Research article

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A posteriori analysis of the spectral element discretization of a non linear heat equation

<https://doi.org/10.1515/anona-2020-0140>

Received May 27, 2020; accepted June 16, 2020.

Abstract: The paper deals with a posteriori analysis of the spectral element discretization of a non linear heat equation. The discretization is based on Euler's backward scheme in time and spectral discretization in space. Residual error indicators related to the discretization in time and in space are defined. We prove that those indicators are upper and lower bounded by the error estimation.

Keywords: Non linear heat equation, Spectral elements discretization, Error indicators, A posteriori analysis

MSC: 35J57, 65M70

1 Introduction

The a posteriori analysis technique presents a very efficient tool for the mesh adaptivity methods. These methods have been widely applied in the context of the finite element discretization (see [2, 4, 5, 14, 17–20]). However, few works considered the discretization by the spectral method (see [1, 8, 10]).

This paper deals with the discretization of a non linear heat equation. We use Euler's implicit scheme with respect to time and spectral element method with respect to space. The spectral element method consists in approaching the solution of a partial differential equation by polynomial function of high degree on each sub-domain of a decomposition.

This work is an extension of the results obtained by Bernardi *et al.* (see [8]) for a discretization based on the finite element method and Chorfi *et al.* (see [10]) for the case of a linear heat equation. Herein, we start by proving that the time semi discrete problem has a unique solution. We define two local families of residual error indicators (see [12]). A first family related to the time discretization and depends only on the discrete solution and the time step. The value of this indicator allows us to choose the next time step. A second family of error indicators concerns the spectral discretization and explicitly depends on the discrete solution and the data of the non linear heat equation. We prove the optimality of those indicators in the sense that their Hilbertian sum is upper and lower bounded by the error estimation with constants independent of the discrete parameter in space and time.

The paper is organized as follows:

- In Section 2, we present the studied non linear heat equation. We prove the existence and uniqueness of the solution of the discrete time and full spectral problems.
- Section 3 is related to the definition of error indicators deduced from the residue of the non linear heat equation. We prove the equivalence between the error estimation and the Hilbertian sum of those error indicators.

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2 A time and space discrete problems

Let Ω be a bounded and simply connected domain of $\mathbb{R}^d$ ($d = 1, 2$ or 3), $\partial\Omega$ is its connected Lipschitz continuous boundary and T is a positive constant. We consider the non linear heat equation: Find φ solution of

$$\begin{cases} \frac{\partial \varphi}{\partial t} - \operatorname{div}(\lambda(\varphi)\nabla \varphi) = f & \text{in } \Omega \times]0, T[\\ \varphi(x, t) = 0 & \text{on } \partial\Omega \times]0, T[\\ \varphi(x, 0) = \varphi_0 & \text{in } \Omega. \end{cases} \quad (2.1)$$

We suppose that λ is a function verifying

$$0 < m_\lambda \leq \lambda(x) \leq M_\lambda \text{ and } |\lambda(x) - \lambda(y)| \leq \kappa_\lambda |x - y|, \quad (2.2)$$

f and φ_0 are given functions and φ is an unknown function.

In order to study the variational formulation of problem (2.1), we define the following spaces

- $H^s(\Omega)$ $s \geq 0$, is the Sobolev space provided with the norm $\|\cdot\|_{s,\Omega}$ and the semi-norm $|\cdot|_{s,\Omega}$,
- $D(\Omega)$ is the space of indefinitely differentiable functions with a compact support in Ω ,
- $H_0^s(\Omega)$ is the closure of $D(\Omega)$ in $H^s(\Omega)$ and $H^{-s}(\Omega)$ its dual space,
- the scalar product in the space $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$,
- $\mathcal{C}^0(0, T; H^s(\Omega))$ is the space of continuous functions, with values in $H^s(\Omega)$,
- $L^2(0, T; H^s(\Omega))$ is the space of square-integrable functions with values in $H^s(\Omega)$,
- $L^2(0, T; H_0^s(\Omega))$ is the space of square-integrable functions with values in $H_0^s(\Omega)$.

Problem (2.1) is equivalent to the following variational formulation:

find $\varphi \in \mathcal{C}^0(0, T, L^2(\Omega)) \cap L^2(0, T, H_0^1(\Omega))$ satisfying

$$\varphi(x, 0) = \varphi_0(x); \quad x \in \Omega \quad (2.3)$$

and $\forall t \in]0, T[, \forall \psi \in H_0^1(\Omega);$

$$\left(\frac{\partial \varphi}{\partial t}(\cdot, t), \psi \right) + (\lambda(\varphi(\cdot, t))\nabla \varphi(\cdot, t), \nabla \psi) = (f(\cdot, t), \psi), \quad (2.4)$$

where $\langle \cdot, \cdot \rangle$ is the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

For a given data f in $L^2(0, T; L^2(\Omega))$ and φ_0 in $L^2(\Omega)$, the proof of existence of φ is based on the fixed point theorem. We refer to ([15], chap 7) and to ([3, 11]) for the detail of this proof.

We remark that by choosing $\psi = \varphi(\cdot, t)$ in (2.4) and integrating between 0 and t , we obtain that the solution φ of problem (2.3)-(2.4) verifies the following stability condition: for $t \in [0, T]$

$$\|\varphi(0, t)\|_{L^2(\Omega)}^2 + m_\lambda \int_0^t \|\nabla \varphi(\cdot, s)\|_{L^2(\Omega)^d}^2 ds \leq \|\varphi_0\|_{L^2(\Omega)}^2 + \frac{1}{m_\lambda} \|f\|_{L^2(0,t;L^2(\Omega))}^2.$$

We define the following norm:

for any $\varphi \in L^2(0, T, H_0^1(\Omega))$,

$$[[\varphi]]^2(t) = \|\varphi\|_{L^2(\Omega)}^2 + m_\lambda \int_0^t \|\nabla \varphi(\cdot, s)\|_{L^2(\Omega)^d}^2 ds. \quad (2.5)$$

2.1 Time discretization

To define the time semi discrete problem, we begin by introducing a partition of the interval $[0, T]$ in sub-intervals $[t_k, t_{k+1}]$, for $1 \leq k \leq K$, such that $0 = t_0 < t_1 < \dots < t_K = T$. Let $\tau_k = t_k - t_{k-1}$, $\tau = (\tau_1, \dots, \tau_K)$, $|\tau| = \max_{1 \leq k \leq K} \tau_k$ and

$$\sigma_k = \max_{2 \leq k \leq K} \frac{\tau_k}{\tau_{k-1}}$$

the regularity parameter.

For the family $(\varphi^k)_{1 \leq k \leq K} = \varphi(\cdot, t_k)$, we associate the function φ_τ , defined on $[0, T]$, affine on each sub-interval $[t_{k-1}, t_k]$; $1 \leq k \leq K$, such that $\varphi_\tau(t_k) = \varphi(t_k)$, then

$$\forall t \in [t_{k-1}, t_k], \quad \varphi_\tau(t) = \varphi^k - \frac{t_k - t}{\tau_k}(\varphi^k - \varphi^{k-1}).$$

By applying the Euler implicit method, we deduce the following time semi-discrete problem:

$$\begin{cases} \frac{\varphi^k - \varphi^{k-1}}{\tau_k} - \operatorname{div}(\lambda(\varphi^k) \nabla \varphi^k) = f^k & \text{in } \Omega, \quad 1 \leq k \leq K \\ \varphi^k = 0 & \text{on } \partial\Omega, \quad 1 \leq k \leq K \\ \varphi^0 = \varphi_0 & \text{in } \Omega, \end{cases} \quad (2.6)$$

where $f^k = f(\cdot, t^k)$.

Problem (2.6) is equivalent to the following variational formulation:

for $f \in \mathcal{C}^0(0, T, H^{-1}(\Omega))$ and $\varphi_0 \in L^2(\Omega)$, find $(\varphi^k)_{0 \leq k \leq K} \in L^2(\Omega) \times (H_0^1(\Omega))^K$ such that for any $1 \leq k \leq K$ and $\psi \in H_0^1(\Omega)$:

$$(\varphi^k, \psi) + \tau_k(\lambda(\varphi^k) \nabla \varphi^k, \nabla \psi) = (\varphi^{k-1}, \psi) + \tau_k \langle f^k, \psi \rangle \quad \text{in } \Omega, \quad (2.7)$$

$$\varphi^0 = \varphi_0, \quad \text{in } \Omega. \quad (2.8)$$

Theorem 2.1. For $f \in \mathcal{C}^0(0, T; H^{-1}(\Omega))$ and $\varphi_0 \in L^2(\Omega)$, problem (2.7)-(2.8) has a unique solution $(\varphi^k)_{0 \leq k \leq K} \in L^2(\Omega) \times (H_0^1(\Omega))^K$ such that

$$\|\varphi^k\|_{L^2(\Omega)}^2 + m_\lambda \sum_{j=1}^k \tau_j \|\nabla \varphi^j\|_{L^2(\Omega)^d}^2 \leq \|\varphi_0\|_{L^2(\Omega)}^2 + \frac{1}{m_\lambda} \sum_{j=1}^k \tau_j \|f^j\|_{H^{-1}(\Omega)}^2. \quad (2.9)$$

Proof 1. We begin by proving the existence of solution using Brouwer fixed point theorem ([13], chap 7).

For $1 \leq k \leq K$, supposing φ^{k-1} is known, we define the application ϕ^k , from $H_0^1(\Omega)$ into $H_0^1(\Omega)$, such that for $\varphi^k \in H_0^1(\Omega)$ and $\psi \in H_0^1(\Omega)$,

$$(\phi^k(\varphi^k)/\psi) = (\varphi^k, \psi) + \tau_k(\lambda(\varphi^k) \nabla \varphi^k, \nabla \psi) - (\varphi^{k-1}, \psi) - \tau_k \langle f^k, \psi \rangle,$$

where $(./.)$ is the scalar product in $H^1(\Omega)$.

Since λ is bounded, we conclude that ϕ^k is continuous in $H_0^1(\Omega)$ and verifies for all $\varphi^k \in H_0^1(\Omega)$ that

$$(\phi^k(\varphi^k)/\varphi^k) \geq \|\varphi^k\|_{L^2(\Omega)}^2 + m_\lambda |\tau| \|\varphi^k\|_{H_0^1(\Omega)}^2 - \|\varphi^{k-1}\|_{L^2(\Omega)} \|\varphi^k\|_{L^2(\Omega)} - |\tau| \|f^k\|_{H^{-1}(\Omega)} \|\varphi^k\|_{H_0^1(\Omega)}.$$

Then, we have

$$(\phi(\varphi^k)/\varphi^k) \geq \min(1, |\tau| m_\lambda) \|\varphi^k\|_{H^1(\Omega)} - (\|\varphi^{k-1}\|_{L^2(\Omega)} + |\tau| \|f^k\|_{H^{-1}(\Omega)}) \|\varphi^k\|_{H^1(\Omega)}.$$

Then, $(\phi(\varphi^k)/\varphi^k)$ is non negative on the sphere of $H_0^1(\Omega)$ with radius

$$r = \frac{\|\varphi^{k-1}\|_{L^2(\Omega)} + |\tau| \|f^k\|_{H^{-1}(\Omega)}}{\min(1, |\tau| m_\lambda)}.$$

Let $(X_n)_n$ a decreasing sequence of sub-spaces of $H^1(\Omega)$ such that $\cup_{n=0}^\infty X_n$ is dense in $H^1(\Omega)$, ϕ^k remains continuous on X_n and verifies the non negative property in X_n . Following Brouwer's fixed point theorem ([13], chap 7, corol. 1.2), there exists φ_n^k in X_n and $\|\varphi_n^k\|_{H^1(\Omega)} \leq r$ such that

$$\forall \theta_m \in H^1(\Omega) \quad (\phi^k(\varphi_n^k)/\theta_m) = 0; \quad m \leq n. \quad (2.10)$$

Equation (2.10) is written as:

for any $\theta_m \in X_m \subset X_n$

$$(\varphi_n^k, \theta_m) + \tau_k(\lambda(\varphi_n^k) \nabla \varphi_n^k, \nabla \theta_m) = (\varphi_n^{k-1}, \theta_m) + \tau_k \langle f^k, \theta_m \rangle. \quad (2.11)$$

Since the sequence (φ_n^k) is bounded in $H^1(\Omega)$, there exists a sub-sequence $(\varphi_{n_p}^k)$ which weakly converges to φ^k in $H^1(\Omega)$. Consequently and according to the properties of the function λ (see (2.2)), we have:
for each θ_m in X_m

$$\int_{\Omega} \lambda(\varphi_{n_p}^k) \nabla \varphi_{n_p}^k \nabla \theta_m \, dx = \int_{\Omega} \nabla \varphi_{n_p}^k (\lambda(\varphi_{n_p}^k) - \lambda(\varphi^k)) \nabla \theta_m \, dx + \int_{\Omega} \nabla \varphi_{n_p}^k \lambda(\varphi^k) \nabla (\theta_m) \, dx.$$

We conclude that

$$\lim_{p \rightarrow +\infty} \int_{\Omega} \lambda(\varphi_{n_p}^k) \nabla \varphi_{n_p}^k \nabla \theta_m \, dx = \int_{\Omega} \lambda(\varphi^k) \nabla \varphi^k \nabla \theta_m \, dx.$$

The convergence of the remaining terms in (2.11) is easy to prove since they are linear.
By the density of $\cup_{m=0}^{\infty} X_m$ in $H^1(\Omega)$, we deduce that φ^k is solution of problem

$$\forall \theta \in H_0^1(\Omega), \quad (\varphi^k, \theta) + \tau_k (\lambda(\varphi^k) \nabla \varphi^k, \nabla \theta) = (\varphi^{k-1}, \theta) + \tau_k (f^k, \theta).$$

Showing now the stability condition (2.9): Let $\psi = \varphi^j$ in (2.7), then,

$$\|\varphi^j\|_{L^2(\Omega)} + m_{\lambda} \tau_j \|\nabla \varphi^j\|_{L^2(\Omega)^d}^2 \leq \|\varphi^{j-1}\|_{L^2(\Omega)} + \frac{\tau_j}{m_{\lambda}} \|f^j\|_{H^{-1}(\Omega)}.$$

Doing sum on j from 0 to k , we conclude the stability condition (2.9). □

Definition 2.1. We define the "local" norm on each ψ^k in $H_0^1(\Omega)$ by

$$[[\psi^k]] = (\|\psi^k\|_{L^2(\Omega)}^2 + \tau_k \|\psi^k\|_{H^1(\Omega)}^2)^{\frac{1}{2}}, \quad (2.12)$$

and the full time discrete norm for all $(\psi^k)_{0 \leq k \leq K} \in L^2(\Omega) \times (H_0^1(\Omega))^K$ by

$$[[\psi^k]]_k = \left(\|\psi_0\|_{L^2(\Omega)}^2 + m_{\lambda} \sum_{j=1}^k \tau_j \|\nabla \psi^j\|_{L^2(\Omega)^d}^2 \right)^{\frac{1}{2}}. \quad (2.13)$$

Lemma 2.1. For each $(\varphi^k)_{0 \leq k \leq K}$ in $L^2(\Omega) \times (H^1(\Omega))^K$, we have

$$\frac{1}{4} [[\varphi^k]]_k^2 \leq [[\varphi_{\tau}]]^2(t_k) \leq \frac{1}{2} (1 + \sigma_{\tau}) [[\varphi_k]]_k^2 + \frac{m_{\lambda}}{2} \tau_1 \|\nabla \varphi_0\|_{L^2(\Omega)^d}^2. \quad (2.14)$$

Proof 2. Let

$$\alpha_k = \int_{t_{k-1}}^{t_k} \|\nabla \varphi_{\tau}(\cdot, s)\|_{L^2(\Omega)^d}^2 \, ds \quad \text{and} \quad \beta_k = \tau_k \|\nabla \varphi^k\|_{L^2(\Omega)^d}^2.$$

Following the definition of φ_{τ} , we deduce that

$$\int_{t_{k-1}}^{t_k} |\nabla \varphi_{\tau}(x, s)|^2 \, ds = \frac{\tau_k}{3} \left(|\nabla \varphi_{\tau}^k(x)|^2 + |\nabla \varphi_{\tau}^{k-1}(x)|^2 + (\nabla \varphi^k(x) / \nabla \varphi^{k-1}(x)) \right),$$

where $(./.)$ is the scalar product in $\mathbb{R}^d$.

Then, we have

$$\alpha_k = \frac{\tau_k}{3} \left(\|\nabla \varphi_{\tau}^k\|_{L^2(\Omega)^d}^2 + \|\nabla \varphi_{\tau}^{k-1}\|_{L^2(\Omega)^d}^2 + (\nabla \varphi^k, \nabla \varphi^{k-1}) \right).$$

Using inequality $ab \geq -\frac{a^2}{4} - b^2$, we deduce

$$\alpha_k \geq \frac{\tau_k}{4} \|\nabla \varphi_{\tau}^k\|_{L^2(\Omega)^d}^2 = \frac{1}{4} \beta_k.$$

To show the other inequality, we use the property $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$. We obtain

$$\alpha_k \leq \frac{\tau_k}{2} \left(\|\nabla \varphi_\tau^k\|_{L^2(\Omega)^d}^2 + \|\nabla \varphi_\tau^{k-1}\|_{L^2(\Omega)^d}^2 \right). \quad (2.15)$$

For $k = 1$, we keep inequality (2.15). For $k \geq 2$ we use the following inequality

$$\alpha_k \leq \frac{\tau_k}{2} \|\nabla \varphi_\tau^k\|_{L^2(\Omega)^d}^2 + \frac{\tau_{k-1}}{2} \sigma_\tau \|\nabla \varphi_\tau^{k-1}\|_{L^2(\Omega)^d}^2.$$

Doing the sum on k , we conclude (2.14). $\square$

2.2 Spectral element discretization

Since the polynomials inverse inequalities are not optimal in dimension $d \geq 2$, herinafter, we consider only the one dimensional case for the a posteriori analysis of the spectral element method applied to the non linear heat equation. We start by describing the discrete problem deduced from the problem (2.7)-(2.8).

Let $\Omega =]-1, 1[$, we perform a partition of Ω such that $1 = \alpha_0 < \alpha_1 < \dots < \alpha_i < \dots < \alpha_I = 1$, where $\Omega_i =]\alpha_{i-1}, \alpha_i[$; for $1 \leq i \leq I$ and $h_i = \alpha_i - \alpha_{i-1}$. Let N_i an integer greater than 2, associated to the sub-domain Ω_i , we define the discrete parameter

$$\delta = ((h_1, N_1), (h_2, N_2), \dots, (h_I, N_I)).$$

Let $\xi_0^N < \xi_1^N < \dots < \xi_N^N$ the zeros of the polynomial $(1 - x^2)L'_N(x)$, where N an integer greater than 2 and L_N the Legendre polynomial defined on Ω .

We recall the following Gauss-Lobatto quadrature formula:

$$\forall \psi \in \mathbb{P}_{2N-1}(\Omega); \quad \int_{-1}^1 \psi(x) dx = \sum_{j=0}^N \psi(\xi_j^N) \rho_j^N \quad (2.16)$$

where ρ_j^N , $0 \leq j \leq N$, represent the weights.

Let $\mathbb{P}_n(\Omega)$ the space of polynomial of degree $\leq n$. We define in $L^2(\Omega)$ the discrete scalar product:

For any continuous functions φ and ψ on $\bar{\Omega}$

$$(\varphi, \psi)_\delta = \sum_{i=1}^I \sum_{j=0}^{N_i} \varphi(\xi_j^{N_i}) \psi(\xi_j^{N_i}) \rho_j^{N_i}, \quad (2.17)$$

where $\xi_j^{N_i} = \mathcal{F}_i^{-1}(\xi_j^N)$ and $\rho_j^{N_i} = (\alpha_i - \alpha_{i-1}) \rho_j^N$, $0 \leq j \leq N$, such that $\mathcal{F}_i$ is the bijection from Ω_i into Ω .

Let the space

$$\mathbb{Z}_\delta = \left\{ \varphi_\delta \in H^1(\Omega); \varphi_\delta|_{\Omega_i} \in \mathbb{P}_{N_i}(\Omega_i), \quad 1 \leq i \leq I \right\},$$

we recall the following property (see [7] for its proof)

$$\forall \varphi_\delta \in \mathbb{Z}_\delta, \quad \|\varphi_\delta\|_{L^2(\Omega)}^2 \leq (\varphi_\delta, \varphi_\delta)_\delta \leq 3 \|\varphi_\delta\|_{L^2(\Omega)}^2. \quad (2.18)$$

We consider i_δ the Lagrange interpolation operator such that for φ continuous on $\bar{\Omega}_i$; $i_\delta(\varphi)|_{\Omega_i} \in \mathbb{P}_{N_i}(\Omega_i)$ and for $0 \leq j \leq N_i$

$$i_\delta(\varphi)|_{\Omega_i}(\xi_j^{N_i}) = \varphi|_{\Omega_i}(\xi_j^{N_i}). \quad (2.19)$$

Let the discrete space

$$\mathbb{X}_\delta = \left\{ \varphi_\delta \in H_0^1(\Omega); \varphi_\delta|_{\Omega_i} \in \mathbb{P}_{N_i}(\Omega_i); \quad 1 \leq i \leq I \right\}.$$

Then, the full discrete problem is: find φ_δ^k in $\mathbb{Z}_\delta \times \Pi_{k=1}^K \mathbb{X}_\delta$ such that,

$$\varphi_\delta^0 = i_\delta(\varphi_0) \quad \text{in} \quad \Omega \quad (2.20)$$

$$\forall \psi_\delta \in \mathbb{X}_\delta, \quad (\varphi_\delta^k, \psi_\delta)_\delta + \tau_k(\lambda_\delta(\varphi_\delta^k) \nabla \varphi_\delta^k, \nabla \psi_\delta)_\delta = (\varphi_\delta^{k-1}, \psi_\delta)_\delta + \tau_k(f^k, \psi_\delta)_\delta, \quad (2.21)$$

where λ_δ is defined, for each φ continuous on $\bar{\Omega}$, by

$$\lambda_\delta(\varphi) = \min \{ \max \{ i_\delta(\lambda(\varphi)); m_\lambda \}; M_\lambda \}. \quad (2.22)$$

Theorem 2.2. For each $1 \leq k \leq K$; we suppose that $f^k \in H^{-1}(\Omega)$, $\varphi^0 \in L^2(\Omega)$ and $\varphi_\delta^{k-1} \in \mathbb{X}_\delta$. Problem (2.20)-(2.21) has a unique solution φ_δ^k in $\mathbb{X}_\delta$ verifying

$$[[\varphi_\delta^k]]_k^2 \leq \|i_\delta \varphi_0\|_{L^2(\Omega)}^2 + \frac{1}{m_\lambda} \sum_{j=1}^k \tau_j \|f^j\|_{H^{-1}(\Omega)}^2.$$

The proof of the above theorem follows exactly the same idea as the proof of Theorem 2.1 by using Brouwer's fixed point Theorem. We simply adjust the discrete norm to the continuous norm using inequality (2.18). We refer to ([16], chap 13) for the a priori analysis of the finite element discretization of this type of problems when the triangulations are independent of time.

Remark 2.1. Calculating the nonlinear term $(\lambda_\delta(\varphi_\delta^k) \nabla \varphi_\delta^k, \nabla \psi_\delta)_\delta$ using the quadrature formula (2.16) is made simpler by choosing a semi-linear λ such that $\lambda_\delta(\varphi_\delta^k) \nabla \varphi_\delta^k \nabla \psi_\delta$ is a polynomial of degree $\leq 2N_i - 1$ on each sub-domain Ω_i .

3 Error indicators, lower and upper bounds

This section deals with the definition of the two families of error indicators. The first indicator is related to time discretization and the second one to spectral element discretization. We prove the equivalence of those indicators with the error estimate.

3.1 The error indicators

The time error indicators are defined by analogy to our previous work in the linear case (see [10, 12]). For each k , $1 \leq k \leq K$,

$$\beta_k = \left(\frac{\tau_k}{3}\right)^{\frac{1}{2}} \|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \frac{d}{dx}(\varphi_\delta^k - \varphi_\delta^{k-1})\|_{L^2(\Omega)}. \quad (3.1)$$

We also define the local indicators, which can be computed explicitly as a function of the discrete solution :

For each k , $1 \leq k \leq K$ and each sub-interval Ω_i

$$\zeta_{k,i} = N_i^{-1} \left\| \left(i_\delta f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \frac{d}{dx}(\lambda_\delta(\varphi_\delta^k) \frac{d\varphi_\delta^k}{dx}) \right) (x - \alpha_{i-1})^{\frac{1}{2}} (\alpha_i - x)^{\frac{1}{2}} \right\|_{L^2(\Omega_i)}.$$

For technical reasons related to forthcoming demonstrations, we define the following discrete space:

$$\mathbb{X}_\delta^- = \left\{ \varphi_\delta \in H_0^1(\Omega); \lambda_\delta(\varphi_\delta) \varphi_\delta|_{\Omega_i} \in \mathbb{P}_{N_i-1}(\Omega_i), 1 \leq i \leq I \right\}.$$

3.2 An upper bound for the error

Hereinafter, to upper-bound the error defined with the norm introduced in (2.5) by the errors indicators and the data function, we apply the triangular inequality :

$$[[\varphi - \varphi_{\delta\tau}]](t_k) \leq [[\varphi - \varphi_\tau]](t_k) + [[\varphi_\tau - \varphi_{\delta\tau}]](t_k). \quad (3.2)$$

For the estimation of the error $[(\varphi - \varphi_\tau)](t_k)$, we consider π_τ the interpolating operator defined as follows:

For any function ψ continuous on $[0, T]$, $\pi_\tau \psi$ is constant on each interval $[t_{k-1}, t_k]$, $1 \leq k \leq K$, equal to $\psi(t_k)$.

Proposition 3.1. Suppose that the data function f belongs to $\mathcal{C}^0(0, T; H^{-1}(\Omega))$ and the function φ_0 belongs to $H_0^1(\Omega)$. We assume that the solution $(\varphi^k)_{0 \leq k \leq K}$ of problem (2.6) satisfies, for $p > 1$,

$$\sup_{0 \leq k \leq K} \left\| \frac{d\varphi^k}{dx} \right\|_{L^p(\Omega)} \leq y, \quad (3.3)$$

where y is a constant depending only on the data f , φ_0 and λ . Then, there exists a positive constant C , depending only on T , m_λ , M_λ , κ_λ and y , such that the following a posteriori error estimate holds between the solution φ of problem (2.1) and the solution $(\varphi^k)_{0 \leq k \leq K}$ of problem (2.6), for all t_k , $1 \leq k \leq K$,

$$\begin{aligned} [(\varphi - \varphi_\tau)](t_k) \leq C \bigg((1 + \sigma_\tau)^{\frac{1}{2}} [(\varphi_\tau - \varphi_{\delta\tau})](t_k) + \left(\sum_{m=1}^k \beta_m^2 \right)^{\frac{1}{2}} \\ + \|f - \pi_\tau f\|_{L^2(0, t_k; H^{-1}(\Omega))} \bigg). \end{aligned} \quad (3.4)$$

Proof 3. By replacing $\varphi = \varphi_\tau$ in (2.4), we obtain, for any $\psi \in H_0^1(\Lambda)$ and $t \in [t_{k-1}, t_k]$,

$$\begin{aligned} (\partial_t \varphi_\tau, \psi) + (\lambda(\varphi_\tau) \partial_x \varphi_\tau, \partial_x \psi) = \left(\frac{\varphi^k - \varphi^{k-1}}{\tau_k}, \psi \right) + \left(\lambda(\varphi_\tau) \partial_x(\varphi_\tau) \right. \\ \left. - \lambda(\varphi^k) \partial_x(\varphi^k), \partial_x \psi \right) + (\lambda(\varphi^k) \partial_x \varphi^k, \partial_x \psi). \end{aligned}$$

Then, considering equation (2.6),

$$(\partial_t \varphi_\tau, \psi) + (\lambda(\varphi_\tau) \partial_x \varphi_\tau, \partial_x \psi) = (f^k, \psi) + \left(\lambda(\varphi_\tau) \partial_x(\varphi_\tau) - \lambda(\varphi^k) \partial_x(\varphi^k), \partial_x \psi \right). \quad (3.5)$$

The difference between equation (3.5) and (2.4) results in:

$$\begin{aligned} (\partial_t(\varphi - \varphi_\tau), \psi) + \left(\lambda(\varphi) \partial_x(\varphi) - \lambda(\varphi_\tau) \partial_x(\varphi_\tau), \partial_x \psi \right) = (f - f^k, \psi) \\ + \left(\lambda(\varphi^k) \partial_x(\varphi^k) - \lambda(\varphi_\tau) \partial_x(\varphi_\tau), \partial_x \psi \right). \end{aligned} \quad (3.6)$$

Since we have, for any u and v ,

$$\lambda(u) \partial_x(u) - \lambda(v) \partial_x(v) = (\lambda(u) - \lambda(v)) \partial_x(u) + \lambda(v) \partial_x(u - v) \quad (3.7)$$

and considering $\psi = (\varphi - \varphi_\tau)$, we obtain:

$$\begin{aligned} (\partial_t(\varphi - \varphi_\tau), (\varphi - \varphi_\tau)) + \left((\lambda(\varphi) - \lambda(\varphi_\tau)) \partial_x \varphi, \partial_x(\varphi - \varphi_\tau) \right) \\ + \left(\lambda(\varphi_\tau) \partial_x(\varphi - \varphi_\tau), \partial_x(\varphi - \varphi_\tau) \right) = (f - f^m, \varphi - \varphi_\tau) \\ + \left((\lambda(\varphi^m) - \lambda(\varphi_\tau)) \partial_x \varphi^m, \partial_x(\varphi - \varphi_\tau) \right) + \left(\lambda(\varphi_\tau) \partial_x(\varphi^m - \varphi_\tau), \partial_x(\varphi - \varphi_\tau) \right). \end{aligned}$$

After integrating between $[t_{m-1}, t_m]$ and doing the sum on m , one can conclude from the properties of the function λ in (2.2) and the Hölder inequality for $\frac{1}{p} + \frac{1}{p^*} = \frac{1}{2}$ the following:

$$\begin{aligned} \frac{1}{2} \|(\varphi - \varphi_\tau)(t_k)\|_{L^2(\Omega)}^2 + m_\lambda \int_0^{t_k} \|\partial_x(\varphi - \varphi_\tau)(s)\|_{L^2(\Omega)}^2 ds \leq \\ \sum_{m=1}^k \left(\int_{t_{m-1}}^{t_m} (f(s) - f^m, (\varphi - \varphi_\tau)(s)) ds \right. \\ \left. - m_\lambda \int_{t_{m-1}}^{t_m} (\partial_x(\varphi_\tau(s) - \varphi^m), \partial_x(\varphi - \varphi_\tau)(s)) ds \right. \\ \left. + \kappa_\lambda \int_{t_{m-1}}^{t_m} \|\varphi^m - \varphi_\tau(s)\|_{L^{p^*}(\Omega)} \|\partial_x \varphi^m\|_{L^p(\Omega)} \|\partial_x(\varphi - \varphi_\tau)(s)\|_{L^2(\Omega)} ds \right). \end{aligned}$$

Let C_s the injection norm of the space $H^1(\Omega)$ in $L^{p^*}(\Omega)$. Then using inequality (3.3), we obtain:

$$\begin{aligned} \frac{1}{2} \|[\varphi - \varphi_\tau]\|^2(t_k) \leq \sum_{m=1}^k \left(\int_{t_{m-1}}^{t_m} (f(s) - f^m, (\varphi - \varphi_\tau)(s)) ds \right. \\ \left. - m_\lambda \int_{t_{m-1}}^{t_m} (\partial_x(\varphi_\tau(s) - \varphi^m), \partial_x(\varphi - \varphi_\tau)(s)) ds \right. \\ \left. + \kappa_\lambda \gamma C_s \int_{t_{m-1}}^{t_m} \|\partial_x(\varphi^m - \varphi_\tau(s))\|_{L^2(\Omega)} \|\partial_x(\varphi - \varphi_\tau)(s)\|_{L^2(\Omega)} ds \right). \end{aligned} \quad (3.8)$$

Now, we proceed by evaluating the three terms in the right-hand side of the inequality (3.8).

a) From the definition of the operator π_τ , we deduce that:

$$\begin{aligned} \left| \int_{t_{m-1}}^{t_m} (f(s) - f^m, (\varphi - \varphi_\tau)(s)) ds \right| \\ \leq \left(\int_{t_{m-1}}^{t_m} \|f - \pi_\tau f\|_{H^{-1}(\Omega)}^2 ds \right)^{\frac{1}{2}} \left(\int_{t_{m-1}}^{t_m} \|\partial_x(\varphi - \varphi_\tau)(s)\|_{L^2(\Omega)}^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

We notice also:

$$\left(\sum_{m=1}^k \int_{t_{m-1}}^{t_m} \|\partial_x(\varphi - \varphi_\tau)(s)\|_{L^2(\Omega)}^2 ds \right)^{\frac{1}{2}} \leq \|[\varphi - \varphi_\tau]\|(t_k).$$

b) Concerning the second and third terms, we proceed in the same way,

$$\begin{aligned} \left| \int_{t_{m-1}}^{t_m} (\partial_x(\varphi_\tau(s) - \varphi^m), \partial_x(\varphi - \varphi_\tau)(s)) ds \right| \\ \leq \left(\int_{t_{m-1}}^{t_m} \|\partial_x(\varphi_\tau(s) - \varphi^m)\|_{L^2(\Omega)}^2 ds \right)^{\frac{1}{2}} \left(\int_{t_{m-1}}^{t_m} \|\partial_x(\varphi - \varphi_\tau)(s)\|_{L^2(\Omega)}^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Then, from the definition of φ_τ , we have:

$$\partial_x(\varphi_\tau - \varphi^m) = \left(-\frac{t_m - s}{\tau_m} \right) \partial_x(\varphi^m - \varphi^{m-1}), \quad (3.9)$$

so,

$$\begin{aligned} \int_{t_{m-1}}^{t_m} \|\partial_x(\varphi_\tau(s) - \varphi^m)\|_{L^2(\Omega)}^2 ds &= \|\partial_x(\varphi^m - \varphi^{m-1})\|_{L^2(\Omega)}^2 \int_{t_{m-1}}^{t_m} \frac{(s - t_m)^2}{\tau_m^2} ds \\ &= \frac{\tau_m}{3} \|\partial_x(\varphi^m - \varphi^{m-1})\|_{L^2(\Omega)}^2. \end{aligned}$$

By the triangular inequality and (3.9), we obtain:

$$\begin{aligned} \left(\int_{t_{m-1}}^{t_m} \|\partial_x(\varphi_\tau(s) - \varphi^m)\|_{L^2(\Omega)}^2 ds \right)^{\frac{1}{2}} &\leq \left(\frac{\tau_m}{3} \right)^{\frac{1}{2}} \|\partial_x(\varphi^m - \varphi_\delta^m)\|_{L^2(\Omega)} \\ &+ \left(\frac{\tau_m}{3} \right)^{\frac{1}{2}} \left(\frac{1}{m_\lambda} \right)^{\frac{1}{2}} \|\lambda_\delta(\varphi_\delta^m)^{\frac{1}{2}} \partial_x(\varphi_\delta^m - \varphi_\delta^{m-1})\|_{L^2(\Omega)} + \left(\frac{\tau_m}{3} \right)^{\frac{1}{2}} \|\partial_x(\varphi^{m-1} - \varphi_\delta^{m-1})\|_{L^2(\Omega)}. \end{aligned}$$

Doing the sum on m , we conclude that there exists a positive constant C , depending only on T , m_λ , M_λ , κ_λ and y , such that:

$$\begin{aligned} \sum_{m=1}^k \int_{t_{m-1}}^{t_m} \|\partial_x(\varphi_\tau(s) - \varphi^m)\|_{L^2(\Omega)}^2 ds \\ \leq C \left(\sum_{m=1}^k \beta_m^2 + \sum_{m=1}^k \frac{\tau_m}{3} (\|\partial_x(\varphi^m - \varphi_\delta^m)\|_{L^2(\Omega)}^2 + \|\partial_x(\varphi^{m-1} - \varphi_\delta^{m-1})\|_{L^2(\Omega)}^2) \right). \end{aligned} \quad (3.10)$$

The property of norms equivalence (2.14) of lemma 2.1, yields that:

$$\begin{aligned} \sum_{m=1}^k \frac{\tau_m}{3} \left(\|\partial_x(\varphi^m - \varphi_\delta^m)\|_{L^2(\Omega)}^2 + \|\partial_x(\varphi^{m-1} - \varphi_\delta^{m-1})\|_{L^2(\Omega)}^2 \right) \\ \leq \frac{\tau_1}{3} \|\partial_x(\varphi^0 - \varphi_\delta^0)\|_{L^2(\Omega)}^2 + \sum_{m=1}^n \frac{\tau_m}{3} \|\partial_x(\varphi^m - \varphi_\delta^m)\|_{L^2(\Omega)}^2 \\ + \sum_{m=2}^k \frac{\tau_{m-1} \sigma_\tau}{3} \|\partial_x(\varphi^{m-1} - \varphi_\delta^{m-1})\|_{L^2(\Omega)}^2. \end{aligned} \quad (3.11)$$

So, we obtain:

$$\begin{aligned} \sum_{m=1}^k \frac{\tau_m}{3} \left(\|\partial_x(\varphi^m - \varphi_\delta^m)\|_{L^2(\Omega)}^2 + \|\partial_x(\varphi^{m-1} - \varphi_\delta^{m-1})\|_{L^2(\Omega)}^2 \right) \\ \leq C(1 + \sigma_\tau)[[\varphi_\tau - \varphi_{\tau\delta}]]^2(t_k), \end{aligned} \quad (3.12)$$

where C is positive constant depending only on T , m_λ , M_λ , κ_λ and y . If we consider (3.8), (3.10), (3.11) and (3.12) together we conclude the result (3.4). $\square$

To finish the global estimation, we have to bound the second term $[[\varphi_\tau - \varphi_{\tau\delta}]](t_k)$, in the right-hand side of the inequality (3.2). The proof of this estimation requires the use of the orthogonal projection operator $\Pi_N^{1,0}$ defined from $H_0^1(\Omega)$ into $\mathbb{P}_N(\Omega) \cap H_0^1(\Omega)$. See ([7], [9]) for more details about this operator and the proof of the next lemma.

Lemma 3.1. Let $\psi \in H^p([-1, 1]) \cap H_0^1([-1, 1])$, $p > 1$. The following estimate holds

$$\left(\int_{-1}^1 (\psi - \Pi_N^{1,0} \psi)^2(x) (1 - x^2)^{-1} dx \right)^{\frac{1}{2}} \leq C N^{-p} \|\psi\|_{H^p(\Omega)},$$

where C is a positive constant independent of N .

Proposition 3.2. Suppose that the data function f belongs to $\mathcal{C}^0(0, T; H^{-1}(\Omega))$ and the function φ_0 belongs to $H_0^1(\Omega)$. We assume that the solution $(\varphi^k)_{0 \leq k \leq K}$ of problem (2.7)-(2.8) satisfies the condition (3.3). Then, there exists a positive constant C , depending only on T , m_λ , M_λ , κ_λ and γ , such that the following a posteriori error estimate holds between the solution $(\varphi^k)_{0 \leq k \leq K}$ of problem (2.7)-(2.8) and the solution $(\varphi_\delta^k)_{0 \leq k \leq K}$ of problem (2.20)-(2.21), for all t_k , $1 \leq k \leq K$,

$$[[\varphi^k - \varphi_\delta^k]]_k \leq C \left(\sum_{m=1}^k \tau_m \sum_{i=1}^I (\zeta_{m,i}^2 + \|f^m - i_\delta f^m\|_{L^2(\Omega_i)}^2) \right)^{\frac{1}{2}} + \|\varphi_0 - i_\delta \varphi_0\|_{L^2(\Omega)}. \quad (3.13)$$

Proof 4. Let $\psi = \psi_\delta \in \mathbb{X}_\delta^-$ in equation (2.7), then,

$$(\varphi^k, \psi_\delta) + \tau_k (\lambda(\varphi^k) \partial_x \varphi^k, \partial_x \psi_\delta) = (\varphi^{k-1}, \psi_\delta) + \tau_k (f^k, \psi_\delta).$$

Likewise, if we consider $\psi_\delta \in \mathbb{X}_\delta^-$ in (2.21), the exactness of the quadrature formula (2.16) permits us to conclude,

$$(\varphi_\delta^k, \psi_\delta) + \tau_k (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k, \partial_x \psi_\delta) = (\varphi_\delta^{k-1}, \psi_\delta) + \tau_k (f^k, \psi_\delta)_\delta.$$

Then, for any $\psi \in H_0^1(\Omega)$, we have:

$$\begin{aligned} (\varphi^k - \varphi_\delta^k, \psi_\delta) + \tau_k (\lambda(\varphi^k) \partial_x (\varphi^k - \varphi_\delta^k), \partial_x \psi_\delta) &= (\varphi^{k-1} - \varphi_\delta^{k-1}, \psi_\delta) + (\varphi^k - \varphi_\delta^k, \psi - \psi_\delta) \\ &+ \tau_k (\lambda(\varphi^k) \partial_x \varphi^k, \partial_x (\psi - \psi_\delta)) - \tau_k (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k, \partial_x (\psi - \psi_\delta)) \\ &- \tau_k ((\lambda_\delta(\varphi_\delta^k) - \lambda(\varphi^k)) \partial_x \varphi^k, \partial_x \psi) \\ &+ \tau_k (f^k, \psi_\delta) - \tau_k (f^k, \psi_\delta)_\delta. \end{aligned} \quad (3.14)$$

Integrating by parts, we obtain:

$$\begin{aligned} (\varphi^k - \varphi_\delta^k, \psi - \psi_\delta) + \tau_k (\lambda(\varphi^k) \partial_x \varphi^k, \partial_x (\psi - \psi_\delta)) - \tau_k (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k, \partial_x (\psi - \psi_\delta)) &= \\ (\varphi^{k-1} - \varphi_\delta^{k-1}, \psi - \psi_\delta) \\ &+ \tau_k \sum_{i=1}^I \int_{\alpha_{i-1}}^{\alpha_i} \left(f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right) (x) (\psi - \psi_\delta)(x) dx \\ &- \tau_k \sum_{i=1}^{I-1} [\partial_x \varphi_\delta^k](\alpha_i) (\psi - \psi_\delta)(\alpha_i), \end{aligned}$$

where $[\cdot]$ is the jump through the point α_i .

Using equality (3.14), we obtain:

$$\begin{aligned} (\varphi^k - \varphi_\delta^k, \psi) + \tau_k (\lambda(\varphi^k) \partial_x \varphi^k - \lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k, \partial_x \psi) &= (\varphi^{k-1} - \varphi_\delta^{k-1}, \psi) \\ &+ \tau_k \sum_{i=1}^I \int_{\alpha_{i-1}}^{\alpha_i} \left(f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right) (x) (\psi - \psi_\delta)(x) dx \\ &- \tau_k \sum_{i=1}^{I-1} [\partial_x \varphi_\delta^k](\alpha_i) (\psi - \psi_\delta)(\alpha_i) + \tau_k (f^k, \psi_\delta) - \tau_k (f^k, \psi_\delta)_\delta. \end{aligned}$$

The definition of the discrete product (2.17) yields:

$$\begin{aligned} (\varphi^k - \varphi_\delta^k, \psi) + \tau_k (\lambda(\varphi^k) \partial_x \varphi^k - \lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k, \partial_x \psi) &= (\varphi^{k-1} - \varphi_\delta^{k-1}, \psi) \\ &+ \tau_k \sum_{i=1}^I \int_{\alpha_{i-1}}^{\alpha_i} \left(f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right) (x) (\psi - \psi_\delta)(x) dx \\ &- \tau_k \sum_{i=1}^{I-1} [\partial_x \varphi_\delta^k](\alpha_i) (\psi - \psi_\delta)(\alpha_i) \\ &+ \tau_k \sum_{i=1}^I \left(\int_{\alpha_{i-1}}^{\alpha_i} f^k(x) \psi_\delta(x) dx - \sum_{j=1}^{N_i} f^k(\xi_j^{N_i}) \psi_\delta(\xi_j^{N_i}) \rho_j^{N_i} \right). \end{aligned}$$

And finally, using (2.19) we conclude,

$$\begin{aligned} (\varphi^k - \varphi_\delta^k, \psi) + \tau_k (\lambda(\varphi^k) \partial_x \varphi^k - \lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k, \partial_x \psi) &= (\varphi^{k-1} - \varphi_\delta^{k-1}, \psi) \\ &+ \tau_k \sum_{i=1}^I \int_{\alpha_{i-1}}^{\alpha_i} \left(f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right) (x) (\psi - \psi_\delta)(x) dx \\ &- \tau_k \sum_{i=1}^{I-1} [\partial_x \varphi_\delta^k](\alpha_i) (\psi - \psi_\delta)(\alpha_i) + \tau_k \sum_{i=1}^I \int_{\alpha_{i-1}}^{\alpha_i} (f^k - i_\delta f^k)(x) \psi_\delta(x) dx. \end{aligned} \quad (3.15)$$

Let z_δ the image of ψ by a local regularization operator (see [6] for the properties of such operator)

$$z_\delta = \sum_{i=1}^I \Pi_{N_i-1}^{1,0} \left(\psi - \psi(\alpha_{i-1}) \theta_{i-1} - \psi(\alpha_i) \theta_i \right) + \sum_{i=0}^I \psi(\alpha_i) \theta_i,$$

where θ_i are continuous functions, affine on each Ω_i , equal to 1 in α_i and to 0 in the other nodes, ($z_\delta \in \mathbb{X}_\delta$ since $\psi \in H_0^1(\Omega)$).

Considering $\psi_\delta = z_\delta$ in equation (3.15), the jump term disappears in:

$$\begin{aligned} (\varphi^k - \varphi_\delta^k, \psi) + \tau_k (\lambda(\varphi^k) \partial_x \varphi^k - \lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k, \partial_x \psi) &= (\varphi^{k-1} - \varphi_\delta^{k-1}, \psi) \\ &+ \tau_k \sum_{i=1}^I \int_{\alpha_{i-1}}^{\alpha_i} \left(f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right) (x) (\psi - z_\delta)(x) dx \\ &+ \tau_k \sum_{i=1}^I \int_{\alpha_{i-1}}^{\alpha_i} (f^k - i_\delta f^k)(x) z_\delta(x) dx. \end{aligned} \quad (3.16)$$

Equation (3.16) can be written, by making appear the term $(x - \alpha_{i-1})^{\frac{1}{2}} (\alpha_i - x)^{\frac{1}{2}}$, as follows:

$$\begin{aligned} (\varphi^k - \varphi_\delta^k, \psi) + \tau_k (\lambda(\varphi^k) \partial_x \varphi^k - \lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k, \partial_x \psi) &= (\varphi^{k-1} - \varphi_\delta^{k-1}, \psi) \\ &+ \tau_k \sum_{i=1}^I \int_{\alpha_{i-1}}^{\alpha_i} \left(f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right) \\ &\quad (x - \alpha_{i-1})^{\frac{1}{2}} (\alpha_i - x)^{\frac{1}{2}} (x - \alpha_{i-1})^{-\frac{1}{2}} (\alpha_i - x)^{-\frac{1}{2}} (\psi - z_\delta)(x) dx \\ &+ \tau_k \sum_{i=1}^I \int_{\alpha_{i-1}}^{\alpha_i} (f^k - i_\delta f^k)(x) z_\delta(x) dx. \end{aligned}$$

Thanks to Cauchy-Schwarz inequality, we have:

$$\begin{aligned}
 (\varphi^k - \varphi_\delta^k, \psi) + \tau_k (\lambda(\varphi^k) \partial_x \varphi^k - \lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k, \partial_x \psi) &\leq (\varphi^{k-1} - \varphi_\delta^{k-1}, \psi) \\
 &+ \tau_k \sum_{i=1}^I \left(\int_{\alpha_{i-1}}^{\alpha_i} (f^k - i_\delta f^k)^2(x) dx \right)^{\frac{1}{2}} \left(\int_{\alpha_{i-1}}^{\alpha_i} \psi^2(x) dx \right)^{\frac{1}{2}} \\
 &+ \tau_k \sum_{i=1}^I \left(\int_{\alpha_{i-1}}^{\alpha_i} \left(i_\delta f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right)^2(x) (x - \alpha_{i-1})(\alpha_i - x) dx \right)^{\frac{1}{2}} \\
 &\left(\int_{\alpha_{i-1}}^{\alpha_i} (\psi - z_\delta)^2(x) (x - \alpha_{i-1})^{-1} (\alpha_i - x)^{-1} dx \right)^{\frac{1}{2}}.
 \end{aligned}$$

Then, using lemma 3.1, we obtain:

$$\begin{aligned}
 (\varphi^k - \varphi_\delta^k, \psi) + \tau_k (\lambda(\varphi^k) \partial_x \varphi^k - \lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k, \partial_x \psi) &\leq (\varphi^{k-1} - \varphi_\delta^{k-1}, \psi) \\
 &+ \tau_k \sum_{i=1}^I \zeta_{k,i} \|\partial_x \psi\|_{L^2(\Omega_i)} + \tau_k \sum_{i=1}^I \|f^k - i_\delta f^k\|_{L^2(\Omega_i)} \|\psi\|_{L^2(\Omega_i)}.
 \end{aligned} \tag{3.17}$$

Choosing $\psi = \varphi^k - \varphi_\delta^k$ and using the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ in (3.17) leads:

$$\begin{aligned}
 \|\varphi^k - \varphi_\delta^k\|_{L^2(\Omega)}^2 + \tau_k (\lambda(\varphi^k) \partial_x \varphi^k - \lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k, \partial_x (\varphi^k - \varphi_\delta^k)) \\
 \leq \frac{\|\varphi^{k-1} - \varphi_\delta^{k-1}\|_{L^2(\Omega)}^2}{2} + \frac{\|\varphi^k - \varphi_\delta^k\|_{L^2(\Omega)}^2}{2} \\
 + \tau_k \sum_{i=1}^I \left(\zeta_{k,i} + \|f^k - i_\delta f^k\|_{L^2(\Omega_i)} \right) \|\partial_x (\varphi^k - \varphi_\delta^k)\|_{L^2(\Omega_i)}^2.
 \end{aligned}$$

Using Cauchy-Schwarz inequality, we have:

$$\begin{aligned}
 \frac{\|\varphi^k - \varphi_\delta^k\|_{L^2(\Omega)}^2}{2} + \tau_k (\lambda(\varphi^k) \partial_x \varphi^k - \lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k, \partial_x (\varphi^k - \varphi_\delta^k)) &\leq \frac{\|\varphi^{k-1} - \varphi_\delta^{k-1}\|_{L^2(\Omega)}^2}{2} \\
 &+ \tau_k \left(\sum_{i=1}^I (\zeta_{k,i} + \|f^k - i_\delta f^k\|_{L^2(\Omega_i)})^2 \right)^{\frac{1}{2}} \\
 &\left(\sum_{i=1}^I \|\partial_x (\varphi^k - \varphi_\delta^k)\|_{L^2(\Omega_i)}^2 \right)^{\frac{1}{2}}.
 \end{aligned}$$

Applying again the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, yields

$$\begin{aligned}
 \frac{\|\varphi^k - \varphi_\delta^k\|_{L^2(\Omega)}^2}{2} + \tau_k (\lambda(\varphi^k) \partial_x \varphi^k - \lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k, \partial_x (\varphi^k - \varphi_\delta^k)) &\leq \frac{\|\varphi^{k-1} - \varphi_\delta^{k-1}\|_{L^2(\Omega)}^2}{2} \\
 &+ \frac{\tau_k}{2} \sum_{i=1}^I (\zeta_{k,i} + \|f^k - i_\delta f^k\|_{L^2(\Omega_i)}) + \frac{\tau_k}{2} \sum_{i=1}^I \|\partial_x (\varphi^k - \varphi_\delta^k)\|_{L^2(\Omega_i)}^2.
 \end{aligned}$$

Then, using equality (3.7) and the properties of the function λ in (2.2), (2.22) and (3.3), we deduce that, there exists a positive constant C only depending on $T, m_\lambda, M_\lambda, \kappa_\lambda$ and y , such that:

$$\begin{aligned}
 \frac{\|\varphi^k - \varphi_\delta^k\|_{L^2(\Omega)}^2}{2} + \frac{\tau_k}{2} \|\partial_x (\varphi^k - \varphi_\delta^k)\|_{L^2(\Omega)}^2 &\leq \frac{\|\varphi^{k-1} - \varphi_\delta^{k-1}\|_{L^2(\Omega)}^2}{2} + \\
 &C \tau_k \sum_{i=1}^I \left(\zeta_{k,i}^2 + \|f^k - i_\delta f^k\|_{L^2(\Omega_i)}^2 \right).
 \end{aligned}$$

We conclude (3.13) by doing the sum on k and applying lemma 2.1. $\square$

The full a posteriori estimate subject of the following theorem is the result of propositions 3.1 and 3.2 combined together.

Theorem 3.1. Suppose that the data function f belongs to $\mathcal{C}^0(0, T; H^{-1}(\Omega))$ and the function φ_0 belongs to $H_0^1(\Omega)$. We assume that the solution $(\varphi^k)_{0 \leq k \leq K}$ of problem (2.7)-(2.8) satisfies the condition (3.3). Then, there exists a positive constant C depending only on $T, m_\lambda, M_\lambda, \kappa_\lambda$ and γ , such that the following a posteriori error holds between the solution φ of problem (2.1) and the solution $(\varphi_\delta^k)_{0 \leq k \leq K}$ of problem (2.21)-(2.20), for all $t_k, 1 \leq k \leq K$,

$$\begin{aligned} \|[\varphi - \varphi_{\delta\tau}](t_k)\| \leq & C \left(\sum_{m=1}^k (\beta_m^2 + \tau_m \sum_{i=1}^I (\zeta_{m,i}^2 + \|f^m - i_\delta f^m\|_{L^2(\Omega_i)}^2)) \right)^{\frac{1}{2}} \\ & + \left(\|\varphi_0 - i_\delta \varphi_0\|_{L^2(\Omega)} + \|f - \pi_\tau f\|_{L^2(0, t_k, H^{-1}(\Omega))} \right). \end{aligned}$$

3.3 An upper bound for the error indicators

In this section, we will focus on the upper bound of the error indicators β_k and $\zeta_{k,i}$ according to the error estimate.

Proposition 3.3. Assume that the data function f belongs to $\mathcal{C}^0(0, T; H^{-1}(\Omega))$ and the function φ_0 belongs to $H_0^1(\Omega)$. We assume also the solution $(\varphi^k)_{0 \leq k \leq K}$ of problem (2.7)-(2.8), such that $\partial_x \varphi^k \in L^p(\Omega)$, $p > 1$, satisfies condition (3.3). Then, there exists a positive constant C depending only on $T, m_\lambda, M_\lambda, \kappa_\lambda, |\tau|$ and γ , such that the following estimate holds for the indicator $\beta_k, 1 \leq k \leq K$:

$$\begin{aligned} \beta_k \leq & C \left(\|[\varphi^k - \varphi_\delta^k]\| + (\sigma_\tau)^{\frac{1}{2}} \|[\varphi^{k-1} - \varphi_\delta^{k-1}]\| + (\tau_k)^{\frac{1}{2}} \|f - \pi_\tau f\|_{L^2(t_{k-1}, t_k; H^{-1}(\Omega))} \right. \\ & + (\tau_k)^{\frac{1}{2}} (\|\partial_t(\varphi - \varphi_\tau)\|_{L^2(t_{k-1}, t_k; H^{-1}(\Omega))} \\ & \left. + \|\partial_x(\varphi - \varphi_\tau)\|_{L^2(t_{k-1}, t_k; L^2(\Omega))}) \right). \end{aligned} \quad (3.18)$$

Proof 5. From the expression of the error indicator in (3.1) and by the triangular inequality, we obtain:

$$\begin{aligned} \beta_k \leq & \left(\frac{\tau_k}{3} \right)^{\frac{1}{2}} \left(\|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi_\delta^k - \varphi^k)\|_{L^2(\Omega)} + \|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi^k - \varphi^{k-1})\|_{L^2(\Omega)} \right. \\ & \left. + \|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi^{k-1} - \varphi_\delta^{k-1})\|_{L^2(\Omega)} \right). \end{aligned}$$

Using the definition of the local norm in (2.12), we conclude:

$$\left(\frac{\tau_k}{3} \right)^{\frac{1}{2}} \|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi_\delta^k - \varphi^k)\|_{L^2(\Omega)} \leq m_\lambda \|[\varphi^k - \varphi_\delta^k]\|$$

and

$$\left(\frac{\tau_k}{3} \right)^{\frac{1}{2}} \|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi^{k-1} - \varphi_\delta^{k-1})\|_{L^2(\Omega)} \leq (\sigma_\tau)^{\frac{1}{2}} m_\lambda \|[\varphi^{k-1} - \varphi_\delta^{k-1}]\|.$$

In order to estimate the term $\|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi^k - \varphi^{k-1})\|_{L^2(\Omega)}$, we consider $\psi = \varphi^k - \varphi^{k-1}$ in (3.6), (3.7) and integrate between t_{k-1}, t_k ,

$$\begin{aligned} & \int_{t_{k-1}}^{t_k} (\partial_t(\varphi - \varphi_\tau)(t), \varphi^k - \varphi^{k-1}) ds + \int_{t_{k-1}}^{t_k} (\lambda(\varphi) \partial_x(\varphi - \varphi_\tau)(t), \partial_x(\varphi^k - \varphi^{k-1})) ds \\ & = \int_{t_{k-1}}^{t_k} (f(s) - f^k, \varphi^k - \varphi^{k-1}) ds - \int_{t_{k-1}}^{t_k} (\lambda_\delta(\varphi_\delta^k) \partial_x(\varphi_\tau(s) - \varphi^k), \partial_x(\varphi^k - \varphi^{k-1})) ds \\ & + \int_{t_{k-1}}^{t_k} ((\lambda(\varphi) - \lambda_\delta(\varphi_\delta^k)) \partial_x \varphi_\tau(s), \partial_x(\varphi^k - \varphi^{k-1})) ds. \end{aligned}$$

When we apply (3.9),

$$\begin{aligned}
 & - \int_{t_{k-1}}^{t_k} (\lambda_\delta(\varphi_\delta^k) \partial_x(\varphi_\tau(s) - \varphi^k), \partial_x(\varphi^k - \varphi^{k-1})) ds \\
 & = \int_{t_{k-1}}^{t_k} \frac{t_k - s}{\tau_k} \|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi^k - \varphi^{k-1})\|_{L^2(\Omega)}^2 ds \\
 & = \|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi^k - \varphi^{k-1})\|_{L^2(\Omega)}^2 \int_{t_{k-1}}^{t_k} \frac{t_k - s}{\tau_k} ds \\
 & = \frac{\tau_k}{2} \|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi^k - \varphi^{k-1})\|_{L^2(\Omega)}^2,
 \end{aligned}$$

we derive:

$$\begin{aligned}
 & \frac{\tau_k}{2} \|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi^k - \varphi^{k-1})\|_{L^2(\Omega)}^2 = \\
 & \int_{t_{k-1}}^{t_k} (\partial_t(\varphi - \varphi_\tau)(t), \varphi^k - \varphi^{k-1}) ds + \int_{t_{k-1}}^{t_k} (\lambda(\varphi) \partial_x(\varphi - \varphi_\tau)(t), \partial_x(\varphi^k - \varphi^{k-1})) ds \\
 & - \int_{t_{k-1}}^{t_k} (f(s) - f^k, \varphi^k - \varphi^{k-1}) ds + \int_{t_{k-1}}^{t_k} ((\lambda(\varphi) - \lambda_\delta(\varphi_\delta^k)) \partial_x \varphi_\tau(s), \partial_x(\varphi^k - \varphi^{k-1})) ds.
 \end{aligned} \tag{3.19}$$

The first term in the right-hand side of the equation (3.19) is obviously bounded as follows (note that this requires that $\varphi^0 = \varphi_0 \in H_0^1(\Omega)$):

$$\begin{aligned}
 & \left| \int_{t_{k-1}}^{t_k} (\partial_t(\varphi - \varphi_\tau)(t), \varphi^k - \varphi^{k-1}) ds \right| \leq \\
 & \left(\frac{\tau_k}{m_\lambda} \right)^{\frac{1}{2}} \|\partial_t(\varphi - \varphi_\tau)\|_{L^2(t_{k-1}, t_k; H^{-1}(\Omega))} \|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi^k - \varphi^{k-1})\|_{L^2(\Omega)}.
 \end{aligned}$$

We also have:

$$\begin{aligned}
 & \left| \int_{t_{k-1}}^{t_k} (\lambda(\varphi) \partial_x(\varphi - \varphi_\tau)(t), \partial_x(\varphi^k - \varphi^{k-1})) \right| \leq \\
 & \left(1 + \frac{\tau_k}{m_\lambda} \right)^{\frac{1}{2}} \tau_k^{\frac{1}{2}} \|\lambda(\varphi)^{\frac{1}{2}} \partial_x(\varphi - \varphi_\tau)\|_{L^2(t_{k-1}, t_k; L^2(\Omega))} \|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi^k - \varphi^{k-1})\|_{L^2(\Omega)},
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| \int_{t_{k-1}}^{t_k} (f(s) - f^k, \varphi^k - \varphi^{k-1}) ds \right| \leq \\
 & \left(\frac{\tau_k}{m_\lambda} \right)^{\frac{1}{2}} \|\partial_t(f - \pi_\tau f)\|_{L^2(t_{k-1}, t_k; H^{-1}(\Omega))} \|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi^k - \varphi^{k-1})\|_{L^2(\Omega)}.
 \end{aligned}$$

The last term in the right-hand side of the equation (3.19) is estimated in the same manner as in the inequality (3.8).

$$\begin{aligned}
 & \left| \int_{t_{k-1}}^{t_k} ((\lambda(\varphi) - \lambda_\delta(\varphi_\delta^k)) \partial_x \varphi_\tau(s), \partial_x(\varphi^k - \varphi^{k-1})) ds \right| \leq \\
 & C \|\lambda(\varphi)^{\frac{1}{2}} \partial_x \varphi_\tau\|_{L^2(t_{k-1}, t_k; L^2(\Omega))} \|\lambda_\delta(\varphi_\delta^k)^{\frac{1}{2}} \partial_x(\varphi^k - \varphi^{k-1})\|_{L^2(\Omega)},
 \end{aligned}$$

where C is a positive constant depending only on $T, m_\lambda, M_\lambda, \kappa_\lambda, |\tau|$ and γ .

We use (2.13) and (2.14) to evaluate the norm of φ_τ . Finally, we conclude the estimation (3.18) by inserting all these estimates into (3.19). $\square$

For the following, we will be interested to upper bound the error indicator $\zeta_{k,i}$, then we need to introduce the following lemma (see [6] for its proof).

Lemma 3.2. For any ψ_N belongs to the polynomial space $\mathbb{P}_N(\Omega)$, the inverse inequalities hold

$$\int_{-1}^1 \psi_N'^2(x)(1-x^2)^2 dx \leq CN^2 \int_{-1}^1 \psi_N^2(x)(1-x^2) dx \quad (3.20)$$

and

$$\int_{-1}^1 \psi_N^2(x) dx \leq CN^2 \int_{-1}^1 \psi_N^2(x)(1-x^2) dx, \quad (3.21)$$

where C is a positive constant independent of N .

Proposition 3.4. Assume that the data function f belongs to $\mathcal{C}^0(0, T; H^{-1}(\Omega))$ and the function φ_0 belongs to $H_0^1(\Omega)$. We assume also the solution $(\varphi^k)_{0 \leq k \leq K}$ of problem (2.7)-(2.8), such that $\partial_x \varphi^k \in L^p(\Omega)$, $p > 1$, satisfies the condition (3.3). Then, there exists a positive constant C depending only on $T, m_\lambda, M_\lambda, \kappa_\lambda, |\tau|$ and γ such that the following estimate holds for the indicator $\zeta_{k,i}$, $1 \leq k \leq K$, $1 \leq i \leq I$,

$$\begin{aligned} \zeta_{k,i} \leq C & \left(\|\partial_x(\varphi^k - \varphi_\delta^k)\|_{L^2(\Omega_i)} + \left\| \frac{(\varphi^k - \varphi_\delta^k) - (\varphi^{k-1} - \varphi_\delta^{k-1})}{\tau_k} \right\|_{H^{-1}(\Omega_i)} \right. \\ & \left. + N_i^{-1} h_i \|f^k - i_\delta f^k\|_{L^2(\Omega_i)} \right). \end{aligned} \quad (3.22)$$

Proof 6. Choosing in equality (3.15), $\psi_\delta = 0$ and

$$\begin{cases} \psi = \left(i_\delta f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x(\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right) (x - \alpha_{i-1})(\alpha_i - x) & \text{in } \Omega_i, \\ \psi = 0 & \text{in } \Omega \setminus \Omega_i, \end{cases}$$

leads

$$\begin{aligned} & \int_{\alpha_{i-1}}^{\alpha_i} \left(i_\delta f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x(\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right)^2 (x - \alpha_{i-1})(\alpha_i - x) dx = \\ & \left(\lambda(\varphi^k) \partial_x(\varphi^k - \varphi_\delta^k), \partial_x \psi \right) - \int_{\alpha_{i-1}}^{\alpha_i} (f^k - i_\delta f^k)(x) \psi(x) dx + \left(\frac{(\varphi^k - \varphi_\delta^k) - (\varphi^{k-1} - \varphi_\delta^{k-1})}{\tau_k}, \psi \right) \\ & + ((\lambda(\varphi^k) - \lambda_\delta(\varphi_\delta^k)) \partial_x(\varphi_\delta^k), \partial_x \psi). \end{aligned}$$

Bounding the terms in the right-hand side follows the same techniques as in the previous proof. Thanks to Cauchy schwarz and the Hölder inequalities ($\frac{1}{p} + \frac{1}{p^*} = \frac{1}{2}$), we obtain

$$\begin{aligned} & \int_{\alpha_{i-1}}^{\alpha_i} \left(i_\delta f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x(\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right)^2 (x - \alpha_{i-1})(\alpha_i - x) dx \leq \\ & \left\| \frac{(\varphi^k - \varphi_\delta^k) - (\varphi^{k-1} - \varphi_\delta^{k-1})}{\tau_k} \right\|_{H^{-1}(\Omega_i)} \|\partial_x \psi\|_{L^2(\Omega_i)} + \|\partial_x(\varphi^k - \varphi_\delta^k)\|_{L^2(\Omega_i)} \|\partial_x \psi\|_{L^2(\Omega_i)} \\ & + \|f^k - i_\delta f^k\|_{L^2(\Omega_i)} \|\psi\|_{L^2(\Omega_i)} + \|\lambda(\varphi^k) - \lambda_\delta(\varphi_\delta^k)\|_{L^{p^*}(\Omega_i)} \|\partial_x(\varphi_\delta^k)\|_{L^p(\Omega_i)} \|\partial_x \psi\|_{L^2(\Omega_i)}. \end{aligned} \quad (3.23)$$

Applying the formula $(a + b)^2 \leq 2a^2 + 2b^2$, we have

$$\begin{aligned} \|\partial_x \psi\|_{L^2(\Omega_i)} &\leq 2 \int_{\alpha_{i-1}}^{\alpha_i} \left(\partial_x (i_\delta f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k)) \right)^2 (x - \alpha_{i-1})^2 (\alpha_i - x)^2 dx \\ &+ 2 \int_{\alpha_{i-1}}^{\alpha_i} \left(i_\delta f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right)^2 (\alpha_i + \alpha_{i-1} - 2x)^2 dx. \end{aligned}$$

Using the two inverse inequalities (3.20) and (3.21) of lemma 3.2, we obtain

$$\|\partial_x \psi\|_{L^2(\Omega_i)} \leq C N_i \left\| \left(i_\delta f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right) (x - \alpha_{i-1})^{\frac{1}{2}} (\alpha_i - x)^{\frac{1}{2}} \right\|_{L^2(\Omega_i)} \quad (3.24)$$

and

$$\|\psi\|_{L^2(\Omega_k)} \leq C h_i \left\| \left(i_\delta f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right) (x - \alpha_{i-1})^{\frac{1}{2}} (\alpha_i - x)^{\frac{1}{2}} \right\|_{L^2(\Omega_i)}. \quad (3.25)$$

The last term in the right-hand side of inequality (3.23) is bounded in the same way as in equation (3.8). Then, there exists a positive constant C depending only on $T, m_\lambda, M_\lambda, \kappa_\lambda, |\tau|$ and y , such that

$$\|\lambda(\varphi^k) - \lambda_\delta(\varphi_\delta^k)\|_{L^{p^*}(\Omega_i)} \|\partial_x(\varphi_\delta^k)\|_{L^p(\Omega_i)} \|\partial_x \psi\|_{L^2(\Omega_i)} \leq C \|\partial_x(\varphi^k - \varphi_\delta^k)\|_{L^2(\Omega_i)} \|\partial_x \psi\|_{L^2(\Omega_i)}. \quad (3.26)$$

Finally, by inserting (3.24), (3.25) and (3.26) in (3.23), simplifying by $\left\| \left(i_\delta f^k - \frac{\varphi_\delta^k - \varphi_\delta^{k-1}}{\tau_k} + \partial_x (\lambda_\delta(\varphi_\delta^k) \partial_x \varphi_\delta^k) \right) (x - \alpha_{i-1})^{\frac{1}{2}} (\alpha_i - x)^{\frac{1}{2}} \right\|_{L^2(\Omega_i)}$ and multiplying by N_i^{-1} , we derive the inequality (3.22). $\square$

Conclusion

The a posteriori analysis of the discretization of a partial differential equations is a very efficient tool for mesh adaptivity. In this paper, we were interested in the a posteriori analysis of the discretization of the non linear heat equation by the spectral element method. We constructed two residual type of indicators and we proved their optimal upper and lower error bounds.

Acknowledgements: Researchers Supporting Project number (RSP-2020/153), King Saud University, Riyadh, Saudi Arabia.

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