Research article

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On a class of critical elliptic systems in $\mathbb{R}^4$ <https://doi.org/10.1515/anona-2020-0136>

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Abstract: In the present paper, we consider the following classes of elliptic systems with Sobolev critical growth:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta uv^2 + \frac{2q}{p} y u^{\frac{2q}{p}-1} v^2 & \text{in } \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v + 2y u^{\frac{2q}{p}} v & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^4$ is a smooth bounded domain with smooth boundary $\partial\Omega$; p, q are positive coprime integers with $1 < \frac{2q}{p} < 2$; $\mu_i > 0$ and $\lambda_i \in \mathbb{R}$ are fixed constants, $i = 1, 2$; $\beta > 0, y > 0$ are two parameters. We prove a nonexistence result and the existence of the ground state solution to the above system under proper assumptions on the parameters. It seems that this system has not been explored directly before.

Keywords: Schrödinger system; Nehari manifold; Ground state

MSC: 35J50, 35J15, 35J60

1 Introduction

In the current paper, we consider the following system:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta uv^2 + \frac{2q}{p} y u^{\frac{2q}{p}-1} v^2 & \text{in } \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v + 2y u^{\frac{2q}{p}} v & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^4$ is a smooth bounded domain with a smooth boundary $\partial\Omega$; p, q are positive coprime integers with $1 < \frac{2q}{p} < 2$, $\mu_i > 0$ and $\lambda_i \in \mathbb{R}$ are fixed constants, $i = 1, 2$; $\beta > 0, y > 0$ are parameters.

In $\mathbb{R}^4$, the Sobolev critical exponent $2^* = 4$, therefore, the system (1.1) has both critical nonlinearities and subcritical coupling perturbations. The study about the solutions of (1.1) when $y = 0$ arises in the solitary wave solutions of the following Schrödinger equations, known as the Gross-Pitaevskii equations(cf.[13, 20]):

$$\begin{cases} -i \frac{\partial}{\partial t} \Psi_1 = \Delta \Psi_1 + \mu_1 |\Psi_1|^2 \Psi_1 + \beta |\Psi_2|^2 \Psi_1, \\ -i \frac{\partial}{\partial t} \Psi_2 = \Delta \Psi_2 + \mu_2 |\Psi_2|^2 \Psi_2 + \beta |\Psi_1|^2 \Psi_2, \\ \Psi_i = \Psi_i(t, x) \in H^1(\mathbb{R}^N; \mathbb{C}), i = 1, 2, N = 1, 2, 3. \end{cases} \quad (1.2)$$

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Here i is the imaginary unit. This system was developed as a mathematical model for Bose-Einstein condensation in hyperfine spin states. To obtain solitary wave solutions, we set $\Psi_i(t, x) = e^{it\lambda_i} u_i(x)$ for $i = 1, 2$, then $u_i(x)$ satisfy the elliptic system:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1 & \text{in } \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 & \text{in } \Omega, \\ u_1, u_2 > 0 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

Since the solitary waves of (1.2) arise in many physical problems, especially in the Hartree-Fock theory and nonlinear optics, the system (1.3) in low dimensions ($1 \leq N \leq 3$) has been well studied. When $N \leq 3$, the nonlinearity and the coupling terms in (1.3) are of subcritical growth. For example, we refer to [11, 14, 16, 20, 21, 25] and the references therein. Recently, multiple results of (1.3) in $\mathbb{R}^N$ with $N \geq 4$ have been derived (cf. [10, 19]). In these cases, the nonlinearities and coupled terms are Sobolev critical for $N = 4$ and even super-critical for $N \geq 5$. Therefore, the study of the cases in high dimensions is much more complicated due to the lack of compactness. In [10], the authors proved that the 2-component system (1.3) has a positive least-energy solution when $N = 4$ for negative β , positive small β and positive large β with $\mu_1, \mu_2 > 0$, $-\alpha_1(\Omega) < \lambda_1, \lambda_2 < 0$, where $\alpha_1(\Omega)$ is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition. In [8, 19], the existence of infinitely many sign-changing solutions for the k -component system (1.3) for $N \geq 2$, $k \geq 2$ was obtained.

From the perspective of mathematics, we also note that the related system as (1.1) was studied by [6] in 1984. Actually, in [6] it was considered the systems of equations

$$-\Delta u_i(x) = g^i(u(x)), \quad i = 1, 2, \dots, N_0,$$

where the functions g^i are the gradients of some functions. Under some hypotheses on g^i , the existence results of the ground state solution were established (see Theorem 2.2 and Theorem 2.3 in [6]).

Furthermore, in [23], the author obtained some results including the existence and nonexistence of the ground states when $N = 4$ to the following Sobolev critical system, which has both linear perturbations and nonlinear couplings arising from (1.3):

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1 + \gamma u_2 & \text{in } \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 + \gamma u_1 & \text{in } \Omega, \\ u_1, u_2 > 0 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

here $\beta > 0$ (attractive), $\gamma > 0$. Such kinds of systems arise in nonlinear optics (cf. [1]). There are also some results on the linearly coupled system like (1.4). For example, in [2] the multi-bump solitons were obtained in the case of the dimension $N \leq 3$. For more general nonlinear terms and $N = 1$, we refer to [3] (see also [4] for $N \geq 2$).

Our interest in (1.1) mainly comes from the challenge in mathematics since this system has not been explored directly before. An idea of dealing with the lack of compactness contributed by Brezis and Nirenberg in [7], is to establish a threshold for the compactness of the (PS) sequence. Recall that the methods in [9, 10] for the critical systems with either only nonlinear couplings or only linear couplings cannot be applied in the present paper, because of the existence with both critical and subcritical nonlinear couplings in our case (1.1). We will devote to obtain a uniqueness result to the related algebraic equation when $\beta > 0$ is properly large, and to estimate the least energy of the related functional by exploring the minimizing problem (3.2) (see Lemma 3.1) below.

Let us present the necessary notations and definitions before stating the main results for (1.1). Set $\mathcal{H} := H_0^1(\Omega) \times H_0^1(\Omega)$. Define the functional $I : \mathcal{H} \rightarrow \mathbb{R}$ by

$$I(u, v) := \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 + \lambda_1 u^2 + \lambda_2 v^2) - \frac{1}{4} \int_{\Omega} (\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2) - y \int_{\Omega} u^{\frac{2q}{p}} v^2. \quad (1.5)$$

Then $I(u, v)$ is of class C^1 in $\mathcal{H}$ and the positive solutions of (1.1) correspond to the positive critical points of $I(u, v)$. We define the Nehari manifold

$$\mathcal{N} := \{(u, v) \in \mathcal{H} \setminus \{(0, 0)\} : F(u, v) := I'(u, v)(u, v) = 0\}, \quad (1.6)$$

where

$$F(u, v) = \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 + \lambda_1 u^2 + \lambda_2 v^2) - \int_{\Omega} (\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2) - \left(\frac{2q}{p} + 2\right)y \int_{\Omega} u^{\frac{2q}{p}} v^2. \quad (1.7)$$

Then any nontrivial positive solution of (1.1) belongs to $\mathcal{N}$. It is not hard to check that $\mathcal{N} \neq \emptyset$. Define

$$\mathcal{B} := \inf_{(u, v) \in \mathcal{N}} I(u, v). \quad (1.8)$$

Since

$$I|_{\mathcal{N}}(u, v) = \frac{q}{2(p+q)} \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 + \lambda_1 u^2 + \lambda_2 v^2) + \frac{p-q}{4(p+q)} \int_{\Omega} (\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2)$$

here $\mu_i > 0$ and $\beta \geq 0$, then I is bounded from below on $\mathcal{N}$ with $0 < \frac{2q}{p} < 2$.

Definition 1.1. The pair (u, v) is called a nontrivial solution iff $u \neq 0, v \neq 0$; we call (u, v) is positive iff $u > 0, v > 0$; the nontrivial solution (u_0, v_0) is called a ground state solution to (1.1) iff $I(u_0, v_0) \leq I(u, v)$ for all nontrivial critical points (u, v) .

Firstly, we consider a little relaxed system:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2 + \frac{2q}{p} y u^{\frac{2q}{p}-1} v^2 & \text{in } \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v + 2y u^{\frac{2q}{p}} v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.9)$$

where $u, v \in H_0^1(\Omega)$ and $y, \lambda_1, \lambda_2 \in \mathbb{R}; \mu_1, \mu_2 > 0$. The first conclusion in this paper is about the nonexistence of solution to the systems (1.1) and (1.9).

Theorem 1.2. Let $\alpha_1(\Omega) > 0$ be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

- 1) Assume $\mu_1, \mu_2, \beta, y > 0$. If $\min(\lambda_1, \lambda_2) \leq -\alpha_1(\Omega)$, then system (1.1) has no solution;
- 2) Assume Ω is star-shaped or $\Omega = \mathbb{R}^4$. If $y \leq 0$ and $\min(\lambda_1, \lambda_2) > 0$, then (1.9) has no nontrivial solution.

Before stating the next main theorem, we recall the Brezis-Nirenberg type critical exponent equation (see [7]):

$$-\Delta u + \lambda_i u = \mu_i u^3, \quad u \geq 0, \quad u \in H_0^1(\Omega),$$

it has a positive least energy solution $u_{\mu_i} \in C^2(\Omega) \cap C(\bar{\Omega})$ when $-\alpha_1(\Omega) < \lambda_i < 0, N = 4$ with energy

$$\begin{aligned} B_{\mu_i} &:= \frac{1}{2} \int_{\Omega} (|\nabla u_{\mu_i}|^2 + \lambda_i u_{\mu_i}^2) - \frac{1}{4} \int_{\Omega} \mu_i u_{\mu_i}^4 \\ &= \frac{1}{4} \int_{\Omega} (|\nabla u_{\mu_i}|^2 + \lambda_i u_{\mu_i}^2) = \frac{1}{4} \int_{\Omega} \mu_i u_{\mu_i}^4 > 0, \end{aligned} \quad (1.10)$$

and there is no solution when $\lambda_i \geq 0$ and Ω is a (smooth) starshaped domain.

Remark 1.3. Assume that $-\alpha_1(\Omega) < \lambda_i < 0$. Let S be the sharp constant of $D^{1,2}(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$. Then B_{μ_i} mentioned in (1.10) satisfies (See [9]):

$$\frac{1}{N} \left(\frac{\alpha_1(\Omega) + \lambda_i}{\alpha_1(\Omega)} \right)^{\frac{N}{2}} \mu_i^{\frac{2-N}{2}} S^{\frac{N}{2}} \leq B_{\mu_i} < \frac{1}{N} \mu_i^{\frac{2-N}{2}} S^{\frac{N}{2}}, \quad i = 1, 2.$$

Define

$$\mathcal{N}_{\mu_i} := \{u \in H_0^1(\Omega) : \int_{\Omega} (|\nabla u|^2 + \lambda_i u^2 - \mu_i u^4) = 0\}. \quad (1.11)$$

Let $C(\Omega)$ be the embedding constant of $H_0^1(\Omega)$ to $L^2(\Omega)$, i.e.,

$$\int_{\Omega} |f|^2 \leq (C(\Omega))^2 \int_{\Omega} |\nabla f|^2. \quad (1.12)$$

Here, $C(\Omega) = \left(\frac{1}{\alpha_1(\Omega)}\right)^{\frac{1}{2}}$.

Theorem 1.4. Let $\lambda_1, \lambda_2 > -\alpha_1(\Omega)$, $\mu_1, \mu_2, \beta, \gamma > 0$. Assume that λ_1 satisfies one of the following two conditions:

$$(1_1) \lambda_1 < 0 \text{ and } \beta > \frac{\left(\int_{\Omega} |\nabla v_0|^2 + \lambda_2^2 v_0^2\right) - 2\gamma \left(\int_{\Omega} u_{\mu_1}^{\frac{2\gamma}{p}} v_0^2\right)}{\int_{\Omega} u_{\mu_1}^2 v_0^2}, \quad \text{for some } 0 \neq v_0 \in H_0^1(\Omega);$$

$(1_2) \lambda_1 > 0$ with Ω is a (smooth) starshaped domain,

and that λ_2 satisfies one of the following two conditions:

$(2_1) \lambda_2 < 0$;

$(2_1) \lambda_2 > 0$ with Ω is a (smooth) starshaped domain.

Then the system (1.1) has a ground state solution provided that one of the following cases holds:

a) $\min(\lambda_1, \lambda_2) < 0 < \max(\lambda_1, \lambda_2)$ and $\beta > \max(\mu_1, \mu_2)$;

b) $\min(\lambda_1, \lambda_2) > 0$ and $\beta > \max(\mu_1, \mu_2)$;

c) $\max(\lambda_1, \lambda_2) < 0$ and $\beta > \sqrt{\mu_1 \mu_2}$.

Remark 1.5. If, for example, (1_1) and (2_1) hold, then only the case (c) occurs by default.

The rest of this paper will be devoted to the proofs of these theorems. We denote the norm of $L^p(\Omega)$ by $\|u\|_p = \left(\int_{\Omega} |u|^p dx\right)^{\frac{1}{p}}$, the norm of $H_0^1(\Omega)$ by $\|u\| = \|\nabla u\|_2$ and positive constants (possibly different) by C . The paper is organized as follows. In section 2, we study the nonexistence of solutions by establishing a kind of Pohozaev's type identity. Then in Section 3, we shall show some preliminary results including energy estimation. In section 4, we will use Ekeland variational principle and the results of Section 3 to prove Theorem 1.4.

2 Nonexistence

Lemma 2.1. [Pohozaev identity] Suppose u, v are (smooth) functions satisfying

$$\begin{cases} -\Delta u = g(u, v) & \text{in } \Omega, \\ -\Delta v = f(u, v) & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where $\Omega \subset \mathbb{R}^n (n \geq 3)$; g and f are continuous on $\mathbb{R}^2$. Then we have

$$\int_{\Omega} \left(n(G + F) + \left(\frac{\partial G}{\partial v} \nabla v + \frac{\partial F}{\partial u} \nabla u \right) \cdot \vec{x} \right) = \frac{n}{2^*} (\|u\|^2 + \|v\|^2) + \frac{1}{2} \int_{\partial\Omega} (|\nabla u|^2 + |\nabla v|^2) (\vec{x} \cdot \vec{v}), \quad (2.2)$$

where $2^* = \frac{2n}{n-2}$, $G(u, v) = \int_0^u g(t, v) dt$, $F(u, v) = \int_0^v f(u, t) dt$ and $\vec{v}$ denotes the outward normal to $\partial\Omega$.

Proof. We multiply the two equations of (2.1) by $(\vec{x} \cdot \nabla u)$, $(\vec{x} \cdot \nabla v)$ respectively and integrate over Ω . Then add them up, we have

$$\int_{\Omega} \left(-(\vec{x} \cdot \nabla u) \Delta u - (\vec{x} \cdot \nabla v) \Delta v \right) = \int_{\Omega} \left(g(u, v)(\vec{x} \cdot \nabla u) + f(u, v)(\vec{x} \cdot \nabla v) \right). \quad (2.3)$$

By Pohozaev identity (cf. [7]), we have

$$\begin{aligned} \int_{\Omega} -\Delta u (\vec{x} \cdot \nabla u) &= \left(1 - \frac{n}{2}\right) \int_{\Omega} |\nabla u|^2 - \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (\vec{x} \cdot \vec{\nu}) ds, \\ \int_{\Omega} -\Delta v (\vec{x} \cdot \nabla v) &= \left(1 - \frac{n}{2}\right) \int_{\Omega} |\nabla v|^2 - \frac{1}{2} \int_{\partial\Omega} |\nabla v|^2 (\vec{x} \cdot \vec{\nu}) ds \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} \int_{\Omega} g(u, v)(\vec{x} \cdot \nabla u) &= \int_{\Omega} \sum_{i=1}^n g(u, v) \cdot x_i \frac{\partial u}{\partial x_i} \\ &= \int_{\Omega} \sum_{i=1}^n \frac{\partial G}{\partial x_i} x_i - \sum_{i=1}^n \frac{\partial G}{\partial v} \frac{\partial v}{\partial x_i} x_i \\ &= -n \int_{\Omega} G(u, v) - \int_{\Omega} \frac{\partial G}{\partial v} (\vec{x} \cdot \nabla v). \end{aligned} \quad (2.5)$$

Similarly, we have

$$\int_{\Omega} f(u, v)(\vec{x} \cdot \nabla v) = -n \int_{\Omega} F(u, v) - \int_{\Omega} \frac{\partial F}{\partial u} (\vec{x} \cdot \nabla u). \quad (2.6)$$

Then plug (2.4), (2.5) and (2.6) into (2.3), we obtain (2.2), which completes the proof. $\square$

Proof of Theorem 1.2. 1) Recall system (1.1), we know that $\beta > 0, \gamma > 0, \mu_1 > 0, \mu_2 > 0$. If $\lambda_1 = \min(\lambda_1, \lambda_2) \leq -\alpha_1(\Omega)$ and let φ be the corresponding eigenfunction of $\alpha_1(\Omega)$. Suppose that (u, v) is a solution of system (1.1), i.e., $u > 0, v > 0$. Then we multiply (1.9) with $(\varphi, 0)$ and integrate by parts, we obtain

$$0 \geq (\lambda_1 + \alpha_1(\Omega)) \int_{\Omega} u \varphi = \int_{\Omega} \nabla u \nabla \varphi + \lambda u \varphi = \int_{\Omega} (\mu_1 u^3 + \beta u v^2 + \frac{2q}{p} \gamma u^{\frac{2q}{p}-1} v^2) \varphi > 0,$$

which is impossible. Similarly, if $\lambda_2 = \min(\lambda_1, \lambda_2) \leq -\alpha_1(\Omega)$ and we multiply (1.1) with $(0, \varphi)$, we have

$$0 \geq (\lambda_2 + \alpha_1(\Omega)) \int_{\Omega} v \varphi = \int_{\Omega} (\mu_2 v^3 + \beta u^2 v + 2\gamma u^{\frac{2q}{p}} v) \varphi > 0,$$

which leads to a contradiction. So system (1.1) has no solution.

2) Let (u, v) be a solution of the system (1.9). We take

$$G(u, v) = -\frac{\lambda_1}{2} u^2 + \frac{\mu_1}{4} u^4 + \frac{\beta}{2} u^2 v^2 + \gamma u^{\frac{2q}{p}} v^2,$$

$$F(u, v) = -\frac{\lambda_2}{2} v^2 + \frac{\mu_2}{4} v^4 + \frac{\beta}{2} u^2 v^2 + \gamma u^{\frac{2q}{p}} v^2.$$

By the Pohozaev identity in Lemma 2.1, we have

$$\int_{\Omega} \left(|\nabla u|^2 + |\nabla v|^2 + 2(\lambda_1 u^2 + \lambda_2 v^2) - (\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 + 4\gamma u^{\frac{2q}{p}} v^2) \right)$$

$$= -\frac{1}{2} \int_{\partial\Omega} (|\nabla u|^2 + |\nabla v|^2)(\vec{x} \cdot \vec{v}) ds. \quad (2.7)$$

One the other hand,

$$\int_{\Omega} \left(|\nabla u|^2 + |\nabla v|^2 + \lambda_1 u^2 + \lambda_2 v^2 - (\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2) - \left(\frac{2q}{p} + 2\right) \gamma u^{\frac{2q}{p}} v^2 \right) = 0. \quad (2.8)$$

Therefore,

$$\int_{\Omega} \left(\lambda_1 u^2 + \lambda_2 v^2 - \left(2 - \frac{2q}{p}\right) \gamma u^{\frac{2q}{p}} v^2 \right) = -\frac{1}{2} \int_{\partial\Omega} (|\nabla u|^2 + |\nabla v|^2)(\vec{x} \cdot \vec{v}) ds.$$

When one of the following two conditions holds: (a) $\Omega = \mathbb{R}^4$ or (b) Ω is star-shaped, i.e., $(\vec{x} \cdot \vec{v}) > 0$ a.e. $\partial\Omega$, then we have $-\frac{1}{2} \int_{\partial\Omega} (|\nabla u|^2 + |\nabla v|^2)(\vec{x} \cdot \vec{v}) ds \leq 0$, it follows that if $\lambda_1, \lambda_2 > 0, \gamma \leq 0$, system (1.9) has no nontrivial solution. $\square$

3 Some Preliminaries

Consider the system

$$\begin{cases} -\Delta u = \mu_1 u^3 + \beta u v^2 & \mathbb{R}^4, \\ -\Delta v = \mu_2 v^3 + \beta u^2 v & \mathbb{R}^4, \end{cases}$$

the corresponding functional is

$$E(u, v) = \frac{1}{2} \int_{\mathbb{R}^4} (|\nabla u|^2 + |\nabla v|^2) - \frac{1}{4} \int_{\mathbb{R}^4} (\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2) \quad (3.1)$$

defined in $\mathcal{D} = (D^{1,2}(\mathbb{R}^4))^2$, and $\mathcal{D}$ is a Hilbert space equipped with the inner product

$$\langle (u_1, v_1), (u_2, v_2) \rangle_{\mathbb{R}^4} = \int_{\mathbb{R}^4} (\nabla u_1 \cdot \nabla u_2 + \nabla v_1 \cdot \nabla v_2).$$

Define

$$\mathcal{M} = \{(u, v) \in \mathcal{D} \setminus \{0\} : E'(u, v)(u, v) = 0\}.$$

We consider the minimizing problem

$$c := \inf_{\mathcal{M}} E(u, v).$$

Define

$$f(t_1, t_2) = \frac{t_1^2}{2} + \frac{t_2^2}{2} - \frac{\mu_1 t_1^4 + \mu_2 t_2^4}{4} - \frac{\beta}{2} t_1^2 t_2^2, \quad (3.2)$$

$$\mathcal{P} = \{(t_1, t_2) \in \mathbb{R}^2 \setminus \{(0, 0)\} : t_1^2 + t_2^2 - (\mu_1 t_1^4 + \mu_2 t_2^4) - 2\beta t_1^2 t_2^2 = 0\}.$$

Repeating the proof of Proposition 3.2 in [23], we have

Lemma 3.1. (see [23]) Let

$$d := \inf_{\mathcal{P}} f(t_1, t_2).$$

Then $c := dS^2$ is attained by (u, v) if and only if $(u, v) = (t_1 U_{\varepsilon, z}, t_2 U_{\varepsilon, z})$, where S is the best Sobolev embedding constant from $D^{1,2}(\mathbb{R}^4)$ to $L^4(\mathbb{R}^4)$ and

$$U_{\varepsilon, z}(x) = \frac{(8\varepsilon^2)^{\frac{1}{2}}}{\varepsilon^2 + |x - z|^2}$$

is the Talanti function satisfying $-\Delta U = U^3$ in $\mathbb{R}^4$ and (t_1, t_2) satisfies

$$\begin{cases} t_1 = \mu_1 t_1^3 + \beta t_1 t_2^2, \\ t_2 = \mu_2 t_2^3 + \beta t_1^2 t_2, \\ t_1, t_2 \geq 0, \\ t_1 + t_2 > 0. \end{cases} \quad (3.3)$$

Moreover, if $\beta > \max\{\mu_1, \mu_2\}$, then $t_1, t_2 > 0$.

Assume $0 \in \Omega$, $\rho > 0$ such that $B(0, 2\rho) \subset \Omega$. Take a radial symmetric cut-off function $\Psi \in C_0^2(B(0, 2\rho))$, $0 \leq \Psi(x) \leq 1$, $\Psi(x) \equiv 1$ in $B(0, \rho)$. Define $V_\varepsilon(x) = \Psi(x) \cdot U_{\varepsilon,0}(x)$, where

$$U_{\varepsilon,0}(x) = \frac{(8\varepsilon^2)^{\frac{1}{2}}}{\varepsilon^2 + |x|^2}.$$

Then, it is well known (cf. [10]) that

$$\begin{cases} \|\nabla V_\varepsilon\|_2^2 = S^2 + o(\varepsilon^2), \\ \|V_\varepsilon\|_4^4 = S^2 + o(\varepsilon^4), \\ \|V_\varepsilon\|_2^2 \geq C\varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2). \end{cases} \quad (3.4)$$

We have to estimate $\mathcal{B}$, which is given by (1.8). To this purpose, we need to prepare Lemma 3.2, Proposition 3.3 and Lemma 3.4. According to the definition of $\mathcal{N}$, if $\exists a_1, a_2 > 0$ such that the following equation

$$\begin{aligned} & \left((\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4) \|V_\varepsilon\|_4^4 \right) k^2 + \left(\left(\frac{2q}{p} + 2 \right) y a_1^{\frac{2q}{p}} a_2^2 \|V_\varepsilon\|^{\frac{2q}{p}+2} \right) k^{\frac{2q}{p}} \\ & - \left((a_1^2 + a_2^2) \|\nabla V_\varepsilon\|_2^2 + (\lambda_1 a_1^2 + \lambda_2 a_2^2) \|V_\varepsilon\|_2^2 \right) = 0 \end{aligned} \quad (3.5)$$

has a solution k , where k is dependent on ε, a_1, a_2 , then $(ka_1 V_\varepsilon, ka_2 V_\varepsilon) \in \mathcal{N}$.

Define

$$\begin{aligned} G_\varepsilon(k) := & \left((\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4) \|V_\varepsilon\|_4^4 \right) k^2 + \left(\left(\frac{2q}{p} + 2 \right) y a_1^{\frac{2q}{p}} a_2^2 \|V_\varepsilon\|^{\frac{2q}{p}+2} \right) k^{\frac{2q}{p}} \\ & - \left((a_1^2 + a_2^2) \|\nabla V_\varepsilon\|_2^2 + (\lambda_1 a_1^2 + \lambda_2 a_2^2) \|V_\varepsilon\|_2^2 \right). \end{aligned} \quad (3.6)$$

Then the equation (3.5) is equivalent to $G_\varepsilon(k) = 0$. Moreover, $G'_\varepsilon(k) \geq 0$, $\forall k \in [0, +\infty)$ as $\mu_i, \beta > 0$, which means $G_\varepsilon(k)$ is monotonically increasing on $[0, +\infty)$.

Lemma 3.2. Assume $\mu_1, \mu_2, \beta, y > 0$.

(a) If $\min(\lambda_1, \lambda_2) < 0 < \max(\lambda_1, \lambda_2)$, then $\exists a_1, a_2 > 0$ such that (3.5) has a unique positive solution k and $\exists Q < 0$ such that

$$\frac{\lambda_1 a_1^2 + \lambda_2 a_2^2}{2} - \frac{\|V_\varepsilon\|^{\frac{2q}{p}+2}}{\|V_\varepsilon\|_2^2} y a_1^{\frac{2q}{p}} a_2^2 k^{\frac{2q}{p}} \leq Q < 0, \quad (3.7)$$

for $p > 0$ and $\varepsilon > 0$ small enough.

(b) If $\min(\lambda_1, \lambda_2) > 0$, then $\forall a_1, a_2 > 0$, (3.5) has a unique positive solution k and $\exists Q < 0$ such that (3.7) holds for $\forall \varepsilon > 0$ small enough.

(c) If $\max(\lambda_1, \lambda_2) < 0$, then $\forall a_1, a_2 > 0$, (3.5) has a unique positive solution k and $\exists Q < 0$ such that (3.7) holds for $\forall p, \varepsilon > 0$ small enough.

Proof. Since $G_\varepsilon(k) \in C^1(\mathbb{R})$ and monotonically increasing on $[0, +\infty)$ with

$$\lim_{k \rightarrow +\infty} G_\varepsilon(k) = +\infty.$$

It is sufficient to verify that the unique solution $k > 0$ exists as long as

$$P_\varepsilon := -G_\varepsilon(0) = (a_1^2 + a_2^2)\|\nabla V_\varepsilon\|_2^2 + (\lambda_1 a_1^2 + \lambda_2 a_2^2)\|V_\varepsilon\|_2^2 > 0. \quad (3.8)$$

(a) Without loss of generality, we assume $\lambda_1 < 0 < \lambda_2$ in this case. Let

$$A_i := \|\nabla V_\varepsilon\|_2^2 + \lambda_i \|V_\varepsilon\|_2^2, \quad i = 1, 2.$$

Then we have $A_1 < A_2$ and $A_2 > 0$. If there exists $a_1, a_2 > 0$ such that

$$\left(\frac{a_2}{a_1}\right)^2 > -\frac{A_1}{A_2}, \quad (3.9)$$

then $P_\varepsilon = a_1^2 \left(A_1 + \left(\frac{a_2}{a_1}\right)^2 A_2 \right) > 0$ and (3.8) holds, which means (3.5) has positive solution k . Next we will show that $\forall a > 0$, $a_1 = a$, $a_2 = \sqrt{-\frac{\lambda_1}{2\lambda_2}}a$ satisfy (3.9). In fact, $-\frac{A_1}{A_2} < 0$ for $\varepsilon > 0$ small enough. Recall that $\lambda_1 < 0 \leq \lambda_2$, $A_2 > 0$ and

$$\|V_\varepsilon\|_2 \leq |B(0, 2\rho)|^{\frac{1}{4}} \cdot \|V_\varepsilon\|_4,$$

here $|B(0, 2\rho)|$ denotes the measure of the ball $B(0, 2\rho)$, then $|B(0, 2\rho)| = 8\pi^2\rho^4$ when $N = 4$, we have

$$\begin{aligned} -\frac{A_1}{A_2} &= -\frac{\|\nabla V_\varepsilon\|_2^2 + \lambda_1 \|V_\varepsilon\|_2^2}{\|\nabla V_\varepsilon\|_2^2 + \lambda_2 \|V_\varepsilon\|_2^2} \\ &\leq -\frac{\mathcal{S}^2 + o(\varepsilon^2) + \lambda_1 (|B(0, 2\rho)|^{\frac{1}{2}} \|V_\varepsilon\|_4^2)}{\|\nabla V_\varepsilon\|_2^2 + \lambda_2 \|V_\varepsilon\|_2^2} \\ &= -\frac{\mathcal{S}^2 + 2\sqrt{2}\pi\rho^2\mathcal{S}\lambda_1 + o(\varepsilon^2)}{\|\nabla V_\varepsilon\|_2^2 + \lambda_2 \|V_\varepsilon\|_2^2}. \end{aligned} \quad (3.10)$$

Take $\rho > 0$ small enough and satisfying

$$\mathcal{S} + 2\sqrt{2}\pi\rho^2\lambda_1 > 0, \quad (3.11)$$

here $\mathcal{S}$ is the best Sobolev embedding constant from $D^{1,2}(\mathbb{R}^N)$ to $L^4(\mathbb{R}^N)$. Thus we have $\mathcal{S}^2 + 2\sqrt{2}\pi\rho^2\lambda_1\mathcal{S} + o(\varepsilon^2) > \frac{1}{2}(\mathcal{S}^2 + 2\sqrt{2}\pi\rho^2\lambda_1\mathcal{S}) > 0$ when $\varepsilon > 0$ small enough and

$$\begin{aligned} -\frac{\mathcal{S}^2 + 2\sqrt{2}\pi\rho^2\lambda_1\mathcal{S} + o(\varepsilon^2)}{\|\nabla V_\varepsilon\|_2^2 + \lambda_2 \|V_\varepsilon\|_2^2} &< -\frac{\frac{1}{2}(\mathcal{S}^2 + 2\sqrt{2}\pi\rho^2\lambda_1\mathcal{S})}{\|\nabla V_\varepsilon\|_2^2 + \lambda_2 \|V_\varepsilon\|_2^2} \\ &\leq -\frac{\frac{1}{2}(\mathcal{S}^2 + 2\sqrt{2}\pi\rho^2\lambda_1\mathcal{S})}{\mathcal{S}^2 + o(\varepsilon^2) + \lambda_2 (|B(0, 2\rho)|^{\frac{1}{2}} \|V_\varepsilon\|_4^2)} \\ &= -\frac{\frac{1}{2}(\mathcal{S}^2 + 2\sqrt{2}\pi\rho^2\lambda_1\mathcal{S})}{\mathcal{S}^2 + 2\sqrt{2}\pi\rho^2\lambda_2\mathcal{S} + o(\varepsilon^2)} \\ &< -\frac{\frac{1}{2}(\mathcal{S}^2 + 2\sqrt{2}\pi\rho^2\lambda_1\mathcal{S})}{\frac{3}{2}(\mathcal{S}^2 + 2\sqrt{2}\pi\rho^2\lambda_2\mathcal{S})} < 0. \end{aligned} \quad (3.12)$$

Together with (3.10), when $\varepsilon > 0$ and $\rho > 0$ small enough, we obtain $-\frac{A_1}{A_2} < 0$. Then $\forall a > 0$, $(a_1, a_2) = (a, \sqrt{-\frac{\lambda_1}{2\lambda_2}}a)$ satisfies (3.9). Therefore (3.5) has a positive solution, and in particular

$$\begin{cases} \frac{\lambda_1 a_1^2 + \lambda_2 a_2^2}{2} < 0, \\ a_1, a_2 > 0. \end{cases}$$

So we may choose $Q = \frac{\lambda_1 a_1^2 + \lambda_2 a_2^2}{2} < 0$ and then (3.7) holds.

(b) For the case $\lambda_1, \lambda_2 > 0$, then obviously $P_\varepsilon > 0$. Therefore, the solution k exists for $\forall a_1, a_2 > 0$. In the following proposition, we will show that if the positive solution k exists, then k has positive lower bound. Note that

$$\begin{aligned} \frac{\|V_\varepsilon\|_{\frac{2q}{p}+2}^{\frac{2q}{p}+2}}{\|V_\varepsilon\|_2^2} &= \frac{\int_{B_{2\rho}(0)} (\Psi(x) \cdot U_{\varepsilon,0}(x))^{\frac{2q}{p}+2} dx}{\int_{B_{2\rho}(0)} (\Psi(x) \cdot U_{\varepsilon,0}(x))^2 dx} \\ &= \frac{\int_0^{\frac{2\rho}{\varepsilon}} r^3 (1+r^2)^{-\frac{2q}{p}+2} dr + o(1)}{8^{-\frac{q}{p}} \varepsilon^{\frac{2q}{p}} \int_0^{\frac{2\rho}{\varepsilon}} r^3 (1+r^2)^{-2} dr + o(1)} \rightarrow +\infty, \quad \varepsilon \rightarrow 0^+, \end{aligned}$$

we can take $G = \frac{\lambda_1 a_1^2 + \lambda_2 a_2^2}{y a_1^{\frac{2q}{p}} a_2^2 k_0^{\frac{2q}{p}}} > 0$, here k_0 will be given by Proposition 3.3. Then we obtain that

$$\frac{\lambda_1 a_1^2 + \lambda_2 a_2^2}{2} - \frac{\|V_\varepsilon\|_{\frac{2q}{p}+2}^{\frac{2q}{p}+2}}{\|V_\varepsilon\|_2^2} y a_1^{\frac{2q}{p}} a_2^2 k^{\frac{2q}{p}} < \frac{\lambda_1 a_1^2 + \lambda_2 a_2^2}{2} - G y a_1 a_2^2 k_0 = -\frac{\lambda_1 a_1^2 + \lambda_2 a_2^2}{2}$$

when $\varepsilon > 0$ is small enough, so we may let $Q = -\frac{\lambda_1 a_1^2 + \lambda_2 a_2^2}{2}$, then $Q < 0$ and satisfies (3.7).

(c) Recall the definition of A_i in the case (a) and $\lambda_1, \lambda_2 < 0$, if ρ satisfies

$$0 < \rho < \left(-\frac{S}{2\sqrt{2}\pi \min(\lambda_1, \lambda_2)} \right)^{\frac{1}{2}},$$

then

$$\begin{aligned} A_i &= \|\nabla V_\varepsilon\|_2^2 + \lambda_i \|V_\varepsilon\|_2^2 \\ &\geq S^2 + 2\sqrt{2}\pi\rho^2 \lambda_i S + o(\varepsilon^2) \\ &> \frac{1}{2}(S^2 + 2\sqrt{2}\pi\rho^2 \lambda_i S) > 0, \quad i = 1, 2 \end{aligned}$$

for $\varepsilon > 0$ small enough. Similar to the proof of the case (a), we obtain $P_\varepsilon = a_1^2 \left(A_1 + \left(\frac{a_2}{a_1}\right)^2 A_2 \right) > 0$ for any $a_1, a_2 > 0$, so (3.5) has a unique positive solution. Further, if we take $Q = \frac{\lambda_1 a_1^2 + \lambda_2 a_2^2}{2}$, then (3.7) holds. $\square$

Proposition 3.3. Assume $\mu_1, \mu_2, \beta, y > 0$, then for all the three cases of Lemma 3.2, $\exists a_1, a_2 > 0$ such that the positive solution k of (3.5) exists and satisfies:

$$0 < k_0 < k < 1,$$

for $\varepsilon > 0$ small enough, here k_0 is a constant depending on $\mu_i, \beta, y, a_1, a_2$.

Proof. Step 1: We are going to show that $0 < k < 1$.

i) For the case $\min(\lambda_1, \lambda_2) < 0 < \max(\lambda_1, \lambda_2)$. According to the proof of (a) in Lemma 3.2, $\forall a > 0$, $\exists (a_1, a_2) = (a, \sqrt{-\frac{\lambda_1}{2\lambda_2}} a)$ such that k exists and $\lambda_1 a_1^2 + \lambda_2 a_2^2 < 0$. Then

$$\begin{aligned} G_\varepsilon(1) &= \left((\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4) \|V_\varepsilon\|_4^4 \right) + \left(\left(\frac{2q}{p} + 2\right) y a_1^{\frac{2q}{p}} a_2^2 \|V_\varepsilon\|_{\frac{2q}{p}+2}^{\frac{2q}{p}+2} \right) \\ &\quad - \left((a_1^2 + a_2^2) \|\nabla V_\varepsilon\|_2^2 + (\lambda_1 a_1^2 + \lambda_2 a_2^2) \|V_\varepsilon\|_2^2 \right) \\ &\geq \left(\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4 \right) \left(S^2 + o(\varepsilon^4) \right) - (a_1^2 + a_2^2) \left(S^2 + o(\varepsilon^2) \right) \\ &\quad - C(\lambda_1 a_1^2 + \lambda_2 a_2^2) \left(\varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2) \right) \end{aligned}$$

$$= (\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4 - (a_1^2 + a_2^2)) \mathcal{S}^2 + o(\varepsilon).$$

We can take $a > 0$ large enough, such that a_1 and a_2 satisfy

$$\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4 - (a_1^2 + a_2^2) > 0.$$

Therefore, $G_\varepsilon(1) > 0$ for $\varepsilon > 0$ small enough, and we observe that $\exists k \in (0, 1)$ such that $G_\varepsilon(k) = 0$, here k depends on a_1, a_2, ε only, therefore it is the unique solution of the algebraic equation (3.5).

ii) For the case $\lambda_1, \lambda_2 > 0$. It follows from $\lambda_1 a_1^2 + \lambda_2 a_2^2 > 0$ and

$$\|V_\varepsilon\|_2 \leq |B(0, 2\rho)|^{\frac{1}{4}} \cdot \|V_\varepsilon\|_4, \quad |B(0, 2\rho)| = \frac{1}{2} \pi^2 (2\rho)^4 = 8\pi^2 \rho^4,$$

that

$$\begin{aligned} G_\varepsilon(1) &= \left((\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4) \|V_\varepsilon\|_4^4 \right) + \left(\left(\frac{2q}{p} + 2 \right) \gamma a_1^{\frac{2q}{p}} a_2^2 \|V_\varepsilon\|^{\frac{2q}{p}+2} \right) \\ &\quad - \left((a_1^2 + a_2^2) \|\nabla V_\varepsilon\|_2^2 + (\lambda_1 a_1^2 + \lambda_2 a_2^2) \|V_\varepsilon\|_2^2 \right) \\ &\geq \left(\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4 \right) (\mathcal{S}^2 + o(\varepsilon^2)) - (a_1^2 + a_2^2) (\mathcal{S}^2 + o(\varepsilon^4)) \\ &\quad - 2\sqrt{2}\pi\rho^2 (\lambda_1 a_1^2 + \lambda_2 a_2^2) (\mathcal{S} + o(\varepsilon^4)) \\ &= \mathcal{S} \left[(\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4 - (a_1^2 + a_2^2)) \mathcal{S} - 2\sqrt{2}\pi\rho^2 (\lambda_1 a_1^2 + \lambda_2 a_2^2) \right] \\ &\quad + o(\varepsilon^2). \end{aligned}$$

According to the case (b) of Lemma 3.2, $\forall a_1, a_2 > 0$, the unique solution k of (3.5) exists. So we can take a_1, a_2 large enough such that

$$(\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4 - (a_1^2 + a_2^2)) \mathcal{S} - 2\sqrt{2}\pi\rho^2 (\lambda_1 a_1^2 + \lambda_2 a_2^2) > 0,$$

and then $G_\varepsilon(1) > 0$ for ε small enough, which means $0 < k < 1$ since $G_\varepsilon(0) < 0$.

iii) For the case of $\max(\lambda_1, \lambda_2) < 0$, we see that $\lambda_1 a_1^2 + \lambda_2 a_2^2 < 0$. We can apply the same calculation as that used in i) to show that for a_1, a_2 large enough and $\rho, \varepsilon > 0$ small enough, there holds $0 < k < 1$.

Step 2: We show that k has a lower bound, i.e., $k > k_0 > 0$.

1) Assume $\lambda_1 < 0 < \lambda_2$. As the Step 1, we consider a_1, a_2 so that $\lambda_1 a_1^2 + \lambda_2 a_2^2 < 0$. Together with

$$\|V_\varepsilon\|^{\frac{2q}{p}+2} \leq |B(0, 2\rho)|^{\frac{p-q}{4(p+q)}} \cdot \|V_\varepsilon\|_4,$$

we have

$$\begin{aligned} G_\varepsilon(k) &\leq (\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4) (\mathcal{S}^2 + o(\varepsilon^4)) k^2 - (\lambda_1 a_1^2 + \lambda_2 a_2^2) |B(0, 2\rho)|^{\frac{1}{2}} (\mathcal{S} + o(\varepsilon^4)) \\ &\quad + \left(\frac{2q}{p} + 2 \right) \gamma a_1^{\frac{2q}{p}} a_2^2 |B(0, 2\rho)|^{\frac{p-q}{2p}} (\mathcal{S}^{\frac{q}{p}+1} + o(\varepsilon^4)) k^{\frac{2q}{p}} - (a_1^2 + a_2^2) (\mathcal{S}^2 + o(\varepsilon^2)) \\ &=: G^*(k) + o(\varepsilon), \end{aligned}$$

here

$$\begin{aligned} G^*(k) &= (\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4) \mathcal{S}^2 k^2 + \left(\frac{2q}{p} + 2 \right) \gamma a_1^{\frac{2q}{p}} a_2^2 |B(0, 2\rho)|^{\frac{p-q}{2p}} \mathcal{S}^{\frac{q}{p}+1} k^{\frac{2q}{p}} \\ &\quad - (a_1^2 + a_2^2) \mathcal{S}^2 - (\lambda_1 a_1^2 + \lambda_2 a_2^2) |B(0, 2\rho)|^{\frac{1}{2}} \mathcal{S}. \end{aligned}$$

As the proof of Lemma 3.2, we take $\rho > 0$ such that $\mathcal{S} + 2\sqrt{2}\pi\rho^2 \lambda_1 > 0$ and $(a_1, a_2) = (a, \sqrt{-\frac{\lambda_1}{2\lambda_2}} a)$. Then we obtain that $-\frac{\mathcal{S}+2\sqrt{2}\pi\rho^2 \lambda_2}{\mathcal{S}+2\sqrt{2}\pi\rho^2 \lambda_1} < 0$, it follows that $(\frac{a_1}{a_2})^2 > -\frac{\mathcal{S}+2\sqrt{2}\pi\rho^2 \lambda_2}{\mathcal{S}+2\sqrt{2}\pi\rho^2 \lambda_1}$ always holds, which means that

$$G^*(0) = -(a_1^2 + a_2^2) \mathcal{S}^2 - (\lambda_1 a_1^2 + \lambda_2 a_2^2) |B(0, 2\rho)|^{\frac{1}{2}} \mathcal{S} < 0.$$

Together with $\lim_{k \rightarrow +\infty} G^*(k) = +\infty$, there exists $k_0 > 0$ dependent on $\mu_i, \beta, \gamma, a_1, a_2$ such that $G^*(k_0) < 0$. Thus

$$G_\varepsilon(k_0) \leq G^*(k_0) + o(\varepsilon) < 0,$$

when ε is small enough and then we obtain $k_0 < k$.

2) Assume that $\lambda_i > 0, i = 1, 2$. In this case, $\lambda_1 a_1^2 + \lambda_2 a_2^2 > 0$, it follows that

$$\begin{aligned} G_\varepsilon(k) &\leq (\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4) (S^2 + o(\varepsilon^4)) k^2 - (\lambda_1 a_1^2 + \lambda_2 a_2^2) (C\varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2)) \\ &\quad - (a_1^2 + a_2^2) (S^2 + o(\varepsilon^2)) + \left(\frac{2q}{p} + 2\right) \gamma a_1^{\frac{2q}{p}} a_2^2 |B(0, 2\rho)|^{\frac{p-q}{2p}} (S^{\frac{q}{p}+1} + o(\varepsilon^4)) k^{\frac{2q}{p}} \\ &\leq (\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4) S^2 k^2 + (8\pi^2 \rho^4)^{\frac{p-q}{2p}} \left(\frac{2q}{p} + 2\right) \gamma a_1^{\frac{2q}{p}} a_2^2 S^{\frac{q}{p}+1} k^{\frac{2q}{p}} \\ &\quad - (a_1^2 + a_2^2) S^2 + o(\varepsilon) \\ &=: G^*(k) + o(\varepsilon), \end{aligned}$$

here

$$\begin{aligned} G^*(k) &= (\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4) S^2 k^2 + (8\pi^2 \rho^4)^{\frac{p-q}{2p}} \left(\frac{2q}{p} + 2\right) \gamma a_1^{\frac{2q}{p}} a_2^2 S^{\frac{q}{p}+1} k^{\frac{2q}{p}} \\ &\quad - (a_1^2 + a_2^2) S^2. \end{aligned}$$

Obviously, $G^*(0) < 0$ and $\lim_{k \rightarrow +\infty} G^*(k) = +\infty$, then $\exists k_0 > 0$ dependent on $\mu_i, \beta, \gamma, a_1, a_2$ such that $G^*(k_0) < 0$. Thus $G_\varepsilon(k_0) \leq G^*(k_0) + o(\varepsilon^2) < 0$ for $\varepsilon > 0$ small enough. Recall that k is the unique positive solution satisfying $G_\varepsilon(k) = 0$, we obtain $k_0 < k$.

3) For the case $\max(\lambda_1, \lambda_2) < 0$, we have $\lambda_1 a_1^2 + \lambda_2 a_2^2 < 0$ as that in case 1). According to the proof of Lemma 3.2, we take $\rho > 0$ satisfying $S + 2\sqrt{2}\pi\rho^2\lambda_i > 0, i = 1, 2$. Thus we can take $a_1, a_2 > 0$, note that $(\frac{a_1}{a_2})^2 > -\frac{S+2\sqrt{2}\pi\rho^2\lambda_2}{S+2\sqrt{2}\pi\rho^2\lambda_1}$, then we have

$$G^*(0) = -(a_1^2 + a_2^2) S^2 - (\lambda_1 a_1^2 + \lambda_2 a_2^2) |B(0, 2\rho)|^{\frac{1}{2}} S < 0.$$

Therefore, similar to the case 1), we get a constant $k_0 > 0$ independent of ε such that $k_0 < k$. $\square$

Lemma 3.4. Assume $\mu_1, \mu_2, \beta, \gamma > 0$. If one of the following cases holds:

- a) $\min(\lambda_1, \lambda_2) < 0 < \max(\lambda_1, \lambda_2)$ and $\beta > \max(\mu_1, \mu_2)$;
- b) $\min(\lambda_1, \lambda_2) > 0$ and $\beta > \max(\mu_1, \mu_2)$;
- c) $\max(\lambda_1, \lambda_2) < 0$;

then $\mathcal{B} < c$.

Proof. The main point of the proof is constructing a test function $(kv_1, kv_2) \in \mathcal{N}$.

a) For the case $\min(\lambda_1, \lambda_2) < 0 < \max(\lambda_1, \lambda_2)$ and $\beta > \max(\mu_1, \mu_2)$, we choose $(kv_1, kv_2) = (ka_1 V_\varepsilon, ka_2 V_\varepsilon)$. We have to determine the values k, a_1, a_2 . By Lemma 3.2, $\exists a_1, a_2 > 0$ such that for ρ satisfying (3.11) and $\varepsilon > 0$ small enough, there exists a unique k which is only dependent on ε, a_1, a_2 such that $(kv_1, kv_2) \in \mathcal{N}$. Moreover, $\exists Q < 0$ such that (3.7) holds. Together with Proposition 3.3, k is bounded, which means

$$k^2(\varepsilon) \cdot o(\varepsilon^2) = o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0 \quad (3.13)$$

and

$$Qk^2(\varepsilon) < Qk_0^2 < 0, \quad Q < 0. \quad (3.14)$$

Then it follows from (3.4) and (3.7) that

$$\mathcal{B} = \inf_{\mathcal{N}} I$$

$$\begin{aligned}
&\leq I(ka_1 V_\varepsilon, ka_2 V_\varepsilon) \\
&= \frac{1}{2}(a_1^2 + a_2^2)k^2 \|\nabla V_\varepsilon\|_2^2 + \frac{1}{2}(\lambda_1 a_1^2 + \lambda_2 a_2^2)k^2 \|V_\varepsilon\|_2^2 \\
&\quad - \frac{1}{4}(\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4)k^4 \|V_\varepsilon\|_4^4 - \gamma a_1^{\frac{2q}{p}} a_2^{\frac{2q}{p}+2} \|V_\varepsilon\|_{\frac{2q}{p}+2}^{\frac{2q}{p}+2} \\
&\leq \frac{1}{2}(a_1^2 + a_2^2)k^2 (S^2 + o(\varepsilon^2)) - \frac{1}{4}(\mu_1 a_1^4 + 2\beta a_1^2 a_2^2 + \mu_2 a_2^4)k^4 (S^2 + o(\varepsilon^4)) \\
&\quad + Qk^2 \|V_\varepsilon\|_2^2 \\
&\leq \left\{ \frac{(ka_1)^2 + (ka_2)^2}{2} - \frac{1}{4}[\mu_1(ka_1)^4 + \mu_2(ka_2)^4 + 2\beta(ka_1)^2(ka_2)^2] \right\} S^2 \\
&\quad + Qk^2 (C\varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2)) + o(\varepsilon^2). \tag{3.15}
\end{aligned}$$

It is standard (cf.[24] lemma 2.2) to show that $f(t_1, t_2)$ given by (3.2) has a global maximum point (t_1, t_2) as (3.3) in $(\mathbb{R}^+)^2$ when $\beta > \max(\mu_1, \mu_2)$. This claims

$$\left\{ \frac{(ka_1)^2 + (ka_2)^2}{2} - \frac{1}{4}[\mu_1(ka_1)^4 + \mu_2(ka_2)^4 + 2\beta(ka_1)^2(ka_2)^2] \right\} S^2 \leq dS^2 = c,$$

combining with (3.13)(3.14), it follows that $\mathcal{B} \leq c - C\varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2)$, which implies that $\mathcal{B} < c$ as $\varepsilon > 0$ small enough.

b) the proof is the same as a) by Lemma 3.2 and Proposition 3.3.

c) For the case $\max(\lambda_1, \lambda_2) < 0$, we take the test function of $\mathcal{B}$ as

$$(kv_1, kv_2) = (kt_1 V_\varepsilon, kt_2 V_\varepsilon) \in \mathcal{N},$$

where t_1, t_2 are constants given by (3.3) and $k = k(\varepsilon)$ satisfies

$$\begin{aligned}
&\left((t_1^2 + t_2^2) \|V_\varepsilon\|_4^4 \right) k^2 + \left(\left(\frac{2q}{p} + 2 \right) \gamma t_1^{\frac{2q}{p}} t_2^{\frac{2q}{p}+2} \|V_\varepsilon\|_{\frac{2q}{p}+2}^{\frac{2q}{p}+2} \right) k^{\frac{2q}{p}} \\
&\quad - \left((t_1^2 + t_2^2) \|\nabla V_\varepsilon\|_2^2 + (\lambda_1 t_1^2 + \lambda_2 t_2^2) \|V_\varepsilon\|_2^2 \right) = 0. \tag{3.16}
\end{aligned}$$

According to Lemma 3.2, such a $k > 0$ exists. Now we define

$$g_\varepsilon(k) := \frac{k^2}{2} \|\nabla V_\varepsilon\|_2^2 - \frac{k^4}{4} \|V_\varepsilon\|_4^4.$$

A direct calculation shows that

$$\frac{dg_\varepsilon(k)}{dk} = \|\nabla V_\varepsilon\|_2^2 \cdot k - \|V_\varepsilon\|_4^4 \cdot k^3,$$

so the function $g_\varepsilon(k)$ is increasing when $0 < k < \frac{\|\nabla V_\varepsilon\|_2^2}{\|V_\varepsilon\|_4^2}$ and decreasing when $k > \frac{\|\nabla V_\varepsilon\|_2^2}{\|V_\varepsilon\|_4^2}$. Recall that

$$\frac{\|\nabla V_\varepsilon\|_2^2}{\|V_\varepsilon\|_4^2} = \frac{S + o(\varepsilon^2)}{S + o(\varepsilon^4)} = 1 + o(\varepsilon^2),$$

we obtain

$$g_\varepsilon(k) \leq g_\varepsilon \left(1 + o(\varepsilon^2) \right) = \frac{1}{2} \|\nabla V_\varepsilon\|_2^2 - \frac{1}{4} \|V_\varepsilon\|_4^4 + o(\varepsilon^2).$$

Then

$$\begin{aligned}
&I(kt_1 V_\varepsilon, kt_2 V_\varepsilon) \\
&= \frac{k^2}{2} \int_{\Omega} (t_1^2 + t_2^2) |\nabla V_\varepsilon|^2 - \frac{k^4}{4} \int_{\Omega} (\mu_1 t_1^4 + \mu_2 t_2^4 + 2\beta t_1^2 t_2^2) V_\varepsilon^4 \\
&\quad + \frac{k^2}{2} \int_{\Omega} (\lambda_1 t_1^2 + \lambda_2 t_2^2) V_\varepsilon^2 - \gamma k^{\frac{2q}{p}+2} \int_{\Omega} t_1^{\frac{2q}{p}} t_2^{\frac{2q}{p}+2} V_\varepsilon^{\frac{2q}{p}+2}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{t_1^2 + t_2^2}{2} k^2 \|\nabla V_\varepsilon\|_2^2 - \frac{t_1^2 + t_2^2}{4} k^4 \|V_\varepsilon\|_4^4 + \frac{k^2}{2} (\lambda_1 t_1^2 + \lambda_2 t_2^2) \|V_\varepsilon\|_2^2 \\
&\leq \frac{t_1^2 + t_2^2}{2} (S^2 + o(\varepsilon^2)) - \frac{t_1^2 + t_2^2}{4} (S^2 + o(\varepsilon^4)) - C\varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2) \\
&= \frac{1}{4} (t_1^2 + t_2^2) S^2 - C\varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2).
\end{aligned} \tag{3.17}$$

On the other hand, note that $\|\nabla U_{\varepsilon,0}\|_2^2 = \|U_{\varepsilon,0}\|_4^4 = S^2$, we have

$$\begin{aligned}
\frac{1}{4} (t_1^2 + t_2^2) S^2 &= \frac{1}{2} (t_1^2 + t_2^2) \|\nabla U_{\varepsilon,0}\|_2^2 - \frac{1}{4} (\mu_1 t_1^4 + \mu_2 t_2^4 + 2\beta t_1^2 t_2^2) \|U_{\varepsilon,0}\|_4^4 \\
&= E(t_1 U_{\varepsilon,0}, t_2 U_{\varepsilon,0}) \\
&= c.
\end{aligned} \tag{3.18}$$

Then we have $\mathcal{B} \leq I(kt_1 V_\varepsilon, kt_2 V_\varepsilon) \leq c - C\varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2)$. Thus we can obtain $\mathcal{B} < c$ by taking $\varepsilon > 0$ small enough. $\square$

In the last section, we will use the Nehari manifold $\mathcal{N}$ given by (1.6) to obtain a (PS)-sequence $\{(u_n, v_n)\} \subset \mathcal{N}$ at the least energy level $\mathcal{B}$ by the Ekeland's variational principle, here $\mathcal{B}$ is given by (1.8). Indeed, we will show that $\mathcal{N}$ is a C^1 -manifold, and it is a natural constraint.

Lemma 3.5. *The Nehari manifold $\mathcal{N}$ of $I(u, v)$ is a natural constraint in $\mathcal{H}$ in the case $\beta \geq -\sqrt{\mu_1 \mu_2}$, $y > 0$.*

Proof. Assume $(u, v) \in \mathcal{N}$ such that $I|_{\mathcal{N}}'(u, v) = 0$. Then there exists a constant $k \in \mathbb{R}$ which is a Lagrange multiplier, such that

$$I'(u, v) + kF'(u, v) = 0,$$

then

$$I'(u, v)(u, v) + kF'(u, v)(u, v) = 0.$$

We have $I'(u, v)(u, v) = 0$, since $(u, v) \in \mathcal{N}$, then

$$kF'(u, v)(u, v) = 0.$$

On the other hand,

$$\begin{aligned}
&F'(u, v)(u, v) \\
&= 2 \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 + \lambda_1 u^2 + \lambda_2 v^2) - 4 \int_{\Omega} (\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2) \\
&\quad - \left(\frac{2q}{p} + 2\right) y \int_{\Omega} u^{\frac{2q}{p}} v^2 \\
&= -2 \int_{\Omega} (\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2) - \frac{2q}{p} \left(\frac{2q}{p} + 2\right) y \int_{\Omega} u^{\frac{2q}{p}} v^2 \\
&\leq -2 \int_{\Omega} (\sqrt{\mu_1} u^2 - \sqrt{\mu_2} v^2)^2 - \frac{2q}{p} \left(\frac{2q}{p} + 2\right) y \int_{\Omega} u^{\frac{2q}{p}} v^2 \\
&< 0,
\end{aligned}$$

since $\beta \geq -\sqrt{\mu_1 \mu_2}$, $y > 0$. Thus we have $k = 0$, which means $I'(u, v) = 0$. $\square$

The following is a version of the Ekeland's variational principle (cf. [22, Theorem 8.5]).

Lemma 3.6. *Let X be a Banach space and let $G \in C^2(X, \mathbb{R})$ be such that, for every $v \in V := \{v \in X | G(v) = 1\}$, $G'(v) \neq 0$. Let $F \in C^1(X, \mathbb{R})$ be bounded below on V , $v \in V$ and $\varepsilon, \delta > 0$. If*

$$F(v) \leq \inf_V F + \varepsilon,$$

then there exists $u \in V$ such that

$$F(u) \leq \inf_V F + 2\varepsilon, \quad \min_{\lambda \in \mathbb{R}} \|F'(u) - \lambda G'(u)\| \leq \frac{8\varepsilon}{\delta}, \quad \|u - v\| \leq 2\delta.$$

Remark 3.7. (cf. [22, Definition 5.10]) Let X be a Banach space, $\psi \in C^2(X, \mathbb{R})$, $V := \{v \in X : \psi(v) = 1\}$ and $\psi'(v) \neq 0$ for every $v \in V$. The norm of the derivative of the restriction of ψ to V at v is defined by $\|\varphi'(v)\|_*$. Moreover, (cf. [22, Proposition 5.12]), if $\varphi \in C^1(X, \mathbb{R})$ and $u \in V$ then

$$\|\varphi'(u)\|_* = \min_{\lambda \in \mathbb{R}} \|\varphi'(u) - \lambda \psi'(u)\|.$$

Lemma 3.8. Assume $\mu_1, \mu_2, \beta, \gamma > 0$, $-\alpha_1(\Omega) < \lambda_1 < 0$, $-\alpha_1(\Omega) < \lambda_2$, u_{μ_1} is given by (1.10). If $\exists v_0 \in H_0^1(\Omega) \setminus \{0\}$, such that β, γ satisfy

$$\left(\int_{\Omega} u_{\mu_1}^2 v_0^2 \right) \beta + \left(2 \int_{\Omega} u_{\mu_1}^{\frac{2q}{p}} v_0^2 \right) \gamma > \left(\int_{\Omega} |\nabla v_0|^2 + \lambda_2 v_0^2 \right), \quad (3.19)$$

then $\mathcal{B} < B_{\mu_1}$.

Proof. $\forall (u, v) \in \mathcal{H}$, assume that $\exists t > 0$ such that $(tu, tv) \in \mathcal{N}$, then t satisfies

$$\begin{aligned} & \left(\int_{\Omega} \mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 \right) t^2 + \left(\left(\frac{2q}{p} + 2 \right) \gamma \int_{\Omega} u^{\frac{2q}{p}} v^2 \right) t^{\frac{2q}{p}} \\ & = \left(\int_{\Omega} |\nabla u|^2 + |\nabla v|^2 + \lambda_1 u^2 + \lambda_2 v^2 \right). \end{aligned} \quad (3.20)$$

Let $(u, v) = (u_{\mu_1}, sv_0)$, here u_{μ_1} is given by (1.10), $0 \neq v_0 \in H_0^1(\Omega)$ and $0 \neq s \in \mathbb{R}$. By (3.20), there exists $t(s) = t(s, u_{\mu_1}, v_0)$ such that $(t(s)u_{\mu_1}, t(s)sv_0) \in \mathcal{N}$, and $t(s)$ is the positive solution of the following equation of t :

$$\begin{aligned} H_s(t) &:= \left(\int_{\Omega} \mu_1 u_{\mu_1}^4 + \mu_2 s^4 v_0^4 + 2\beta u_{\mu_1}^2 v_0^2 s^2 \right) t^2 + \left(\left(\frac{2q}{p} + 2 \right) \gamma s^2 \int_{\Omega} u_{\mu_1}^{\frac{2q}{p}} v_0^2 \right) t^{\frac{2q}{p}} \\ &\quad - \left(\int_{\Omega} |\nabla u_{\mu_1}|^2 + s^2 |\nabla v_0|^2 + \lambda_1 u_{\mu_1}^2 + s^2 \lambda_2 v_0^2 \right) = 0. \end{aligned} \quad (3.21)$$

In fact, $H_s(t)$ is continuous and monotonically increasing on $[0, +\infty)$ with

$$\lim_{k \rightarrow +\infty} H_s(k) = +\infty$$

and $\int_{\Omega} |\nabla v_0|^2 + \lambda_2 v_0^2 \geq 2\sqrt{B_{\mu_2}} \left(\int_{\Omega} v_0^4 \right)^{\frac{1}{2}} > 0$, when $\lambda_2 < 0$; $\int_{\Omega} |\nabla v_0|^2 + \lambda_2 v_0^2 \geq 0$, when $\lambda_2 \geq 0$. So

$$\begin{aligned} H_s(0) &= - \int_{\Omega} (|\nabla u_{\mu_1}|^2 + \lambda_1 u_{\mu_1}^2 + s^2 |\nabla v_0|^2 + s^2 \lambda_2 v_0^2) \\ &= -(4B_{\mu_1} + s^2 \int_{\Omega} |\nabla v_0|^2 + \lambda_2 v_0^2) < 0. \end{aligned}$$

Then $\exists$ unique $t = t(s) > 0$ such that (3.21) holds. Moreover, $t(s) \rightarrow 1$, when $s \rightarrow 0$. In order to simplify the calculation process, we define

$$\begin{aligned} m_1 &:= \int_{\Omega} \mu_2 v_0^4, \quad m_2 := \int_{\Omega} 2\beta u_{\mu_1}^2 v_0^2, \quad m_3 := \left(\frac{2q}{p} + 2 \right) \gamma \int_{\Omega} u_{\mu_1}^{\frac{2q}{p}} v_0^2, \\ m_4 &:= 4B_{\mu_1}, \quad m_5 := \int_{\Omega} |\nabla v_0|^2 + \lambda_2 v_0^2. \end{aligned}$$

We get $t'(s)$ from (3.21) by implicit function differentiation

$$\begin{aligned}
 t'(s) &= -\frac{2m_1s^3t^2 + m_2st^2 + m_3st^{\frac{2q}{p}} - m_5s}{m_1s^4t + m_2s^2t + \frac{q}{p}m_3s^2t^{\frac{2q}{p}-1} + m_4t}, \\
 \frac{t'(s)}{s} &= -\frac{2m_1s^2t^2 + m_2t^2 + m_3t^{\frac{2q}{p}} - m_5}{m_1s^4t + m_2s^2t + \frac{q}{p}m_3s^2t^{\frac{2q}{p}-1} + m_4t} \\
 &\rightarrow -\frac{m_2 + m_3 - m_5}{m_4} \\
 &= \frac{(\int_{\Omega} |\nabla v_0|^2 + \lambda_2 v_0^2) - (\frac{2q}{p} + 2)y(\int_{\Omega} u_{\mu_1}^{\frac{2q}{p}} v_0^2) - 2\beta \int_{\Omega} u_{\mu_1}^2 v_0^2}{4B_{\mu_1}}, \quad \text{as } s \rightarrow 0.
 \end{aligned} \tag{3.22}$$

Let

$$k := \frac{(\int_{\Omega} |\nabla v_0|^2 + \lambda_2 v_0^2) - (\frac{2q}{p} + 2)y(\int_{\Omega} u_{\mu_1}^{\frac{2q}{p}} v_0^2) - 2\beta \int_{\Omega} u_{\mu_1}^2 v_0^2}{4B_{\mu_1}}. \tag{3.23}$$

Then

$$\begin{aligned}
 t'(s) &= ks + o(s), \quad t(s) = 1 + \frac{k}{2}s^2 + o(s^2), \quad t^2(s) = 1 + ks^2 + o(s^2), \\
 t^{\frac{2q}{p}+2}(s) &= 1 + (\frac{q}{p} + 1)ks^2 + o(s^2).
 \end{aligned}$$

Therefore we have

$$\begin{aligned}
 I(tu_{\mu_1}, ts v_0) &= \frac{t^2}{4} \int_{\Omega} (|\nabla u_{\mu_1}|^2 + \lambda_1 u_{\mu_1}^2 + s^2(|\nabla v_0|^2 + \lambda_2 v_0^2)) \\
 &\quad - (\frac{p-q}{2p})s^2 t^{\frac{2q}{p}+2} y \int_{\Omega} u_{\mu_1}^{\frac{2q}{p}} v_0^2 \\
 &= t^2 B_{\mu_1} + \frac{1}{4} t^2 s^2 \left(\int_{\Omega} |\nabla v_0|^2 + \lambda_2 v_0^2 \right) - (\frac{p-q}{2p})s^2 t^{\frac{2q}{p}+2} y \int_{\Omega} u_{\mu_1}^{\frac{2q}{p}} v_0^2 \\
 &= (1 + ks^2 + o(s^2)) B_{\mu_1} + \frac{s^2}{4} (1 + ks^2 + o(s^2)) \left(\int_{\Omega} |\nabla v_0|^2 + \lambda_2 v_0^2 \right) \\
 &\quad - (\frac{p-q}{2p})s^2 \left(1 + (\frac{q}{p} + 1)ks^2 + o(s^2) \right) y \int_{\Omega} u_{\mu_1}^{\frac{2q}{p}} v_0^2 \\
 &= B_{\mu_1} + \left(kB_{\mu_1} + \frac{1}{4} \int_{\Omega} (|\nabla v_0|^2 + \lambda_2 v_0^2) - (\frac{p-q}{2p})y \int_{\Omega} u_{\mu_1}^{\frac{2q}{p}} v_0^2 \right) s^2 + o(s^2) \\
 &< B_{\mu_1},
 \end{aligned} \tag{3.24}$$

as $|s| > 0$ small enough, here (3.19) and (3.23) are applied. Then

$$\mathcal{B} \leq I(t(s)u_{\mu_1}, t(s)s v_0) < B_{\mu_1}.$$

This completes the proof. $\square$

Lemma 3.9. Assume $\mu_1, \mu_2, \gamma, \beta > 0$. If $-\alpha_1(\Omega) < \lambda_2 < 0$, then $\mathcal{B} < B_{\mu_2}$.

Proof. According to the proof of Lemma 3.8, we take the test function as (tsu, tu_{μ_2}) , here $0 < s, t(s) \in \mathbb{R}, 0 \neq u \in H_0^1(\Omega)$ and u_{μ_2} is given by (1.10). In order to ensure $(tsu, tu_{\mu_2}) \in \mathcal{N}$, we have

$$L_s(t) := \left(\int_{\Omega} \mu_1 s^4 u^4 + \mu_2 u_{\mu_2}^4 + 2\beta u^2 u_{\mu_2}^2 s^2 \right) t^2 + \left(\left(\frac{2q}{p} + 2 \right) y s^{\frac{2q}{p}} \int_{\Omega} u^{\frac{2q}{p}} u_{\mu_2}^2 \right) t^{\frac{2q}{p}}$$

$$-\left(\int_{\Omega} |\nabla u_{\mu_2}|^2 + s^2 |\nabla u|^2 + \lambda_2 u_{\mu_2}^2 + s^2 \lambda_1 u^2\right) = 0. \quad (3.25)$$

Then as the same as Lemma 3.8, $\exists$ unique $t = t(s) > 0$ such that (3.25) holds and $t(s) \rightarrow 1$, when $s \rightarrow 0$. We define

$$w_1 := \int_{\Omega} \mu_1 u^4, \quad w_2 := \int_{\Omega} 2\beta u^2 u_{\mu_2}^2, \quad w_3 := \left(\frac{2q}{p} + 2\right) \gamma \int_{\Omega} u^{\frac{2q}{p}} u_{\mu_2}^2, \\ w_4 := 4B_{\mu_2}, \quad w_5 := \int_{\Omega} |\nabla u|^2 + \lambda_1 u^2.$$

By the implicit function differentiation, we have

$$t'(s) = -\frac{2w_1 s^3 t^2 + w_2 s t^2 + \frac{q}{p} w_3 s^{\frac{2q}{p}-1} t^{\frac{2q}{p}} - w_5 s}{w_1 s^4 t + w_2 s^2 t + \frac{q}{p} w_3 s^{\frac{2q}{p}} t^{\frac{2q}{p}-1} + w_4 t}, \\ \frac{t'(s)}{s^{\frac{2q}{p}-1}} = -\frac{2w_1 s^{4-\frac{2q}{p}} t^2 + w_2 s^{2-\frac{2q}{p}} t^2 + \frac{q}{p} w_3 t^{\frac{2q}{p}} - w_5 s^{2-\frac{2q}{p}}}{w_1 s^4 t + w_2 s^2 t + \frac{q}{p} w_3 s^{\frac{2q}{p}} t^{\frac{2q}{p}-1} + w_4 t} \\ \rightarrow -\frac{q w_3}{p w_4} \\ = -\frac{\frac{q}{p} \left(\frac{2q}{p} + 2\right) \gamma \int_{\Omega} u^{\frac{2q}{p}} u_{\mu_2}^2}{4B_{\mu_2}}, \quad \text{as } s \rightarrow 0. \quad (3.26)$$

Let

$$d := -\frac{\frac{q}{p} \left(\frac{2q}{p} + 2\right) \gamma \int_{\Omega} u^{\frac{2q}{p}} u_{\mu_2}^2}{4B_{\mu_2}}. \quad (3.27)$$

Then

$$t'(s) = ds^{\frac{2q}{p}-1} + o(s^{\frac{2q}{p}-1}), \quad t(s) = 1 + \frac{p}{2q} ds^{\frac{2q}{p}} + o(s^{\frac{2q}{p}}), \\ t^2(s) = 1 + \frac{p}{q} ds^{\frac{2q}{p}} + o(s^{\frac{2q}{p}}), \quad t^{\frac{2q}{p}+2}(s) = 1 + \left(\frac{q}{p} + 1\right) \frac{p}{q} ds^{\frac{2q}{p}} + o(s^{\frac{2q}{p}}).$$

Therefore we have

$$I(tsu, tu_{\mu_2}) = \frac{t^2}{4} \int_{\Omega} \left(s^2 (|\nabla u|^2 + \lambda_1 u^2) + (|\nabla u_{\mu_2}|^2 + \lambda_2 u_{\mu_2}^2) \right) \\ - \left(\frac{p-q}{2p} \right) s^{\frac{2q}{p}} t^{\frac{2q}{p}+2} \gamma \int_{\Omega} u^{\frac{2q}{p}} u_{\mu_2}^2 \\ = t^2 B_{\mu_2} + \frac{1}{4} t^2 s^2 \left(\int_{\Omega} |\nabla u|^2 + \lambda_1 u^2 \right) - \left(\frac{p-q}{2p} \right) s^{\frac{2q}{p}} t^{\frac{2q}{p}+2} \gamma \int_{\Omega} u^{\frac{2q}{p}} u_{\mu_2}^2 \\ = \left(1 + \frac{p}{q} ds^{\frac{2q}{p}} + o(s^{\frac{2q}{p}}) \right) B_{\mu_2} + \frac{s^2}{4} \left(1 + \frac{p}{q} ds^{\frac{2q}{p}} + o(s^{\frac{2q}{p}}) \right) \left(\int_{\Omega} |\nabla u|^2 + \lambda_1 u^2 \right) \\ - \left(\frac{p-q}{2p} \right) s^{\frac{2q}{p}} \left(1 + \left(\frac{q}{p} + 1 \right) \frac{p}{q} ds^{\frac{2q}{p}} + o(s^{\frac{2q}{p}}) \right) \gamma \int_{\Omega} u^{\frac{2q}{p}} u_{\mu_2}^2 \\ = B_{\mu_2} + \left(\frac{pd}{q} B_{\mu_2} - \left(\frac{p-q}{2p} \right) \gamma \int_{\Omega} u^{\frac{2q}{p}} u_{\mu_2}^2 \right) s^{\frac{2q}{p}} + o(s^{\frac{2q}{p}}) \\ = B_{\mu_2} - \left(\gamma \int_{\Omega} u^{\frac{2q}{p}} u_{\mu_2}^2 \right) s^{\frac{2q}{p}} + o(s^{\frac{2q}{p}}) \\ < B_{\mu_2}, \quad (3.28)$$

as $|s| > 0$ small enough, here $y > 0$ and (3.27) are applied. Then

$$\mathcal{B} \leq I(t(s)su, t(s)u_{\mu_2}) < B_{\mu_2}.$$

This completes the proof. $\square$

Lemma 3.10. Assume $\beta, \gamma, \mu_1, \mu_2 > 0$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_i > -\alpha_1(\Omega)$, $i = 1, 2$. Suppose that λ_1 satisfies one of the following two conditions:

- (1₁) $\lambda_1 < 0$ and β, γ satisfy (3.19);
 - (1₂) $\lambda_1 \geq 0$ with Ω is a (smooth) starshaped domain,
- and that λ_2 satisfies one of the following two conditions:
- (2₁) $\lambda_2 < 0$;
 - (2₂) $\lambda_2 \geq 0$ with Ω is a (smooth) starshaped domain.

If (u_1, v_1) is the minimizer of $I(u, v)$ on $\mathcal{N}$, then $(u_2, v_2) = (|u_1|, |v_1|)$ is a ground state solution of system (1.1): $u_2 > 0, v_2 > 0$.

Proof. Since $\forall u \in H_0^1(\Omega)$, $|\nabla|u|| \leq |\nabla u|$ a.e. $\in \mathbb{R}^4$, we obtain that $(u_2, v_2) \in \mathcal{N}$ and $I(|u_1|, |v_1|) \leq I(u_1, v_1) = \min_{\mathcal{N}} I$. Thus (u_2, v_2) is also a minimizer of $I(u, v)$ on $\mathcal{N}$. Since $\mathcal{N}$ is a natural constraint by Lemma 3.5, (u_2, v_2) satisfies the system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2 + \frac{2q}{p} \gamma u^{\frac{2q}{p}-1} v^2 & \text{in } \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v + 2\gamma u^{\frac{2q}{p}} v & \text{in } \Omega, \\ u, v \geq 0 & \text{in } \Omega, \\ u_2, v_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle and $0 \neq (u_2, v_2) \in \mathcal{N}$, there exists the following three possibilities: $u_2 = 0, v_2 > 0$; $u_2 > 0, v_2 > 0$ or $u_2 > 0, v_2 = 0$. Suppose that (u_2, v_2) is not a solution of (1.1), then $u_2 > 0, v_2 = 0$ or $u_2 = 0, v_2 > 0$. First, we suppose $u_2 > 0, v_2 = 0$ if $\lambda_1 < 0$, which leads to a contradiction since $\mathcal{B} = \inf_{\mathcal{N}} I(u, v) < B_{\mu_1} = \inf_{\mathcal{N}_{\mu_1}} I(u, 0)$ by Lemma 3.8; if $\lambda_1 \geq 0$, then u_2 satisfies the equation following:

$$\begin{cases} -\Delta u_2 + \lambda_1 u_2 = \mu_1 u_2^3 & \text{in } \Omega, \\ u_2 > 0 & \text{in } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

which leads to a contradiction for that Brezis-Nirenberg equation has no solution when $\lambda_1 \geq 0$. Suppose $u_2 = 0, v_2 > 0$ and prove as the last assumption by Lemma 3.9, we obtain a contradiction. Thus $u_2 > 0, v_2 > 0$ and (u_2, v_2) is a ground state of system (1.1). $\square$

4 Proof of Theorem 1.4

Define the function

$$z(x, y) := \frac{2q}{p-q}(\lambda_1 x + \lambda_2 y) + \mu_1 x^2 + \mu_2 y^2 + 2\beta xy, \quad x, y \in \mathbb{R}. \quad (4.1)$$

Lemma 4.1. If $\mu_1, \mu_2 > 0$, $\beta > \sqrt{\mu_1 \mu_2}$ or $\beta < -\sqrt{\mu_1 \mu_2}$, then $z(x, y) \geq 0$.

Proof. Let

$$\begin{cases} \frac{\partial z}{\partial x} = \frac{2q}{p-q} \lambda_1 + 2\mu_1 x + 2\beta y = 0 \\ \frac{\partial z}{\partial y} = \frac{2q}{p-q} \lambda_2 + 2\mu_2 y + 2\beta x = 0, \end{cases}$$

we have critical point (x_0, y_0) as

$$(x_0, y_0) = \frac{q}{p-q} \left(\frac{\lambda_1 \mu_2 - \beta \lambda_2}{\beta^2 - \mu_1 \mu_2}, \frac{\lambda_2 \mu_1 - \beta \lambda_1}{\beta^2 - \mu_1 \mu_2} \right).$$

Then, $\frac{\partial^2 z}{\partial x^2}|_{(x_0, y_0)} = 2\mu_1$, $\frac{\partial^2 z}{\partial y^2}|_{(x_0, y_0)} = 2\mu_2$, $\frac{\partial^2 z}{\partial x \partial y}|_{(x_0, y_0)} = 2\beta$. Since $(2\beta)^2 - (2\mu_1)(2\mu_2) > 0$, $z(x, y)_{\min} = z(x_0, y_0) = (\frac{q}{p-q})^2 (\frac{\lambda_1^2 \mu_2 + \lambda_2^2 \mu_1 - 2\lambda_1 \lambda_2 \beta}{\beta^2 - \mu_1 \mu_2}) \geq 0$. The last inequality holds since the function $h(x, y) := \mu_2 x^2 + \mu_1 y^2 - 2\beta xy$ has minimum 0 at $(0, 0)$ by the same calculation as above. $\square$

Lemma 4.2. Assume that $\beta, y > 0$ and $\lambda_1, \lambda_2 > -\alpha_1(\Omega)$, where $\alpha_1(\Omega) > 0$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Then there exists $C > 0$ such that $\forall u, v \in \mathcal{N}$,

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \geq C.$$

Proof. By the Hölder and Young inequalities, $\forall \varepsilon > 0, \exists C_\varepsilon > 0$, such that

$$\int_{\Omega} u^{\frac{4q}{p}} \leq \left(\int_{\Omega} u^2 \right)^{2-\frac{2q}{p}} \left(\int_{\Omega} u^4 \right)^{\frac{2q}{p}-1} \leq \varepsilon \left(\int_{\Omega} u^2 \right) + C_\varepsilon \left(\int_{\Omega} u^4 \right).$$

For $u, v \in \mathcal{N}$, we have

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 + |\nabla v|^2 &= \int_{\Omega} \mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 + \left(\frac{2q}{p} + 2 \right) y u^{\frac{2q}{p}} v^2 - \lambda_1 u^2 - \lambda_2 v^2 \\ &\leq \int_{\Omega} (\mu_1 + \beta) u^4 + \left(\mu_2 + \beta + \left(\frac{q}{p} + 1 \right) y \right) v^4 + \left(\frac{q}{p} + 1 \right) y u^{\frac{4q}{p}} \\ &\quad - \lambda_1 u^2 - \lambda_2 v^2 \\ &\leq \left(\mu_1 + \beta + C_\varepsilon \left(\frac{q}{p} + 1 \right) y \right) \int_{\Omega} u^4 + \left(\left(\frac{q}{p} + 1 \right) y \varepsilon - \lambda_2 \right) \int_{\Omega} v^2 \\ &\quad + \left(\mu_2 + \beta + \left(\frac{q}{p} + 1 \right) y \right) \int_{\Omega} v^4 - \lambda_1 \int_{\Omega} u^2, \end{aligned}$$

that is,

$$\begin{aligned} &\int_{\Omega} |\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \left(\lambda_2 - \left(\frac{q}{p} + 1 \right) y \varepsilon \right) \int_{\Omega} v^2 \\ &\leq C \int_{\Omega} u^4 + v^4 \\ &\leq C \left(\int_{\Omega} |\nabla u|^2 + |\nabla v|^2 \right)^2. \end{aligned} \tag{4.2}$$

By the Poincaré inequality

$$\|f\|_{L^2(\Omega)} \leq \frac{1}{\alpha_1(\Omega)} \|\nabla f\|_{L^2(\Omega)}, \quad \forall f \in H_0^1(\Omega)$$

we obtain

$$\int_{\Omega} |\nabla u|^2 + \lambda_1 u^2 \geq C_1 \int_{\Omega} |\nabla u|^2,$$

and

$$\int_{\Omega} |\nabla v|^2 + \left(\lambda_2 - \left(\frac{q}{p} + 1 \right) y \varepsilon \right) v^2 \geq C_2 \int_{\Omega} |\nabla v|^2,$$

for ε small enough, here C_1, C_2 are positive constants. Together with (4.2), we obtain $\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \geq C$. $\square$

Proof of Theorem 1.4. First, we verify the assumptions of Ekeland's variational principle, i.e., $I|_{\mathcal{N}}(u, v)$ is bounded below on $\mathcal{N}$.

$$\begin{aligned} I|_{\mathcal{N}}(u, v) &= \frac{q}{2(p+q)} \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 + \lambda_1 u^2 + \lambda_2 v^2) \\ &\quad + \frac{p-q}{4(p+q)} \int_{\Omega} (\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2) \\ &= \frac{p-q}{4(p+q)} \int_{\Omega} \left(\frac{2q}{p-q} (|\nabla u|^2 + |\nabla v|^2) + z(u^2, v^2) \right), \end{aligned}$$

where $z(x, y)$ is defined in (4.1), we obtain that $I|_{\mathcal{N}}(u_n, v_n) \geq 0$. Then according to the Ekeland's variational principle, there is a (PS) sequence $\{(u_n, v_n)\} \subset \mathcal{N}$ of $I(u, v)$ at the least energy level $\mathcal{B}$. Since $\mathcal{B} = \lim_{n \rightarrow \infty} I|_{\mathcal{N}}(u_n, v_n) \geq 0$ and $z(u^2, v^2) \geq 0$, we have

$$\frac{q}{2(p+q)} \int_{\Omega} (|\nabla u_n|^2 + |\nabla v_n|^2) \leq I|_{\mathcal{N}}(u_n, v_n) \leq M.$$

So $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$, we may assume that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$. Next, we will prove that $(u, v) \neq (0, 0)$. Suppose to the contrary that $(u, v) = (0, 0)$, then up to a subsequence

$$\begin{aligned} u_n &\rightarrow 0, \quad v_n \rightarrow 0 \quad \in L^2(\Omega), \\ u_n &\rightarrow 0, \quad v_n \rightarrow 0 \quad \in L^{\frac{2q}{p}+2}(\Omega). \end{aligned} \quad (4.3)$$

Therefore, we have

$$|u_n|^{\frac{2q}{p}+2} \rightarrow 0, \quad |v_n|^{\frac{2q}{p}+2} \rightarrow 0 \in L^1(\Omega),$$

as $n \rightarrow \infty$ and

$$\int_{\Omega} |u_n|^{\frac{2q}{p}} |v_n|^2 \leq \left(\int_{\Omega} |u_n|^{\frac{2q}{p}+2} \right)^{\frac{q}{p+q}} \left(\int_{\Omega} |v_n|^{\frac{2q}{p}+2} \right)^{\frac{p}{p+q}} = o(1), \quad (4.4)$$

$$\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 = \mu_1 \|u_n\|_4^4 + \mu_2 \|v_n\|_4^4 + 2\beta \int_{\Omega} u_n^2 v_n^2 + o(1). \quad (4.5)$$

Take t_n as following such that $(t_n u_n, t_n v_n) \in \mathcal{M}$ and with (4.5), we have

$$\begin{aligned} t_n &= \frac{\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2}{\mu_1 \|u_n\|_4^4 + \mu_2 \|v_n\|_4^4 + 2\beta \int_{\Omega} u_n^2 v_n^2} \\ &= \frac{\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2}{(\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2) + o(1)} \\ &= 1 + o(1), \end{aligned} \quad (4.6)$$

since $\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 \geq C > 0$ by Lemma 4.2. On the other hand, by (4.3)-(4.4)

$$I(u_n, v_n) = E(u_n, v_n) + \frac{1}{2} \int_{\Omega} \lambda_1 u_n^2 + \lambda_2 v_n^2 - 2y u_n^{\frac{2q}{p}} v_n^2 = o(1),$$

here $E(u, v)$ is given in (3.1). Therefore, together with (4.6) we have

$$\mathcal{B} + o(1) = I(u_n, v_n) = E(t_n u_n, t_n v_n) + o(1) \geq c + o(1).$$

Thus we obtain that $\mathcal{B} \geq c$, which contradicts Lemma 3.4. Thus $(u, v) \neq (0, 0)$.

Since $I|'_{\mathcal{N}}(u_n, v_n) = o(1)$ and according to Lemma 3.6 and Remark 3.7, we have

$$I'(u_n, v_n) = o(1). \quad (4.7)$$

As $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$, passing to a subsequence, we assume that

$$\begin{aligned} u_n &\rightharpoonup u, & v_n &\rightharpoonup v, & \text{weakly in } L^4(\Omega), \\ u_n &\rightarrow u, & v_n &\rightarrow v, & \text{a.e. } x \in \Omega, \\ u_n &\rightarrow u, & v_n &\rightarrow v, & \text{strongly in } L^2(\Omega). \end{aligned}$$

Thus by (4.7), we have $I'(u, v) = 0$. Together with $(u, v) \neq (0, 0)$, we obtain $(u, v) \in \mathcal{N}$. Then by the Fatou Lemma, we have

$$\mathcal{B} = \lim_{n \rightarrow \infty} I(u_n, v_n) \geq I(u, v) \geq \inf_{\mathcal{N}} I = \mathcal{B}.$$

Thus we obtain that (u, v) is the minimizer of $I(u, v)$ in $\mathcal{N}$. Then by Lemma 3.10, system (1.1) has a positive ground state solution: $u > 0$ and $v > 0$. $\square$

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