}^n) \hookrightarrow L_K^r(\mathbb{R}^n)$ is compact.

Proof. The proof is essentially the same as that of Theorem 4.2. Keep in mind $q > r \geq p_s^* \geq \tilde{p}$ when ensuring $\mathfrak{w}, \mathfrak{r}, \mathfrak{y}, \mathfrak{z} \in (0, 1)$. Actually, to have $\mathfrak{z} > 0$, one sees $\kappa > \frac{\beta(p_s^*-\tilde{p})+p_s^*}{q-p_s^*}$, while to get $\mathfrak{y} > 0$, one has either $0 < \kappa < \frac{\beta(r-\tilde{p})+r}{q-r}$ or $\kappa > \frac{\beta(r-\tilde{p})+r}{q-r}$; so, $\kappa \geq \frac{\beta(r-\tilde{p})+r}{q-r}$ after taking $\mathfrak{y} = 0$. Recall $M_{E,V}^{s;q,p}(\mathbb{R}^n) \rightarrow L_K^r(\mathbb{R}^n)$ is continuous with no need of $K(x) \in L_{loc}^t(\mathbb{R}^n)$. $\square$

Several remarks are in order: first, Theorem 4.2 merged [21, Theorem 1] (the *upper-triangle* case $r \geq \tilde{p}$) and [47, Theorem 2.3] (the *lower-triangle* case $r < q$) in one result, while $\beta = \infty$ or $\kappa = 0$ yield versions of Theorem 4.1 via integrability conditions; second, Theorem 4.2 is useful particularly for Theorem 1.2, but the case $\beta = 0$ contributes to condition **(f3)** of Theorem 1.1; finally, Theorem 4.1 is intended for Theorem 1.1, but the case $\tau = 0$ is used in condition **(EK)** of Theorem 1.2. As such, it is probably helpful to summarize them as follows.

Proposition 4.4. Assume that $n \geq 2$, $p = \tilde{p} = 2$ and $0 < s < 1$.

(i). Suppose further $1 \leq r < \min\{2_s^*, q\}$, and $K(x), V(x) > 0$ satisfy $K^{-\frac{2_s^* + \kappa(2_s^* - q)}{2_s^* - r}}(x)V^{-\kappa}(x) \in L^1(\mathbb{R}^n)$ for some $\kappa \in [0, \frac{r}{q-r}]$; then, the embedding $M_V^{s;q,2}(\mathbb{R}^n) \rightarrow L_K^r(\mathbb{R}^n)$ is continuous, where $M_V^{s;q,2}(\mathbb{R}^n)$ is the function space defined via (4.1) without the term Eu^2 .

(ii). Suppose further $2 \leq r < 2_s^*$, $\inf_{\mathbb{B}_R} E(x) \geq E_{\mathbb{B}_R} > 0$ for each $R > 0$, $K(x) \in L^t(\Omega)$ for some $t \in [\frac{p_s^*}{p_s^* - r}, \infty]$ on each set Ω of $\mathbb{R}^n$ with $\mathcal{L}(\Omega) < \infty$, and $K(x)E^{-\frac{2_s^* - r}{2_s^* - 2}}(x)$ vanishes at infinity; then, the embedding $H_E^s(\mathbb{R}^n) \rightarrow L_K^r(\mathbb{R}^n)$ is continuous, where $H_E^s(\mathbb{R}^n)$ is the Hilbert function space defined through (4.1) with the term Vu^q dropped.

We can take $p = \tilde{p}$ and $V(x) \equiv 0$ to have $W_E^{s,p}(\mathbb{R}^n) \hookrightarrow L_K^p(\mathbb{R}^n)$ or $W_E^{s,p}(\mathbb{R}^n) \hookrightarrow L_K^1(\mathbb{R}^n)$, or take $E(x) \equiv V(x) \equiv 0$ to have $D^{s,p}(\mathbb{R}^n) \hookrightarrow L_K^p(\mathbb{R}^n)$, for suitable $E(x)$ and $K(x)$, which seems to provide a complement to [14, Condition (h_1)] in terms of compact embedding.

Finally, following Servadei and Valdinoci [48, Section 3] or [49], or applying Auchmuty [6] (see also [37, Section 3.1]), one observes the eigenvalue problems

$$(-\Delta)^s f + Ef = \varrho Kf \text{ and } (-\Delta)^s g = \xi Kg \text{ in } \mathbb{R}^n, \quad (4.6)$$

when $n \geq 2$, admit families $\{\varrho_k > 0 : k \geq 1\}$ and $\{\xi_l > 0 : l \geq 1\}$ of eigenvalues, and sequences $\{f_k \in H_E^s(\mathbb{R}^n) : k \geq 1\}$ and $\{g_l \in D^s(\mathbb{R}^n) : l \geq 1\}$ of associated eigenfunctions, respectively, in view of Theorem 4.1 (when taking $V(x) \equiv 0$ and $p = \tilde{p} = r = 2$ to work in Hilbert spaces) and Theorem 4.2 (when taking $E(x) \equiv V(x) \equiv 0$ and $p = r = 2$ to work in Hilbert spaces).

ϱ_1, ξ_1 are simple, each ϱ_k, ξ_l are of finite multiplicities, and the sequences $\{\varrho_k : k \geq 1\}$ and $\{\xi_l : l \geq 1\}$ of eigenvalues increase without bound.

f_1, g_1 are nonnegative, and the sequences $\{f_k : k \geq 1\}$ and $\{g_l : l \geq 1\}$ provide orthogonal bases to the spaces $H_E^s(\mathbb{R}^n), L_K^2(\mathbb{R}^n)$ and $D^s(\mathbb{R}^n), L_K^2(\mathbb{R}^n)$, respectively.

The procedure is standard, with details left for the interested reader.

5 Proof of Theorem 1.1

First, recall for a weakly convergent sequence $\{u_k : k \geq 1\} \rightharpoonup D^s(\mathbb{R}^n)$ having the weak limit u , there are bounded measures μ, ν and constants $\mu_i, \nu_i, \mu_\infty, \nu_\infty \geq 0$ such that

$$\begin{aligned} (\nabla^s u_k)^2 \, \mathbf{d}x &\rightharpoonup \mathbf{d}\mu \geq (\nabla^s u)^2 \, \mathbf{d}x + \sum_{i \in \mathbb{N}} \mu_i \delta_{x_i} + \mu_\infty \delta_\infty \text{ and} \\ |u_k|^{2_s^*} \, \mathbf{d}x &\rightharpoonup \mathbf{d}\nu = |u|^{2_s^*} \, \mathbf{d}x + \sum_{i \in \mathbb{N}} \nu_i \delta_{x_i} + \nu_\infty \delta_\infty \end{aligned} \quad (5.1)$$

in the sense of weakly* convergence of measures, where

$$\begin{aligned} (\nabla^s v)^2 &= (\nabla^s v(x))^2 := \int_{\mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x - y|^{n+2s}} \, \mathbf{d}y \text{ for each } v \in D^s(\mathbb{R}^n), \\ \mu_\infty &:= \lim_{R \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_{\mathbb{B}_R^c} (\nabla^s u_k)^2 \, \mathbf{d}x \text{ and } \nu_\infty := \lim_{R \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_{\mathbb{B}_R^c} |u_k|^{2_s^*} \, \mathbf{d}x, \end{aligned}$$

δ_{x_i} denotes the Dirac delta at $x_i \in \mathbb{R}^n$, and $Sv_i^{2/2_s^*} \leq \mu_i$ for $i \in \mathbb{N}$ and $Sv_\infty^{2/2_s^*} \leq \mu_\infty$.

We seek a positive solution to (1.1) as the critical point of its energy functional

$$\begin{aligned} \mathcal{F}(u) := & \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + \frac{1}{2} \int_{\mathbb{R}^n} E u^2 \, \mathbf{d}x + \frac{1}{q} \int_{\mathbb{R}^n} V |u|^q \, \mathbf{d}x \\ & - \int_{\mathbb{R}^n} K F_+(u) \, \mathbf{d}x - \frac{1}{2_s^*} \int_{\mathbb{R}^n} (u^+)^{2_s^*} \, \mathbf{d}x, \end{aligned} \quad (5.2)$$

where $F_+(u) := \int_0^u f_+(v) \, \mathbf{d}v \geq 0$ for $f_+(u) := f(u)$ if $u \geq 0$ and $f_+(u) := 0$ otherwise through condition **(f2)**. Then, u is a positive weak solution to equation (1.1), provided

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + \int_{\mathbb{R}^n} E u v \, \mathbf{d}x + \int_{\mathbb{R}^n} V |u|^{q-2} u v \, \mathbf{d}x \\ & = \int_{\{x: u(x) \geq 0\}} K f_+(u) v \, \mathbf{d}x + \int_{\mathbb{R}^n} (u^+)^{2_s^*-1} v \, \mathbf{d}x, \quad \forall v \in M_{E,V}^{s;q,2}(\mathbb{R}^n). \end{aligned} \quad (5.3)$$

To see this fact, take $v = u^- := \min\{u, 0\} \in M_{E,V}^{s;q,2}(\mathbb{R}^n)$. Notice for each u , $u = u^+ + u^-$ with $u^+ := \max\{u, 0\} \in M_{E,V}^{s;q,2}(\mathbb{R}^n)$ and $u^+ \cdot u^- \equiv 0$. Then, we get from (5.3) that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(u^-(x) - u^-(y))}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + \int_{\mathbb{R}^n} E (u^-)^2 \, \mathbf{d}x + \int_{\mathbb{R}^n} V |u^-|^q \, \mathbf{d}x \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u^-(x))^2 + (u^-(y))^2 - 2u^-(x)u^-(y)}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + \int_{\mathbb{R}^n} E (u^-)^2 \, \mathbf{d}x + \int_{\mathbb{R}^n} V |u^-|^q \, \mathbf{d}x \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u^-(x) - u^-(y))^2 - 2u^+(x)u^-(y)}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + \int_{\mathbb{R}^n} E (u^-)^2 \, \mathbf{d}x + \int_{\mathbb{R}^n} V |u^-|^q \, \mathbf{d}x \\ &\geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u^-(x) - u^-(y))^2}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + \int_{\mathbb{R}^n} E (u^-)^2 \, \mathbf{d}x + \int_{\mathbb{R}^n} V |u^-|^q \, \mathbf{d}x \geq 0, \end{aligned} \quad (5.4)$$

which further implies $\|u^-\|_{M_{E,V}^{s;q,2}(\mathbb{R}^n)} = 0$; so, $u = u^+ \geq 0$ follows immediately.

Now, via condition **(f1)**, $f_+(u) \leq \varepsilon |u|^{q-1} + a_\varepsilon |u|^{2_s^*-1}$ for each $\varepsilon > 0$ and a sufficiently large constant $a_\varepsilon > 0$ depending on ε ; thus, we have

$$\int_{\mathbb{R}^n} K F_+(u) \, \mathbf{d}x \leq \frac{\varepsilon}{q} \int_{\mathbb{R}^n} K |u|^q \, \mathbf{d}x + \frac{a_\varepsilon}{2_s^*} \|K\|_{\infty, \mathbb{R}^n} \int_{\mathbb{R}^n} |u|^{2_s^*} \, \mathbf{d}x.$$

From condition **(EKV1)** and Theorem 4.1, one accordingly derives

$$\begin{aligned} \mathcal{F}(u) &\geq \frac{1}{2} [u]_{s,2,\mathbb{R}^n}^2 - C_K \frac{\varepsilon}{q} [u]_{s,2,\mathbb{R}^n}^q + \frac{1}{2} \|u\|_{L_E^2(\mathbb{R}^n)}^2 - C_K \frac{\varepsilon}{q} \|u\|_{L_E^2(\mathbb{R}^n)}^q \\ &\quad + \frac{1}{q} \|u\|_{L_V^q(\mathbb{R}^n)}^q - C_K \frac{\varepsilon}{q} \|u\|_{L_V^q(\mathbb{R}^n)}^q - \frac{1}{2_s^*} (1 + a_{\varepsilon,K}) \|u\|_{2_s^*, \mathbb{R}^n}^{2_s^*} \\ &\geq [u]_{s,2,\mathbb{R}^n}^2 \left(\frac{1}{2} - C_K \frac{\varepsilon}{q} [u]_{s,2,\mathbb{R}^n}^{q-2} - C_1 \frac{1 + a_{\varepsilon,K}}{2_s^*} [u]_{s,2,\mathbb{R}^n}^{2_s^*-2} \right) \\ &\quad + \|u\|_{L_E^2(\mathbb{R}^n)}^2 \left(\frac{1}{2} - C_K \frac{\varepsilon}{q} \|u\|_{L_E^2(\mathbb{R}^n)}^{q-2} \right) + \frac{1}{q} (1 - \varepsilon C_K) \|u\|_{L_V^q(\mathbb{R}^n)}^q, \end{aligned}$$

so that $\mathcal{F}(u) \geq \rho_0 > 0$ when $0 < \frac{\rho}{2} \leq \|u\|_{M_{E,V}^{s;q,2}(\mathbb{R}^n)} \leq \rho$ for sufficiently small ε , where $C_K > 0$ depends on the embedding $M_{E,V}^{s;q,2}(\mathbb{R}^n) \rightarrow L_K^q(\mathbb{R}^n)$ and $a_{\varepsilon,K} := a_\varepsilon \|K\|_{\infty, \mathbb{R}^n} > 0$.

$\mathcal{F}(0) = 0$ and $\mathcal{F}(w_0) < 0$ for a $w_0 \in M_{E,V}^{s;q,2}(\mathbb{R}^n)$ with $\|w_0\|_{M_{E,V}^{s;q,2}(\mathbb{R}^n)} > \rho$. So, the standard mountain pass structure is satisfied, and $\mathcal{F}$ attains a Palais-Smale sequence at level c for $c = \inf_{h \in \mathcal{H}} \max_{t \in [0,1]} \mathcal{F}(h(t)) > 0$ with

$$\mathcal{H} := \{h \in C([0, 1]; M_{E,V}^{s;q,2}(\mathbb{R}^n)) : h(0) = 0, h(1) = w_0\}.$$

Below, we find an upper bound for $c > 0$ via the mountain pass solution $v_0 > 0$ to equation (1.3) seeing $M_{E,V}^{s;q,2}(\mathbb{R}^n) \hookrightarrow L_K^g(\mathbb{R}^n)$. For $t \geq 0$, condition **(f2)** and (1.5) yield

$$\begin{aligned} \mathcal{F}(tv_0) &\leq \frac{t^2}{2} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_0(x) - v_0(y))^2}{|x - y|^{n+2s}} \mathbf{d}x \mathbf{d}y + \int_{\mathbb{R}^n} E v_0^2 \mathbf{d}x \right) + \frac{t^q}{q} \int_{\mathbb{R}^n} V v_0^q \mathbf{d}x \\ &\quad - A_g \frac{t^g}{g} \int_{\mathbb{R}^n} K v_0^g \mathbf{d}x \leq \left\{ \frac{t^2}{2} + \frac{t^q}{q} - A_g \frac{t^g}{g} \right\} \int_{\mathbb{R}^n} K v_0^g \mathbf{d}x \\ &\leq \sup_{t \geq 0} \left\{ t^2 - A_g \frac{t^g}{g} \right\} \int_{\mathbb{R}^n} K v_0^g \mathbf{d}x < \frac{s}{2n} S^{\frac{n}{2s}} \end{aligned} \quad (5.5)$$

using a relation similar to (5.3) fulfilled by $u = v = v_0$; thus, $0 < c < \frac{s}{2n} S^{\frac{n}{2s}}$.

Now, take $\{u_k : k \geq 1\}$ to be a Palais-Smale sequence of $\mathcal{F}$ at level c in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$; that is, $\mathcal{F}(u_k) \rightarrow c$ and $\mathcal{F}'(u_k) \rightarrow 0$ in the dual space of $M_{E,V}^{s;q,2}(\mathbb{R}^n)$ as $k \rightarrow \infty$. Then, one has

$$\begin{aligned} c + o(1) &= \mathcal{F}(u_k) - \frac{1}{\theta} \mathcal{F}'(u_k)(u_k) \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta} \right) [u_k]_{s,2,\mathbb{R}^n}^2 + \left(\frac{1}{q} - \frac{1}{\theta} \right) \|u_k\|_{L_V^q(\mathbb{R}^n)}^q + \left(\frac{1}{\theta} - \frac{1}{2_s^*} \right) \|u_k\|_{2_s^*,\mathbb{R}^n}^{2_s^*} \\ &\quad - \frac{\eta_1}{\theta} \int_{\mathbb{R}^n} K_1 |u_k|^{r_1} \mathbf{d}x - \frac{\eta_2}{\theta} \int_{\mathbb{R}^n} K_2 |u_k|^{r_2} \mathbf{d}x + \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u_k\|_{L_E^2(\mathbb{R}^n)}^2 \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta} \right) [u_k]_{s,2,\mathbb{R}^n}^2 + \left(\frac{1}{q} - \frac{1}{\theta} \right) \|u_k\|_{L_V^q(\mathbb{R}^n)}^q + \left(\frac{1}{\theta} - \frac{1}{2_s^*} \right) \|u_k\|_{2_s^*,\mathbb{R}^n}^{2_s^*} \\ &\quad - \frac{\eta_1}{\theta} C_{K_1} \|u_k\|_{2_s^*,\mathbb{R}^n}^{r_1} - \frac{\eta_2}{\theta} C_{K_2} [u_k]_{s,2,\mathbb{R}^n}^{r_2} - \frac{\eta_2}{\theta} C_{K_2} \|u_k\|_{L_V^q(\mathbb{R}^n)}^{r_2} \end{aligned}$$

from condition **(f3)** using Proposition 4.4 (i)¹, with $C_{K_2} > 0$ depending only on $M_V^{s;q,2}(\mathbb{R}^n) \rightarrow L_{K_2}^{r_2}(\mathbb{R}^n)$. Thus, $\{u_k : k \geq 1\}$ is bounded in $M_{E,V}^{s;q,2}(\mathbb{R}^n) \subsetneq D^s(\mathbb{R}^n)$ and $L^{2_s^*}(\mathbb{R}^n)$, and there is a subsequence, still denoted by $\{u_k : k \geq 1\}$, and a function $u \in M_{E,V}^{s;q,2}(\mathbb{R}^n)$ such that both the estimates in (5.1) hold, and $u_k \rightarrow u$ in $L_K^q(\mathbb{R}^n)$ and $L_V^q(\mathbb{B}_R)$ for each $R > 0$.

Fix an increasing function χ_R with $\chi_R \equiv 0$ on $|x| \leq R$, $\chi_R \equiv 1$ on $|x| \geq R^2$, and $1 - \chi_R \in C_c^1(\mathbb{R}^n)$ (see, for instance, [28, Appendix I]). Next, from the discussions in [14, Section 2], and noting condition **(f1)** yields $f_+(u) \leq b_\varepsilon |u|^{q-1} + \varepsilon |u|^{2_s^*-1}$ for each $\varepsilon > 0$ and a sufficiently large constant $b_\varepsilon > 0$ depending on ε , it follows that

$$\begin{aligned} 0 = \lim_{k \rightarrow \infty} \mathcal{F}'(u_k)(u_k \chi_R) &\geq \limsup_{k \rightarrow \infty} \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_k(x) - u_k(y))^2}{|x - y|^{n+2s}} \chi_R(x) \mathbf{d}x \mathbf{d}y \right. \\ &\quad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_k(x) - u_k(y))(\chi_R(x) - \chi_R(y))}{|x - y|^{n+2s}} u_k(y) \mathbf{d}x \mathbf{d}y + \int_{\mathbb{R}^n} E u_k^2 \chi_R \mathbf{d}x \\ &\quad \left. + \int_{\mathbb{R}^n} V |u_k|^q \chi_R \mathbf{d}x - b_\varepsilon \int_{\mathbb{R}^n} K |u_k|^q \chi_R \mathbf{d}x - (1 + \varepsilon \|K\|_{\infty, \mathbb{R}^n}) \int_{\mathbb{R}^n} |u_k|^{2_s^*} \chi_R \mathbf{d}x \right\}. \end{aligned} \quad (5.6)$$

By virtue of Lemma 3.6 in [14], one has

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_k(x) - u_k(y))^2}{|x - y|^{n+2s}} \chi_R(x) \mathbf{d}x \mathbf{d}y &\rightarrow \mu_\infty, \quad \int_{\mathbb{R}^n} |u_k|^{2_s^*} \chi_R \mathbf{d}x \rightarrow \nu_\infty, \quad \text{and} \\ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_k(x) - u_k(y))(\chi_R(x) - \chi_R(y))}{|x - y|^{n+2s}} u_k(y) \mathbf{d}x \mathbf{d}y &\rightarrow 0 \quad \text{when } R, k \rightarrow \infty; \end{aligned}$$

¹ Note we unfortunately cannot apply Theorem 4.2 directly, since $1 \leq r_2 < 2$ is needed here; of course, if we instead assume $\inf_{\mathbb{B}_R} E(x) \geq E_{\mathbb{B}_R} > 0$ for each $R > 0$, then $H_E^s(\mathbb{R}^n)$ can be used in place of $M_V^{s;q,2}(\mathbb{R}^n)$.

moreover, $\int_{\mathbb{R}^n} K|u_k|^q \chi_R \, \mathbf{d}x \rightarrow 0$ when $R, k \rightarrow \infty$ as $M_{E,V}^{s;q,2}(\mathbb{R}^n) \hookrightarrow L_K^q(\mathbb{R}^n)$. So, one sees from (5.6) that $(1 + \varepsilon \|K\|_{\infty, \mathbb{R}^n}) v_\infty \geq \mu_\infty + v_\infty \geq \mu_\infty$ for $v_\infty := \lim_{R \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} V|u_k|^q \chi_R \, \mathbf{d}x \geq 0$; a similar but simpler analysis shows $(1 + \varepsilon \|K\|_{\infty, \mathbb{R}^n}) v_i \geq \mu_i$ for each $x_i \in \mathbb{R}^n$. Thus, $\mu_i, \mu_\infty \geq S^{\frac{n}{2s}} \left(\frac{1}{1 + \varepsilon \|K\|_{\infty, \mathbb{R}^n}} \right)^{\frac{2}{2s-2}}$ and $v_i, v_\infty \geq S^{\frac{n}{2s}} \left(\frac{1}{1 + \varepsilon \|K\|_{\infty, \mathbb{R}^n}} \right)^{\frac{2s^*}{2s-2}}$; so, $\mu_i, \mu_\infty, v_i, v_\infty \geq S^{\frac{n}{2s}}$, if they are not zero, by the arbitrariness of ε , which however would lead to a contradiction as follows.

In fact, set $\ell := \mu(\mathbb{R}^n) - [u]_{s,2,\mathbb{R}^n}^2 - \sum_{i \in \mathbb{N}} \mu_i - \mu_\infty \geq 0$ to observe that

$$\begin{aligned} c &= \lim_{k \rightarrow \infty} \left\{ \mathcal{F}(u_k) - \frac{1}{\theta} \mathcal{F}'(u_k)(u_k) \right\} \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta} \right) \left([u]_{s,2,\mathbb{R}^n}^2 + \sum_{i \in \mathbb{N}} \mu_i + \mu_\infty + \ell \right) - \frac{\eta_2}{\theta} C_{K_2} \left([u]_{s,2,\mathbb{R}^n}^2 + \sum_{i \in \mathbb{N}} \mu_i + \mu_\infty + \ell \right)^{\frac{r_2}{2}} \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{2s^*} \right) \left(\|u\|_{2s^*,\mathbb{R}^n}^{2s^*} + \sum_{i \in \mathbb{N}} v_i + v_\infty \right) - \frac{\eta_1}{\theta} C_{K_1} \left(\|u\|_{2s^*,\mathbb{R}^n}^{2s^*} + \sum_{i \in \mathbb{N}} v_i + v_\infty \right)^{\frac{r_1}{2s^*}} \\ &\quad + \left(\frac{1}{q} - \frac{1}{\theta} \right) \left(\|u\|_{L_V^q(\mathbb{R}^n)}^q + v_\infty \right) - \frac{\eta_2}{\theta} C_{K_2} \left(\|u\|_{L_V^q(\mathbb{R}^n)}^q + v_\infty \right)^{\frac{r_2}{q}}, \end{aligned}$$

which certainly implies only finitely many of $\mu_i, \mu_\infty, v_i, v_\infty (\geq S^{\frac{n}{2s}})$ do not vanish. To illustrate the contradiction, assume without loss of generality that $\mu_\infty, v_\infty \geq S^{\frac{n}{2s}}$. Then, we have

$$\begin{aligned} c &= \lim_{k \rightarrow \infty} \left\{ \mathcal{F}(u_k) - \frac{1}{\theta} \mathcal{F}'(u_k)(u_k) \right\} \geq \frac{5}{6} \left(\frac{1}{2} - \frac{1}{\theta} \right) \mu_\infty + \frac{5}{6} \left(\frac{1}{\theta} - \frac{1}{2s^*} \right) v_\infty \\ &\quad + \frac{\theta - 2}{6 \cdot 2\theta} \left([u]_{s,2,\mathbb{R}^n}^2 + \sum_{i \in \mathbb{N}} \mu_i + \mu_\infty + \ell \right) - \frac{\eta_2}{\theta} C_{K_2} \left([u]_{s,2,\mathbb{R}^n}^2 + \sum_{i \in \mathbb{N}} \mu_i + \mu_\infty + \ell \right)^{\frac{r_2}{2}} \\ &\quad + \frac{2s^* - \theta}{6 \cdot 2s^* \theta} \left(\|u\|_{2s^*,\mathbb{R}^n}^{2s^*} + \sum_{i \in \mathbb{N}} v_i + v_\infty \right) - \frac{\eta_1}{\theta} C_{K_1} \left(\|u\|_{2s^*,\mathbb{R}^n}^{2s^*} + \sum_{i \in \mathbb{N}} v_i + v_\infty \right)^{\frac{r_1}{2s^*}} \\ &\quad + \frac{\theta - q}{q\theta} \left(\|u\|_{L_V^q(\mathbb{R}^n)}^q + v_\infty \right) - \frac{\eta_2}{\theta} C_{K_2} \left(\|u\|_{L_V^q(\mathbb{R}^n)}^q + v_\infty \right)^{\frac{r_2}{q}} \\ &\geq \frac{5}{6} \left(\frac{1}{2} - \frac{1}{2s^*} \right) S^{\frac{n}{2s}} - 3 \cdot \frac{s}{9n} S^{\frac{n}{2s}} = \frac{5s}{6n} S^{\frac{n}{2s}} - \frac{s}{3n} S^{\frac{n}{2s}} = \frac{s}{2n} S^{\frac{n}{2s}}, \end{aligned}$$

contradicting (5.5); parallel discussions yield contradictions again on μ_i, v_i for all $x_i \in \mathbb{R}^n$.

So, there occurs no evanescence phenomenon and we get $u_k \rightarrow u \in M_{E,V}^{s;q,2}(\mathbb{R}^n)$.

6 Proof of Theorem 1.2

In this section, we combine the strategies from [7, 8] and Han [31] to find nontrivial positive solutions to (1.6) as the critical points of its energy functional $\mathcal{F}_\lambda(u)$ defined by

$$\begin{aligned} \mathcal{F}_\lambda(u) &:= \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + \frac{1}{2} \int_{\mathbb{R}^n} E u^2 \, \mathbf{d}x + \frac{1}{q} \int_{\mathbb{R}^n} V |u|^q \, \mathbf{d}x \\ &\quad - \frac{\lambda}{r} \int_{\mathbb{R}^n} K (u^+)^r \, \mathbf{d}x. \end{aligned} \tag{6.1}$$

2 $v_\infty := \lim_{R \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} E u_k^2 \chi_R \, \mathbf{d}x \geq 0$ if $H_E^s(\mathbb{R}^n)$ is used with $\inf_{\mathbb{B}_R} E(x) \geq E_{\mathbb{B}_R} > 0$ for each $R > 0$.

Like the analysis in (5.4), u is a positive weak solution to equation (1.6) provided

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + \int_{\mathbb{R}^n} Eu v \, \mathbf{d}x + \int_{\mathbb{R}^n} V|u|^{q-2} u v \, \mathbf{d}x \\ &= \lambda \int_{\mathbb{R}^n} K(u^+)^{r-1} v \, \mathbf{d}x, \quad \forall v \in M_{E,V}^{s;q,2}(\mathbb{R}^n). \end{aligned} \quad (6.2)$$

Lemma 6.1. *Under the assumptions (EKV2-1) or (EKV2-2), the functional $\mathcal{F}_\lambda$ is C^1 and is coercive in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$, so that each sequence $\{u_k : k \geq 1\}$ of functions with $\mathcal{F}_\lambda(u_k)$ bounded has a weakly convergent subsequence in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$. Moreover, $\mathcal{F}_\lambda$ is sequentially weakly lower semi-continuous in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$; that is, when $u_k \rightharpoonup u \in M_{E,V}^{s;q,2}(\mathbb{R}^n)$, for a subsequence relabeled using the same notation, one has $\mathcal{F}_\lambda(u) \leq \liminf_{k \rightarrow \infty} \mathcal{F}_\lambda(u_k)$.*

Proof. The proof of showing $\mathcal{F}_\lambda$ C^1 is standard³, and the claim regarding the boundedness of $\mathcal{F}_\lambda(u_k)$ implying the existence of a weakly convergent subsequence in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$ follows from the coercivity of $\mathcal{F}_\lambda$ and the reflexivity of $M_{E,V}^{s;q,2}(\mathbb{R}^n)$; for the latter, see [7, 8] for details, since now $M_{E,V}^{s;q,2}(\mathbb{R}^n)$ is a Banach subspace of the Hilbert function space $H_E^s(\mathbb{R}^n)$.

Next, for $r_1 := \mathfrak{w}^{-1}$, $r_2 := \mathfrak{r}^{-1}$, $r_3 := \frac{2}{2s} \mathfrak{r}^{-1}$ and $r_4 := \frac{1}{1 - \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3}} > 0$ (being true under our conditions (EKV2-1) or (EKV2-2)) in (4.4) of Theorem 4.2, one has

$$\begin{aligned} & \frac{\lambda}{r} \int_{\mathbb{R}^n} K(u^+)^r \, \mathbf{d}x \leq \frac{\lambda}{r} \int_{\mathbb{R}^n} K|u|^r \, \mathbf{d}x \\ & \leq \left(\frac{r_1}{4} \int_{\mathbb{R}^n} Eu^2 \, \mathbf{d}x \right)^{\frac{1}{r_1}} \left(\frac{r_2}{2q} \int_{\mathbb{R}^n} V|u|^q \, \mathbf{d}x \right)^{\frac{1}{r_2}} \left(\frac{r_3}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y \right)^{\frac{1}{r_3}} \\ & \quad \times \left\{ \left(\frac{\lambda C'_1}{r} \right)^{r_4} \left(\frac{4}{r_3} \right)^{\frac{r_4}{r_3}} \left(\frac{2q}{r_2} \right)^{\frac{r_4}{r_2}} \left(\frac{4}{r_1} \right)^{\frac{r_4}{r_1}} \left(\int_{\mathbb{R}^n} K^{\psi_{2s^*}(\beta, \kappa)} E^{-\beta} V^{-\kappa} \, \mathbf{d}x \right)^{\frac{r_4}{\psi_{2s^*}(\beta, \kappa)}} \right\}^{\frac{1}{r_4}} \\ & \leq \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + \frac{1}{4} \int_{\mathbb{R}^n} Eu^2 \, \mathbf{d}x + \frac{1}{2q} \int_{\mathbb{R}^n} V|u|^q \, \mathbf{d}x + \lambda^{r_4} C_{EKV} \end{aligned} \quad (6.3)$$

by Young's inequality for an absolute constant $C_{EKV} > 0$ independent of λ, u . So,

$$\mathcal{F}_\lambda(u) \geq C_4 \|u\|_{M_{E,V}^{s;q,2}(\mathbb{R}^n)}^2 - \lambda^{r_4} C_{EKV} - C'_4$$

for some constants $C_4, C'_4 > 0$ as $q > 2$, which implies the coercivity of $\mathcal{F}$.

Since $\mathcal{F}_\lambda$ is coercive, each sequence $\{u_k : k \geq 1\}$ of functions in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$ with bounded $\mathcal{F}_\lambda(u_k)$ has a weakly convergent subsequence, written again as $\{u_k : k \geq 1\}$, such that $u_k \rightharpoonup u \in M_{E,V}^{s;q,2}(\mathbb{R}^n)$. The weakly lower semi-continuity of norms in $D^s(\mathbb{R}^n)$, $L_E^2(\mathbb{R}^n)$ and $L_V^q(\mathbb{R}^n)$, along with the compact embedding $M_{E,V}^{s;q,2}(\mathbb{R}^n) \hookrightarrow L_K^r(\mathbb{R}^n)$ by Theorems 4.2 and 4.3, leads to the sequentially weak lower semi-continuity of $\mathcal{F}_\lambda$, which concludes the proof. $\square$

Define

$$\tilde{\lambda} := \inf_{\substack{u \in M_{E,V}^{s;q,2}(\mathbb{R}^n) \\ \|u^+\|_{L_K^r(\mathbb{R}^n)} = 1}} \left\{ \frac{r}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + \frac{r}{2} \int_{\mathbb{R}^n} Eu^2 \, \mathbf{d}x + \frac{r}{q} \int_{\mathbb{R}^n} V|u|^q \, \mathbf{d}x \right\}. \quad (6.4)$$

Then, one has $\tilde{\lambda} > 0$. Actually, if not, there would exist a sequence $\{u_l : l \geq 1\} \subsetneq M_{E,V}^{s;q,2}(\mathbb{R}^n)$ with $\|u_l^+\|_{L_K^r(\mathbb{R}^n)} = 1$ such that

$$\frac{r}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_l(x) - u_l(y))^2}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + \frac{r}{2} \int_{\mathbb{R}^n} Eu_l^2 \, \mathbf{d}x + \frac{r}{q} \int_{\mathbb{R}^n} V|u_l|^q \, \mathbf{d}x \rightarrow 0;$$

³ See [8, Lemma 3.2] for the term $\frac{1}{2} \|u\|_{s,2,\mathbb{R}^n}^2$, where the estimate $|a^2 - b^2| \leq 2\{(a-b)^2 + \max\{|a|, |b|\}|a-b|\}$ should have been used.

so, $\|u_l\|_{M_{E,V}^{s;q,2}(\mathbb{R}^n)} \rightarrow 0$, contradicting the compact embedding $M_{E,V}^{s;q,2}(\mathbb{R}^n) \hookrightarrow L_K^r(\mathbb{R}^n)$.

Define λ^* to be the supremum of λ such that equation (1.6) only has the trivial nonnegative solution for each $\iota < \lambda$, and λ^{**} to be the infimum of λ such that equation (1.6) has at least a nontrivial positive solution at λ . Then, $0 \leq \lambda^* = \lambda^{**} \leq \tilde{\lambda}$. As a matter of fact,

$$\lambda \int_{\mathbb{R}^n} K(v_\lambda^+)^r \, dx > \frac{r}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_\lambda(x) - v_\lambda(y))^2}{|x - y|^{n+2s}} \, dx dy + \frac{r}{2} \int_{\mathbb{R}^n} E v_\lambda^2 \, dx + \frac{r}{q} \int_{\mathbb{R}^n} V |v_\lambda|^q \, dx$$

holds for all $\lambda > \tilde{\lambda}$ with some $v_\lambda \in M_{E,V}^{s;q,2}(\mathbb{R}^n)$ by homogeneity, and consequently $\mathcal{F}_\lambda(v_\lambda) < 0$, which along with Lemma 6.1 leads to $\mathcal{F}_\lambda(u_\lambda) = \inf_{u \in M_{E,V}^{s;q,2}(\mathbb{R}^n)} \mathcal{F}_\lambda(u) \leq \mathcal{F}_\lambda(v_\lambda) < 0$ for an $u_\lambda \geq 0$ in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$ that is a nontrivial

positive solution to equation (1.6), since $\mathcal{F}_\lambda(|u_\lambda|) \leq \mathcal{F}_\lambda(u_\lambda)$. So, we have $\lambda^{**} \leq \tilde{\lambda}$. On the other hand, if $\lambda^* > \lambda^{**}$, we would find a $\lambda' \in [\lambda^{**}, \lambda^*)$ such that problem (1.6) has at least a nontrivial positive solution at λ' according to the definition of λ^{**} , which however is against the definition of λ^* ; if $\lambda^* < \lambda^{**}$, we would find a $\lambda' \in (\lambda^*, \lambda^{**}]$ such that problem (1.6) has at least a nontrivial positive solution at some $\iota' (< \lambda')$ according to the definition of λ^* , which however is against the definition of λ^{**} . Thus, $\lambda^* = \lambda^{**}$.

Write $\lambda_1 := \lambda^* = \lambda^{**}$ in the sequel.

Proposition 6.2. *Under our conditions (EKV2-1) or (EKV2-2), $\lambda_1 \geq 0$ and equation (1.6) has a nontrivial positive solution $u_\lambda \geq 0$ in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$ for all $\lambda > \lambda_1$.*

Proof. By definition, $\lambda_1 = \lambda^*$; so, if u_λ is a nontrivial positive solution to (1.6) in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$, then $\lambda \geq \lambda_1$. Now, we show (1.6) has at least one nontrivial solution $u_\lambda \geq 0$ in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$ for all $\lambda > \lambda_1$ combining Struwe [50, Theorem 2.4], [8, Theorem 4.2] and [31, Proposition 3.3].

By definition, $\lambda_1 = \lambda^{**}$; so, there is an $\iota \in [\lambda_1, \lambda)$ such that (1.6) has a nontrivial solution $u_\iota \geq 0$ in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$, which of course is a subsolution to (1.6) at λ . Consider the constrained minimization problem $\inf_{u \in \mathcal{M}} \mathcal{F}_\lambda(u)$ for $\mathcal{M} := \{u \in M_{E,V}^{s;q,2}(\mathbb{R}^n) : u \geq u_\iota \geq 0\}$. Notice $\mathcal{M}$ is closed and convex, so that it is weakly closed in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$. Thus, Lemma 6.1 ensures the attainment of a minimizer of $\mathcal{F}_\lambda$ in $\mathcal{M}$; that is, there is an $u_\lambda (\geq u_\iota \geq 0)$ in $\mathcal{M}$ with $\mathcal{F}_\lambda(u_\lambda) = \inf_{u \in \mathcal{M}} \mathcal{F}_\lambda(u)$. Take $\varphi \in C_c^1(\mathbb{R}^n)$, and set $\varphi_\varepsilon := \max\{0, u_\iota - u_\lambda + \varepsilon\varphi\} \geq 0$ and $v_\varepsilon := \varphi_\varepsilon + u_\lambda - \varepsilon\varphi (\geq u_\iota \geq 0)$ in $\mathcal{M}$ for all $\varepsilon > 0$. So, one has $\mathcal{F}'_\lambda(u_\lambda)(u_\lambda) \leq \mathcal{F}'_\lambda(u_\lambda)(v_\varepsilon)$ that further implies

$$\mathcal{F}'_\lambda(u_\lambda)(\varphi) \leq \frac{1}{\varepsilon} \mathcal{F}'_\lambda(u_\lambda)(\varphi_\varepsilon). \quad (6.5)$$

Put $\Omega_\varepsilon := \{x \in \mathbb{R}^n : \varphi_\varepsilon(x) > 0\} = \{x \in \mathbb{R}^n : u_\lambda(x) - \varepsilon\varphi(x) < u_t(x)\} \subsetneq \text{supp}(\varphi^+)$. Since u_t is a subsolution to (1.6) at λ and $\varphi_\varepsilon \geq 0$, $\mathcal{F}'_\lambda(u_t)(\varphi_\varepsilon) \leq 0$ follows and one has⁴

$$\begin{aligned}
& \mathcal{F}'_\lambda(u_\lambda)(\varphi_\varepsilon) \leq \mathcal{F}'_\lambda(u_\lambda)(\varphi_\varepsilon) - \mathcal{F}'_\lambda(u_t)(\varphi_\varepsilon) \\
&= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(\omega(x) - \omega(y))(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y \\
&\quad + \int_{\mathbb{R}^n} E(u_\lambda - u_t)\varphi_\varepsilon \, \mathbf{d}x + \int_{\mathbb{R}^n} V(u_\lambda^{q-1} - u_t^{q-1})\varphi_\varepsilon \, \mathbf{d}x - \lambda \int_{\mathbb{R}^n} K(u_\lambda^{r-1} - u_t^{r-1})\varphi_\varepsilon \, \mathbf{d}x \\
&= \iint_{\{\Omega_\varepsilon \times \Omega_\varepsilon\}^c} \frac{(\omega(x) - \omega(y))(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y \\
&\quad + \int_{\Omega_\varepsilon} E(u_\lambda - u_t)\varphi_\varepsilon \, \mathbf{d}x + \int_{\Omega_\varepsilon} V(u_\lambda^{q-1} - u_t^{q-1})\varphi_\varepsilon \, \mathbf{d}x - \lambda \int_{\Omega_\varepsilon} K(u_\lambda^{r-1} - u_t^{r-1})\varphi_\varepsilon \, \mathbf{d}x \\
&\leq \varepsilon \iint_{\{\Omega_\varepsilon \times \Omega_\varepsilon\}^c} \frac{(\omega(x) - \omega(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y - \iint_{\{\Omega_\varepsilon \times \Omega_\varepsilon\}^c} \frac{(\omega(x) - \omega(y))^2}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y \\
&\quad + \int_{\Omega_\varepsilon} E u_\lambda \varphi_\varepsilon \, \mathbf{d}x + \int_{\Omega_\varepsilon} E u_t \varphi_\varepsilon \, \mathbf{d}x + \int_{\Omega_\varepsilon} V u_\lambda^{q-1} \varphi_\varepsilon \, \mathbf{d}x + \int_{\Omega_\varepsilon} V u_t^{q-1} \varphi_\varepsilon \, \mathbf{d}x \\
&\leq \varepsilon \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(\omega(x) - \omega(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + 2\varepsilon \iint_{\Omega_\varepsilon \times \Omega_\varepsilon^c} \frac{(\omega(x) - \omega(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y \\
&\quad + \varepsilon \left\{ \int_{\Omega_\varepsilon} E u_\lambda |\varphi| \, \mathbf{d}x + \int_{\Omega_\varepsilon} E u_t |\varphi| \, \mathbf{d}x + \int_{\Omega_\varepsilon} V u_\lambda^{q-1} |\varphi| \, \mathbf{d}x + \int_{\Omega_\varepsilon} V u_t^{q-1} |\varphi| \, \mathbf{d}x \right\} \\
&\leq 2\varepsilon \iint_{\Omega_\varepsilon \times \mathbb{R}^n} \frac{(\omega(x) - \omega(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y \\
&\quad + \varepsilon \left\{ \|\varphi\|_{L^2_\mathbb{E}(\Omega_\varepsilon)} (\|u_\lambda\|_{L^2_\mathbb{E}(\Omega_\varepsilon)} + \|u_t\|_{L^2_\mathbb{E}(\Omega_\varepsilon)}) + \|\varphi\|_{L^q_\mathbb{V}(\Omega_\varepsilon)} (\|u_\lambda\|_{L^q_\mathbb{V}(\Omega_\varepsilon)}^{q-1} + \|u_t\|_{L^q_\mathbb{V}(\Omega_\varepsilon)}^{q-1}) \right\} \\
&= 2\varepsilon \iint_{\Omega_\varepsilon \times \mathbb{B}_R} \frac{(\omega(x) - \omega(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + 2\varepsilon \iint_{\Omega_\varepsilon \times \mathbb{B}_R^c} \frac{(\omega(x) - \omega(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y \\
&\quad + \varepsilon \left\{ \|\varphi\|_{L^2_\mathbb{E}(\Omega_\varepsilon)} (\|u_\lambda\|_{L^2_\mathbb{E}(\Omega_\varepsilon)} + \|u_t\|_{L^2_\mathbb{E}(\Omega_\varepsilon)}) + \|\varphi\|_{L^q_\mathbb{V}(\Omega_\varepsilon)} (\|u_\lambda\|_{L^q_\mathbb{V}(\Omega_\varepsilon)}^{q-1} + \|u_t\|_{L^q_\mathbb{V}(\Omega_\varepsilon)}^{q-1}) \right\} \\
&\leq 2\varepsilon \iint_{\Omega_\varepsilon \times \mathbb{B}_R} \frac{(\omega(x) - \omega(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y + 2\varepsilon \iint_{\{\mathbb{B}_R \times \mathbb{B}_R\}^c} \frac{(\omega(x) - \omega(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, \mathbf{d}x \mathbf{d}y \\
&\quad + \varepsilon \left\{ \|\varphi\|_{L^2_\mathbb{E}(\Omega_\varepsilon)} (\|u_\lambda\|_{L^2_\mathbb{E}(\Omega_\varepsilon)} + \|u_t\|_{L^2_\mathbb{E}(\Omega_\varepsilon)}) + \|\varphi\|_{L^q_\mathbb{V}(\Omega_\varepsilon)} (\|u_\lambda\|_{L^q_\mathbb{V}(\Omega_\varepsilon)}^{q-1} + \|u_t\|_{L^q_\mathbb{V}(\Omega_\varepsilon)}^{q-1}) \right\}
\end{aligned}$$

with $\omega := u_\lambda - u_t \geq 0$, seeing $0 < \varphi_\varepsilon \leq \varepsilon|\varphi|$ on Ω_ε ; that is, we have $\mathcal{F}'_\lambda(u_\lambda)(\varphi_\varepsilon) \rightarrow o(\varepsilon)$ when $R \rightarrow \infty$ and $\varepsilon \rightarrow 0^+$. This combined with (6.5) yields $\mathcal{F}'_\lambda(u_\lambda)(\varphi) \leq 0$ for all $\varphi \in C^1_c(\mathbb{R}^n)$; so, we also get $\mathcal{F}'_\lambda(u_\lambda)(-\varphi) \leq 0$ via symmetry. By density, it follows that $\mathcal{F}'_\lambda(u_\lambda)(v) = 0$ for each $v \in M^{s;q,2}_{\mathbb{E},\mathbb{V}}(\mathbb{R}^n)$, and hence $u_\lambda (\geq u_t \geq 0)$ is a nontrivial solution to (1.6) at λ . $\square$

Proposition 6.2 obviously provides the proof for the first part of Theorem 1.2. Now, we will proceed to discuss the second part of Theorem 1.2 by first giving some estimates for solutions to equation (1.6) in general, which are presented as the result below.

⁴ In the subsequent estimate, the fact that $\Omega_\varepsilon \times \mathbb{R}^n = \{\Omega_\varepsilon \times \mathbb{B}_R\} \cup \{\Omega_\varepsilon \times \mathbb{B}_R^c\} \subsetneq \{\Omega_\varepsilon \times \mathbb{B}_R\} \cup \{\mathbb{B}_R \times \mathbb{B}_R\}^c$ for sufficiently large R as well as that the function $\frac{(\omega(x) - \omega(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \in L^1(\mathbb{R}^{2n})$ will be used.

Lemma 6.3. Under the assumptions **(EKV2-1)** or **(EKV2-2)**, each solution u_λ to equation (1.6) in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_\lambda(x) - u_\lambda(y))^2}{|x - y|^{n+2s}} \mathbf{d}x \mathbf{d}y + \int_{\mathbb{R}^n} E u_\lambda^2 \mathbf{d}x + \int_{\mathbb{R}^n} V |u_\lambda|^q \mathbf{d}x \leq \lambda^{r_4} C'_{EKV}, \quad (6.6)$$

where $C'_{EKV} > 0$ is an absolute constant independent of λ, u . Moreover, when the assumption **(EK)** also holds, there are absolute constants $C_{EK}, \tilde{C}_{EKV} > 0$ such that $\lambda_1 \geq \tilde{C}_{EKV} > 0$ and such that each nontrivial solution u_λ to equation (1.6) in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} K |u_\lambda|^r \mathbf{d}x \geq (\lambda C_{EK}^{\frac{2}{r}})^{\frac{r}{2-r}}. \quad (6.7)$$

Proof. First, (6.6) follows from the fact that each solution u_λ to (1.6) satisfies

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_\lambda(x) - u_\lambda(y))^2}{|x - y|^{n+2s}} \mathbf{d}x \mathbf{d}y + \int_{\mathbb{R}^n} E u_\lambda^2 \mathbf{d}x + \int_{\mathbb{R}^n} V |u_\lambda|^q \mathbf{d}x = \lambda \int_{\mathbb{R}^n} K |u_\lambda|^r \mathbf{d}x \quad (6.8)$$

and an estimate akin to (6.3), which is left as an easy exercise for the reader.

Now, when our condition **(EK)** also holds, from Proposition 4.4 (ii) and for $u \in M_{E,V}^{s;q,2}(\mathbb{R}^n)$ with an absolute constant $C_{EK} > 0$, one has

$$\int_{\mathbb{R}^n} K |u|^r \mathbf{d}x \leq C_{EK} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \mathbf{d}x \mathbf{d}y + \int_{\mathbb{R}^n} E u^2 \mathbf{d}x \right)^{\frac{r}{2}}. \quad (6.9)$$

Combining (6.8) and (6.9), one observes, for each nontrivial solution u_λ to (1.6),

$$\left(\int_{\mathbb{R}^n} K |u_\lambda|^r \mathbf{d}x \right)^{\frac{2}{r}} \leq \lambda C_{EK}^{\frac{2}{r}} \int_{\mathbb{R}^n} K |u_\lambda|^r \mathbf{d}x,$$

which in turn yields (6.7). Also, by (6.6), (6.7) and (6.9), we get $(\lambda C_{EK}^{\frac{2}{r}})^{\frac{2}{2-r}} \leq \lambda^{r_4} C_{EK}^{\frac{2}{r}} C'_{EKV}$, so that $\lambda \geq \tilde{C}_{EKV} > 0$ as $2 < r$, which in particular leads to $\lambda_1 \geq \tilde{C}_{EKV} > 0$. $\square$

Finally, we are ready to discuss the second twofold claim of Theorem 1.2.

Proposition 6.4. Under our conditions **(EKV2-1)** and **(EK)**, we have $\lambda_1 > 0$ and equation (1.6) has a nontrivial positive solution $u_\lambda \geq 0$ in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$ if and only if $\lambda \geq \lambda_1$; besides, for $\lambda_2 := \tilde{\lambda}(\geq \lambda_1 > 0)$ as given in (6.4), equation (1.6) has at least two distinct nontrivial positive solutions $u_\lambda, \tilde{u}_\lambda \geq 0$ in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$ for all $\lambda > \lambda_2$.

Proof. Note if $u_\lambda \geq 0$ is a nontrivial positive solution to (1.6) in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$, then $\lambda \geq \lambda_1 > 0$; so, we are left to show (1.6) has a nontrivial positive solution at λ_1 . Let $\{\lambda(k) > \lambda_1 : k \geq 1\}$ decrease to λ_1 , with $\{u_{\lambda(k)} \geq 0 : k \geq 1\}$ an associated sequence of nontrivial solutions to (1.6) that is bounded in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$ by (6.6). Then, there exists a subsequence $\{u_{\lambda(k)} \geq 0 : k \geq 1\}$, using the same notation, such that $u_{\lambda(k)} \rightharpoonup w$ in $H_E^s(\mathbb{R}^n)$, $u_{\lambda(k)} \rightharpoonup w$ in $L_V^q(\mathbb{R}^n)$, $u_{\lambda(k)} \rightarrow w$ in $L_K^r(\mathbb{R}^n)$, and $u_{\lambda(k)} \rightarrow w$ a.e. on $\mathbb{R}^n$ for a function $w \geq 0$ in $M_{E,V}^{s;q,2}(\mathbb{R}^n)$. Thus, one has

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} K u_{\lambda(k)}^{r-1} v \mathbf{d}x = \int_{\mathbb{R}^n} K w^{r-1} v \mathbf{d}x, \quad \forall v \in M_{E,V}^{s;q,2}(\mathbb{R}^n).$$

Keep in mind $\mathcal{F}'_{\lambda(k)}(u_{\lambda(k)}) = 0$; that is, (6.2) holds for $u = u_{\lambda(k)}$ and $\lambda = \lambda(k)$. Since $\mathcal{F}'_{\lambda}(w)$ is a continuous linear functional on $M^{s;q,2}_{E,V}(\mathbb{R}^n)$, it follows that

$$\begin{aligned} 0 &\leftarrow \mathcal{F}'_{\lambda(k)}(u_{\lambda(k)})(u_{\lambda(k)} - w) + \lambda(k) \int_{\mathbb{R}^n} Ku_{\lambda(k)}^{r-1}(u_{\lambda(k)} - w) \, dx \\ &\quad - \mathcal{F}'_{\lambda}(w)(u_{\lambda(k)} - w) - \lambda \int_{\mathbb{R}^n} Kw^{r-1}(u_{\lambda(k)} - w) \, dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\varpi(x) - \varpi(y))^2}{|x - y|^{n+2s}} \, dx dy + \int_{\mathbb{R}^n} E(u_{\lambda(k)} - w)^2 \, dx \\ &\quad + \int_{\mathbb{R}^n} V(u_{\lambda(k)}^{q-1} - w^{q-1})(u_{\lambda(k)} - w) \, dx \\ &\geq [u_{\lambda(k)} - w]_{s,2,\mathbb{R}^n}^2 + \|u_{\lambda(k)} - w\|_{L^2_E(\mathbb{R}^n)}^2 + C_q \|u_{\lambda(k)} - w\|_{L^q_V(\mathbb{R}^n)}^q \geq 0 \text{ as } k \rightarrow \infty \end{aligned}$$

for $\varpi := u_{\lambda(k)} - w$ and an absolute constant $C_q > 0$ (see Han [29, Lemma 3.2]); so, $u_{\lambda(k)} \rightarrow w$ in $H^s_E(\mathbb{R}^n)$ and $L^q_V(\mathbb{R}^n)$. Therefore, (6.2) is satisfied by w and λ for all $v \in M^{s;q,2}_{E,V}(\mathbb{R}^n)$ upon taking the limit $k \rightarrow \infty$. Using (6.7), one sees $\int_{\mathbb{R}^n} Kw^r \, dx \geq (\lambda(1)C_{EK}^{\frac{2}{r-2}})^{\frac{r}{r-2}} > 0$ by virtue of the compact embedding $M^{s;q,2}_{E,V}(\mathbb{R}^n) \hookrightarrow L^r_K(\mathbb{R}^n)$. So, $w \geq 0$ is a nontrivial positive solution in $M^{s;q,2}_{E,V}(\mathbb{R}^n)$ to equation (1.6); that is, $w = u_{\lambda_1}$ in common practice notation.

On the other hand, for each $\lambda > \lambda_2$, there is a solution $u_{\lambda} \geq 0$ to (1.6) with $\mathcal{F}_{\lambda}(u_{\lambda}) < 0$. Furthermore, it is easily seen from (6.9) that, for some constant $C'_{EK} > 0$,

$$\mathcal{F}_{\lambda}(u) \geq \frac{1}{4} \|u\|_{H^s_E(\mathbb{R}^n)}^2 - \frac{\lambda}{r} C'_{EK} \|u\|_{H^s_E(\mathbb{R}^n)}^r \geq \|u\|_{H^s_E(\mathbb{R}^n)}^2 \left(\frac{1}{4} - \frac{\lambda}{r} C'_{EK} \|u\|_{H^s_E(\mathbb{R}^n)}^{r-2} \right),$$

and thus $\mathcal{F}_{\lambda}(u) \geq \tilde{\rho}_0 > 0$ provided $\|u\|_{H^s_E(\mathbb{R}^n)} = \tilde{\rho} < \min \left\{ \|u_{\lambda}\|_{H^s_E(\mathbb{R}^n)}, \left(\frac{r}{4\lambda C'_{EK}} \right)^{\frac{1}{r-2}} \right\}$. So, via the mountain pass theorem of Candela and Palmieri [19, Theorem 2.5] (see also [7, Theorem A.3]), there exists a sequence $\{u_l : l \geq 1\}$ in $M^{s;q,2}_{E,V}(\mathbb{R}^n)$ with $\mathcal{F}_{\lambda}(u_l) \rightarrow c > 0$ and $\mathcal{F}'_{\lambda}(u_l) \rightarrow 0$ in the dual space of $M^{s;q,2}_{E,V}(\mathbb{R}^n)$ when $l \rightarrow \infty$ for $c = \inf_{h \in \mathcal{H}_{\lambda}} \max_{t \in [0,1]} \mathcal{F}_{\lambda}(h(t)) > 0$ with $\mathcal{H}_{\lambda} := \{h \in C([0,1]; M^{s;q,2}_{E,V}(\mathbb{R}^n)) : h(0) = 0, h(1) = u_{\lambda}\}$. Lemma 6.1 provides a subsequence $\{u_l : l \geq 1\}$, using the same notation, such that $u_l \rightharpoonup \tilde{w}$ in $H^s_E(\mathbb{R}^n)$, $u_l \rightharpoonup \tilde{w}$ in $L^q_V(\mathbb{R}^n)$, $u_l \rightarrow \tilde{w}$ in $L^r_K(\mathbb{R}^n)$, and $u_l \rightarrow \tilde{w}$ a.e. on $\mathbb{R}^n$ for a function $\tilde{w} \in M^{s;q,2}_{E,V}(\mathbb{R}^n)$. Similar analysis then yields $u_l \rightarrow \tilde{w}$ in $M^{s;q,2}_{E,V}(\mathbb{R}^n)$; that is, $\tilde{u}_{\lambda} := \tilde{w} \geq 0$ is a nontrivial solution to (1.6) in $M^{s;q,2}_{E,V}(\mathbb{R}^n)$, as $\mathcal{F}'_{\lambda}(\tilde{u}_{\lambda}) = 0$ and thus (6.2) is satisfied, which is distinct from u_{λ} since $\mathcal{F}_{\lambda}(\tilde{u}_{\lambda}) = c > 0$. $\square$

A Appendix

This section provides an additional analysis for Subsection 2.1. Recall Ω denotes a bounded domain having a compact, Lipschitz boundary $\partial\Omega$. Note $L^q(\Omega) \rightarrow L^1(\Omega)$. Using Minkowski's inequality and Lemma 3.1 of Bellido and Mora-Corral [11], we have

$$\begin{aligned} \|u\|_{p,\Omega} &\leq \|u - \bar{u}\|_{p,\Omega} + \|\bar{u}\|_{p,\Omega} \leq C_5([u]_{s,p,\Omega} + \|u\|_{1,\Omega}) \\ &\leq C'_5([u]_{s,p,\Omega} + \|u\|_{q,\Omega}), \quad \forall u \in M^{s;q,p}(\Omega) \text{ and } 1 \leq q \leq \infty \end{aligned} \quad (\text{A.1})$$

with $\bar{u} := \frac{1}{|\Omega|} \int_{\Omega} u \, dx$ and $C_5, C'_5 > 0$ some constants depending on n, p, q, s, Ω . Thus, (2.3) follows from (A.1) and (2.1) consequently; that is, $M^{s;q,p}(\Omega) \rightarrow W^{s,p}(\Omega)$.

On the other hand, seeing $W^{s,p}(\Omega) \rightarrow L^q(\Omega)$ for $1 \leq q \leq p_s^*$ when $sp < n$, one accordingly has $W^{s,p}(\Omega) \rightarrow M^{s;q,p}(\Omega)$, which together with the preceding analysis yields (2.2). Moreover, noticing $W^{s,p}(\Omega) \rightarrow L^q(\Omega)$ for $1 \leq q < \infty$ or $1 \leq q \leq \infty$ when $sp = n$ or $sp > n$, the preceding analysis then leads to (2.4) and (2.6) respectively.

Conflict of interest statement: There is no conflict of interest.

Acknowledgement: The author is greatly indebted to the anonymous referees for helpful suggestions. The author gratefully acknowledges the partial financial support of the Research Council Grant #21870120011 plus the 2017–2021 faculty research fellowships from the College of Arts and Sciences at Texas A&M University-San Antonio.

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