Research article

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Convex solutions of Monge-Ampère equations and systems: Existence, uniqueness and asymptotic behavior

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Abstract: In this paper, the equations and systems of Monge-Ampère with parameters are considered. We first show the uniqueness of nontrivial radial convex solution of Monge-Ampère equations by using sharp estimates. Then we analyze the existence and nonexistence of nontrivial radial convex solutions to Monge-Ampère systems, which includes some new ingredients in the arguments. Furthermore, the asymptotic behavior of nontrivial radial convex solutions for Monge-Ampère systems is also considered. Finally, as an application, we obtain sufficient conditions for the existence of nontrivial radial convex solutions of the power-type system of Monge-Ampère equations.

Keywords: system of Monge-Ampère equations; convex solutions; sharp estimates and uniqueness; existence and asymptotic behavior; multiparameter

MSC: primary 35J60; secondary 35J66, 35J96

1 Introduction

The Monge-Ampère equations come from geometric problems, fluid mechanics and other applied subjects. For example, in [51], Trudinger and Wang pointed out that Monge-Ampère equation can describe reflector shape design, or Weingarten curvature. In recent years, increasing attention has been paid to the study of the Monge-Ampère equations by different methods (see [3,4,7,11,13,22,23,25,34,37,38,42,45,47,52,56,57,60]). In particular in [62], Zhang and Wang studied the following Monge-Ampère equation

$$\begin{cases} \det D^2 u = e^{-u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded convex domain in $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary, and $D^2 u$ denotes the Hessian of u , $\det D^2 u$ is Monge-Ampère operator. Applying the argument of moving plane, the authors firstly verified that any solution on a ball is radially symmetric. Then the authors showed there exists a critical radius such that when the radius of a ball is smaller than this critical value there exists a solution, and the converse is also true.

Recently, using standard approximation arguments, Philippis and Figalli [44] showed that convex Alexandrov solutions of

$$\begin{cases} \det D^2 u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

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with $0 < \lambda \leq f \leq \Lambda$, are $W_{loc}^{2,1}$.

In [59], Zhang and Du proved sharp conditions of $K(x)$ and $f(u)$ on the existence of strictly convex solutions to the Monge-Ampère problem

$$\begin{cases} M[u](x) = K(x)f(u) \text{ for } x \in \Omega, \\ u(x) \rightarrow +\infty \text{ as } \text{dist}(x, \partial\Omega) \rightarrow 0, \end{cases} \quad (1.3)$$

where $M[u] = \det(u_{x_i x_j})$ is the Monge-Ampère operator, and Ω is a smooth, bounded, strictly convex domain in $\mathbb{R}^N$ ($N \geq 2$). The approach is largely based on the construction of suitable sub- and super-solutions.

This result of problem (1.3) has also been improved in Zhang and Feng [58], where it is actually verified that problem (1.3) admits a strictly convex solution if and only if f satisfies Keller-Osserman type condition. The asymptotic behavior of strictly convex solutions to (1.3) is also considered under weaker conditions than previous references. However, the most significant one is that if f does not satisfy Keller-Osserman type condition, the authors obtained the existence results of strictly convex solutions under appropriate conditions on $K(x)$. The proof combines the sub-supersolution method with non-standard arguments and Karamata regular variation theory.

At the same time, we notice that many authors have paid more attention to various of system problems, for example, see [2,10,14-16,20,21,29,30,32,35,40,46,49,50]. Specially, Lair and Wood [36] analyzed the existence of entire positive solutions of system

$$\begin{cases} \Delta u = p(|x|)v^\alpha, \\ \Delta v = q(|x|)u^\beta, \end{cases}$$

where p and q are nonnegative, continuous, c -positive, and satisfy the decay conditions $\int_0^\infty tp(t)dt < \infty$ and $\int_0^\infty tq(t)dt < \infty$ for α and β greater than unity, and $\int_0^\infty tp(t)dt = \infty$ and $\int_0^\infty tq(t)dt = \infty$ if neither α nor β is greater than one. In [11], Cîrstea and Rădulescu generalized the results of Lair and Wood [36] to the following system

$$\begin{cases} \Delta u = p(x)f(v), \\ \Delta v = q(x)f(u), \end{cases} \quad (1.4)$$

for $p, q \in C_{loc}^{0,\alpha}(\mathbb{R}^n)$ under the condition

$$\lim_{t \rightarrow \infty} \frac{f(cg(t))}{t} = 0$$

for all $c > 0$.

Recently, by utilizing the same method as papers [11], Ghergu and Rădulescu [8] improved the results of problem (1.4). Very recently, Mavinga and Pardo [41] provided a-priori L^∞ bounds for classical positive solutions of semilinear elliptic systems

$$\begin{cases} -\Delta u = \frac{v^p}{[\ln(e+v)]^\alpha} \text{ in } \Omega, \\ -\Delta v = \frac{u^q}{[\ln(e+u)]^\beta} \text{ in } \Omega, \\ u = 0, v = 0 \text{ on } \partial\Omega, \end{cases} \quad (1.5)$$

where $1 < p, q < \infty$ and $\alpha, \beta > 0$. Applying moving planes method, Rellich-Pohozaev type identities for systems, and local and global bifurcation techniques, they proved the existence of positive solutions for system (1.5).

On Monge-Ampère system problems, only a few results can be found, for example see Wang, An [53] and Wang [54,55]. Specially, Zhang and Qi [61] considered the following elliptic system coupled by Monge-Ampère equations

$$\begin{cases} \det D^2 u_1 = (-u_2)^\alpha \text{ in } \Omega, \\ \det D^2 u_2 = (-u_1)^\beta \text{ in } \Omega, \\ u_1 < 0, u_2 < 0 \text{ in } \Omega, \\ u_1 = u_2 = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a smooth, bounded and strictly convex domain in $\mathbb{R}^N$, $N \geq 2$, $\alpha > 0$, $\beta > 0$. When Ω is the unit ball in $\mathbb{R}^N$, applying the index theory of fixed points for completely continuous operators, the authors obtained

the existence, uniqueness results and nonexistence of radial convex solutions under some corresponding assumptions on α, β . When $\alpha > 0, \beta > 0$ and $\alpha\beta = N^2$ they also considered a corresponding eigenvalue problem in more general domains.

Moreover, we notice that many authors have paid more attention to existence and uniqueness problems, for example, see [1,5,6,9,17-19,27,28,31,39,48]. Specially, we would like to mention some results of Hai and Shivaji [33] and Mohammed, Rădulescu and Vitolo [43]. In [33], Hai and Shivaji considered the following p -Laplacian equation

$$\begin{cases} -\Delta_p z = \lambda g(z) & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.6)$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $p > 1$, λ is a positive parameter, Ω is the open unit ball in $\mathbb{R}^N$. By the subsolution method together with sharp estimates near the boundary, the authors proved the existence and uniqueness of positive radial solutions to problem (1.6).

In [43], Mohammed, Rădulescu and Vitolo considered the infinite boundary value problem

$$\begin{cases} H[u] = f(u) + h(x) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \quad (1.7)$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded open set with C^2 boundary $\partial\Omega$, $H[u] := H(x, u, Du, D^2u)$ is a fully nonlinear uniformly elliptic operator. The authors obtained existence and uniqueness results of solution to problem (1.7) by using the maximum principle. They also studied the asymptotic boundary estimates of solutions to problem (1.7).

Inspired by the above works, we are interested in the existence and uniqueness of nontrivial radial convex solutions to the following Monge-Ampère equation

$$\begin{cases} \det D^2 u = \lambda f(-u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (P_\lambda)$$

Here $\lambda > 0$ is a parameter, $D^2 u$ denotes the Hessian of u , $\det(D^2 u)$ is Monge-Ampère operator, $\Omega = \{x \in \mathbb{R}^N : |x| < 1\}$.

We also give new existence results for system

$$\begin{cases} \det D^2 u_1 = \lambda_1 f_1(-u_2) & \text{in } \Omega, \\ \det D^2 u_2 = \lambda_2 f_2(-u_3) & \text{in } \Omega, \\ \vdots \\ \det D^2 u_n = \lambda_n f_n(-u_1) & \text{in } \Omega, \\ u_1 = u_2 = \dots = u_n = 0 & \text{on } \partial\Omega, \end{cases} \quad (S_{\lambda_1, \dots, \lambda_n})$$

$\lambda_i (i = 1, 2, \dots, n)$ are positive parameters. Our main tool is the eigenvalue theory in cones. However, based on the idea of decoupling method we will investigate composite operators. Besides, the exactly determined intervals of positive parameter $\lambda_1 \times \lambda_2 \times \dots \times \lambda_n$ are established. Here we extend the study in Zhang and Qi [61] from a power-type coupled system to a more general system with n parameters, and also generalize that in Hai and Shivaji [33] from a quasilinear problem to a fully nonlinear system problem. Meanwhile, we obtain some new existence results by defining composite operators and using the eigenvalue theory in cones. Moreover, we also analyze the asymptotic behavior of nontrivial radial convex solutions to system $(S_{\lambda_1, \dots, \lambda_n})$.

The rest of the present article is organized as follows. In Section 2, we give some preliminary and known results to be used for the proof of our main results, which mainly includes the eigenvalue theory. Section 3 is devoted to establish the uniqueness result of nontrivial radial convex solution to problem (P_λ) . In Section 4, we analyze the exactly determined intervals of positive parameter $\lambda_i (i \in \{1, 2, \dots, n\})$ in which we get the existence and nonexistence results of nontrivial radial convex solutions for system $(S_{\lambda_1, \dots, \lambda_n})$. In Section 5, as an application, some new sufficient conditions for the existence of nontrivial radial convex solutions of the power-type system of Monge-Ampère equations are given. Finally in Section 6, we will discuss the asymptotic behavior of nontrivial radial convex solutions to system $(S_{\lambda_1, \dots, \lambda_n})$ on the parameters $\lambda_1, \lambda_2, \dots, \lambda_n$.

2 Preliminaries

Let us search radial convex solutions of (P_λ) and $(S_{\lambda_1, \dots, \lambda_n})$. It is similar to that of Appendix A.2 of [24], one can convert (P_λ) and $(S_{\lambda_1, \dots, \lambda_n})$ to the following equation

$$\begin{cases} ((u')^N)' = \lambda N r^{N-1} f(-u), & 0 < r < 1, \\ u'(0) = u(1) = 0, \end{cases} \quad (2.1)$$

and system

$$\begin{cases} ((u'_1)^N)' = \lambda_1 N r^{N-1} f_1(-u_2), & 0 < r < 1, \\ ((u'_2)^N)' = \lambda_2 N r^{N-1} f_2(-u_3), & 0 < r < 1, \\ \vdots \\ ((u'_n)^N)' = \lambda_n N r^{N-1} f_n(-u_1), & 0 < r < 1, \\ u'_i(0) = u_i(1) = 0, & i = 1, 2, \dots, n. \end{cases} \quad (2.2)$$

Solutions of (2.1) and (2.2) equal to fixed points of certain operators, and we can handle more general equations and systems. Equivalently, we can search positive concave solutions for convenience by letting $v = -u$ and $v_i = -u_i (i=1, 2, \dots, n)$, and then we can transform (2.1) and (2.2) to

$$\begin{cases} ((-v')^N)' = \lambda N t^{N-1} f(v), & 0 < t < 1, \\ v'(0) = v(1) = 0, \end{cases} \quad (\mathbf{P}_\lambda)$$

and

$$\begin{cases} ((-v'_1)^N)' = \lambda_1 N t^{N-1} f_1(v_2), & 0 < t < 1, \\ ((-v'_2)^N)' = \lambda_2 N t^{N-1} f_2(v_3), & 0 < t < 1, \\ \vdots \\ ((-v'_n)^N)' = \lambda_n N t^{N-1} f_n(v_1), & 0 < t < 1, \\ v'_i(0) = v_i(1) = 0, & i = 1, 2, \dots, n. \end{cases} \quad (\mathbf{S}_{\lambda_1, \dots, \lambda_n})$$

For $i = 1, 2, \dots, n$, we assume that f_i satisfies

(C₀) $f_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\mathbb{R}_+ = [0, +\infty)$;

and f satisfies

(C₁) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ is strictly increasing and C^1 on $(0, \infty)$;

(C₂)

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x^N} = 0, \quad \lim_{x \rightarrow 0^+} \frac{f(x)}{x^N} > 0;$$

(C₃)

$$\lim_{x \rightarrow +\infty} \frac{x f'(x)}{f(x)} < N, \quad \lim_{x \rightarrow 0^+} x f'(x) < \infty.$$

Let $E = C[0, 1]$. Then E is a real Banach space with the norm $\|\cdot\|$ defined by

$$\|x\| = \max_{t \in J} |x(t)|.$$

Let $P \subset E$ be

$$P := \{v \in E : v(t) \geq 0, t \in J, v(t) \geq \theta \|v\|, t \in J_\theta\}, \quad (2.3)$$

where

$$\theta \in (0, \frac{1}{2}), J_\theta = [\theta, 1 - \theta]. \quad (2.4)$$

It is easy to see that P is a normal cone of E .

For $v \in P$, define $T_i : P \rightarrow E (i = 1, 2, \dots, n)$ to be

$$(T_1 v)(t) = \int_t^1 \left(\int_0^\tau N s^{N-1} f_1(v(s)) ds \right)^{\frac{1}{N}} d\tau,$$

$$\begin{aligned}
 (T_2 v)(t) &= \lambda_2^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_2(v(s)) ds \right)^{\frac{1}{N}} d\tau, \\
 &\vdots \\
 (T_n v)(t) &= \lambda_n^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_n(v(s)) ds \right)^{\frac{1}{N}} d\tau.
 \end{aligned}$$

It follows from Lemma 2.2 in [44] that $T_i (i = 1, 2, \dots, n)$ maps P into itself. Moreover, for $i = 1, 2, \dots, n$, T_i are completely continuous by standard arguments.

Define a composite operator $\widetilde{T}_1 = T_1 T_2 \cdots T_n$, which is also completely continuous from P to itself. For the case $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$, similar to what Zhang and Qi [61] pointed out that

$$(v_1, v_2, \dots, v_n) \in \underbrace{C^1[0, 1] \times C^1[0, 1] \times \cdots \times C^1[0, 1]}_n$$

solves system $(S_{\lambda_1, \dots, \lambda_n})$ if and only if $(v_1, v_2, \dots, v_n)$ belongs to

$$\underbrace{P \setminus \{0\} \times P \setminus \{0\} \times \cdots \times P \setminus \{0\}}_n$$

satisfying

$$v_1 = T_1 v_2, v_2 = T_2 v_3, \dots, v_n = T_n v_1.$$

This shows that if $v_1 \in P \setminus \{0\}$ is a fixed point of $\widetilde{T}_1$, define $v_2 = T_2 v_3, \dots, v_n = T_n v_1$, then $v_1 \in P \setminus \{0\}$ such that

$$(v_1, v_2, \dots, v_n) \in \underbrace{C^1[0, 1] \times C^1[0, 1] \times \cdots \times C^1[0, 1]}_n$$

solves system $(S_{\lambda_1, \dots, \lambda_n})$; on the contrary, if

$$(v_1, v_2, \dots, v_n) \in \underbrace{C^1[0, 1] \times C^1[0, 1] \times \cdots \times C^1[0, 1]}_n$$

solves system $(S_{\lambda_1, \dots, \lambda_n})$, then v_1 is certainly a nonzero fixed point of $\widetilde{T}_1$ in P . Therefore the task of this paper is to search nonzero fixed points of operator $\widetilde{T}_1$.

We can also define other composite operators

$$\begin{aligned}
 \widetilde{T}_2 &= T_2 T_3 \cdots T_n T_1, \\
 \widetilde{T}_3 &= T_3 T_4 \cdots T_n T_1 T_2, \\
 &\vdots \\
 \widetilde{T}_n &= T_n T_1 \cdots T_{n-2} T_{n-1},
 \end{aligned}$$

which have the same meaning as $\widetilde{T}_1$.

Next, we consider the existence results of system $(S_{\lambda_1, \dots, \lambda_n})$ by using a completely different method from that of [2,8,10,11,14-16,20,21,29,30,32,35,36,40,41,46,49,50,53-55,61], namely the eigenvalue theory:

Theorem 2.1. (See [26]) Suppose D is an open subset of an infinite-dimensional real Banach space E , $\theta \in D$, and P is a cone of E . If the operator $\Gamma : P \cap D \rightarrow P$ is completely continuous with $\Gamma\theta = \theta$ and satisfies

$$\inf_{x \in P \cap \partial D} \|\Gamma x\| > 0,$$

then Γ has a proper element on $P \cap \partial D$ associated with a positive eigenvalue. That is, there exist $x_0 \in P \cap \partial D$ and $\mu_0 > 0$ such that $\Gamma x_0 = \mu_0 x_0$.

3 Uniqueness of positive concave solution for (P_λ)

In this section, we apply sharp estimates near the boundary to establish the uniqueness results of positive concave solution to system (P_λ) .

Lemma 3.1. Suppose that $(C_1) - (C_3)$ hold and let θ_0 be a number in $(0,1)$. Then there exists positive constants ε and l such that for $x \geq l$

- (i) $\frac{\theta^N f(x)}{f(\theta x)} \geq \theta^\varepsilon$ if $\theta \geq 1$;
- (ii) $\frac{\theta^N f(x)}{f(\theta x)} \leq \theta^\varepsilon$ if $\theta_0 \leq \theta \leq 1$;
- (iii) For each $L > 0$,

$$L \frac{x^N}{f(x)} \leq \frac{(L_1 x)^N}{f(L_1 x)},$$

where $L_1 = \max\{1, L^{\frac{1}{\varepsilon}}\}$.

Proof. Letting $H(\theta, x) = \frac{\theta^N f(x)}{f(\theta x)}$, then

$$\frac{d}{d\theta} H(\theta, x) = \frac{\theta^{N-1} f(x)}{f(\theta x)} \left(N - \frac{\theta x f'(\theta x)}{f(\theta x)} \right).$$

Because $\theta \geq \theta_0$, there is $l > 0$ such that

$$\frac{\theta x f'(\theta x)}{f(\theta x)} \leq l^* < N$$

for $x \geq l$. This shows that

$$\frac{d}{d\theta} H(\theta, x) \geq \frac{\varepsilon}{\theta} H(\theta, x), \quad x \geq l,$$

where $\varepsilon = N - l^* > 0$. We hence get

$$\frac{d}{d\theta} \left(\frac{H(\theta, x)}{\theta^\varepsilon} \right) \geq 0, \quad x \geq l.$$

Therefore, we get $\frac{H(\theta, x)}{\theta^\varepsilon} \geq H(1, x) = 1$ for $\theta \geq 1$, and $\frac{H(\theta, x)}{\theta^\varepsilon} \leq 1$ for $\theta_0 \leq \theta \leq 1$. This gives the proof of (i) and (ii).

Obviously, (iii) holds when $L \leq 1$. For $L > 1$, using (i) with $\theta = L^{\frac{1}{\varepsilon}}$ shows

$$L \frac{x^N}{f(x)} \leq \frac{L^{\frac{N}{\varepsilon}} f(x)}{f(L^{\frac{1}{\varepsilon}} x)} \frac{x^N}{f(x)} = \frac{L^{\frac{N}{\varepsilon}} x^N}{f(L^{\frac{1}{\varepsilon}} x)}.$$

This finishes the proof of Lemma 3.1. □

The next result provides sharp upper and lower estimates for positive concave solutions of (P_λ) .

Lemma 3.2. Suppose that $(C_1) - (C_3)$ hold. Then for large $\lambda > 0$ there exists A_λ with $\lim_{\lambda \rightarrow +\infty} A_\lambda = +\infty$ so that any positive concave solution v of (P_λ) satisfies

$$M_1(\lambda f(A_\lambda))^{\frac{1}{N}}(1-t) \leq v(t) \leq M_2(\lambda f(A_\lambda))^{\frac{1}{N}}(1-t), \quad 0 < t < 1,$$

where M_1 and M_2 are positive constants independent of λ .

Proof. Letting v be a positive concave solution of (P_λ) , then v is decreasing, and a calculation proves that

$$-v'(t) = \left(\int_0^t \lambda N S^{N-1} f(v(s)) ds \right)^{\frac{1}{N}}. \quad (3.1)$$

So for $t \geq \frac{1}{2}$ we get

$$-v'(t) \geq \left(\int_0^{\frac{1}{2}} \lambda N S^{N-1} f(v(s)) ds \right)^{\frac{1}{N}} > \left(\lambda C f(v(\frac{1}{2})) \right)^{\frac{1}{N}}, \quad (3.2)$$

where $C = \frac{1}{2^{N+1}}$.

Then we have by integrating on $(\frac{1}{2}, 1)$

$$v(\frac{1}{2}) > \frac{1}{2} \left(\lambda C f(v(\frac{1}{2})) \right)^{\frac{1}{N}},$$

or

$$\frac{v^N(\frac{1}{2})}{f(v(\frac{1}{2}))} > \frac{\lambda C}{2^N}. \quad (3.3)$$

Let $A_\lambda > 0$ be such that

$$\sup_{0 < z \leq A_\lambda} \frac{v^N(z)}{f(z)} = \frac{\lambda C}{2^N}. \quad (3.4)$$

It follows from (C_2) that the function $H(x) = \sup_{0 < z \leq x} \frac{z^N}{f(z)}$ satisfies $\lim_{x \rightarrow 0^+} H(x) < \infty$ and $\lim_{x \rightarrow \infty} H(x) = \infty$. This shows that A_λ exists for λ large. Obviously,

$$\lim_{\lambda \rightarrow \infty} A_\lambda = \infty.$$

We hence get from (3.2)-(3.3) that

$$v(\frac{1}{2}) > A_\lambda. \quad (3.5)$$

By (3.2) and (3.5), we obtain

$$-v'(t) \geq \left(\lambda C f(A_\lambda) \right)^{\frac{1}{N}}, \quad t > \frac{1}{2}. \quad (3.6)$$

From (3.6), we get by integration that

$$v(t) \geq \left(\lambda C f(A_\lambda) \right)^{\frac{1}{N}} (1-t), \quad t > \frac{1}{2}.$$

For $t \leq \frac{1}{2}$, we get

$$v(t) \geq v(\frac{1}{2}) \geq \frac{1}{2} \left(\lambda C f(A_\lambda) \right)^{\frac{1}{N}} (1-t).$$

This proves the left-hand inequality in Lemma 3.2.

Next we verify the right-hand inequality in Lemma 3.2. Since

$$v(t) = \lambda^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f(v(s)) ds \right)^{\frac{1}{N}} d\tau,$$

we get

$$\|v\| \leq \lambda^{\frac{1}{N}} \left(f(\|v\|) \right)^{\frac{1}{N}}.$$

This, together with (3.3) and Lemma 3.1 (iii), shows that

$$\frac{\|v\|^N}{f(\|v\|)} \leq \lambda = \frac{2^N}{C} \frac{B_\lambda^N}{f(B_\lambda)} \leq \frac{(C_2 B_\lambda)^N}{f(C_2 B_\lambda)},$$

where $B_\lambda \in (0, A_\lambda]$ is such that $\frac{B_\lambda^N}{f(B_\lambda)} = \frac{C\lambda}{2^N}$, and $C_2 > 1$ is independent of v and λ . Thus we get

$$\|v\| \leq C_2 B_\lambda \leq C_2 A_\lambda \quad (3.7)$$

for large λ . It follows from (C_3) that $\frac{f(x)}{x^N}$ is decreasing for large x , so we obtain by (3.7) and (3.1)

$$-v'(t) \leq \left(\lambda N f(C_2 A_\lambda) \right)^{\frac{1}{N}} \leq \left(\lambda N C_2^N f(A_\lambda) \right)^{\frac{1}{N}}. \quad (3.8)$$

Therefore we get the right-hand inequality by integrating (3.8) on $(r, 1)$. This gives the proof of Lemma 3.2. $\square$

Theorem 3.1. Suppose that $(C_1) - (C_3)$ hold. Then system (P_λ) admits at most one positive concave solution for large λ .

Proof. Let u and v be two positive concave solutions of system (P_λ) . It follows from Lemma 3.2 that

$$\frac{M_1}{M_2}v \leq u \leq \frac{M_2}{M_1}v.$$

Letting $\theta = \inf \left\{ \frac{u(x)}{v(x)}, x \in [0, 1] \right\}$, then $u \geq \theta v$, and $\theta \geq \frac{M_1}{M_2} \equiv \theta_0$. We assert that $\theta \geq 1$. In fact, we can assume by contradiction that $\theta < 1$.

It follows from the equations for u and v that

$$\left((u')^N - (\theta v')^N \right)' = -\lambda N t^{N-1} \left(f(u) - \theta^N f(v) \right).$$

By integrating, we get

$$(u'(t))^N - (\theta v'(t))^N = -\lambda \int_0^t N s^{N-1} \left(f(u(s)) - \theta^N f(v(s)) \right) ds.$$

Fix $t_0 \in (0, 1)$. By Lemma 3.2, we get that for $t \leq t_0$

$$v(t) \geq M_1 (\lambda f(A_\lambda))^{\frac{1}{N}} (1 - t_0),$$

and so by Lemma 3.1 (ii)

$$\begin{aligned} & \int_0^t N s^{N-1} \left(f(u(s)) - \theta^N f(v(s)) \right) ds \\ & \geq \int_0^t N s^{N-1} \left(f(\theta v(s)) - \theta^N f(v(s)) \right) ds \\ & = \int_0^t N s^{N-1} f(\theta v(s)) \left(1 - \frac{\theta^N f(v(s))}{f(\theta v(s))} \right) ds \\ & \geq \int_0^t N s^{N-1} f(\theta v(s)) ds (1 - \theta^\varepsilon) \\ & \geq t^N f(\theta_0 v(t_0)) m_{1\varepsilon} (1 - \theta) \end{aligned} \quad (3.9)$$

for large λ , where $m_{1\varepsilon} = \min\{1, \varepsilon\}$. Here we use the inequality

$$1 - \theta^\varepsilon \geq \min\{1, \varepsilon\}(1 - \theta) \text{ for } 0 < \theta < 1.$$

For $t > t_0$, we have

$$\begin{aligned} & \int_0^t N s^{N-1} \left(f(u(s)) - \theta^N f(v(s)) \right) ds \\ & = \int_0^{t_0} N s^{N-1} \left(f(u(s)) - \theta^N f(v(s)) \right) ds + \int_{t_0}^t N s^{N-1} \left(f(u(s)) - \theta^N f(v(s)) \right) ds \\ & \geq t_0^N f(\theta_0 v(t_0)) m_{1\varepsilon} (1 - \theta) - \int_T \left| N s^{N-1} \left(f(u(s)) - \theta^N f(v(s)) \right) \right| ds, \end{aligned} \quad (3.10)$$

where $T = \{t \in (t_0, 1) : f(u(s)) - \theta^N f(v(s)) < 0\}$. For $t \in T$, we get

$$\left| N s^{N-1} \left(f(u(s)) - \theta^N f(v(s)) \right) \right| \leq \left| N s^{N-1} \left(f(\theta v(s)) - \theta^N f(v(s)) \right) \right|,$$

and also $v(t) \leq d$, where $d > 0$ is so that $f(\theta x) - \theta^N f(x) > 0$ for $x > d$.

Fix $x \in (0, d)$ and let $h(\theta) = f(\theta x) - \theta^N f(x)$. Then by the mean value theorem we know that there are $\xi \in (\theta, 1)$ and $M_4 > 0$ so that

$$\begin{aligned} |h(\theta)| &= |h(\theta) - h(1)| = (1 - \theta) |h'(\xi)| \\ &= (1 - \theta) |x f'(\theta x) - N \theta^{N-1} f(x)| \\ &\leq M_4 (1 - \theta), \end{aligned} \quad (3.11)$$

where M_{i4} is independent of $x \in (0, \xi)$. Here we use (C_3) and the fact that $\theta \geq \theta_0$.

When t_0 is sufficiently close to 1, we hence from (3.9), (3.10) and (3.11) obtain that

$$\begin{aligned} & \int_0^t N s^{N-1} \left(f(u(s)) - \theta^N f(v(s)) \right) ds \\ & \geq \left[t_0^N f(\theta_0 v(t_0)) m_{1\varepsilon} - M_4(1 - t_0) \right] (1 - \theta) \\ & > 0. \end{aligned}$$

This shows that $u' - \theta v' < 0$ on $(0, 1]$, and hence there is a constant $\theta_1 > \theta$ such that $u \geq \theta_1 v$ on $(0, 1]$, which is a contradiction. This finishes the proof of Theorem 3.1. $\square$

4 New existence and nonexistence results

In this section, we apply Theorems 2.1 to establish the existence and nonexistence of positive radial concave solutions for system $(S_{\lambda_1, \dots, n})$.

For $i = 1, 2, \dots, n$, let

$$f_i^\infty := \lim_{v \rightarrow \infty} \frac{f_i(v)}{v^N}, \quad f_i^0 := \lim_{v \rightarrow 0} \frac{f_i(v)}{v^N}.$$

Theorem 4.1. Suppose that (C_0) holds. If $0 < f_i^\infty < +\infty$ ($i = 1, 2, \dots, n$), then there exists $\beta_0 > 0$ such that, for every $R > \beta_0$, system $(S_{\lambda_1, \dots, n})$ admits a positive radial concave solution $v_R = (v_{1R}, v_{2R}, \dots, v_{nR})$ satisfying $\|v_{iR}\| = R$ for any

$$\lambda_{iR} \in [\lambda_R, \bar{\lambda}_R] \quad (4.1)$$

under the case $\lambda_j = 1$, $j, i \in \{1, 2, \dots, n\}$, $j \neq i$, where λ_R and $\bar{\lambda}_R$ are positive finite numbers.

Proof. It follows from $0 < f_i^\infty < +\infty$ that there exist $0 < l_1 < l_2$, $\mu > 0$ such that

$$\begin{aligned} l_1 v_2^N &< f_1(v_2) < l_2 v_2^N, \quad \forall v_2 \geq \mu, \\ l_1 v_3^N &< f_2(v_3) < l_2 v_3^N, \quad \forall v_3 \geq \mu, \\ &\vdots \\ l_1 v_n^N &< f_{n-1}(v_n) < l_2 v_n^N, \quad \forall v_n \geq \mu, \\ l_1 v_1^N &< f_n(v_1) < l_2 v_1^N, \quad \forall v_1 \geq \mu. \end{aligned}$$

Now, we prove that $\beta_0 = \frac{\mu}{\theta}$ is required. Let

$$\Omega_R = \{x \in E : \|x\| < R\}.$$

Then Ω_R is a bounded open subset of Banach space E and $0 \in \Omega_R$.

Noticing $R > \beta_0$, for any $v_i \in P \cap \partial\Omega_R$, we have

$$v_i(t) \geq \theta \|v_i\| = \theta R, \quad t \in J_\theta,$$

and then

$$v_i(t) \geq \theta \|v_i\| > \theta \beta_0 = \mu, \quad t \in J_\theta.$$

Therefore, for any $v_2 \in P \cap \partial\Omega_R$, we get

$$\begin{aligned}
 (T_1 v_2)(t) &\geq \int_{1-\theta}^1 \left(\int_0^{1-\theta} N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_1 v_2^N(s) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_1 \theta^N \|v_2\|^N ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \theta^2 \|v_2\| l_1^{\frac{1}{N}} \left(\int_\theta^{1-\theta} N s^{N-1} ds \right)^{\frac{1}{N}} \\
 &\geq \theta^2 \|v_2\| \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J.
 \end{aligned}$$

Similarly, for $v_i \in P \cap \partial\Omega_R$, $i = 1, 3, 4, \dots, n$, we get

$$\begin{aligned}
 (T_2 v_3)(t) &\geq \int_{1-\theta}^1 \left(\int_0^{1-\theta} N s^{N-1} f_2(v_3(s)) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} f_2(v_3(s)) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_1 v_3^N(s) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_1 \theta^N \|v_3\|^N ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \theta^2 \|v_3\| l_1^{\frac{1}{N}} \left(\int_\theta^{1-\theta} N s^{N-1} ds \right)^{\frac{1}{N}} \\
 &\geq \theta^2 \|v_3\| \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J,
 \end{aligned}$$

⋮

$$\begin{aligned}
 (T_{n-1} v_n)(t) &\geq \int_{1-\theta}^1 \left(\int_0^{1-\theta} N s^{N-1} f_{n-1}(v_n(s)) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} f_{n-1}(v_n(s)) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_1 v_n^N(s) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_1 \theta^N \|v_n\|^N ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \theta^2 \|v_n\| l_1^{\frac{1}{N}} \left(\int_\theta^{1-\theta} N s^{N-1} ds \right)^{\frac{1}{N}} \\
 &\geq \theta^2 \|v_n\| \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J,
 \end{aligned}$$

$$\begin{aligned}
(T_n v_1)(t) &\geq \int_{1-\theta}^1 \left(\int_0^{1-\theta} N s^{N-1} f_n(v_1(s)) ds \right)^{\frac{1}{N}} d\tau \\
&\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} f_n(v_1(s)) ds \right)^{\frac{1}{N}} d\tau \\
&\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_1 v_1^N(s) ds \right)^{\frac{1}{N}} d\tau \\
&\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_1 \theta^N \|v_1\|^N ds \right)^{\frac{1}{N}} d\tau \\
&\geq \theta^2 \|v_1\| l_1^{\frac{1}{N}} \left(\int_\theta^{1-\theta} N s^{N-1} ds \right)^{\frac{1}{N}} \\
&\geq \theta^2 \|v_1\| \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\widetilde{T}_1 v_1)(t) &= (T_1 T_2 \dots T_n v_1)(t) \\
&\geq \theta^2 \|T_2 T_3 \dots T_n v_1\| \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}} \\
&\geq \theta^4 \|T_3 T_4 \dots T_n v_1\| \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{2}{N}} \\
&\quad \vdots \\
&\geq \theta^{2(n-1)} \|T_n v_1\| \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{(n-1)}{N}} \\
&\geq \theta^{2n} \|v_1\| \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
(\widetilde{T}_2 v_2)(t) &= (T_2 T_3 \dots T_n T_1 v_2)(t) \\
&\geq \theta^{2n} \|v_2\| \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}}, \\
&\quad \vdots \\
(\widetilde{T}_n v_n)(t) &= (T_n T_1 \dots T_{n-2} T_{n-1} v_n)(t) \\
&\geq \theta^{2n} \|v_n\| \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}}.
\end{aligned}$$

This shows that

$$\begin{aligned}
\inf_{v_1 \in P \cap \partial \Omega_R} \widetilde{T}_1 v_1 &\geq \theta^{2n} \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}} R > 0, \\
\inf_{v_2 \in P \cap \partial \Omega_R} \widetilde{T}_2 v_2 &\geq \theta^{2n} \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}} R > 0, \\
&\quad \vdots \\
\inf_{v_n \in P \cap \partial \Omega_R} \widetilde{T}_n v_n &\geq \theta^{2n} \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}} R > 0.
\end{aligned}$$

For any $R > \beta_0$, Theorem 2.1 yields that operator $\widetilde{T}_1$ has a proper element $v_{1R} \in P$ associated with the eigenvalue $\mu_{1R} > 0$, further v_{1R} satisfies $\|v_{1R}\| = R$. For any $R > \beta_0$, Theorem 2.1 also yields that operator $\widetilde{T}_2, \dots, \widetilde{T}_n$ have proper elements $v_{2R} \in P, \dots, v_{nR} \in P$ associated with the eigenvalue $\mu_{2R} > 0, \dots, \mu_{nR} > 0$, further v_{iR} satisfies $\|v_{iR}\| = R, i = 2, 3, \dots, n$.

For operator $\widetilde{T}_1$, one can denote

$$v_{nR} = T_n v_{1R}, v_{(n-1)R} = T_{n-1} v_{nR}, \dots, v_{2R} = T_2 v_{3R};$$

for operator $\widetilde{T}_2$, one can denote

$$v_{1R} = T_1 v_{2R}, v_{nR} = T_n v_{1R}, v_{(n-1)R} = T_{n-1} v_{nR}, \dots, v_{3R} = T_3 v_{4R};$$

$$\vdots$$

for operator $\widetilde{T}_n$, one can denote

$$v_{(n-1)R} = T_{n-1} v_{nR}, v_{(n-2)R} = T_{n-2} v_{(n-1)R}, \dots, v_{2R} = T_2 v_{3R}, v_{1R} = T_1 v_{2R},$$

then $(v_{1R}, v_{2R}, \dots, v_{nR})$ is the solution of system $(\mathbf{S}_{\lambda_1, \dots, n})$.

For $i = 1, 2, \dots, n$, let $\lambda_{iR} = \frac{1}{\mu_{iR}^{\frac{1}{N}}}$. Then we have

$$\widetilde{T}_1 v_{1R} = \mu_{1R} v_{1R} = \lambda_{1R}^{-\frac{1}{N}} v_{1R}, \quad (4.2)$$

$$\widetilde{T}_2 v_{2R} = \mu_{2R} v_{2R} = \lambda_{2R}^{-\frac{1}{N}} v_{2R}, \quad (4.3)$$

$$\vdots$$

$$\widetilde{T}_n v_{nR} = \mu_{nR} v_{nR} = \lambda_{nR}^{-\frac{1}{N}} v_{nR}. \quad (4.4)$$

From the proof above, for any $R > \beta_0$ and $i = 1, 2, \dots, n$, system $(\mathbf{S}_{\lambda_1, \dots, n})$ admits a positive solution $v = (v_{1R}, v_{2R}, \dots, v_{nR})$ with $v_{iR} \in P \cap \partial\Omega_R$ associated with $\lambda_i = \lambda_{iR} > 0$. Thus, it respectively follows from (4.2), (4.3) and (4.4) that

$$v_{1R}(t) = \lambda_{1R}^{\frac{1}{N}} \widetilde{T}_1 v_{1R},$$

$$v_{2R}(t) = \lambda_{2R}^{\frac{1}{N}} \widetilde{T}_2 v_{2R},$$

$$\vdots$$

$$v_{nR}(t) = \lambda_{nR}^{\frac{1}{N}} \widetilde{T}_n v_{nR},$$

and hence

$$\begin{aligned} v_{1R}(t) &= \lambda_{1R}^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_1((T_2 T_3 \dots T_n) v_{1R}(s)) ds \right)^{\frac{1}{N}} d\tau \\ &= \lambda_{1R}^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_1((T_2 T_3 \dots T_{n-1}) v_{nR}(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\quad \vdots \\ &= \lambda_{1R}^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_1((T_2 T_3) v_{4R}(s)) ds \right)^{\frac{1}{N}} d\tau \\ &= \lambda_{1R}^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_1((T_2) v_{3R}(s)) ds \right)^{\frac{1}{N}} d\tau \\ &= \lambda_{1R}^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_1(v_{2R}(s)) ds \right)^{\frac{1}{N}} d\tau. \end{aligned}$$

Similarly, we have

$$v_{2R}(t) = \lambda_{2R}^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_2(v_{3R}(s)) ds \right)^{\frac{1}{N}} d\tau,$$

$$\vdots$$

$$v_{nR}(t) = \lambda_{nR}^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_n(v_{1R}(s)) ds \right)^{\frac{1}{N}} d\tau,$$

with $\|v_{iR}\| = R$, $i = 1, 2, \dots, n$.

On the one hand,

$$\begin{aligned} v_{1R}(t) &= \lambda_{1R}^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\leq \lambda_{1R}^{\frac{1}{N}} \int_0^1 \left(\int_0^\tau N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\leq \lambda_{1R}^{\frac{1}{N}} \int_0^1 \left(\int_0^\tau N s^{N-1} l_2 \|v_2\|^N ds \right)^{\frac{1}{N}} d\tau \\ &\leq (\lambda_{1R} l_2)^{\frac{1}{N}} \|v_{2R}\|, \quad \forall t \in J. \end{aligned}$$

Next, by $v_2 = T_2 v_3, \dots, v_n = T_n v_1$, we similarly get

$$\begin{aligned} v_{2R}(t) &\leq l_2^{\frac{1}{N}} \|v_{3R}\|, \quad \forall t \in J, \\ &\vdots \\ v_{nR}(t) &\leq l_2^{\frac{1}{N}} \|v_{1R}\|, \quad \forall t \in J. \end{aligned}$$

This shows that

$$\|v_{1R}\| = R \leq (\lambda_{1R} l_2^n)^{\frac{1}{N}} \|v_{1R}\|,$$

and hence,

$$\lambda_{1R} \geq l_2^{-n} = \lambda_R.$$

On the other hand,

$$\begin{aligned} (v_{1R})(t) &= \lambda_{1R}^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_1(v_{2R}(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_{1R}^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_0^{1-\theta} N s^{N-1} f_1(v_{2R}(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_{1R}^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} f_1(v_{2R}(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_{1R}^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_1 v_{2R}^N(s) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_{1R}^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_1 \theta^N \|v_{2R}\|^N ds \right)^{\frac{1}{N}} d\tau \\ &\geq \theta^2 \|v_{2R}\| \left(\lambda_{1R} l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J. \end{aligned}$$

Similarly, by $v_2 = T_2 v_3, \dots, v_n = T_n v_1$, one can prove that

$$\begin{aligned} (v_{2R})(t) &\geq \theta^2 \|v_{3R}\| \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J, \\ &\vdots \\ (v_{nR})(t) &\geq \theta^2 \|v_{1R}\| \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J. \end{aligned}$$

This yields that

$$\|v_{1R}\| \geq \theta^{2n} \|v_{1R}\| (\lambda_{1R})^{\frac{1}{N}} \left(l_1 ((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}},$$

and hence,

$$\lambda_{1R} \leq \theta^{-2nN} \left(l_1 ((1-\theta)^N - \theta^N) \right)^{-n} = \bar{\lambda}_R. \quad (4.5)$$

In conclusion, $\lambda_{1R} \in [\lambda_R, \bar{\lambda}_R]$.

Similarly, we can prove $\lambda_{iR} \in [\lambda_R, \bar{\lambda}_R]$ for $i \in \{2, 3, \dots, n\}$. This gives the proof of Theorem 4.1. $\square$

Theorem 4.2. Suppose that (C_0) holds. If $0 < f_i^0 < +\infty$ ($i = 1, 2, \dots, n$), then there exists $\beta_0^* > 0$ such that, for

every $0 < r < \beta_0^*$, system $(S_{\lambda_1, \dots, n})$ admits a positive radial concave solution $v_r = (v_{1r}, v_{2r}, \dots, v_{nr})$ satisfying $\|v_{ir}\| = r$ for any

$$\lambda_{ir} \in [\lambda_r, \bar{\lambda}_r]$$

under the case $\lambda_j = 1$, $j, i \in \{1, 2, \dots, n\}$, $j \neq i$, where λ_r and $\bar{\lambda}_r$ are positive finite numbers.

Proof. The proof is similar to that of Theorem 4.1, so we omit it here. $\square$

Theorem 4.3. Suppose that (C_0) holds. If $f_i^\infty = +\infty$ ($i = 1, 2, \dots, n$), then there exists $\bar{\beta}_0 > 0$ such that, for every $R^* > \bar{\beta}_0$, system $(S_{\lambda_1, \dots, n})$ admits a nontrivial radial solution $v_{R^*} = (v_{1R^*}, v_{2R^*}, \dots, v_{nR^*})$ satisfying $\|v_{iR^*}\| = R^*$ for any

$$\lambda_{iR^*} \in (0, \lambda_{R^*}] \quad (4.6)$$

under the case $\lambda_j = 1$, $j, i \in \{1, 2, \dots, n\}$, $j \neq i$, where λ_{R^*} is a positive finite number.

Proof. It follows from $f_i^\infty = +\infty$ that there exist $l^* > 0$, $\mu^* > 0$ such that

$$f_1(v_2) > l^* v_2^N, \quad \forall v_2 \geq \mu^*,$$

$$f_2(v_3) > l^* v_3^N, \quad \forall v_3 \geq \mu^*,$$

$$\vdots$$

$$f_n(v_1) > l^* v_1^N, \quad \forall v_1 \geq \mu^*.$$

Now, we prove that $\bar{\beta}_0 = \frac{\mu^*}{\theta}$ is required. Let

$$\Omega_{R^*} = \{x \in E : \|x\| < R^*\}.$$

Noticing $R^* > \bar{\beta}_0$, for any $v_i \in P \cap \partial\Omega_{R^*}$, we have

$$v_i(t) \geq \theta \|v_i\| = \theta R^*, \quad t \in J_\theta,$$

and then

$$v_i(t) \geq \theta \|v_i\| > \theta \bar{\beta}_0 = \mu^*, \quad t \in J_\theta.$$

Therefore, for any $v_2 \in P \cap \partial\Omega_{R^*}$, we get

$$\begin{aligned} (T_1 v_2)(t) &\geq \int_{1-\theta}^1 \left(\int_0^{1-\theta} N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l^* v_2^N(s) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l^* \theta^N \|v_2\|^N ds \right)^{\frac{1}{N}} d\tau \\ &\geq \theta^2 \|v_2\| l^{\frac{1}{N}} \left(\int_\theta^{1-\theta} N s^{N-1} ds \right)^{\frac{1}{N}} \\ &\geq \theta^2 \|v_2\| \left(l^*((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J. \end{aligned}$$

Similarly, for $v_i \in P \cap \partial\Omega_{R^+}$, $i = 1, 3, 4, \dots, n$, we get

$$\begin{aligned}
 (T_2 v_3)(t) &\geq \int_{1-\theta}^1 \left(\int_0^{1-\theta} N s^{N-1} f_2(v_3(s)) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} f_2(v_3(s)) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_* v_3^N(s) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_* \theta^N \|v_3\|^N ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \theta^2 \|v_3\| l_*^{\frac{1}{N}} \left(\int_\theta^{1-\theta} N s^{N-1} ds \right)^{\frac{1}{N}} \\
 &\geq \theta^2 \|v_3\| \left(l_* ((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J,
 \end{aligned}$$

$\vdots$

$$\begin{aligned}
 (T_{n-1} v_n)(t) &\geq \int_{1-\theta}^1 \left(\int_0^{1-\theta} N s^{N-1} f_{n-1}(v_n(s)) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} f_{n-1}(v_n(s)) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_* v_n^N(s) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_* \theta^N \|v_n\|^N ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \theta^2 \|v_n\| l_*^{\frac{1}{N}} \left(\int_\theta^{1-\theta} N s^{N-1} ds \right)^{\frac{1}{N}} \\
 &\geq \theta^2 \|v_n\| \left(l_* ((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J,
 \end{aligned}$$

$$\begin{aligned}
 (T_n v_1)(t) &\geq \int_{1-\theta}^1 \left(\int_0^{1-\theta} N s^{N-1} f_n(v_1(s)) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} f_n(v_1(s)) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_* v_1^N(s) ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} l_* \theta^N \|v_1\|^N ds \right)^{\frac{1}{N}} d\tau \\
 &\geq \theta^2 \|v_1\| l_*^{\frac{1}{N}} \left(\int_\theta^{1-\theta} N s^{N-1} ds \right)^{\frac{1}{N}} \\
 &\geq \theta^2 \|v_1\| \left(l_* ((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (\widetilde{T}_1 v_1)(t) &= (T_1 T_2 \dots T_n v_1)(t) \\
 &\geq \theta^2 \|T_2 T_3 \dots T_n v_1\| \left(l_*((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}} \\
 &\geq \theta^4 \|T_3 T_4 \dots T_n v_1\| \left(l_*((1-\theta)^N - \theta^N) \right)^{\frac{2}{N}} \\
 &\vdots \\
 &\geq \theta^{2(n-1)} \|T_n v_1\| \left(l_*((1-\theta)^N - \theta^N) \right)^{\frac{(n-1)}{N}} \\
 &\geq \theta^{2n} \|v_1\| \left(l_*((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}}.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 (\widetilde{T}_2 v_2)(t) &= (T_2 T_3 \dots T_n T_1 v_2)(t) \\
 &\geq \theta^{2n} \|v_2\| \left(l_*((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}}, \\
 &\vdots \\
 (\widetilde{T}_n v_n)(t) &= (T_n T_1 \dots T_{n-2} T_{n-1} v_n)(t) \\
 &\geq \theta^{2n} \|v_n\| \left(l_*((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}}.
 \end{aligned}$$

This shows that

$$\begin{aligned}
 \inf_{v_1 \in P \cap \partial \Omega_{R_*}} \widetilde{T}_1 v_1 &\geq \theta^{2n} \left(l_*((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}} R_* > 0, \\
 \inf_{v_2 \in P \cap \partial \Omega_{R_*}} \widetilde{T}_2 v_2 &\geq \theta^{2n} \left(l_*((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}} R_* > 0, \\
 &\vdots \\
 \inf_{v_n \in P \cap \partial \Omega_{R_*}} \widetilde{T}_n v_n &\geq \theta^{2n} \left(l_*((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}} R_* > 0.
 \end{aligned}$$

For any $R_* > \beta_0$, Theorem 2.1 yields that operator $\widetilde{T}_1$ has a proper element $v_{1R_*} \in P$ associated with the eigenvalue $\mu_{1R_*} > 0$, further v_{1R_*} satisfies $\|v_{1R_*}\| = R_*$. For any $R_* > \beta_0$, Theorem 2.1 also yields that operator $\widetilde{T}_2, \dots, \widetilde{T}_n$ have proper elements $v_{2R_*} \in P, \dots, v_{nR_*} \in P$ associated with the eigenvalue $\mu_{2R_*} > 0, \dots, \mu_{nR_*} > 0$, further v_{iR_*} satisfies $\|v_{iR_*}\| = R_*$, $i = 2, 3, \dots, n$.

For operator $\widetilde{T}_1$, one can denote

$$v_{nR_*} = T_n v_{1R_*}, v_{(n-1)R_*} = T_{n-1} v_{nR_*}, \dots, v_{2R_*} = T_2 v_{3R_*};$$

for operator $\widetilde{T}_2$, one can denote

$$v_{1R_*} = T_1 v_{2R_*}, v_{nR_*} = T_n v_{1R_*}, v_{(n-1)R_*} = T_{n-1} v_{nR_*}, \dots, v_{3R_*} = T_3 v_{4R_*};$$

$\vdots$

for operator $\widetilde{T}_n$, one can denote

$$v_{(n-1)R_*} = T_{n-1} v_{nR_*}, v_{(n-2)R_*} = T_{n-2} v_{(n-1)R_*}, \dots, v_{2R_*} = T_2 v_{3R_*}, v_{1R_*} = T_1 v_{2R_*}.$$

then $(v_{1R_*}, v_{2R_*}, \dots, v_{nR_*})$ is the solution of system $(S_{\lambda_1, \dots, \lambda_n})$.

For $i = 1, 2, \dots, n$, let $\lambda_{ir} = \frac{1}{\mu_{ir}^N}$. Next, similar to the proof of (4.5), one can prove that (4.6) holds. This finishes the proof of Theorem 4.3. $\square$

Theorem 4.4. Suppose that (C_0) holds. If $f_i^0 = +\infty$ ($i = 1, 2, \dots, n$), then there exists $\beta_1 > 0$ such that, for every $0 < r^* < \beta_1$, system $(S_{\lambda_1, \dots, n})$ admits a positive radial concave solution $v_{r^*} = (v_{1r^*}, v_{2r^*}, \dots, v_{nr^*})$ satisfying $\|v_{ir^*}\| = r^*$ for any

$$\lambda_{ir^*} \in (0, \lambda^{**}]$$

under the case $\lambda_j = 1$, $j, i \in \{1, 2, \dots, n\}$, $j \neq i$, where λ^{**} is a positive finite number.

Proof. The proof is similar to that of Theorem 4.3, we omit it here. $\square$

For ease of exposition, we set

$$m_{f_i}(r^{**}) = \min \left\{ \frac{f_1(v)}{r^{**N}} : v_2 \in [\theta r^{**}, r^{**}] \right\}, \quad i = 1, 2, \dots, n,$$

where θ is defined in (2.3).

Theorem 4.5. Suppose that (C_0) holds. If there exist $r^{**} > 0$ and $\beta_{r^{**}} > 0$ such that $m_{f_i}(r^{**}) \geq \beta_{r^{**}}$ ($i = 1, 2, \dots, n$), then system $(S_{\lambda_1, \dots, n})$ admits a positive radial concave solution $v_{r^{**}} = (v_{1r^{**}}, v_{2r^{**}}, \dots, v_{nr^{**}})$ satisfying $\|v_{ir^{**}}\| = r^{**}$ for any

$$\lambda_{ir^{**}} \in (0, \bar{\lambda}^{**}]$$

under the case $\lambda_j = 1$, $j, i \in \{1, 2, \dots, n\}$, $j \neq i$, where $\bar{\lambda}^{**}$ is a positive finite number.

Proof. In fact, for any $v_i \in P \cap \partial\Omega_{r^{**}}$, we get $\theta r^{**} \leq v_i(t) \leq r^{**}$, $t \in J_\theta$.

Noticing that $m_{f_i}(r^{**}) \geq \beta_{r^{**}} > 0$ ($i = 1, 2$), we get

$$f_1(v_2(t)) \geq m_{f_1}(r^{**})r^{**N} \geq r^{**N}\beta_{r^{**}} \geq \beta_{r^{**}}v_1^N(t), \quad \forall t \in J_\theta, \quad v_2 \in [\theta r^{**}, r^{**}],$$

$$f_2(v_3(t)) \geq m_{f_2}(r^{**})r^{**N} \geq r^{**N}\beta_{r^{**}} \geq \beta_{r^{**}}v_2^N(t), \quad \forall t \in J_\theta, \quad v_3 \in [\theta r^{**}, r^{**}],$$

$$\vdots$$

$$f_n(v_1(t)) \geq m_{f_n}(r^{**})r^{**N} \geq r^{**N}\beta_{r^{**}} \geq \beta_{r^{**}}v_1^N(t), \quad \forall t \in J_\theta, \quad v_1 \in [\theta r^{**}, r^{**}].$$

The following proof is similar to that of Theorem 4.3. This finishes the proof of Theorem 4.5. $\square$

Next, consider the nonexistence of positive radial concave solution of system $(S_{\lambda_1, \dots, n})$.

Theorem 4.6. Assume that (C_0) holds. If $f_i^0 = f_i^\infty = 0$, $i = 1, 2, \dots, n$, then there exists $\underline{\lambda} > 0$ such that system $(S_{\lambda_1, \dots, n})$ admits no positive radial concave solution for

$$\lambda_{iR} \in (0, \underline{\lambda})$$

under the case $\lambda_j = 1$, $j, i \in \{1, 2, \dots, n\}$, $j \neq i$.

Proof. It follows from $f_i^0 = f_i^\infty = 0$ ($i = 1, 2, \dots, n$) that there exists $\bar{v}_0 > 0$ such that

$$\frac{f_i(\bar{v}_0)}{\bar{v}_0^N} = \max_{v>0} \frac{f_i(v)}{v^N}, \quad i = 1, 2, \dots, n.$$

For $i = 1, 2, \dots, n$, let

$$M = \max \left\{ \frac{f_i(\bar{v}_0)}{\bar{v}_0^N} \right\} + 1.$$

Then $M > 0$ and

$$f_i(v) \leq Mv^N, \quad i = 1, 2, \dots, n, \quad v > 0. \quad (4.7)$$

Assume that $(v_1, v_2, \dots, v_n)$ is a radial concave solution of system $(S_{\lambda_1, \dots, n})$. We will show that this leads to a contradiction for $\lambda_{1R} < \underline{\lambda}$, where $\underline{\lambda} = M^{-n}$.

For $i = 1, 2, \dots, n$, let $\lambda_i = \frac{1}{\mu_i^N}$. Then we have

$$\widetilde{T_1} v_1 = \mu_1 v_1 = \lambda_1^{-\frac{1}{N}} v_1,$$

$$\widetilde{T_2} v_2 = \mu_2 v_2 = \lambda_2^{-\frac{1}{N}} v_2,$$

$$\begin{array}{c} \vdots \\ \widetilde{T}_n v_n = \mu_2 v_2 = \lambda_n^{-\frac{1}{N}} v_2, \end{array}$$

and then

$$\begin{aligned} v_1(t) &= \lambda_1^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\leq \lambda_1^{\frac{1}{N}} \int_0^1 \left(\int_0^1 N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\leq \lambda_1^{\frac{1}{N}} \int_0^1 \left(\int_0^1 N s^{N-1} M \|v_2\|^N ds \right)^{\frac{1}{N}} d\tau \\ &\leq (\lambda_1 M)^{\frac{1}{N}} \|v_2\|, \quad \forall t \in J. \end{aligned}$$

Next, by $v_2 = T_2 v_3, \dots, v_n = T_n v_1$, we similarly get

$$\begin{array}{c} v_2(t) \leq M^{\frac{1}{N}} \|v_3\|, \quad \forall t \in J, \\ \vdots \\ v_n(t) \leq M^{\frac{1}{N}} \|v_1\|, \quad \forall t \in J. \end{array}$$

This shows that

$$\|v_1\| \leq (\lambda_1 M^n)^{\frac{1}{N}} \|v_1\| < (\lambda M^n)^{\frac{1}{N}} \|v_1\| = \|v_1\|,$$

which is a contradiction. This completes the proof. $\square$

5 On a power-type system of Monge-Ampère equations

In this section, as an application, we consider the power-type system of Monge-Ampère equations

$$\begin{cases} \det D^2 u_1 = \lambda_1 (-u_2)^{\alpha_2} & \text{in } \Omega, \\ \det D^2 u_2 = \lambda_2 (-u_3)^{\alpha_3} & \text{in } \Omega, \\ \vdots \\ \det D^2 u_{n-1} = \lambda_{n-1} (-u_1)^{\alpha_n} & \text{in } \Omega, \\ \det D^2 u_n = \lambda_n (-u_1)^{\alpha_1} & \text{in } \Omega, \\ u_1 < 0, u_2 < 0, \dots, u_n < 0 & \text{in } \Omega, \\ u_1 = u_2 = \dots = u_n = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.1)$$

where Ω is the unit ball in $\mathbb{R}^N$, $N \geq 2$, $\alpha_i > 0$, $i = 1, 2, \dots, n$.

Similar to $(S_{\lambda_1, \dots, \lambda_n})$, we have

$$\begin{cases} ((u'_1)^N)' = \lambda_1 N r^{N-1} (-u_2)^{\alpha_2}, & 0 < r < 1, \\ ((u'_2)^N)' = \lambda_2 N r^{N-1} (-u_3)^{\alpha_3}, & 0 < r < 1, \\ \vdots \\ ((u'_n)^N)' = \lambda_n N r^{N-1} (-u_1)^{\alpha_1}, & 0 < r < 1, \\ u'_i(0) = u_i(1) = 0, & i = 1, 2, \dots, n. \end{cases} \quad (5.2)$$

For $i = 1, 2, \dots, n$, let $v_i = -u_i$. Then it follows from $(S_{\lambda_1, \dots, \lambda_n})$ that

$$\begin{cases} ((-v'_1)^N)' = \lambda_1 N t^{N-1} v_2^{\alpha_2}, & 0 < t < 1, \\ ((-v'_2)^N)' = \lambda_2 N t^{N-1} v_3^{\alpha_3}, & 0 < t < 1, \\ \vdots \\ ((-v'_n)^N)' = \lambda_n N t^{N-1} v_1^{\alpha_1}, & 0 < t < 1, \\ v'_i(0) = v_i(1) = 0, & i = 1, 2, \dots, n. \end{cases} \quad (5.3)$$

Theorem 5.1. For $i = 1, 2, \dots, n$, let $\alpha_i > 0$. Then

(1) if $\alpha_1 \alpha_2 \cdots \alpha_n > N^n$, then for every $r \in (0, 1)$, system (5.3) admits a positive radial concave solution $v_r = (v_{1r}, v_{2r}, \dots, v_{nr})$ satisfying $\|u_{ir}\| = r$ for any

$$\lambda_{ir}^{\frac{1}{N}} \in [1, +\infty)$$

under the case $\lambda_j = 1$, $j, i \in \{1, 2, \dots, n\}$, $j \neq i$;

(2) if $\alpha_1 \alpha_2 \cdots \alpha_n < N^n$, then for every $r \in (1, +\infty)$, system (5.3) admits a positive radial concave solution $v_r = (v_{1r}, v_{2r}, \dots, v_{nr})$ satisfying $\|v_{ir}\| = r$ for any

$$\lambda_{ir}^{\frac{1}{N}} \in (0, \lambda_{N\alpha}]$$

under the case $\lambda_j = 1$, $j, i \in \{1, 2, \dots, n\}$, $j \neq i$, where $\lambda_{N\alpha}$ is a positive finite number;

(3) if $\alpha_1 \alpha_2 \cdots \alpha_n = N^n$, then for every $r > 0$, system (5.3) admits a positive radial concave solution $v_r = (v_{1r}, v_{2r}, \dots, v_{nr})$ satisfying $\|v_{ir}\| = r$ for any

$$\lambda_{ir}^{\frac{1}{N}} \in [1, \lambda_{N\alpha}]$$

under the case $\lambda_j = 1$, $j, i \in \{1, 2, \dots, n\}$, $j \neq i$.

Proof. For $r > 0$, let

$$\Omega_r = \{x \in E : \|x\| < r\}.$$

Then, for any $v_i \in P \cap \partial\Omega_r$, we have

$$v_i(t) \geq \theta \|v_i\| = \theta r, \quad t \in J_\theta,$$

and then, for any $v_2 \in P \cap \partial\Omega_r$, we get

$$\begin{aligned} (T_1 v_2)(t) &\geq \int_{1-\theta}^1 \left(\int_0^{1-\theta} N s^{N-1} v_2^{\alpha_2}(s) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} v_2^{\alpha_2}(s) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} \theta^{\alpha_2} \|v_2\|^{\alpha_2} ds \right)^{\frac{1}{N}} d\tau \\ &\geq \theta (\theta \|v_2\|)^{\frac{\alpha_2}{N}} \left(\int_\theta^{1-\theta} N s^{N-1} ds \right)^{\frac{1}{N}} \\ &\geq \theta (\theta \|v_2\|)^{\frac{\alpha_2}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}}, \quad \forall t \in J. \end{aligned}$$

Similarly, for $v_i \in P \cap \partial\Omega_r$, $i = 1, 3, 4, \dots, n$, we get

$$\begin{aligned} (T_2 v_3)(t) &\geq \theta (\theta \|v_3\|)^{\frac{\alpha_3}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}}, \quad \forall t \in J, \\ &\vdots \\ (T_{n-1} v_n)(t) &\geq \theta (\theta \|v_n\|)^{\frac{\alpha_n}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}}, \quad \forall t \in J, \\ (T_n v_1)(t) &\geq \theta (\theta \|v_1\|)^{\frac{\alpha_1}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}}, \quad \forall t \in J. \end{aligned}$$

Therefore,

$$\begin{aligned} (\widetilde{T}_1 v_1)(t) &= (T_1 T_2 \dots T_n v_1)(t) \\ &\geq \theta (\theta \|T_2 T_3 \dots T_n v_1\|)^{\frac{\alpha_2}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}} \\ &\geq \theta \left(\theta^{\frac{N+\alpha_2}{N}} \|T_3 T_4 \dots T_n v_1\|^{\frac{\alpha_3}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}} \right)^{\frac{\alpha_2}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}} \\ &\vdots \\ &\geq h(\theta; n, N, \alpha_1, \alpha_2, \dots, \alpha_n) \|v_1\|^{\frac{\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n}{N^n}}, \end{aligned}$$

where h is a power function of θ , and the power exponent is a function of n, N, α_i , $i = 1, 2, \dots, n$. Since $0 < \theta < \frac{1}{2}$, $\alpha_i > 0$, $h(\theta; n, N, \alpha_1, \alpha_2, \dots, \alpha_n) > 0$.

$$\begin{aligned}(\widetilde{T_2 v_2})(t) &= (T_2 T_3 \dots T_n T_1 v_2)(t) \\ &\geq h(\theta; n, N, \alpha_1, \alpha_2, \dots, \alpha_n) \|v_2\|^{\frac{\alpha_1 \alpha_2 \alpha_3 \dots \alpha_n}{N^n}}, \\ &\vdots \\ (\widetilde{T_n v_n})(t) &= (T_n T_1 \dots T_{n-2} T_{n-1} v_n)(t) \\ &\geq h(\theta; n, N, \alpha_1, \alpha_2, \dots, \alpha_n) \|v_n\|^{\frac{\alpha_1 \alpha_2 \alpha_3 \dots \alpha_n}{N^n}}.\end{aligned}$$

This shows that

$$\begin{aligned}\inf_{v_1 \in P \cap \partial \Omega_r} \widetilde{T_1 v_1} &\geq h(\theta; n, N, \alpha_1, \alpha_2, \dots, \alpha_n) r^{\frac{\alpha_1 \alpha_2 \alpha_3 \dots \alpha_n}{N^n}} > 0, \\ \inf_{v_2 \in P \cap \partial \Omega_r} \widetilde{T_2 v_2} &\geq h(\theta; n, N, \alpha_1, \alpha_2, \dots, \alpha_n) r^{\frac{\alpha_1 \alpha_2 \alpha_3 \dots \alpha_n}{N^n}} > 0, \\ &\vdots \\ \inf_{v_n \in P \cap \partial \Omega_r} \widetilde{T_n v_n} &\geq h(\theta; n, N, \alpha_1, \alpha_2, \dots, \alpha_n) r^{\frac{\alpha_1 \alpha_2 \alpha_3 \dots \alpha_n}{N^n}} > 0.\end{aligned}$$

For any $r > 0$, Theorem 2.1 yields that operator $\widetilde{T_1}$ has a proper element $v_{1r} \in P$ associated with the eigenvalue $\mu_{1r} > 0$, further v_{1r} satisfies $\|v_{1r}\| = r$. For any $r > \beta_0$, Theorem 2.1 also yields that operator $\widetilde{T_2}, \dots, \widetilde{T_n}$ have proper elements $v_{2r} \in P, \dots, v_{nr} \in P$ associated with the eigenvalue $\mu_{2r} > 0, \dots, \mu_{nr} > 0$, further v_{ir} satisfies $\|v_{ir}\| = r$, $i = 2, 3, \dots, n$.

For operator $\widetilde{T_1}$, one can denote

$$v_{nr} = T_n v_{1r}, v_{(n-1)r} = T_{n-1} v_{nr}, \dots, v_{2r} = T_2 v_{3r};$$

for operator $\widetilde{T_2}$, one can denote

$$v_{1r} = T_1 v_{2r}, v_{nr} = T_n v_{1r}, v_{(n-1)r} = T_{n-1} v_{nr}, \dots, v_{3r} = T_3 v_{4r};$$

$\vdots$

for operator $\widetilde{T_n}$, one can denote

$$v_{(n-1)r} = T_{n-1} v_{nr}, v_{(n-2)r} = T_{n-2} v_{(n-1)r}, \dots, v_{2r} = T_2 v_{3r}, v_{1r} = T_1 v_{2r}.$$

then $(v_{1r}, v_{2r}, \dots, v_{nr})$ is the solution of system (5.3).

For $i = 1, 2, \dots, n$, let $\lambda_{ir} = \frac{1}{\mu_{ir}^{\frac{1}{N}}}$. Then we have

$$\widetilde{T_1 v_{1r}} = \mu_{1r} v_{1r} = \lambda_{1r}^{-\frac{1}{N}} v_{1r}, \quad (5.4)$$

$$\widetilde{T_2 v_{2r}} = \mu_{2r} v_{2r} = \lambda_{2r}^{-\frac{1}{N}} v_{2r}, \quad (5.5)$$

$\vdots$

$$\widetilde{T_n v_{nr}} = \mu_{nr} v_{nr} = \lambda_{nr}^{-\frac{1}{N}} v_{nr}. \quad (5.6)$$

From the proof above, for any $r > 0$ and $i = 1, 2, \dots, n$, system (5.3) admits a positive solution $v = (v_{1r}, v_{2r}, \dots, v_{nr})$ with $v_{ir} \in P \cap \partial \Omega_r$ associated with $\lambda_i = \lambda_{ir} > 0$. Thus, it respectively follows from (5.4), (5.5) and (5.6) that

$$v_{1r}(t) = \lambda_{1r}^{\frac{1}{N}} \widetilde{T_1 v_{1r}}, \quad v_{2r}(t) = \lambda_{2r}^{\frac{1}{N}} \widetilde{T_2 v_{2r}}, \quad \dots, \quad v_{nr}(t) = \lambda_{nr}^{\frac{1}{N}} \widetilde{T_n v_{nr}}$$

and hence, we get

$$v_{1r}(t) = \lambda_{1r}^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} v_{2r}^{\alpha_2}(s) ds \right)^{\frac{1}{N}} d\tau,$$

$$\begin{aligned}
v_{2r}(t) &= \lambda_{2r}^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} v_{3r}^{\alpha_3}(s) ds \right)^{\frac{1}{N}} d\tau, \\
&\vdots \\
v_{nr}(t) &= \lambda_{nr}^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} v_{1r}^{\alpha_1}(s) ds \right)^{\frac{1}{N}} d\tau
\end{aligned}$$

with $\|v_{ir}\| = r$, $i = 1, 2, \dots, n$.

On the one hand,

$$\begin{aligned}
v_{1r}(t) &= \lambda_{1r}^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} v_{2r}^{\alpha_2}(s) ds \right)^{\frac{1}{N}} d\tau \\
&\leq \lambda_{1r}^{\frac{1}{N}} \int_0^1 \left(\int_0^1 N s^{N-1} \|v_{2r}\|^{\alpha_2} ds \right)^{\frac{1}{N}} d\tau \\
&\leq \lambda_{1r}^{\frac{1}{N}} \|v_{2r}\|^{\frac{\alpha_2}{N}}, \quad \forall t \in J.
\end{aligned}$$

Similarly, by $v_2 = T_2 v_3, \dots, v_n = T_n v_1$, we get

$$\begin{aligned}
v_{2r}(t) &\leq \|v_{3r}\|^{\frac{\alpha_3}{N}}, \quad \forall t \in J, \\
&\vdots \\
v_{nr}(t) &\leq \|v_{1r}\|^{\frac{\alpha_1}{N}}, \quad \forall t \in J.
\end{aligned}$$

This shows that

$$\|v_{1r}\| = r \leq \lambda_{1r}^{\frac{1}{N}} \|v_{1r}\|^{\frac{\alpha_1 \alpha_2 \dots \alpha_n}{Nn}}. \quad (5.7)$$

On the other hand,

$$\begin{aligned}
(v_{1r})(t) &\geq \lambda_{1r}^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_0^{1-\theta} N s^{N-1} v_{2r}^{\alpha_2}(s) ds \right)^{\frac{1}{N}} d\tau \\
&\geq \lambda_{1r}^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} v_{2r}^{\alpha_2}(s) ds \right)^{\frac{1}{N}} d\tau \\
&\geq \lambda_{1r}^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} (\theta \|v_{2r}\|)^{\alpha_2} ds \right)^{\frac{1}{N}} d\tau \\
&\geq \lambda_{1r}^{\frac{1}{N}} (\theta \|v_{2r}\|)^{\frac{\alpha_2}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} N s^{N-1} ds \right)^{\frac{1}{N}} d\tau \\
&\geq \lambda_{1r}^{\frac{1}{N}} \theta^{\frac{\alpha_2+N}{N}} \|v_{2r}\|^{\frac{\alpha_2}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}}, \quad \forall t \in J.
\end{aligned}$$

Similarly, by $v_2 = T_2 v_3, \dots, v_n = T_n v_1$, one can prove that

$$\begin{aligned}
(v_{2r})(t) &\geq \theta^{\frac{\alpha_3+N}{N}} \|v_{3r}\|^{\frac{\alpha_3}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}}, \quad \forall t \in J, \\
&\vdots \\
(v_{nr})(t) &\geq \theta^{\frac{\alpha_1+N}{N}} \|v_{1r}\|^{\frac{\alpha_1}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}}, \quad \forall t \in J.
\end{aligned}$$

This yields that

$$\begin{aligned}
 \|v_{1r}\| &\geq \lambda_{1r}^{\frac{1}{N}} \theta^{\frac{\alpha_2+N}{N}} \|v_{2r}\|^{\frac{\alpha_2}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}} \\
 &\geq \lambda_{1r}^{\frac{1}{N}} \theta^{\frac{\alpha_2+N}{N}} \theta^{\frac{\alpha_2(\alpha_3+N)}{N^2}} \|v_{3r}\|^{\frac{\alpha_2\alpha_3}{N^2}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2}{N^2}} \\
 &\geq \lambda_{1r}^{\frac{1}{N}} \theta^{\frac{\alpha_2+N}{N}} \theta^{\frac{\alpha_2(\alpha_3+N)}{N^2}} \theta^{\frac{\alpha_2\alpha_3(\alpha_4+N)}{N^3}} \|v_{3r}\|^{\frac{\alpha_2\alpha_3}{N^2}} \\
 &\quad \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2}{N^2}} \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2\alpha_3}{N^3}} \\
 &\quad \vdots \\
 &\geq \lambda_{1r}^{\frac{1}{N}} \theta^{\frac{\alpha_2+N}{N}} \theta^{\frac{\alpha_2(\alpha_3+N)}{N^2}} \theta^{\frac{\alpha_2\alpha_3(\alpha_4+N)}{N^3}} \dots \theta^{\frac{\alpha_2\alpha_3\dots\alpha_{n-1}(a_n+N)}{N^n}} \\
 &\quad \|v_{nr}\|^{\frac{\alpha_1\alpha_2\alpha_3\dots\alpha_n}{N^n}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2}{N^2}} \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2\alpha_3}{N^3}} \\
 &\quad \dots \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2\alpha_3\dots\alpha_n}{N^n}}.
 \end{aligned} \tag{5.8}$$

(1) For $0 < r < 1$, if $\alpha_1\alpha_2\alpha_3\dots\alpha_n > N^n$, then (5.7) yields that

$$\begin{aligned}
 \|v_{1r}\| &= r \leq \lambda_{1r}^{\frac{1}{N}} \|v_{1r}\|^{\frac{\alpha_1\alpha_2\dots\alpha_n}{N^n}} \\
 &\leq \lambda_{1r}^{\frac{1}{N}} r,
 \end{aligned}$$

which shows that

$$\lambda_{1r}^{\frac{1}{N}} \geq 1. \tag{5.9}$$

(2) For $r > 1$, if $\alpha_1\alpha_2\alpha_3\dots\alpha_n < N^n$, then (5.8) yields that

$$\begin{aligned}
 \|v_{1r}\| &= r \geq \lambda_{1r}^{\frac{1}{N}} \theta^{\frac{\alpha_2+N}{N}} \theta^{\frac{\alpha_2(\alpha_3+N)}{N^2}} \theta^{\frac{\alpha_2\alpha_3(\alpha_4+N)}{N^3}} \dots \theta^{\frac{\alpha_2\alpha_3\dots\alpha_{n-1}(a_n+N)}{N^n}} \\
 &\quad \|v_{nr}\|^{\frac{\alpha_1\alpha_2\alpha_3\dots\alpha_n}{N^n}} \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2}{N^2}} \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2\alpha_3}{N^3}} \\
 &\quad \dots \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2\alpha_3\dots\alpha_n}{N^n}} \\
 &\geq \lambda_{1r}^{\frac{1}{N}} \theta^{\frac{\alpha_2+N}{N}} \theta^{\frac{\alpha_2(\alpha_3+N)}{N^2}} \theta^{\frac{\alpha_2\alpha_3(\alpha_4+N)}{N^3}} \dots \theta^{\frac{\alpha_2\alpha_3\dots\alpha_{n-1}(a_n+N)}{N^n}} \\
 &\quad r \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2}{N^2}} \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2\alpha_3}{N^3}} \\
 &\quad \dots \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2\alpha_3\dots\alpha_n}{N^n}},
 \end{aligned}$$

and hence,

$$\begin{aligned}
 \lambda_{1r}^{\frac{1}{N}} &\leq \left\{ \theta^{\frac{\alpha_2+N}{N}} \theta^{\frac{\alpha_2(\alpha_3+N)}{N^2}} \theta^{\frac{\alpha_2\alpha_3(\alpha_4+N)}{N^3}} \dots \theta^{\frac{\alpha_2\alpha_3\dots\alpha_{n-1}(a_n+N)}{N^n}} \right. \\
 &\quad \left((1-\theta)^N - \theta^N \right)^{\frac{1}{N}} \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2}{N^2}} \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2\alpha_3}{N^3}} \\
 &\quad \left. \dots \left((1-\theta)^N - \theta^N \right)^{\frac{\alpha_2\alpha_3\dots\alpha_n}{N^n}} \right\}^{-1} = \lambda_{N\alpha}.
 \end{aligned} \tag{5.10}$$

(3) For $r > 0$, if $\alpha_1\alpha_2\alpha_3\dots\alpha_n = N^n$, then (5.7) and (5.8) respectively yield that (5.9) and (5.10) hold.

The proof of Theorem 4.1 is complete. $\square$

6 Asymptotic behavior of positive radial concave solutions

In this section, we consider the dependence of positive concave solutions on parameters λ_i ($i = 1, 2, \dots, n$) for system $(S_{\lambda_1, \dots, \lambda_n})$ by making use of the following fixed point theorem of cone expansion and compression

of norm type.

Lemma 6.1. (Theorem 2.3.4 of [26]) (Fixed point theorem of cone expansion and compression of norm type) Let Ω_1 and Ω_2 be two bounded open sets in a real Banach space E such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let operator $T : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow P$ be completely continuous, where P is a cone in E . Suppose that one of the two conditions

$$(i) \|Tx\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1 \text{ and } \|Tx\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2,$$

and

$$(ii) \|Tx\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1, \text{ and } \|Tx\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$$

is satisfied. Then T has at least one fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Letting P be defined as (2.3), then for $v \in P$, define $T_{i*} : P \rightarrow E$ ($i = 1, 2, \dots, n$) to be

$$\begin{aligned} (T_{1*}v)(t) &= \lambda_1^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_1(v(s)) ds \right)^{\frac{1}{N}} d\tau, \\ (T_{2*}v)(t) &= \lambda_2^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_2(v(s)) ds \right)^{\frac{1}{N}} d\tau, \\ &\vdots \\ (T_{n*}v)(t) &= \lambda_n^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_n(v(s)) ds \right)^{\frac{1}{N}} d\tau. \end{aligned}$$

It follows from Lemma 2.2 in [54] that T_i ($i = 1, 2, \dots, n$) maps P into itself. Moreover, for $i = 1, 2, \dots, n$, T_i are completely continuous by standard arguments.

Define a composite operator $\widetilde{T}_{1*} = T_{1*}T_{2*} \cdots T_{n*}$, which is also completely continuous from P to itself. So the operator $\widetilde{T}_{1*}$ also maps P into P . For the case $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$, Zhang and Qi [61] pointed out that

$$(v_1, v_2, \dots, v_n) \in \underbrace{C^1[0, 1] \times C^1[0, 1] \times \cdots \times C^1[0, 1]}_n$$

solves system $(S_{\lambda_1, \dots, \lambda_n})$ if and only if $(v_1, v_2, \dots, v_n)$ belongs to

$$\underbrace{P \setminus \{0\} \times P \setminus \{0\} \times \cdots \times P \setminus \{0\}}_n$$

satisfying

$$v_1 = T_1 v_2, v_2 = T_2 v_3, \dots, v_n = T_n v_1.$$

This shows that if $v_1 \in P \setminus \{0\}$ is a fixed point of $\widetilde{T}_{1*}$, define $v_2 = T_{2*}v_3, \dots, v_n = T_{n*}v_1$, then $v_1 \in \setminus\{0\}$ such that

$$(v_1, v_2, \dots, v_n) \in \underbrace{C^1[0, 1] \times C^1[0, 1] \times \cdots \times C^1[0, 1]}_n$$

solves system $(S_{\lambda_1, \dots, \lambda_n})$; on the contrary, if

$$(v_1, v_2, \dots, v_n) \in \underbrace{C^1[0, 1] \times C^1[0, 1] \times \cdots \times C^1[0, 1]}_n$$

solves system $(S_{\lambda_1, \dots, \lambda_n})$, then v_1 is certainly a nonzero fixed point of $\widetilde{T}_{1*}$ in P . Therefore the task of this paper is to search nonzero fixed points of operator $\widetilde{T}_{1*}$.

We also define another composite operator

$$\widetilde{T}_{2*} = T_{2*}T_{3*} \cdots T_{n*}T_{1*},$$

$$\widetilde{T}_{3*} = T_{3*}T_{4*} \cdots T_{n*}T_{1*}T_{2*},$$

$$\begin{aligned} & \vdots \\ \widetilde{T_{n^*}} &= T_{n^*} T_{1^*} \cdots T_{(n-2)^*} T_{(n-1)^*}, \end{aligned}$$

which have the same meaning as $\widetilde{T_{1^*}}$.

Theorem 6.1. Suppose that (C_0) holds. For $i \in \{1, 2, \dots, n\}$, then we have the following two conclusions.

(C₄) If $f_i^0 = 0$ and $f_i^\infty = \infty$, then for every $\lambda_i > 0$ system $(S_{\lambda_1, \dots, \lambda_n})$ admits a positive radial concave solution $v = (v_{\lambda_1}, v_{\lambda_2}, \dots, v_{\lambda_n})$ with $v_{\lambda_i}(t)$ satisfying $\lim_{\lambda_i \rightarrow 0^+} \|v_{\lambda_i}\| = \infty$;

(C₅) If $f_i^0 = \infty$ and $f_i^\infty = 0$, then for every $\lambda_i > 0$ system $(S_{\lambda_1, \dots, \lambda_n})$ admits a positive radial concave solution $v = (v_{\lambda_1}, v_{\lambda_2}, \dots, v_{\lambda_n})$ with $v_{\lambda_i}(t)$ satisfying $\lim_{\lambda_i \rightarrow 0^+} \|v_{\lambda_i}\| = 0$.

Proof. Here we only prove this theorem under condition (C_4) since the proof is similar when (C_5) holds. For $i \in \{1, 2, \dots, n\}$, let $\lambda_i > 0$. Considering $f_i^0 = 0$, there exists $r > 0$ such that

$$f_1(v_2) \leq \frac{1}{\lambda_1} v_2^N, \quad \forall \quad 0 \leq v_2 \leq r,$$

$$f_2(v_3) \leq \frac{1}{\lambda_2} v_3^N, \quad \forall \quad 0 \leq v_3 \leq r,$$

$$\vdots$$

$$f_n(v_1) \leq \frac{1}{\lambda_n} v_1^N, \quad \forall \quad 0 \leq v_1 \leq r.$$

Thus, for $i = \{1, 2, \dots, n\}$ and $v_i \in P \cap \partial\Omega_r$, we have

$$\begin{aligned} (T_{1^*} v_2)(t) &= \lambda_1^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\leq \lambda_1^{\frac{1}{N}} \int_0^1 \left(\int_0^1 N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\leq \|v_2\|, \quad \forall t \in J, \end{aligned}$$

$$\begin{aligned} (T_{2^*} v_3)(t) &= \lambda_2^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_2(v_3(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\leq \lambda_2^{\frac{1}{N}} \int_0^1 \left(\int_0^1 N s^{N-1} f_2(v_3(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\leq \|v_3\|, \quad \forall t \in J. \end{aligned}$$

$$\vdots$$

$$\begin{aligned} (T_{n^*} v_1)(t) &= \lambda_n^{\frac{1}{N}} \int_t^1 \left(\int_0^\tau N s^{N-1} f_n(v_1(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\leq \lambda_n^{\frac{1}{N}} \int_0^1 \left(\int_0^1 N s^{N-1} f_n(v_1(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\leq \|v_1\|, \quad \forall t \in J. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\widetilde{T_{1^*} v_1}\| &= \|T_{1^*} T_{2^*} \cdots T_{n^*} v_1\| \\ &\leq \|T_{2^*} T_{3^*} \cdots T_{n^*} v_1\| \\ &\leq \|T_{3^*} T_{4^*} \cdots T_{n^*} v_1\| \\ &\vdots \\ &\leq \|T_{n^*} v_1\| \\ &\leq \|v_1\|. \end{aligned} \tag{6.1}$$

Next, turning to $f_i^\infty = \infty$, there exists $\hat{R}$ satisfying $0 < r < \hat{R}$ such that

$$f_1(v_2) \geq \varepsilon v_2^N, \quad \forall v_2 \geq \hat{R},$$

$$f_2(v_3) \geq \varepsilon v_3^N, \quad \forall v_3 \geq \hat{R},$$

$$\vdots$$

$$f_n(v_1) \geq \varepsilon v_1^N, \quad \forall v_1 \geq \hat{R},$$

where $\varepsilon > 0$ satisfies

$$(\lambda_1 \lambda_2 \dots \lambda_n)^{\frac{1}{N}} \theta^{2n} \left(\varepsilon((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}} \geq 1.$$

Let $R > \max\{\hat{R}, \frac{\hat{R}}{\theta}\}$. Then, for $v_i \in P \cap \partial\Omega_R$, we have

$$v_i(t) \geq \theta \|v_i\| \geq \hat{R}, \quad t \in J_\theta,$$

and then

$$\begin{aligned} (T_{1*}v_2)(t) &\geq \lambda_1^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_0^{1-\theta} Ns^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_1^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} Ns^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_1^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} Ns^{N-1} \varepsilon v_2^N(s) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_1^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} Ns^{N-1} \varepsilon \theta^N \|v_2\|^N ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_1^{\frac{1}{N}} \theta^2 \|v_2\| \varepsilon^{\frac{1}{N}} \left(\int_\theta^{1-\theta} Ns^{N-1} ds \right)^{\frac{1}{N}} \\ &\geq \lambda_1^{\frac{1}{N}} \theta^2 \|v_2\| \left(\varepsilon((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J, \end{aligned}$$

$$\begin{aligned} (T_{2*}v_3)(t) &\geq \lambda_2^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_0^{1-\theta} Ns^{N-1} f_2(v_3(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_2^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} Ns^{N-1} f_2(v_3(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_2^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} Ns^{N-1} \varepsilon v_3^N(s) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_2^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} Ns^{N-1} \varepsilon \theta^N \|v_3\|^N ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_2^{\frac{1}{N}} \theta^2 \|v_3\| \varepsilon^{\frac{1}{N}} \left(\int_\theta^{1-\theta} Ns^{N-1} ds \right)^{\frac{1}{N}} \\ &\geq \lambda_2^{\frac{1}{N}} \theta^2 \|v_3\| \left(\varepsilon((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J, \end{aligned}$$

$$\vdots$$

$$\begin{aligned} (T_{n*}v_1)(t) &\geq \lambda_n^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_0^{1-\theta} Ns^{N-1} f_n(v_1(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_n^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} Ns^{N-1} f_n(v_1(s)) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_n^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} Ns^{N-1} \varepsilon v_1^N(s) ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_n^{\frac{1}{N}} \int_{1-\theta}^1 \left(\int_\theta^{1-\theta} Ns^{N-1} \varepsilon \theta^N \|v_1\|^N ds \right)^{\frac{1}{N}} d\tau \\ &\geq \lambda_n^{\frac{1}{N}} \theta^2 \|v_1\| \varepsilon^{\frac{1}{N}} \left(\int_\theta^{1-\theta} Ns^{N-1} ds \right)^{\frac{1}{N}} \\ &\geq \lambda_n^{\frac{1}{N}} \theta^2 \|v_1\| \left(\varepsilon((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}}, \quad \forall t \in J. \end{aligned}$$

Therefore,

$$\begin{aligned}
 (\widetilde{T_1} v_1)(t) &= (T_1 T_2 \dots T_n v_1)(t) \\
 &\geq \lambda_1^{\frac{1}{N}} \theta^2 \|T_2 T_3 \dots T_n v_1\| \left(\varepsilon((1-\theta)^N - \theta^N) \right)^{\frac{1}{N}} \\
 &\geq (\lambda_1 \lambda_2)^{\frac{1}{N}} \theta^4 \|T_3 T_4 \dots T_n v_1\| \left(\varepsilon((1-\theta)^N - \theta^N) \right)^{\frac{2}{N}} \\
 &\vdots \\
 &\geq (\lambda_1 \lambda_2 \dots \lambda_{n-1})^{\frac{1}{N}} \theta^{2(n-1)} \|T_n v_1\| \left(\varepsilon((1-\theta)^N - \theta^N) \right)^{\frac{(n-1)}{N}} \\
 &\geq (\lambda_1 \lambda_2 \dots \lambda_n)^{\frac{1}{N}} \theta^{2n} \|v_1\| \left(\varepsilon((1-\theta)^N - \theta^N) \right)^{\frac{n}{N}} \\
 &\geq \|v_1\|.
 \end{aligned} \tag{6.2}$$

Applying (i) of Lemma 6.1 to (6.1) and (6.2) yields that operator $\widetilde{T_1^*}$ admits a fixed point $v_1 \in P \cap (\bar{\Omega}_R \setminus \Omega_r)$. Denote $v_2 = T_2^* v_3, \dots, v_n = T_n^* v_1$, then $(v_1, v_2, \dots, v_n)$ is the desired solution of system $(S_{\lambda_1, \dots, \lambda_n})$.

Similarly, (i) of Lemma 6.1 also yields that $\widetilde{T_2^*}$ admits a fixed point $v_2 \in P \cap (\bar{\Omega}_R \setminus \Omega_r)$, $\dots$, and $\widetilde{T_n^*}$ admits a fixed point $v_n \in P \cap (\bar{\Omega}_R \setminus \Omega_r)$.

Next, for $i \in \{1, 2, \dots, m\}$, we prove that $\|v_{\lambda_i}\| = +\infty$ as $\lambda_i \rightarrow +\infty$. In fact, if not, there exist a number $\zeta_i > 0$ and a sequence $\lambda_{im} \rightarrow +\infty$ such that

$$\|v_{\lambda_{im}}\| \leq \zeta_i \quad (m = 1, 2, 3, \dots).$$

Furthermore, the sequence $\{\|v_{\lambda_{im}}\|\}$ contains a subsequence that converges to a number $\eta_i (0 \leq \eta_i \leq \zeta_i)$. For simplicity, suppose that $\{\|v_{\lambda_{im}}\|\}$ itself converges to η_i .

If $\eta_i > 0$, then $\|v_{i\lambda_{im}}\| > \frac{\eta_i}{2}$ for sufficiently large m ($m > N$), and therefore

$$\begin{aligned}
 \frac{1}{\lambda_{1m}^{\frac{1}{N}}} &= \frac{\| \int_t^1 \left(\int_0^\tau N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \|}{\|v_{1\lambda_{1m}}\|} \\
 &\leq \frac{\| \int_0^1 \left(\int_0^\tau N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \|}{\|v_{1\lambda_{1m}}\|} \\
 &\leq \frac{M_1}{\|v_{1\lambda_{1m}}\|} \\
 &< \frac{2M_1}{\eta_1} \quad (m > N), \\
 &\vdots \\
 \frac{1}{\lambda_{nm}^{\frac{1}{N}}} &= \frac{\| \int_t^1 \left(\int_0^\tau N s^{N-1} f_n(v_1(s)) ds \right)^{\frac{1}{N}} d\tau \|}{\|v_{n\lambda_{nm}}\|} \\
 &\leq \frac{\| \int_0^1 \left(\int_0^\tau N s^{N-1} f_n(v_1(s)) ds \right)^{\frac{1}{N}} d\tau \|}{\|v_{n\lambda_{nm}}\|} \\
 &\leq \frac{M_n}{\|v_{n\lambda_{nm}}\|} \\
 &< \frac{2M_n}{\eta_n} \quad (m > N),
 \end{aligned}$$

where,

$$\begin{aligned}
 M_1 &= \max \left\{ f_1(v_2), \quad r_2 \leq \|v_2\| \leq R_2 \right\}, \\
 &\vdots \\
 M_n &= \max \left\{ f_n(v_1), \quad r_m \leq \|v_1\| \leq R_2 \right\}.
 \end{aligned}$$

This contradicts $\lambda_{im} \rightarrow 0^+$ for $i \in \{1, 2, \dots, n\}$.

If $\eta_i = 0$, then $\|v_{i\lambda_{im}}\| \rightarrow 0$ for sufficiently large m ($m > N$), and therefore it follows from (C_4) that for any $\varepsilon > 0$ there exists $r^* > 0$ such that

$$\begin{aligned} f_1(v_{2\lambda_{2n}}) &\leq \varepsilon v_{2\lambda_{2n}}^N, \quad \forall 0 \leq v_{2\lambda_{2n}} \leq r^*, \\ f_2(v_{3\lambda_{3n}}) &\leq \varepsilon v_{3\lambda_{3n}}^N, \quad \forall 0 \leq v_{3\lambda_{3n}} \leq r^*, \\ &\vdots \\ f_n(v_{1\lambda_{1n}}) &\leq \varepsilon v_{1\lambda_{1n}}^N, \quad \forall 0 \leq v_{1\lambda_{1n}} \leq r^*. \end{aligned}$$

Then, for $v_{i\lambda_{im}} \in P \cap \partial\Omega_{r^*}$ and $\|v_{i\lambda_{im}}\| = r^*$, we get

$$\begin{aligned} \frac{1}{\lambda_{1m}^{\frac{1}{N}}} &= \frac{\left\| \int_t^1 \left(\int_0^\tau N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \right\|}{\|v_{1\lambda_{1m}}\|} \\ &\leq \frac{\left\| \int_0^1 \left(\int_0^1 N s^{N-1} f_1(v_2(s)) ds \right)^{\frac{1}{N}} d\tau \right\|}{\|v_{1\lambda_{1m}}\|} \\ &\leq \frac{\varepsilon \|v_2\|}{\|v_{1\lambda_{1m}}\|}, \\ &\vdots \\ \frac{1}{\lambda_{nm}^{\frac{1}{N}}} &= \frac{\left\| \int_t^1 \left(\int_0^\tau N s^{N-1} f_n(v_1(s)) ds \right)^{\frac{1}{N}} d\tau \right\|}{\|v_{n\lambda_{nm}}\|} \\ &\leq \frac{\left\| \int_0^1 \left(\int_0^1 N s^{N-1} f_n(v_1(s)) ds \right)^{\frac{1}{N}} d\tau \right\|}{\|v_{n\lambda_{nm}}\|} \\ &\leq \frac{\varepsilon \|v_1\|}{\|v_{n\lambda_{nm}}\|}, \end{aligned}$$

Since ε is arbitrary, we have $\lambda_{im} \rightarrow +\infty$ ($m \rightarrow +\infty$) in contradiction with $\lambda_{im} \rightarrow 0^+$. Therefore, $\|v_{i\lambda_i}\| \rightarrow +\infty$ as $\lambda_i \rightarrow 0^+$. Since $(v_1, v_2, \dots, v_n)$ is the desired solution of system $(S_{\lambda_1, \dots, \lambda_n})$, the proof of Theorem 6.1 is complete. $\square$

Remark 6.2. Theorems 4.1-4.6 and Theorem 6.1 are also held for problem (P_λ) , but the approach to prove Theorem 3.1 can not be applied to system $(S_{\lambda_1, \dots, \lambda_n})$.

Remark 6.1. If

$$\begin{aligned} f_1(-u_2) &= f_1(-u_1, -u_2, \dots, -u_n), \\ f_2(-u_3) &= f_2(-u_1, -u_2, \dots, -u_n), \\ &\dots, \\ f_n(-u_1) &= f_n(-u_1, -u_2, \dots, -u_n), \end{aligned}$$

we conjecture that Theorem 3.1, Theorems 4.1-4.6 and Theorem 6.1 also hold, but we can not give a proof right now.

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