Research article

Xingchang Wang* and Runzhang Xu

Global existence and finite time blowup for a nonlocal semilinear pseudo-parabolic equation

<https://doi.org/10.1515/anona-2020-0141>

Received November 16, 2019; accepted December 17, 2019.

Abstract: In this paper, the initial boundary value problem for a nonlocal semilinear pseudo-parabolic equation is investigated, which was introduced to model phenomena in population dynamics and biological sciences where the total mass of a chemical or an organism is conserved. The existence, uniqueness and asymptotic behavior of the global solution and the blowup phenomena of solution with subcritical initial energy are established. Then these results are extended parallelly to the critical initial energy. Further the blowup phenomena of solution with supercritical initial energy is proved, but the existence, uniqueness and asymptotic behavior of the global solution with supercritical initial energy are still open.

Keywords: Semilinear parabolic equation; nonlocal source; potential well

MSC: 35K10; 35K20

1 Introduction

In this paper, we consider the initial boundary value problem of semilinear pseudo-parabolic equation with Neumann boundary condition

$$\begin{cases} u_t - \Delta u - \Delta u_t = |u|^{p-1}u - f_\Omega |u|^{p-1}u dx, & \text{in } \Omega \times (0, T), \\ u(x, 0) = u_0(x) \neq 0, & \text{in } \Omega, \\ f_\Omega u_0 dx = \frac{1}{|\Omega|} \int_\Omega u_0 dx = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}(x, t) = 0, & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (1.1)$$

where $u_0 \in H^1(\Omega)$, $T \in (0, \infty]$ and p satisfies

$$(H) \quad 1 < p < \infty \text{ if } n = 1, 2; \quad 1 < p < \frac{n+2}{n-2} \text{ if } n \geq 3.$$

Problem (1.1) can be used to model phenomena in population dynamics and biological sciences where the total mass of a chemical or an organism is conserved [1–4]. The nonlocal term acts to conserve the spatial integral of the unknown function as time evolves. Such equations give insight into biological and chemical problems where conservation properties predominate. So the model including this nonlocal term distinguishes the classical heat equation [5–17], the parabolic systems [18], the fractional Laplacian parabolic

*Corresponding Author: Xingchang Wang, College of Automation, Harbin Engineering University, 150001, People's Republic of China

Runzhang Xu, College of Automation, Harbin Engineering University, 150001, People's Republic of China
College of Mathematical Sciences, Harbin Engineering University, 150001, People's Republic of China,
E-mail: xurunzh@163.com

equation [19–21], variable exponent parabolic equation [22] and the pseudo-parabolic equation with singular potential [23].

Budd et al. in [1] studied the 1-dimensional nonlinear parabolic equation

$$u_t = u_{xx} + u^2 - \int_0^1 u^2 dx, \quad (1.2)$$

with Neumann boundary condition and proved that the solution blows up for some special initial value. Fila and Levine [24] considered the n -dimensional case of (1.2)

$$u_t - \Delta u = f(u) - \int_{\Omega} f(u) dx \text{ in } \Omega \times (0, T), \quad (1.3)$$

they showed that the global solution is bounded in some space with $f(u) = |u|^{p-1}u$. The initial boundary problem (1.3) with Neumann boundary and $f(u) = |u|^p$ was considered in [25], and the finite time blowup was proved for negative initial energy, i.e. $J(u_0) < 0$ and $1 < p \leq 2$. Later this result was extended to $p > 1$ in [26]. Then for $f(u) = |u|^{p-1}u$, the above blowup result was further extended to the positive initial energy case, i.e. $J(u_0) < d$ in [27] by the similar potential well method employed in [28]. In [29], the above restriction on the initial energy was relaxed to $J(u_0) > 0$ and the finite time blowup was derived for some large initial data.

For problem

$$u_t - \eta \Delta u_t - \Delta u = f(u), \quad (1.4)$$

Showalter and Ting [30] investigated the initial boundary value problem with $f(u) = 0$. They proved global existence, uniqueness and regularity of solutions. Xu and Liu [31–34] studied the initial and boundary value problem of (1.4) for $f(u) = u^p$. They proved the existence, asymptotic behavior of the global solutions and global nonexistence of solutions with subcritical and critical initial energy $J(u_0) \leq d$, and also obtained the global nonexistence of solutions with supercritical initial energy $J(u_0) > d$ by comparison principle, further estimated the upper bound of the blowup time for supercritical initial energy.

Due to the role of the corresponding conservation properties in the real world and its connect to the biological and chemical model equation (1.4), it is interesting to consider (1.1) by investigating the effect of the nonlocal term $\int_{\Omega} |u|^{p-1}u dx$. Another tough task is to “keep balance” between the bad side and the good side of the strongly dissipative term Δu_t , which helps the global existence by decaying the energy, meanwhile impedes blowup and delay the blowup time. As the goal of this paper is not to treat just one side of above, but to find out for which initial data the solution exists globally, and for which the solution blows up in finite time, it is important but not easy to deal with the blowup case and keep such balance. As we know that the global well-posedness of solution to the evolution equation strongly relies on the initial data, especially the initial energy, we plan to conduct a comprehensive study in this paper on the global well-posedness of solution at subcritical and critical initial energy $J(u_0) \leq d$, and supercritical initial energy $J(u_0) > 0$. For each initial energy level, we aim to obtain the global existence, asymptotic behavior of the solution as the time goes to infinity and finite time blow up of solution. But we do not succeed for all the cases. We summarize the main conclusions of this paper in the following Table 1, where the mark “ $\checkmark$ ” denotes the results we get in this paper and the question mark “?” indicates the problem unsolved still.

Table 1: Main Results

Initial data	Global existence	Asymptotic behavior	Blow up
$J(u_0) < d$	$\checkmark$	$\checkmark$	$\checkmark$
$J(u_0) = d$	$\checkmark$	$\checkmark$	$\checkmark$
$J(u_0) > 0$	?	?	$\checkmark$

This article is structured as follows:

- (i) In Section 2, we introduce some functionals and the potential well. Also, we give several preliminary definitions and lemmas.
- (ii) In Section 3, we prove the local existence of solution by the standard Galerkin method.
- (iii) In Section 4, we prove the existence, uniqueness, decay estimate of the global solution and global nonexistence of solution with $J(u_0) < d$ are inspired by [31].
- (iv) In Section 5, we extend all the conclusions in Section 4 parallelly to the initial energy $J(u_0) = d$ are inspired by [15].
- (v) In Section 6, we prove the global nonexistence of solution with $J(u_0) > 0$ by using the two different methods in [29, 35, 36], but the global existence and asymptotic behavior of solution at supercritical initial energy level are still open.

2 Preliminaries

In this section, we introduce some definition, functionals and the potential well, and also establish some properties of them.

Throughout the paper, we assume that $\Omega \subset \mathbb{R}^n$ is an open bounded smooth domain. We denote by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \leq q \leq \infty$ and by $\|\nabla \cdot\|_2$ the Dirichlet norm in $H^1(\Omega)$. And for $u, v \in H^1(\Omega)$, denote by

$$(u, v)_* = \int_{\Omega} uv dx + \int_{\Omega} \nabla u \nabla v dx$$

the inner product in $H^1(\Omega)$. The inner product endow with the norm

$$\|u\|_{H^1} = \left(\|u\|_2^2 + \|\nabla u\|_2^2 \right)^{\frac{1}{2}}$$

for every $u \in H^1(\Omega)$, which is equivalent to $\|\nabla u\|_2$.

Next, the energy functional $J(u)$ and the Nehari functional $I(u)$ are defined as

$$J(u) = \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{p+1} \|u\|_{p+1}^{p+1} \quad (2.5)$$

and

$$I(u) = \|\nabla u\|_2^2 - \|u\|_{p+1}^{p+1}. \quad (2.6)$$

By $I(u)$ and $J(u)$, we introduce the stable set

$$W = \left\{ u \in H^1(\Omega) \mid J(u) < d, I(u) > 0 \right\} \cup \{0\},$$

the unstable set

$$V = \left\{ u \in H^1(\Omega) \mid J(u) < d, I(u) < 0 \right\},$$

where d is the depth of potential well defined as (the so-called mountain pass level in [37])

$$d = \inf_{u \in \mathcal{N}} J(u)$$

and the Nehari manifold is

$$\mathcal{N} = \left\{ u \in H^1(\Omega) \mid I(u) = 0, \|u\|_{H^1} \neq 0 \right\}.$$

Next, we give the following definition of weak solution.

Definition 1 (Weak solution). Function $u = u(x, t)$ is called a weak solution of problem (1.1) on $\Omega \times [0, T]$, if $u \in L^\infty(0, T; H^1(\Omega))$ with $u_t \in L^2(0, T; H^1(\Omega))$ satisfies

$$(i) \quad \int_0^t \int_{\Omega} \left(u_{\tau} \varphi + \nabla u \nabla \varphi + \nabla u_{\tau} \nabla \varphi - \left(|u|^{p-1} u - \int_{\Omega} |u|^{p-1} u dx \right) \varphi \right) dx d\tau = 0 \quad (2.7)$$

for any $\varphi \in L^{\infty}(0, T; H^1(\Omega))$;

(ii) $u(x, 0) = u_0(x)$ in $H^1(\Omega)$;

(iii) for $0 \leq t < T$,

$$\int_0^t \|u_{\tau}\|_{H^1}^2 d\tau + J(u) = J(u_0). \quad (2.8)$$

Here, we show that the nonlocal term can give us a conservation law.

Lemma 2 (Conservation Law). Let u be the weak solution of problem (1.1). If $\int_{\Omega} u_0(x) dx = 0$, then the integral of u is conserved, that is $\int_{\Omega} u(x, t) dx = 0$.

Proof. For the weak solution defined by (2.7), we can take $\varphi = 1$ as the test function, then by noticing

$$\begin{aligned} \int_0^t \int_{\Omega} \left(|u|^{p-1} u - \int_{\Omega} |u|^{p-1} u dx \right) dx d\tau &= \int_0^t \left(\int_{\Omega} |u|^{p-1} u dx - \int_{\Omega} \int_{\Omega} |u|^{p-1} u dx dx \right) d\tau \\ &= \int_0^t \left(\int_{\Omega} |u|^{p-1} u dx - \int_{\Omega} |u|^{p-1} u dx \right) d\tau = 0, \end{aligned}$$

we can obtain $\int_{\Omega} u(x, t) dx = \int_{\Omega} u_0(x) dx = 0$. $\square$

Definition 3 (Maximal existence time). Let $u(t)$ be a weak solution of problem (1.1). We define the maximal existence time T of $u(t)$ as follows

- (i) If $u(t)$ exists for $0 \leq t < \infty$, then $T = +\infty$.
- (ii) If there exists a $t_0 \in (0, \infty)$ such that $u(t)$ exists for $0 \leq t < t_0$, but doesn't exist at $t = t_0$, then $T = t_0$.

Here, we give a estimate of nonlinear term $|u|^{p-1}u$ by Gâteaux derivative.

Lemma 4. If p satisfies (H), for any $u_1(x, t), u_2(x, t)$ with $(x, t) \in \Omega \times [0, T]$, $|u_1| + |u_2| > 0$ and $u_1 \neq u_2$, there holds

$$|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2 \leq p(|u_1| + |u_2|)^{p-1}|u_1 - u_2|.$$

Proof. Let $F(u) := |u|^{p-1}u$. Define $\psi(x, t) := u_1 - u_2$ for $(x, t) \in \Omega \times [0, T]$. By the property of Gâteaux derivative, we know

$$F(u_1) - F(u_2) = F(u_2 + \psi) - F(u_2) = \int_0^1 dF(u_2 + s\psi; \psi) ds$$

for $s \in (0, 1)$, which together with the definition of Gâteaux derivative

$$dF(u_2 + s\psi; \psi) = \lim_{\tau \rightarrow 0} \frac{F(u_2 + s\psi + \tau\psi) - F(u_2 + s\psi)}{\tau} = \frac{d}{d\tau} F(u_2 + s\psi + \tau\psi) \Big|_{\tau=0}$$

gives

$$F(u_1) - F(u_2) = \int_0^1 \frac{d}{d\tau} \left(|u_2 + s\psi + \tau\psi|^{p-1}(u_2 + s\psi + \tau\psi) \right) \Big|_{\tau=0} ds$$

$$\begin{aligned}
&= \int_0^1 p|u_2 + s\psi + \tau\psi|^{p-1}\psi \Big|_{\tau=0} ds \\
&= \int_0^1 p|u_2 + s\psi|^{p-1}\psi ds \\
&= \int_0^1 p|u_2 + su_1 - su_2|^{p-1}(u_1 - u_2) ds \\
&= \int_0^1 p|(1-s)u_2 + su_1|^{p-1}(u_1 - u_2) ds \\
&\leq \int_0^1 p(|u_2| + |u_1|)^{p-1}|u_1 - u_2| ds \\
&= p(|u_1| + |u_2|)^{p-1}|u_1 - u_2|.
\end{aligned}$$

□

For the Nehari functional $I(u)$, we have the following properties.

Lemma 5 (Relations between $I(u)$ and $\|\nabla u\|_2$). Assume $u \in H^1(\Omega)$.

- (i) If $0 < \|\nabla u\|_2 < r_0$, then $I(u) > 0$;
- (ii) If $I(u) < 0$, then $\|\nabla u\|_2 > r_0$;
- (iii) If $I(u) = 0$ and $\|\nabla u\|_2 \neq 0$, then $\|\nabla u\|_2 \geq r_0$,

where

$$r_0 = \left(\frac{1}{C_\star^{p+1}} \right)^{\frac{1}{p-1}},$$

$C_\star$ is the best Sobolev constant for $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and

$$C_\star = \sup_{u \in H^1, u \neq 0} \frac{\|u\|_{p+1}}{\|\nabla u\|_2}. \quad (2.9)$$

Proof.

- (i) From $0 < \|\nabla u\|_2 < r_0$, we have

$$\|u\|_{p+1}^{p+1} \leq C_\star^{p+1} \|\nabla u\|_2^{p+1} = C_\star^{p+1} \|\nabla u\|_2^{p-1} \|\nabla u\|_2^2 < \|\nabla u\|_2^2,$$

which implies $I(u) > 0$.

- (ii) From $I(u) < 0$ it implies that $\|\nabla u\|_2 \neq 0$. Thus

$$\|\nabla u\|_2^2 < \|u\|_{p+1}^{p+1} \leq C_\star^{p+1} \|\nabla u\|_2^{p+1} = C_\star^{p+1} \|\nabla u\|_2^{p-1} \|\nabla u\|_2^2$$

gives $\|\nabla u\|_2 > r_0$.

- (iii) If $I(u) = 0$, $\|\nabla u\|_2 \neq 0$, then from

$$\|\nabla u\|_2^2 = \|u\|_{p+1}^{p+1} \leq C_\star^{p+1} \|\nabla u\|_2^{p+1} = C_\star^{p+1} \|\nabla u\|_2^{p-1} \|\nabla u\|_2^2,$$

we get $\|\nabla u\|_2 \geq r_0$.

□

Lemma 6 (Properties of $J(\lambda u)$, [31]). Let $u \in H^1(\Omega)$ and $\|\nabla u\|_2^2 \neq 0$. Then

- (i) $\lim_{\lambda \rightarrow 0} J(\lambda u) = 0$, $\lim_{\lambda \rightarrow +\infty} J(\lambda u) = -\infty$;
 (ii) for $\lambda \in (0, \infty)$, there exists a unique $\lambda^* = \lambda^*(u)$, such that

$$\frac{d}{d\lambda} J(\lambda u) \big|_{\lambda=\lambda^*} = 0;$$

- (iii) the point $\lambda = \lambda^*$ is the maximum point of $J(\lambda u)$, $J(\lambda u)$ is increasing on $\lambda \in [0, \lambda^*]$ and decreasing on $\lambda \in [\lambda^*, \infty)$;
 (iv) $I(\lambda^* u) = 0$, $I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$ and $I(\lambda u) < 0$ for $\lambda^* < \lambda < \infty$.

Lemma 7 (Depth d of potential well). Suppose that (H) holds. Then the potential well depth

$$d = \frac{p-1}{2(p+1)} \left(\frac{1}{C_*^{p+1}} \right)^{\frac{2}{p-1}} > 0.$$

Proof. By the definition of $J(u)$, $I(u)$ and the fact that $I(u) = 0$, we have

$$J(u) = \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{p+1} \|u\|_{p+1}^{p+1} = \left(\frac{1}{2} - \frac{1}{p+1} \right) \|\nabla u\|_2^2 + \frac{1}{p+1} I(u) = \left(\frac{1}{2} - \frac{1}{p+1} \right) \|\nabla u\|_2^2,$$

which implies $J(u) > 0$.

By Lemma 5 (iii), there holds

$$J(u) = \frac{p-1}{2(p+1)} \|\nabla u\|_2^2 \geq \frac{p-1}{2(p+1)} \left(\frac{1}{C_*^{p+1}} \right)^{\frac{2}{p-1}},$$

which says

$$d = \frac{p-1}{2(p+1)} \left(\frac{1}{C_*^{p+1}} \right)^{\frac{2}{p-1}}.$$

□

Remark 1. For condition $J(u_0) < d$, a simple computation induces that

$$C_*^{p+1} \left(\frac{2(p+1)}{p-1} J(u_0) \right)^{\frac{p-1}{2}} < 1. \quad (2.10)$$

3 Local solution

In this section, by the standard Galerkin method we prove the local existence of solution for problem (1.1). From now on, the constants in the Sobolev, Poincaré and Young's inequalities, and other constants that may appear in calculation, are denoted uniformly by $C > 0$.

Theorem 8 (Local solution). Suppose that (H) and $u_0 \in H^1(\Omega)$. Then there exist $T > 0$ and a unique solution of (1.1) over $[0, T]$. Moreover, if

$$T_{\max} = \sup \{ T > 0 \mid u = u(t) \text{ exists on } (0, T] \} < \infty,$$

then

$$\lim_{t \rightarrow T_{\max}} \|u(t)\|_s = \infty \text{ for } s \geq 1 \text{ such that } s > \frac{n(p-1)}{2}. \quad (3.11)$$

Proof. The proof of this theorem is divided into three steps. The first step is to show that the existence and uniqueness of solution to the linear problem corresponding to problem (1.1) by Galerkin method. The second step is to prove the existence and uniqueness of the local solution to problem (1.1) based on first step by the Contraction Mapping Principle. And the third step is to get the assertion (3.11).

Step I: The existence and uniqueness of solution to the linear problem. Take $u_0 \in H^1(\Omega)$ and let $R^2 = 2\|\nabla u_0\|_2^2$. For every $T > 0$ consider the space $\mathcal{H} = C([0, T]; H^1(\Omega))$ endowed with the norm

$$\|u\|_{\mathcal{H}}^2 = \max_{t \in [0, T]} \|\nabla u\|_2^2 \quad (3.12)$$

and the set

$$\mathcal{H}_T := \left\{ u \in \mathcal{H} \mid \|u\|_{\mathcal{H}} \leq R \right\}.$$

Next for every $T > 0$ and $u \in \mathcal{H}_T$, we shall respectively prove the existence and uniqueness of solution to the following linear problem by Galerkin method

$$\begin{cases} v_t - \Delta v - \Delta v_t = |u|^{p-1}u - f_\Omega |u|^{p-1}u dx, & \text{in } \Omega \times (0, T), \\ v(x, 0) = u_0(x) \neq 0, & \text{in } \Omega, \\ \int_\Omega v_0 dx = 0, & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu}(x, t) = 0, & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (3.13)$$

Existence. Fix a positive integer m , assume that $\{\omega_j(x)\}$ is an orthogonal complete basis in $H^1(\Omega)$ and $L^2(\Omega)$ such that $\mathcal{W}_m = \text{Span}\{\omega_1, \dots, \omega_m\}$ and $\|\omega_j\|_2 = 1$ for all j . Denote by λ_j the related eigenvalues of $\omega_j(x)$. Let

$$u_{m0}(x) = \sum_{j=1}^m \left(\int_\Omega \nabla u_0 \nabla \omega_j dx \right) \omega_j$$

such that $u_{m0} \in \mathcal{W}_m$ and $u_{m0} \rightarrow u_0$ in $H^1(\Omega)$ as $m \rightarrow \infty$, then $u_{m0} \in \mathcal{H}_T$. We seek m functions $g_{1m}(t), \dots, g_{mm}(t) \in C^1[0, T]$ to construct the approximate solutions to problem (3.13)

$$v_m(x, t) = \sum_{j=1}^m g_{jm}(t) \omega_j(x), \quad m = 1, 2, \dots \quad (3.14)$$

satisfying

$$\begin{cases} \int_\Omega (v_{mt} - \Delta v_m - \Delta v_{mt} - f(u)) \mu dx = 0, \\ \int_\Omega v_m(0) dx = 0, \quad v_m(0) = u_{m0}(x) \end{cases} \quad (3.15)$$

for any $\mu \in \mathcal{W}_m$ and $f(u) = |u|^{p-1}u - f_\Omega |u|^{p-1}u dx$. Taking $\mu = \omega_j$ in (3.15) we have the following Cauchy problem of system of linear ordinary differential equations with the unknown $g_{jm}(t)$

$$\begin{cases} (1 + \lambda_j) \dot{g}_{jm}(t) + \lambda_j g_{jm}(t) = \int_\Omega f(u) \omega_j dx := \xi(t), \\ g_{jm}(0) = \int_\Omega u_0 \omega_j dx. \end{cases} \quad (3.16)$$

Since $\xi(t) \in C[0, T]$, for all j , according to standard existence theory for ordinary differential equations, the Cauchy problem (3.16) admits unique global solution $g_{jm}(t) \in C^1[0, T]$. Then the unique $v_m(x, t)$ defined by (3.14) solves (3.15).

Taking $\mu = v_{mt}$ in (3.15) and integrating the both sides over $[0, t] \subset [0, T]$ with respect to t , we have

$$\begin{aligned} \int_0^t \|v_{m\tau}\|_{H^1}^2 d\tau + \frac{1}{2} \|\nabla v_m\|_2^2 &= \frac{1}{2} \|\nabla u_{m0}\|_2^2 + \int_0^t \int_\Omega |u|^{p-1} u v_{m\tau} dx d\tau - \int_0^t \int_\Omega \left(\int_\Omega |u|^{p-1} u dx \right) v_{m\tau} dx d\tau \\ &= \frac{1}{2} \|\nabla u_{m0}\|_2^2 + \int_0^t \int_\Omega |u|^{p-1} u v_{m\tau} dx d\tau - \int_0^t \left(\int_\Omega |u|^{p-1} u dx \right) \int_\Omega v_{m\tau} dx d\tau \end{aligned} \quad (3.17)$$

$$= \frac{1}{2} \|\nabla u_{m0}\|_2^2 + \int_0^t \int_{\Omega} |u|^{p-1} u v_{m\tau} dx d\tau$$

by $\int_{\Omega} v_m(0) dx = 0$ and Lemma 2, i.e. $\int_{\Omega} v_m dx = 0$.

For $u \in \mathcal{H}_T$, i.e. $\|\nabla u\|_2 \leq R$, we estimate the last term in the right hand side of (3.17) by Hölder, Sobolev and Young's inequalities as follows

$$\begin{aligned} \int_0^t \int_{\Omega} |u|^{p-1} u v_{m\tau} dx d\tau &\leq \int_0^t \int_{\Omega} |u|^p |v_{m\tau}| dx d\tau \\ &\leq \int_0^t \left(\int_{\Omega} |u|^{\frac{2np}{n+2}} dx \right)^{\frac{n+2}{2n}} \left(\int_{\Omega} |v_{m\tau}|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{2n}} d\tau \\ &= \int_0^t \|u\|_{\frac{2np}{n+2}}^p \|v_{m\tau}\|_{\frac{2n}{n-2}} d\tau \\ &\leq C \int_0^t \|u\|_{H^1}^p \|v_{m\tau}\|_{H^1} d\tau \\ &\leq C \int_0^t \|u\|_{H^1}^{2p} d\tau + \frac{1}{2} \int_0^t \|v_{m\tau}\|_{H^1}^2 d\tau \\ &\leq CTR^{2p} + \frac{1}{2} \int_0^t \|v_{m\tau}\|_{H^1}^2 d\tau. \end{aligned} \quad (3.18)$$

Then combining (3.17), (3.18) and $u_{m0} \in \mathcal{H}_T$, we obtain

$$\int_0^t \|v_{m\tau}\|_{H^1}^2 d\tau + \|\nabla v_m\|_2^2 \leq C \quad (3.19)$$

for every $m \geq 1$, where C is independent of m . Therefore, we see that the sequence $\{v_m\}$ is bounded in $L^\infty([0, T]; H^1(\Omega))$ and $\{v_{mt}\}$ is bounded in $L^2([0, T]; H^1(\Omega))$.

Consequently, there exists a v and a subsequence $\{v_k\}$ of $\{v_m\}$ such that

$$\begin{cases} v_k \rightarrow v \text{ in } L^\infty([0, T]; H^1(\Omega)) \text{ weak star,} \\ v_{kt} \rightarrow v_t \text{ in } L^2([0, T]; H^1(\Omega)) \text{ weakly.} \end{cases}$$

In (3.15) letting $m = k \rightarrow \infty$, we conclude the existence of a weak solution v of (3.13) with the above regularity. Moreover, observing (3.19) we get $v \in H^1([0, T]; H^1(\Omega))$, then $v \in C([0, T]; H^1(\Omega))$.

Uniqueness. Arguing by contradiction, if v_1 and v_2 are two weak solutions of (3.13) with the same initial datum, by subtracting the two equations corresponding to v_1 and v_2 respectively, and testing it with $v_{1t} - v_{2t}$, we get

$$\int_0^t \|v_{1\tau} - v_{2\tau}\|_{H^1}^2 d\tau + \frac{1}{2} \|\nabla v_1 - \nabla v_2\|_2^2 = 0, \quad (3.20)$$

which implies $v_1 \equiv v_2$ in $H^1(\Omega)$. Then the uniqueness is proved.

Step II: The existence and uniqueness of local solution to the nonlinear problem (1.1). By Step I, for any $u \in \mathcal{H}_T$ and the unique solution v to problem (3.13) we can define $v := \Phi(u)$. We claim that $\Phi(\mathcal{H}_T) \subset \mathcal{H}_T$ is a contractive map. First for given $u \in \mathcal{H}_T$, similar to (3.17), we have

$$\int_0^t \|v_\tau\|_{H^1}^2 d\tau + \frac{1}{2} \|\nabla v\|_2^2 = \frac{1}{2} \|\nabla u_0\|_2^2 + \int_0^t \int_{\Omega} |u|^{p-1} u v_\tau dx d\tau. \quad (3.21)$$

For the last term in above equality, by similar estimates as (3.18) we get

$$\int_0^t \int_{\Omega} |u|^{p-1} u v_{\tau} dx d\tau \leq CTR^{2p} + \int_0^t \|v_{\tau}\|_{H^1}^2 d\tau \quad (3.22)$$

for all $t \in (0, T]$. Combining (3.21) and (3.22) gives

$$\|v\|_{\mathcal{H}}^2 \leq \frac{1}{2}R^2 + CTR^{2p}.$$

For sufficiently small T , we see $\|v\|_{\mathcal{H}} \leq R$, which implies that $\Phi(\mathcal{H}_T) \subset \mathcal{H}_T$. Next we prove such map is contractive. Taking $u_1, u_2 \in \mathcal{H}_T$ to be the known functions in the linear terms of (3.13) respectively, subtracting the two equations in form of (3.13) for $v_1 = \Phi(u_1)$ and $v_2 = \Phi(u_2)$ respectively, setting $v = v_1 - v_2$ and testing the both sides by v_t , we can arrive at

$$\int_0^t \|v_{\tau}\|_{H^1}^2 d\tau + \frac{1}{2} \|\nabla v\|_2^2 = \int_0^t \int_{\Omega} \left(|u_1|^{p-1} u_1 - |u_2|^{p-1} u_2 \right) v_{\tau} dx d\tau. \quad (3.23)$$

For the last term of (3.23), by Lemma 4, Hölder, Sobolev and Young's inequalities, we get

$$\begin{aligned} \int_0^t \int_{\Omega} \left(|u_1|^{p-1} u_1 - |u_2|^{p-1} u_2 \right) v_{\tau} dx d\tau &\leq \int_0^t \int_{\Omega} p(|u_1| + |u_2|)^{p-1} |u_1 - u_2| v_{\tau} dx d\tau \\ &\leq C \int_0^t \|(|u_1| + |u_2|)^{p-1}\|_{A_1} \|u_1 - u_2\|_{A_2} \|v_{\tau}\|_{A_3} d\tau \\ &= C \int_0^t \| |u_1| + |u_2| \|_{(p-1)A_1}^{p-1} \|u_1 - u_2\|_{A_2} \|v_{\tau}\|_{A_3} d\tau \\ &\leq C \int_0^t \| |u_1| + |u_2| \|_{H^1}^{p-1} \|u_1 - u_2\|_{H^1} \|v_{\tau}\|_{H^1} d\tau \\ &\leq C \int_0^T \| |u_1| + |u_2| \|_{H^1}^{2(p-1)} \|u_1 - u_2\|_{H^1}^2 d\tau + \int_0^t \|v_{\tau}\|_{H^1}^2 d\tau \\ &\leq CR^{2p-2} T \|u_1 - u_2\|_{H^1}^2 + \int_0^t \|v_{\tau}\|_{H^1}^2 d\tau, \end{aligned} \quad (3.24)$$

where $(p-1)A_1 = \frac{2n}{n-2}$ and $A_2 = A_3 = \frac{4n}{3n-(n-2)p-2} < \frac{2n}{n-2}$ by (H). Combining (3.23) and (3.24), for any $t \in [0, T]$ we obtain

$$\|\nabla v\|_2^2 \leq CR^{2p-2} T \|u_1 - u_2\|_{H^1}^2,$$

which by (3.12) gives

$$\begin{aligned} \|\Phi(u_1) - \Phi(u_2)\|_{\mathcal{H}}^2 &= \|v\|_{\mathcal{H}}^2 = \max_{t \in [0, T]} \|\nabla v\|_2^2 \\ &\leq CR^{2p-2} T \|u_1 - u_2\|_{H^1}^2 \\ &\leq CR^{2p-2} T \|\nabla(u_1 - u_2)\|_2^2 \\ &\leq CR^{2p-2} T \|u_1 - u_2\|_{\mathcal{H}}^2 \\ &:= \delta_T \|u_1 - u_2\|_{\mathcal{H}}^2 \end{aligned}$$

for some $\delta_T = CR^{2p-2} T < 1$ as long as T is sufficiently small. Therefore, the map $v = \Phi(u)$ is contractive. By the Contraction Mapping Principle, there exists a unique weak local solution to (1.1) defined on $[0, T]$.

Step III: Finite time blowup of local solution. By the arguments above, especially the requirements on δ_T , the sufficient conditions ensuring the contraction indicate that the existence time of the local solution only depends on the scale of the norm of the initial data by the connection between R and $\|\nabla u_0\|_2$. Therefore, the local solution u is continued as long as $\|u\|_{\mathcal{H}}$ remains bounded, see also Theorem 1 in [38] and Theorem 3.1 in [39] for similar argument. Hence, if the maximum existence time of local solution is $T_{\max} < \infty$, we have

$$\lim_{t \rightarrow T_{\max}} \|\nabla u\|_2^2 = \infty. \quad (3.25)$$

By (2.5) and (2.8) we deduce

$$\frac{1}{2} \|\nabla u\|_2^2 \leq \frac{1}{p+1} \|u\|_{p+1}^{p+1} + J(u_0) \text{ for all } t \in [0, T_{\max}),$$

which together with the following Gagliardo-Nirenberg interpolation inequality

$$\|u\|_{B_1} \leq C \|\nabla u\|_{B_2}^\alpha \|u\|_{B_3}^{1-\alpha} \text{ for } \frac{1}{B_1} = \left(\frac{1}{B_2} - \frac{1}{n} \right) \alpha + \frac{1-\alpha}{B_3} \text{ and } 0 < \alpha < 1$$

gives

$$\frac{1}{2} \|\nabla u\|_2^2 - J(u_0) \leq \frac{1}{p+1} \|u\|_{p+1}^{p+1} \leq C \|\nabla u\|_2^{(p+1)\alpha} \|u\|_s^{(p+1)(1-\alpha)}, \quad (3.26)$$

where $B_1 = p+1$, $B_2 = 2$ and $B_3 = s$, and $\frac{n(p-1)}{2} < s < p+1$ in order to ensure $0 < \alpha = \frac{2n(p+1-s)}{(p+1)(2n+2s-ns)} < \frac{2}{p+1}$. Then

$$\frac{1}{2} \|\nabla u\|_2^{2-(p+1)\alpha} - J(u_0) \|\nabla u\|_2^{-(p+1)\alpha} \leq C \|u\|_s^{(p+1)(1-\alpha)}, \quad (3.27)$$

which together with (3.25) implies that

$$\lim_{t \rightarrow T_{\max}} \|u\|_s = \infty.$$

Combining Step I, Step II and Step III we complete the proof of Theorem 8. $\square$

4 Subcritical initial energy $J(u_0) < d$

In this section, we shall prove the global existence, asymptotic behavior and nonexistence of solution for problem (1.1) with the subcritical initial energy $J(u_0) < d$. First, we show that the invariance of some sets of problem (1.1).

Lemma 9 (Invariant sets for $J(u_0) < d$). Let p satisfy (H), $u_0 \in H^1(\Omega)$, T be the maximal existence time of solution, then

- (i) the solution u of (1.1) with $J(u_0) < d$ belongs to W for $0 \leq t < T$, provided $I(u_0) > 0$;
- (ii) the solution u of (1.1) with $J(u_0) < d$ belongs to V for $0 \leq t < T$, provided $I(u_0) < 0$.

Proof.

- (i) Arguing by contradiction, by the continuity of $I(u)$ respect to t , we suppose that $t_0 \in (0, T)$ is the first time such that $u(t_0) \in \partial W$, that is $I(u(t_0)) = 0$, $\|u(t_0)\|_{H^1} \neq 0$ or $J(u(t_0)) = d$. From

$$\int_0^t \|u_\tau\|_{H^1}^2 d\tau + J(u) = J(u_0) < d, \quad 0 \leq t < T, \quad (4.28)$$

we know that $J(u(t_0)) \neq d$. If $I(u(t_0)) = 0$, $\|u(t_0)\|_{H^1} \neq 0$, then by the definition of d we have $J(u(t_0)) \geq d$, which contradicts (4.28).

(ii) The proof is similar to (i).

□

Next, we prove the global existence and uniqueness of solution with $J(u_0) < d$, also show the asymptotic behavior of global solution.

Theorem 10 (Global existence, uniqueness and asymptotic behavior for $J(u_0) < d$). *Supposed that p satisfies (H). If $u_0 \in H^1(\Omega)$ and $u_0 \in W$, then problem (1.1) admits a unique global weak solution $u \in L^\infty(0, \infty; H^1(\Omega))$ with $u_t \in L^2(0, \infty; H^1(\Omega))$ which does not vanish in finite time and there exist a constant $\delta > 0$ such that $\|u\|_{H^1}^2 < \|u_0\|_{H^1}^2 e^{-2\delta t}$.*

Proof. By Theorem 8, we know that the problem (1.1) admits a unique local solution $u \in C([0, T]; H^1(\Omega))$. Next we will prove the existence time T of solution $u(t)$ is infinite with $u_0 \in W$.

Global existence. Assume $\{\omega_j(x)\}$ is an orthogonal basis of $H^1(\Omega)$. We give the construction of approximate solutions to problem (1.1)

$$u_m(x, t) = \sum_{j=1}^m g_{jm}(t) \omega_j(x), \quad m = 1, 2, \dots$$

satisfying

$$(u_{mt}, \omega_s) + (\nabla u_{mt}, \nabla \omega_s) + (\nabla u_m, \nabla \omega_s) = (f(u_m), \omega_s), \quad s = 1, 2, \dots, m \quad (4.29)$$

and

$$u_m(x, 0) = \sum_{j=1}^m a_{jm} \omega_j(x) \rightarrow u_0(x) \text{ in } H^1(\Omega), \quad (4.30)$$

where

$$f(u) = |u|^{p-1}u - \int_{\Omega} |u|^{p-1}u dx \text{ and } a_{jm} = g_{jm}(0).$$

Multiplying (4.29) by $g'_{sm}(t)$, integrating over $[0, t]$ and summing for s gives

$$\int_0^t \|u_{m\tau}\|_{H^1}^2 d\tau + J(u_m) = J(u_m(0)), \quad 0 \leq t < \infty. \quad (4.31)$$

By (4.30) we know that $J(u_m(0)) \rightarrow J(u_0) < d$ and $I(u_m(0)) \rightarrow I(u_0) > 0$, which combining (4.31) implies that

$$\int_0^t \|u_{m\tau}\|_{H^1}^2 d\tau + J(u_m) < d, \quad 0 \leq t < \infty, \quad (4.32)$$

and by Lemma 9, for $0 \leq t < \infty$ and sufficiently large m we have $u_m(t) \in W$. Combining (4.32) and

$$J(u_m) = \frac{p-1}{2(p+1)} \|\nabla u_m\|_2^2 + \frac{1}{p+1} I(u_m),$$

gives

$$\int_0^t \|u_{m\tau}\|_{H^1}^2 d\tau + \frac{p-1}{2(p+1)} \|\nabla u_m\|_2^2 < d, \quad 0 \leq t < \infty$$

for sufficiently large m , which yields a priori estimate

$$\int_0^t \|u_{m\tau}\|_{H^1}^2 d\tau < d, \quad 0 \leq t < \infty, \quad (4.33)$$

$$\|\nabla u_m\|_2^2 < \frac{2(p+1)}{p-1}d, \quad 0 \leq t < \infty, \quad (4.34)$$

$$\|u_m^p\|_q^q = \|u_m\|_{p+1}^{p+1} \leq C_*^{p+1} \left(\frac{2(p+1)}{p-1}d \right)^{\frac{p+1}{2}}, \quad q = \frac{p+1}{p}, \quad 0 \leq t < \infty. \quad (4.35)$$

Therefore, there exists a u and a subsequence $\{u_v\}$ of $\{u_m\}$ such that

$$\begin{cases} u_v \rightarrow u \text{ in } L^\infty(0, \infty; H^1(\Omega)) \text{ weak star and a.e. in } Q = \Omega \times (0, \infty), \\ u_v^p \rightarrow u^p \text{ in } L^\infty(0, \infty; L^q(\Omega)) \text{ weak star,} \\ u_{vt} \rightarrow u_t \text{ in } L^2(0, \infty; H^1(\Omega)) \text{ weakly,} \\ |u_v|^{p-1}u_v \rightarrow |u|^{p-1}u \text{ in } L^\infty(0, \infty; L^{\frac{p+1}{p}}(\Omega)) \text{ weak star.} \end{cases}$$

In (4.29) we fixed s , letting $m = v \rightarrow \infty$, we conclude that

$$(u_t, \omega_s) + (\nabla u_t, \nabla \omega_s) + (\nabla u, \nabla \omega_s) = (f(u), \omega_s) \text{ for all } s,$$

and

$$(u_t, v) + (\nabla u_t, \nabla v) + (\nabla u, \nabla v) = (f(u), v) \text{ for all } v \in H^1(\Omega), \quad t \in (0, \infty).$$

Moreover, (4.30) gives $u(x, 0) = u_0(x)$ in $H^1(\Omega)$.

Assume that there exists a $t_0 > 0$ such that $u(t_0) = 0$. Then we get $I(u(t_0)) = 0$, which is contradictory to $I(u) > 0$. Thus the solution does not vanish in finite time.

Uniqueness. If u_1 and u_2 are two solutions of (1.1) with the same initial data, then functions u_1 and u_2 satisfy the following equations for any $\phi \in L^2(0, \infty; H^1(\Omega))$

$$(u_{1t}, \phi) + (\nabla u_{1t}, \nabla \phi) + (\nabla u_1, \nabla \phi) = (f(u_1), \phi) \quad (4.36)$$

and

$$(u_{2t}, \phi) + (\nabla u_{2t}, \nabla \phi) + (\nabla u_2, \nabla \phi) = (f(u_2), \phi), \quad (4.37)$$

where $f(u) = |u|^{p-1}u - \int_\Omega |u|^{p-1}u dx$. Let $v = u_1 - u_2$. By subtracting the above two equations, take $\phi = v$ in (4.36) and (4.37), integrating on $(0, t)$, then

$$\int_0^t \int_\Omega (v_\tau v + \nabla v_\tau \nabla v + |\nabla v|^2) dx d\tau = \int_0^t \int_\Omega (|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2) v dx d\tau. \quad (4.38)$$

By Lemma 4, (3.24) and the boundedness of the solutions, we reach the inequality

$$\begin{aligned} & \int_0^t \int_\Omega (|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2) v dx d\tau \\ & \leq \int_0^t \int_\Omega p(|u_1| + |u_2|)^{p-1} |u_1 - u_2| v dx d\tau \\ & \leq C \int_0^t \|(|u_1| + |u_2|)^{p-1}\|_{A_1} \|u_1 - u_2\|_{A_2} \|v\|_{A_3} d\tau \\ & = C \int_0^t \| |u_1| + |u_2| \|_{(p-1)A_1}^{p-1} \|u_1 - u_2\|_{A_2} \|v\|_{A_3} d\tau \\ & \leq C \int_0^t \| |u_1| + |u_2| \|_{H^1}^{p-1} \|u_1 - u_2\|_{H^1} \|v\|_{H^1} d\tau \end{aligned} \quad (4.39)$$

$$\leq C \int_0^t \|v\|_{H^1}^2 d\tau,$$

where $(p-1)A_1 = \frac{2n}{n-2}$ and $A_2 = A_3 = \frac{4n}{3n-(n-2)p-2} < \frac{2n}{n-2}$ by (H). As $v(x, 0) = 0$, both (4.38) and (4.39) give

$$\begin{aligned} \int_0^t \int_{\Omega} (v_{\tau} v + \nabla v_{\tau} \nabla v) dx d\tau &= \frac{1}{2} \int_{\Omega} \left(v^2(\tau) \Big|_{v(0)}^{v(t)} + |\nabla v(\tau)|^2 \Big|_{v(0)}^{v(t)} \right) dx \\ &= \frac{1}{2} \int_{\Omega} \left(v^2(t) + |\nabla v(t)|^2 \right) dx \\ &= \frac{1}{2} \|v\|_{H^1}^2 \\ &\leq C \int_0^t \|v\|_{H^1}^2 d\tau, \end{aligned} \quad (4.40)$$

i.e.

$$\|v\|_{H^1}^2 \leq C \int_0^t \|v\|_{H^1}^2 d\tau.$$

By Gronwall's inequality, we get

$$\|v\|_{H^1}^2 \leq 0 \cdot \exp \left(\int_0^t C ds \right) = 0,$$

that is $\|v\|_{H^1}^2 = \|u_1 - u_2\|_{H^1}^2 = 0$. Thus $u_1 = u_2 = 0$ a.e. in $\Omega \times (0, \infty)$.

Asymptotic behavior. Setting $\varphi = u$ in (2.7) and noting that

$$\int_0^t \int_{\Omega} \left(\int_{\Omega} |u|^{p-1} u dx \right) u dx d\tau = \int_0^t \left(\int_{\Omega} |u|^{p-1} u dx \right) \int_{\Omega} u dx d\tau = 0,$$

we have

$$\frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{2} \|u_0\|_{H^1}^2 + \int_0^t \left(\|\nabla u\|_2^2 - \|u\|_{p+1}^{p+1} \right) ds = 0,$$

that is

$$\frac{1}{2} \frac{d}{dt} \|u\|_{H^1}^2 = -I(u). \quad (4.41)$$

According to $0 < J(u_0) < d$, $I(u_0) > 0$ and Lemma 9, we have $I(u) > 0$ for $0 < t < \infty$. Then by (2.9) and the norm $\|u\|_{H^1}$ is equivalent to the norm $\|\nabla u\|_2$ on $H^1(\Omega)$, there holds

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{H^1}^2 &= -I(u) = -\|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1} \\ &\leq -\|\nabla u\|_2^2 + C_{\star}^{p+1} \|\nabla u\|_2^{p+1} \\ &= \left(C_{\star}^{p+1} \|\nabla u\|_2^{p-1} - 1 \right) \|\nabla u\|_2^2 \\ &\leq C \left(C_{\star}^{p+1} \|\nabla u\|_2^{p-1} - 1 \right) \|u\|_{H^1}^2, \end{aligned} \quad (4.42)$$

and we shall show that $C_{\star}^{p+1} \|\nabla u\|_2^{p-1} - 1 < 0$. By (2.5), (2.6), (2.8) and $I(u) > 0$ we deduce that

$$J(u) > \frac{p-1}{2(p+1)} \|\nabla u\|_2^2,$$

that is

$$\|\nabla u\|_2^2 < \frac{2(p+1)}{p-1} J(u) \leq \frac{2(p+1)}{p-1} J(u_0). \quad (4.43)$$

From (4.43) and (2.10), we reach the following inequality

$$C_*^{p+1} \|\nabla u\|_2^{p-1} < C_*^{p+1} \left(\frac{2(p+1)}{p-1} J(u_0) \right)^{\frac{p-1}{2}} < 1. \quad (4.44)$$

From (4.42) there holds

$$\frac{1}{2} \frac{d}{dt} \|u\|_{H^1}^2 < C \left(C_*^{p+1} \left(\frac{2(p+1)}{p-1} J(u_0) \right)^{\frac{p-1}{2}} - 1 \right) \|u\|_{H^1}^2.$$

Note that (4.44), we can see that there exists $\delta > 0$ such that

$$-\delta := C \left(C_*^{p+1} \left(\frac{2(p+1)}{p-1} J(u_0) \right)^{\frac{p-1}{2}} - 1 \right).$$

By Gronwall's inequality, we have for $\delta > 0$

$$\|u\|_{H^1}^2 < \|u_0\|_{H^1}^2 e^{-2\delta t}, \quad 0 \leq t < \infty.$$

The proof of this theorem is complete. $\square$

Here, we show a relationship between $\|\nabla u\|_2^2$ and the depth of potential well d with the initial value condition $u_0 \in V$, which is helpful for the proof of finite time blowup of solution.

Lemma 11. If $u_0 \in V$, then

$$\|\nabla u\|_2^2 > \frac{2(p+1)}{p-1} d.$$

Proof. Suppose that $u(t)$ is a weak solution of (1.1) with $J(u_0) < d$ and $I(u_0) < 0$, T be the maximal existence time. From Lemma 9, we can see that $u(x, t) \in V$, that is $I(u) < 0$ for $0 < t < T$. By Lemma 5, we obtain

$$\|\nabla u\|_2^2 > \left(\frac{1}{C_*^{p+1}} \right)^{\frac{2}{p-1}}.$$

Therefore, Lemma 7 implies that

$$d < \frac{p-1}{2(p+1)} \|\nabla u\|_2^2.$$

$\square$

Theorem 12 (Blow up for $J(u_0) < d$). Assume that p satisfies (H). If $u_0 \in H^1(\Omega)$, then for $u_0 \in V$ the weak solution $u(t)$ of (1.1) blows up in finite time.

Proof. First, Theorem 8 shows that the problem (1.1) admits a unique local solution $u \in C([0, T]; H^1(\Omega))$. We will prove the existence time T of solution $u(t)$ is finite with $u_0 \in V$. Arguing by contradiction, we assume that the solution is global in time, i.e. $T = +\infty$.

By $u_0 \in V$ and Lemma 9, for $t \in [0, +\infty)$ we get $u \in V$. We introduce an auxiliary function as

$$F(t) := \int_0^t \|u\|_{H^1}^2 d\tau + (T_0 - t) \|u_0\|_{H^1}^2, \quad t \in [0, T_0],$$

where $0 < T_0 < +\infty$. Obviously, $F(t) > 0$ for any $t \in [0, T_0]$. By the continuity of $F(t)$ with respect to t , it follows that there exists a constant $\varsigma > 0$, such that $F(t) \geq \varsigma$ for $t \in [0, T_0]$. Then

$$\begin{aligned} F'(t) &= \|u\|_{H^1}^2 - \|u_0\|_{H^1}^2 = \|u\|_2^2 + \|\nabla u\|_2^2 - \|u_0\|_2^2 - \|\nabla u_0\|_2^2 \\ &= 2 \int_0^t (u, u_\tau) d\tau + 2 \int_0^t (\nabla u, \nabla u_\tau) d\tau = 2 \int_0^t (u, u_\tau)_* d\tau \end{aligned} \quad (4.45)$$

and (4.41) gives

$$F''(t) = 2(u, u_t) + 2(\nabla u, \nabla u_t) = -2I(u). \quad (4.46)$$

From (4.45) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} (F'(t))^2 &= 4 \left(\int_0^t (u, u_\tau) d\tau \right)^2 = 4 \left(\int_0^t (u, u_\tau) d\tau + \int_0^t (\nabla u, \nabla u_\tau) d\tau \right)^2 \\ &= 4 \left(\left(\int_0^t (u, u_\tau) d\tau \right)^2 + \left(\int_0^t (\nabla u, \nabla u_\tau) d\tau \right)^2 + 2 \int_0^t (u, u_\tau) d\tau \int_0^t (\nabla u, \nabla u_\tau) d\tau \right) \\ &\leq 4 \left(\int_0^t \|u\|_2^2 d\tau \int_0^t \|u_\tau\|_2^2 d\tau + \int_0^t \|\nabla u\|_2^2 d\tau \int_0^t \|\nabla u_\tau\|_2^2 d\tau \right. \\ &\quad \left. + 2 \left(\int_0^t \|u\|_2^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|u_\tau\|_2^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\nabla u\|_2^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\nabla u_\tau\|_2^2 d\tau \right)^{\frac{1}{2}} \right) \\ &\leq 4 \left(\int_0^t \|u\|_2^2 d\tau \int_0^t \|u_\tau\|_2^2 d\tau + \int_0^t \|\nabla u\|_2^2 d\tau \int_0^t \|\nabla u_\tau\|_2^2 d\tau \right. \\ &\quad \left. + \int_0^t \|u\|_2^2 d\tau \int_0^t \|\nabla u_\tau\|_2^2 d\tau + \int_0^t \|\nabla u\|_2^2 d\tau \int_0^t \|u_\tau\|_2^2 d\tau \right) \\ &\leq 4 \left(\int_0^t \|u\|_2^2 d\tau + \int_0^t \|\nabla u\|_2^2 d\tau \right) \left(\int_0^t \|u_\tau\|_2^2 d\tau + \int_0^t \|\nabla u_\tau\|_2^2 d\tau \right) \\ &= 4 \int_0^t \|u\|_{H^1}^2 d\tau \int_0^t \|u_\tau\|_{H^1}^2 d\tau \leq 4F(t) \int_0^t \|u_\tau\|_{H^1}^2 d\tau. \end{aligned} \quad (4.47)$$

By (4.47) and (4.46), we can get

$$\begin{aligned} F''(t)F(t) - \frac{p+3}{4}(F'(t))^2 &\geq F(t) \left(F''(t) - (p+3) \int_0^t \|u_\tau\|_{H^1}^2 d\tau \right) \\ &= F(t) \left(-2I(u) - (p+3) \int_0^t \|u_\tau\|_{H^1}^2 d\tau \right). \end{aligned} \quad (4.48)$$

Let

$$\xi(t) := -2I(u) - (p+3) \int_0^t \|u_\tau\|_{H^1}^2 d\tau. \quad (4.49)$$

By (2.8) and

$$J(u) = \frac{p-1}{2(p+1)} \|\nabla u\|_2^2 + \frac{1}{p+1} I(u), \quad (4.50)$$

we get

$$\xi(t) = (p-1) \|\nabla u\|_2^2 - 2(p+1)J(u_0) + (p-1) \int_0^t \|u_\tau\|_{H^1}^2 d\tau. \quad (4.51)$$

Now, we discuss the following two cases.

(i) If $0 < J(u_0) < d$, then Lemma 11 gives

$$\xi(t) > 2(p+1)d - 2(p+1)J(u_0) + (p-1) \int_0^t \|u_\tau\|_{H^1}^2 d\tau = \varrho > 0. \quad (4.52)$$

Hence we have

$$F''(t)F(t) - \frac{p+3}{4}(F'(t))^2 \geq \varsigma\varrho > 0, \quad t \in [0, T_0],$$

which leads to

$$\left(F^{-\vartheta}(t)\right)'' = -\frac{\vartheta}{F^{\vartheta+2}(t)} \left(F''(t)F(t) - (\vartheta+1)(F'(t))^2\right) < 0, \quad \vartheta = \frac{p-1}{4}.$$

Therefore, by [5] there exists a $T > 0$ such that

$$\lim_{t \rightarrow T} F^{-\vartheta}(t) = 0$$

and

$$\lim_{t \rightarrow T} F(t) = +\infty,$$

which is a contradiction with $T = +\infty$.

(ii) If $J(u_0) \leq 0$, we can get (4.52) directly. The remaining proved similar to case (i).

The proof is completed. $\square$

5 Critical initial energy $J(u_0) = d$

In this section, we shall extend parallelly the global existence, asymptotic behavior and blowup of solution for problem (1.1) to the critical initial energy $J(u_0) = d$.

Theorem 13 (Global existence and asymptotic behavior for $J(u_0) = d$). *Let p satisfy (H), $u_0 \in H^1(\Omega)$ and $J(u_0) = d$. If $I(u_0) \geq 0$, then (1.1) admits a unique global weak solution $u(t) \in L^\infty(0, \infty; H^1(\Omega))$ with $u_t(t) \in L^2(0, \infty; H^1(\Omega))$. If $I(u_0) > 0$, there exist constants $C_1 > 0$, $t_1 > 0$ and $\kappa > 0$ such that $\|u\|_{H^1}^2 < \|u(t_1)\|_{H^1}^2 e^{-2\kappa(t-t_1)}$ for $t \in (t_1, +\infty)$.*

Proof. Global existence and uniqueness. First the condition $J(u_0) = d$ implies that $\|u_0\|_{H^1} \neq 0$. We define $\chi_s = 1 - \frac{1}{s}$ and $u_{0s}(x) = \chi_s u_0(x)$ for $s = 2, 3, \dots$, which implies that $0 < \chi_s < 1$ and $\chi_s \rightarrow 1$ as $s \rightarrow \infty$. Next, we study the problem (1.1) with the condition

$$u(x, 0) = u_{0s}(x), \quad s = 2, 3, \dots. \quad (5.53)$$

Combining Lemma 6 and the initial data $I(u_0) \geq 0$, we get $\chi^* = \chi^*(u_0) \geq 1$. Further, we derive $I(u_{0s}) = I(\chi_s u_0) > 0$ and $J(u_{0s}) = J(\chi_s u_0) < J(u_0) = d$. By Theorem 10, for each $s = 2, 3, \dots$ the problem (1.1) with the initial condition (5.53) admits a unique global solution $u_s(t) \in L^\infty(0, \infty; H^1(\Omega))$ with $u_{st}(t) \in L^2(0, \infty; H^1(\Omega))$. Similar to the proof of Theorem 10, we obtain

$$\int_0^t \|u_{s\tau}\|_{H^1}^2 d\tau + J(u_s) < d$$

and

$$\int_0^t \|u_{s\tau}\|_{H^1}^2 d\tau + \frac{p-1}{2(p+1)} \|\nabla u_s\|_2^2 < d.$$

The remainder of the proof is similar to Theorem 10.

Asymptotic behavior. From above we have proved that the solution $u(t)$ of (1.1) is global, then we claim that $u \in W$ for $t > 0$. Arguing by contradiction, we suppose that $t_0 > 0$ is the first time that $I(u(t_0)) = 0$. By the definition of d , we see that $J(u(t_0)) \geq d$. By (2.8) which implies that

$$0 < J(u(t_0)) = d - \int_0^{t_0} \|u_\tau\|_{H^1}^2 d\tau = d_1 \leq d \quad (5.54)$$

for any $t_0 > 0$. Hence we deduce

$$J(u(t_0)) = d,$$

which means (by (5.54))

$$\int_0^{t_0} \|u_\tau\|_{H^1}^2 d\tau = 0,$$

that is $u_t \equiv 0$ for $0 \leq t \leq t_0$, which contradicts $I(u_0) > 0$ (by (4.46)). Hence we have $u \in W$ for $0 < t < \infty$.

By the continuity of $J(u)$ and $I(u)$ with respect to t , we take a $t_1 > 0$ as the initial time, then $u(x, t) \in W$ for $t > t_1$. Hence from (4.42)-(4.44) we know that there exists a constant $\kappa > 0$ as

$$-\kappa := C \left(C_*^{p+1} \left(\frac{2(p+1)}{p-1} J(u(t_1)) \right)^{\frac{p-1}{2}} - 1 \right)$$

such that

$$\|u\|_{H^1}^2 < \|u(t_1)\|_{H^1}^2 e^{-2\kappa(t-t_1)}, \quad t \in (t_1, +\infty).$$

The proof of this theorem is complete. $\square$

Theorem 14 (Blow up for $J(u_0) = d$). Assume that p satisfies (H). If $u_0 \in H^1(\Omega)$, $J(u_0) = d$ and $I(u_0) < 0$, then the weak solution $u(t)$ of (1.1) blows up in finite time.

Proof. From the continuity of $I(u)$ and $J(u)$ with respect to the time t , for $I(u_0) < 0$ and $J(u_0) = d > 0$, which means there exists a sufficiently small $t_0 > 0$ such that $I(u(t_0)) < 0$ and $J(u(t_0)) > 0$. Combining (4.46), we have $u_t \neq 0$ for $0 < t \leq t_0$. Hence, by (2.8) there holds

$$J(u(t_0)) = d - \int_0^{t_0} \|u_\tau\|_{H^1}^2 d\tau = d_0 < d. \quad (5.55)$$

By Lemma 9 and by taking $t = t_0$ as the initial time, we get $u(x, t) \in V$ for $t > t_0$.

The remainder of the proof is the same as Theorem 12. $\square$

6 Supercritical initial energy blowup $J(u_0) > 0$

In this section, we prove the blowup of solution to problem (1.1) with supercritical initial energy and suitable initial data by two different methods.

We will use the following lemmas to estimate the blowup time and prove the blowup in finite time.

Lemma 15 ([35]). If the function $\omega(t) > 0$ is twice-differentiable and for the constant $\theta > 0$ and any $t > 0$ which satisfies

$$\omega''(t)\omega(t) - (1 + \theta)(\omega'(t))^2 \geq 0$$

and $\omega'(0) > 0$, then there exists $0 < t_1 \leq \frac{\omega(0)}{\theta\omega'(0)}$ such that $\omega(t)$ tends to infinity as $t \rightarrow t_1$.

Lemma 16. Assume that $u_0 \in H^1(\Omega)$ satisfies

$$\frac{2(p+1)}{p-1}J(u_0) < \frac{C}{1+C}\|u_0\|_{H^1}^2 \leq \|\nabla u_0\|_2^2, \quad (6.56)$$

then $u \in \mathcal{N}_- := \{u \in H^1(\Omega) | I(u) < 0\}$, where $C > 0$ satisfies $C\|u\|_2^2 \leq \|\nabla u\|_2^2$.

Proof. Let $u(t)$ be the weak solution of problem (1.1). Multiplying the first equation in (1.1) by $u_t(t)$ and integrating on Ω , we deduce that

$$\|u_t\|_2^2 + \|\nabla u_t\|_2^2 = -\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 + \frac{1}{p+1} \frac{d}{dt} \|u\|_{p+1}^{p+1},$$

which implies that

$$-\|u_t\|_{H^1}^2 = \frac{d}{dt} \left(\frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{p+1} \|u\|_{p+1}^{p+1} \right),$$

i.e.

$$\frac{d}{dt} J(u) = -\|u_t\|_{H^1}^2 < 0. \quad (6.57)$$

By (2.5) and (2.6), we get

$$J(u_0) = \frac{p-1}{2(p+1)} \|\nabla u_0\|_2^2 + \frac{1}{p+1} I(u_0), \quad (6.58)$$

and (6.56) gives $I(u_0) < 0$.

Next, for $t \in [0, T)$ we shall prove $u(t) \in \mathcal{N}_-$. If it is false, according to the continuity of $I(u)$ with respect to t , there must exist a $\bar{t} \in (0, T)$ such that $u(\bar{t}) \in \mathcal{N}$ and for $0 \leq t < \bar{t}$ we have $u(t) \in \mathcal{N}_-$, then (4.41) gives

$$\frac{d}{dt} \|u(t)\|_{H^1}^2 = -2I(u) > 0 \text{ for any } t \in [0, \bar{t}), \quad (6.59)$$

which says

$$\|u_0\|_{H^1}^2 < \|u(\bar{t})\|_{H^1}^2. \quad (6.60)$$

By (6.57), we derive the inequality

$$J(u(\bar{t})) < J(u_0). \quad (6.61)$$

Combining (2.5), (2.6) and $u(\bar{t}) \in \mathcal{N}$, which yields

$$J(u(\bar{t})) = \frac{p-1}{2(p+1)} \|\nabla u(\bar{t})\|_2^2 + \frac{1}{p+1} I(u(\bar{t})) = \frac{p-1}{2(p+1)} \|\nabla u(\bar{t})\|_2^2.$$

Then (6.61) and (6.56) give

$$\frac{p-1}{2(p+1)} \|\nabla u(\bar{t})\|_2^2 < J(u_0) < \frac{p-1}{2(p+1)} \|\nabla u_0\|_2^2,$$

i.e.

$$\|u_0\|_{H^1}^2 > \|u(\bar{t})\|_{H^1}^2,$$

which contradicts (6.60). $\square$

Theorem 17 (Blow up for $J(u_0) > 0$). Let $u(x, t)$ be a weak solution to problem (1.1). Assume that (H) and $u_0 \in H^1(\Omega)$. If $J(u_0) > 0$ and (6.56) hold, then the solution $u(x, t)$ blows up in finite time.

Method I. (Inspired by [29] and [35])

Proof. First, Theorem 8 shows that the problem (1.1) admits a unique local solution $u \in C([0, T); H^1(\Omega))$. We will prove the existence time T of solution $u(t)$ is finite with (6.56) and there exists a

$$0 < t_* \leq \frac{2G(0)}{(\alpha-1)G'(0)}$$

such that

$$\lim_{t \rightarrow t^*} \int_0^t \|u\|_{H^1}^2 d\tau = +\infty, \quad (6.62)$$

where

$$G(t) = \left(\int_0^t \|u\|_{H^1}^2 d\tau \right)^2 + \varepsilon^{-1} \|u_0\|_{H^1}^2 \int_0^t \|u\|_{H^1}^2 d\tau + c, \quad (6.63)$$

$$1 < \alpha < \frac{(p-1)C\|u_0\|_{H^1}^2}{2(p+1)(1+C)J(u_0)}, \quad (6.64)$$

$$0 < \varepsilon < \frac{1}{2\alpha\|u_0\|_{H^1}^2} \left(\frac{2(p-1)C}{(p+1)(1+C)} \|u_0\|_{H^1}^2 - 4\alpha J(u_0) \right) \quad (6.65)$$

and

$$c > \frac{1}{4} \varepsilon^{-2} \|u_0\|_{H^1}^4. \quad (6.66)$$

If it is false, we suppose that the solution $u(x, t)$ is a global one, that is $T = +\infty$. From (6.59) we deduce that

$$\begin{aligned} \frac{d}{dt} \|u\|_{H^1}^2 &= -2I(u) = -2 \left(\|\nabla u\|_2^2 - \|u\|_{p+1}^{p+1} \right) \\ &= -4 \left(\frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{p+1} \|u\|_{p+1}^{p+1} \right) + \left(2 - \frac{4}{p+1} \right) \|u\|_{p+1}^{p+1} \\ &= -4J(u) + \frac{2(p-1)}{p+1} \|u\|_{p+1}^{p+1}. \end{aligned} \quad (6.67)$$

Next, we will complete the proof in the following two cases.

Case (i): $J(u) \geq 0$ for $t \in (0, \infty)$.

According to (6.56), we select α satisfying (6.64). Combining (2.8) and (6.67), by the fact $J(u) \geq 0$ gives

$$\begin{aligned} \frac{d}{dt} \|u\|_{H^1}^2 &= 4(\alpha-1)J(u) - 4\alpha J(u) + \frac{2(p-1)}{p+1} \|u\|_{p+1}^{p+1} \\ &\geq -4\alpha J(u_0) + 4\alpha \int_0^t \|u_\tau\|_{H^1}^2 d\tau + \frac{2(p-1)}{p+1} \|u\|_{p+1}^{p+1}. \end{aligned} \quad (6.68)$$

From Lemma 16, there holds

$$\|u\|_{p+1}^{p+1} > \|\nabla u\|_2^2 \geq \frac{C}{1+C} \|u\|_{H^1}^2, \quad (6.69)$$

which makes (6.68) become

$$\frac{d}{dt} \|u\|_{H^1}^2 > -4\alpha J(u_0) + 4\alpha \int_0^t \|u_\tau\|_{H^1}^2 d\tau + \frac{2(p-1)C}{(p+1)(1+C)} \|u\|_{H^1}^2, \quad (6.70)$$

i.e.

$$\frac{d}{dt} \|u\|_{H^1}^2 - \frac{2(p-1)C}{(p+1)(1+C)} \|u\|_{H^1}^2 > -4\alpha J(u_0), \quad (6.71)$$

further

$$\|u\|_{H^1}^2 > \|u_0\|_{H^1}^2 e^{\frac{2(p-1)C}{(p+1)(1+C)} t} + \frac{2(p+1)(1+C)}{(p-1)C} \alpha J(u_0) \left(1 - e^{\frac{2(p-1)C}{(p+1)(1+C)} t} \right). \quad (6.72)$$

Now, we define

$$g(t) := \int_0^t \|u(\tau)\|_{H^1}^2 d\tau.$$

Since we have assumed the existence time of solution to be $+\infty$, $g(t)$ is bounded for all $t \geq 0$. Then we get directly

$$g'(t) = \|u(t)\|_{H^1}^2 \quad (6.73)$$

and

$$g''(t) = \frac{d}{dt} \|u(t)\|_{H^1}^2.$$

By (6.65), we reach

$$\frac{2(p-1)C}{(p+1)(1+C)} \|u_0\|_{H^1}^2 - 4\alpha J(u_0) > 2\alpha\varepsilon \|u_0\|_{H^1}^2. \quad (6.74)$$

Substituting (6.72) into (6.70), by (6.74) and the fact $e^{\frac{2(p-1)C}{(p+1)(1+C)}t} \geq 1$ gives

$$\begin{aligned} g''(t) &> 4\alpha \int_0^t \|u_\tau\|_{H^1}^2 d\tau + \left(\frac{2(p-1)C}{(p+1)(1+C)} \|u_0\|_{H^1}^2 - 4\alpha J(u_0) \right) e^{\frac{2(p-1)C}{(p+1)(1+C)}t} \\ &> 4\alpha \int_0^t \|u_\tau\|_{H^1}^2 d\tau + 2\alpha\varepsilon \|u_0\|_{H^1}^2. \end{aligned} \quad (6.75)$$

By the definition of $G(t)$ in (6.63), we get

$$G'(t) = \left(2g(t) + \varepsilon^{-1} \|u_0\|_{H^1}^2 \right) g'(t) \quad (6.76)$$

and

$$G''(t) = \left(2g(t) + \varepsilon^{-1} \|u_0\|_{H^1}^2 \right) g''(t) + 2(g'(t))^2. \quad (6.77)$$

We can define μ by (6.66) as

$$\mu := 4c - \varepsilon^{-2} \|u_0\|_{H^1}^4 > 0,$$

then (6.76) becomes

$$\begin{aligned} (G'(t))^2 &= \left(4g^2(t) + 4\varepsilon^{-1} \|u_0\|_{H^1}^2 g(t) + \varepsilon^{-2} \|u_0\|_{H^1}^4 \right) (g'(t))^2 \\ &= \left(4g^2(t) + 4\varepsilon^{-1} \|u_0\|_{H^1}^2 g(t) + 4c - \mu \right) (g'(t))^2 \\ &= (4G(t) - \mu) (g'(t))^2, \end{aligned} \quad (6.78)$$

which means

$$4G(t)(g'(t))^2 = (G'(t))^2 + \mu(g'(t))^2. \quad (6.79)$$

Integrating the following identity on $[0, t]$

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^1}^2 = (u, u_t)_*,$$

gives

$$\frac{1}{2} \left(\|u(t)\|_{H^1}^2 - \|u_0\|_{H^1}^2 \right) = \int_0^t (u, u_\tau)_* d\tau,$$

which says

$$\|u(t)\|_{H^1}^2 = \|u_0\|_{H^1}^2 + 2 \int_0^t (u, u_\tau)_* d\tau. \quad (6.80)$$

By (6.80), Hölder and Young's inequalities, (6.73) becomes

$$(g'(t))^2 = \|u(t)\|_{H^1}^4 = \left(\|u_0\|_{H^1}^2 + 2 \int_0^t (u, u_\tau)_* d\tau \right)^2 \quad (6.81)$$

$$\begin{aligned}
&\leq \left(\|u_0\|_{H^1}^2 + 2 \left(\int_0^t \|u\|_{H^1}^2 d\tau \right)^{1/2} \left(\int_0^t \|u_\tau\|_{H^1}^2 d\tau \right)^{1/2} \right)^2 \\
&= \|u_0\|_{H^1}^4 + 4 \|u_0\|_{H^1}^2 (g(t))^{1/2} \left(\int_0^t \|u_\tau\|_{H^1}^2 d\tau \right)^{1/2} + 4g(t) \int_0^t \|u_\tau\|_{H^1}^2 d\tau \\
&\leq \|u_0\|_{H^1}^4 + 4g(t) \int_0^t \|u_\tau\|_{H^1}^2 d\tau + 2\varepsilon \|u_0\|_{H^1}^2 g(t) + 2\varepsilon^{-1} \|u_0\|_{H^1}^2 \int_0^t \|u_\tau\|_{H^1}^2 d\tau \\
&:= A(t).
\end{aligned}$$

Combining (6.77) and (6.79) gives

$$\begin{aligned}
2G(t)G''(t) &= 2 \left((2g(t) + \varepsilon^{-1} \|u_0\|_{H^1}^2) g''(t) + 2(g'(t))^2 \right) G(t) \\
&= 2 \left(2g(t) + \varepsilon^{-1} \|u_0\|_{H^1}^2 \right) g''(t)G(t) + 4(g'(t))^2 G(t) \\
&= 2 \left(2g(t) + \varepsilon^{-1} \|u_0\|_{H^1}^2 \right) g''(t)G(t) + (G'(t))^2 + \mu(g'(t))^2.
\end{aligned} \tag{6.82}$$

From (6.82), (6.78), (6.75) and (6.81), we deduce that

$$\begin{aligned}
&2G(t)G''(t) - (1 + \alpha)(G'(t))^2 \\
&= 2 \left(2g(t) + \varepsilon^{-1} \|u_0\|_{H^1}^2 \right) g''(t)G(t) - \alpha(G'(t))^2 + \mu(g'(t))^2 \\
&= 2 \left(2g(t) + \varepsilon^{-1} \|u_0\|_{H^1}^2 \right) g''(t)G(t) - \alpha(4G(t) - \mu)(g'(t))^2 + \mu(g'(t))^2 \\
&= 2 \left(2g(t) + \varepsilon^{-1} \|u_0\|_{H^1}^2 \right) g''(t)G(t) - 4\alpha G(t)(g'(t))^2 + \mu(1 + \alpha)(g'(t))^2 \\
&> 4\alpha G(t) \left(2g(t) + \varepsilon^{-1} \|u_0\|_{H^1}^2 \right) \left(2 \int_0^t \|u_\tau\|_{H^1}^2 d\tau + \varepsilon \|u_0\|_{H^1}^2 \right) - 4\alpha G(t)A(t) \\
&= 4\alpha G(t)A(t) - 4\alpha G(t)A(t) = 0,
\end{aligned} \tag{6.83}$$

which says for $t \in [0, +\infty)$

$$G(t)G''(t) - \frac{1 + \alpha}{2}(G'(t))^2 > 0,$$

that is

$$\left(G^{-\theta}(t) \right)'' = -\frac{\theta}{G^{\theta+2}(t)} \left(G''(t)G(t) - (\theta + 1)(G'(t))^2 \right) < 0, \quad \theta = \frac{\alpha - 1}{2} > 0.$$

By $G(0) = c > 0$ and $G'(0) = \varepsilon^{-1} \|u_0\|_{H^1}^4 > 0$, Lemma 15 shows that there exists a

$$0 < t_* \leq \frac{2G(0)}{(\alpha - 1)G'(0)}$$

such that

$$\lim_{t \rightarrow t_*} G^{-\theta}(t) = 0$$

and

$$\lim_{t \rightarrow t_*} G(t) = +\infty,$$

which contradicts $T = +\infty$. Now, by considering the continuity of G in t , we can derive $g(t) \rightarrow \infty$ as $t \rightarrow t_*$.

Case (ii): $J(u(\tilde{t})) < 0$ for some $\tilde{t} > 0$.

Since $J(u)$ is continuous with respect to t , for $J(u_0) > 0$ and (6.57) there must exist a time $t_1 > 0$ such that $J(u) < 0$ for $t > t_1$ and $J(u(t_1)) = 0$. We choose $u(t_1)$ as a new initial datum of problem (1.1), then Lemma 16 gives $u \in \mathcal{N}_-$ for $t > t_1$. Similar to the proof of Theorem 12, we obtain the blowup of solution in finite time.

Combining the Case (i) and Case (ii), we conclude that the blowup of solution in finite time. $\square$

Method II. (Inspired by [36])

Proof. First, Theorem 8 shows that the problem (1.1) admits a unique local solution $u \in C([0, T]; H^1(\Omega))$. We show that the solution $u(t)$ blows up in finite time $T_{\max}$. If it is false, we suppose that the existence time $T = +\infty$. By Lemma 16, we have $u \in \mathcal{N}_-$, i.e. $J(u) < 0$ for all $t \in [0, +\infty)$. Next, we consider the following two cases.

Case I. $J(u) \geq 0$ for all $t > 0$. In this case, we divide the proof into two steps.

Step 1. Let

$$M(t) := \frac{1}{2} \|u\|_{H^1}^2 - \frac{(p+1)(1+C)}{(p-1)C} J(u), \quad t \in [0, +\infty).$$

By (6.57), the first equation in (1.1), (2.5) and (6.56), we get

$$\begin{aligned} M'(t) &= (u, u_t)_* + \frac{(p+1)(1+C)}{(p-1)C} \|u_t\|_{H^1}^2 \\ &= (u, u_t) - (u, \Delta u_t) + \frac{(p+1)(1+C)}{(p-1)C} \|u_t\|_{H^1}^2 \\ &= \int_{\Omega} u \left(\Delta u + |u|^{p-1}u - \int_{\Omega} |u|^{p-1}u dx \right) dx + \frac{(p+1)(1+C)}{(p-1)C} \|u_t\|_{H^1}^2 \\ &= \left(- \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |u|^{p+1} dx \right) + \frac{(p+1)(1+C)}{(p-1)C} \|u_t\|_{H^1}^2 \\ &= \left(\frac{p-1}{2} \|\nabla u\|_2^2 - (p+1)J(u) \right) + \frac{(p+1)(1+C)}{(p-1)C} \|u_t\|_{H^1}^2 \\ &= (p-1) \left(\frac{1}{2} \|\nabla u\|_2^2 - \frac{p+1}{p-1} J(u) \right) + \frac{(p+1)(1+C)}{(p-1)C} \|u_t\|_{H^1}^2 \\ &\geq (p-1) \left(\frac{C}{2(1+C)} \|u\|_{H^1}^2 - \frac{p+1}{p-1} J(u) \right) + \frac{(p+1)(1+C)}{(p-1)C} \|u_t\|_{H^1}^2 \\ &= \frac{(p-1)C}{1+C} \left(\frac{1}{2} \|u\|_{H^1}^2 - \frac{(p+1)(1+C)}{(p-1)C} J(u) \right) + \frac{(p+1)(1+C)}{(p-1)C} \|u_t\|_{H^1}^2 \\ &\geq \frac{(p-1)C}{1+C} M(t). \end{aligned} \quad (6.84)$$

Combining (6.59), (6.57) and (6.56), we derive

$$M(t) = \frac{1}{2} \|u\|_{H^1}^2 - \frac{(p+1)(1+C)}{(p-1)C} J(u) \geq \frac{1}{2} \|u_0\|_{H^1}^2 - \frac{(p+1)(1+C)}{(p-1)C} J(u_0) = M(0) > 0, \quad (6.85)$$

which combining (6.84) gives

$$\frac{1}{2} \|u\|_{H^1}^2 > M(t) \geq M(0) e^{\frac{(p-1)C}{1+C} t}. \quad (6.86)$$

On the other hand, by triangle inequality, Hölder's inequality, (2.8), (6.57) and $J(u) \geq 0$, we have

$$\begin{aligned} \|u\|_{H^1} &= \left\| u_0 + \int_0^t u_{\tau} d\tau \right\|_{H^1} \leq \|u_0\|_{H^1} + \left\| \int_0^t u_{\tau} d\tau \right\|_{H^1} \\ &= \|u_0\|_{H^1} + \left(\left\| \int_0^t u_{\tau} d\tau \right\|_2^2 + \left\| \int_0^t \nabla u_{\tau} d\tau \right\|_2^2 \right)^{\frac{1}{2}} \\ &= \|u_0\|_{H^1} + \left(\int_{\Omega} \left(\int_0^t u_{\tau} d\tau \right)^2 dx + \int_{\Omega} \left(\int_0^t \nabla u_{\tau} d\tau \right)^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
& \leq \|u_0\|_{H^1} + \left(\int_{\Omega} \left(\int_0^t 1^2 d\tau \right) \left(\int_0^t |u_{\tau}|^2 d\tau \right) dx \right. \\
& \quad \left. + \int_{\Omega} \left(\int_0^t 1^2 d\tau \right) \left(\int_0^t |\nabla u_{\tau}|^2 d\tau \right) dx \right)^{\frac{1}{2}} \\
& \leq \|u_0\|_{H^1} + t^{\frac{1}{2}} \left(\int_0^t \|u_{\tau}\|_{H^1}^2 d\tau \right)^{\frac{1}{2}} \\
& = \|u_0\|_{H^1} + t^{\frac{1}{2}} (J(u_0) - J(u))^{\frac{1}{2}} \\
& \leq \|u_0\|_{H^1} + J^{\frac{1}{2}}(u_0) t^{\frac{1}{2}}.
\end{aligned} \tag{6.87}$$

Therefore, combining (6.86) and (6.87), we deduce

$$(2M(0))^{\frac{1}{2}} e^{\frac{(p-1)C}{2(1+C)}t} < \|u\|_{H^1} \leq \|u_0\|_{H^1} + J^{\frac{1}{2}}(u_0)t^{\frac{1}{2}} \text{ for all } t \in [0, \infty),$$

that is

$$ae^{bt} < k + rt^{\frac{1}{2}} \text{ for all } t \in [0, \infty), \tag{6.88}$$

where $a = (2M(0))^{\frac{1}{2}} > 0$, $b = \frac{(p-1)C}{2(1+C)} > 0$, $k = \|u_0\|_{H^1} > 0$ and $r = J^{\frac{1}{2}}(u_0) > 0$. It is obvious that (6.88) does not hold as $t \rightarrow \infty$, hence the solution $u(t)$ blows up in finite time.

Step 2. We estimate the upper bound of blow up time.

We introduce an auxiliary function as

$$H(t) := \int_0^t \|u\|_{H^1}^2 d\tau + (T_{\max} - t)\|u_0\|_{H^1}^2 + \eta(t + \xi)^2, \quad t \in [0, T_{\max}), \tag{6.89}$$

where the two positive constants $\eta, \xi > 0$ will be determined later. By (6.59), we have

$$H(0) = T_{\max}\|u_0\|_{H^1}^2 + \eta\xi^2 > 0, \tag{6.90}$$

$$H'(t) = \|u\|_{H^1}^2 - \|u_0\|_{H^1}^2 + 2\eta(t + \xi) > 0, \quad t \in [0, T_{\max}) \tag{6.91}$$

and

$$H'(0) = 2\eta\xi > 0. \tag{6.92}$$

By (6.59), (2.5), (2.6), (2.8) and (6.56), we deduce

$$\begin{aligned}
H''(t) &= -2I(u) + 2\eta = (p-1)\|\nabla u\|_2^2 - 2(p+1)J(u) + 2\eta \\
&= \frac{(p-1)C}{(1+C)}\|u\|_{H^1}^2 - 2(p+1)J(u_0) + 2(p+1)\int_0^t \|u_{\tau}\|_{H^1}^2 d\tau + 2\eta \\
&\geq \frac{(p-1)C}{(1+C)}\|u_0\|_{H^1}^2 - 2(p+1)J(u_0) + 2(p+1)\int_0^t \|u_{\tau}\|_{H^1}^2 d\tau + 2\eta > 0.
\end{aligned} \tag{6.93}$$

By Cauchy-Schwarz inequality, similar to (4.47), we derive

$$\int_0^t (u, u_{\tau})_* d\tau \leq \left(\int_0^t \|u\|_{H^1}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|u_{\tau}\|_{H^1}^2 d\tau \right)^{\frac{1}{2}}. \tag{6.94}$$

Define

$$\mu(t) := \left(\int_0^t \|u\|_{H^1}^2 d\tau \right)^{\frac{1}{2}} \quad \text{and} \quad y(t) := \left(\int_0^t \|u_\tau\|_{H^1}^2 d\tau \right)^{\frac{1}{2}},$$

by (6.94) we have the following estimates

$$\begin{aligned} & \left(\int_0^t \|u\|_{H^1}^2 d\tau + \eta(t+\xi)^2 \right) \left(\int_0^t \|u_\tau\|_{H^1}^2 d\tau + \eta \right) - \left(\frac{1}{2} (\|u\|_{H^1}^2 - \|u_0\|_{H^1}^2) + \eta(t+\xi) \right)^2 \\ &= \left(\mu^2(t) + \eta(t+\xi)^2 \right) \left(y^2(t) + \eta \right) - \left(\int_0^t (u, u_\tau) d\tau + \eta(t+\xi) \right)^2 \\ &\geq \left(\mu^2(t) + \eta(t+\xi)^2 \right) \left(y^2(t) + \eta \right) - \left(\left(\int_0^t \|u\|_{H^1}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|u_\tau\|_{H^1}^2 d\tau \right)^{\frac{1}{2}} + \eta(t+\xi) \right)^2 \\ &= \left(\mu^2(t) + \eta(t+\xi)^2 \right) \left(y^2(t) + \eta \right) - \left(\mu(t)y(t) + \eta(t+\xi) \right)^2 \\ &= \eta \mu^2(t) + \eta(t+\xi)^2 y^2(t) - 2\mu(t)y(t)\eta(t+\xi) \\ &= (\sqrt{\eta}\mu(t))^2 - 2\mu(t)y(t)\eta(t+\xi) + (\sqrt{\eta}(t+\xi)y(t))^2 \\ &= \left(\sqrt{\eta}\mu(t) - \sqrt{\eta}(t+\xi)y(t) \right)^2 \geq 0. \end{aligned} \tag{6.95}$$

Combining (6.89) and (6.95), (6.91) gives

$$\begin{aligned} & - (H'(t))^2 \\ &= -4 \left(\frac{1}{2} (\|u\|_{H^1}^2 - \|u_0\|_{H^1}^2) + \eta(t+\xi) \right)^2 \\ &= 4 \left(\int_0^t \|u\|_{H^1}^2 d\tau + \eta(t+\xi)^2 \right) \left(\int_0^t \|u_\tau\|_{H^1}^2 d\tau + \eta \right) - 4 \left(\frac{1}{2} (\|u\|_{H^1}^2 - \|u_0\|_{H^1}^2) + \eta(t+\xi) \right)^2 \\ &\quad - 4 \left(H(t) - (T_{\max} - t) \|u_0\|_{H^1}^2 \right) \left(\int_0^t \|u_\tau\|_{H^1}^2 d\tau + \eta \right) \\ &\geq -4 \left(H(t) - (T_{\max} - t) \|u_0\|_{H^1}^2 \right) \left(\int_0^t \|u_\tau\|_{H^1}^2 d\tau + \eta \right) \geq -4H(t) \left(\int_0^t \|u_\tau\|_{H^1}^2 d\tau + \eta \right). \end{aligned} \tag{6.96}$$

By (6.56), we let

$$0 < \eta \leq \rho := \frac{(p-1)C}{2p(1+C)} \|u_0\|_{H^1}^2 - \frac{p+1}{p} J(u_0), \tag{6.97}$$

then combining (6.93) and (6.96), we get

$$\begin{aligned} & H(t)H''(t) - \frac{p+1}{2} (H'(t))^2 \\ &\geq H(t) \left(\frac{(p-1)C}{(1+C)} \|u_0\|_{H^1}^2 - 2(p+1)J(u_0) + 2(p+1) \int_0^t \|u_\tau\|_{H^1}^2 d\tau + 2\eta \right. \\ &\quad \left. - 2(p+1) \left(\int_0^t \|u_\tau\|_{H^1}^2 d\tau + \eta \right) \right) \end{aligned}$$

$$= H(t) \left(\frac{(p-1)C}{(1+C)} \|u_0\|_{H^1}^2 - 2(p+1)J(u_0) - 2p\eta \right) \geq 0, \quad (6.98)$$

which means that for $\sigma = \frac{p+1}{2}$

$$\left(H^{-(\sigma-1)}(t) \right)'' = -\frac{\sigma-1}{F^{\sigma+1}(t)} \left(H(t)H''(t) - \sigma(H'(t))^2 \right) \leq 0.$$

Hence, it follows that there exists a $T_{\max} > 0$ such that

$$\lim_{t \rightarrow T_{\max}} H^{-(\sigma-1)}(t) = 0,$$

i.e.,

$$\lim_{t \rightarrow T_{\max}} H(t) = +\infty.$$

Further by (6.98), Lemma 15, (6.90) and (6.92), we get

$$0 < T_{\max} \leq \frac{2H(0)}{(p-1)H'(0)} = \frac{T_{\max}\|u_0\|_{H^1}^2 + \eta\xi^2}{(p-1)\eta\xi} = \frac{T_{\max}\|u_0\|_{H^1}^2}{(p-1)\eta\xi} + \frac{\xi}{p-1}. \quad (6.99)$$

We choose

$$\frac{\|u_0\|_{H^1}^2}{(p-1)\eta} < \xi < \infty \quad (6.100)$$

to make (6.99) be

$$0 < T_{\max} \leq \frac{\eta\xi^2}{(p-1)\eta\xi - \|u_0\|_{H^1}^2}. \quad (6.101)$$

Combining (6.97) and (6.100), we deduce

$$\frac{\|u_0\|_{H^1}^2}{(p-1)\xi} < \eta \leq \rho, \quad (6.102)$$

then

$$\frac{\|u_0\|_{H^1}^2}{(p-1)\rho} < \xi < \infty. \quad (6.103)$$

We define

$$f(\eta, \xi) := \frac{\eta\xi^2}{(p-1)\eta\xi - \|u_0\|_{H^1}^2}, \quad (6.104)$$

which is a continuous function of variables $\eta \in \left(\frac{\|u_0\|_{H^1}^2}{(p-1)\xi}, \rho \right]$ and $\xi \in \left(\frac{\|u_0\|_{H^1}^2}{(p-1)\rho}, \infty \right)$ due to (6.102) and (6.103).

Next, we estimate the upper bounded of the blowup time $T_{\max}$.

Differentiating (6.104) with respect to η , we have

$$f'_\eta(\eta, \xi) = \frac{-\xi^2\|u_0\|_{H^1}^2}{((p-1)\eta\xi - \|u_0\|_{H^1}^2)^2} < 0,$$

which says that $f(\eta, \xi)$ is decreasing with η , then for any $\xi \in \left(\frac{\|u_0\|_{H^1}^2}{(p-1)\rho}, \infty \right)$, (6.102) gives

$$\inf_{\eta, \xi} f(\eta, \xi) = \inf_{\xi} f(\rho, \xi) = \inf_{\xi} \frac{\rho\xi^2}{(p-1)\rho\xi - \|u_0\|_{H^1}^2}. \quad (6.105)$$

By differentiating $f(\rho, \xi)$ with respect to ξ and letting $f'_\xi(\rho, \xi) = 0$, we can get the minimum point $\xi_{\min} = \frac{2\|u_0\|_{H^1}^2}{(p-1)\rho}$ satisfying (6.103), then for any ρ satisfying (6.97) we have

$$\inf_{\xi} f(\rho, \xi) = f(\rho, \xi_{\min}) = \frac{4\|u_0\|_{H^1}^2}{(p-1)^2\rho},$$

which says

$$\inf_{\eta, \xi} f(\eta, \xi) = f(\rho, \xi_{\min}) = \frac{4\|u_0\|_{H^1}^2}{(p-1)^2\rho}. \quad (6.106)$$

Therefore, by (6.101), (6.104)-(6.106), we have

$$T_{\max} \leq \inf_{\eta, \xi} f(\eta, \xi) = \frac{4\|u_0\|_{H^1}^2}{(p-1)^2\rho}.$$

Case II. Similar to the proof of (ii) in Method I.

Combining the above Case I and Case II, we conclude that the solution will blow up in finite time. $\square$

Remark 2 (Difference from [31]). Although in [31] Xu and Su obtained the finite time blowup of solution for the arbitrary high initial energy level, i.e. $J(u_0) \geq M$, the blowup result obtained in the present paper is different from that in [31], as the main blowup theorem in [31] was proved by comparison principle, which does not work for the problem considered in the present paper. In order to apply the comparison principle, we need distinguish two different initial data $w_0(x)$ and $v_0(x)$. Without loss of generality, we can suppose $w_0(x) < v_0(x)$ satisfying $\int_{\Omega} w_0(x)dx = \int_{\Omega} v_0(x)dx = 0$, which directly leads to that $w_0(x)$ and $v_0(x)$ must intersect at least at one point x_0 , then the failure of the comparison principle. Hence in the present paper we employ the energy estimate and concavity method to replace the comparison principle to overcome this difficulty.

Remark 3 (Comparison between the results for $J(u_0) < d$ and $J(u_0) > 0$). Comparing Theorem 10 (global existence for sub-critical energy), Theorem 12 (finite time blowup for sub-critical energy) and Theorem 17 (finite time blowup for positive high energy), if we only focus on the initial condition about energy, we will find that the conditions $J(u_0) < d$ and $J(u_0) > 0$ have overlapping parts, that is $0 < J(u_0) < d$. Therefore, it is necessary to clarify their relationship.

As the depth of potential well d , so-called the mountain pass level, plays a very important role in characterizing the initial data manifolds in terms of the signs of the Nehari functional i.e. $I(u_0) > 0$, $I(u_0) < 0$ or the radius of the ball in $H^1(\Omega)$ space, i.e., $\|\nabla u\|_2 < r_0 = \left(\frac{1}{C_*^{p+1}}\right)^{\frac{1}{p-1}}$ and $\|\nabla u\|_2 > r_0$ only under the condition $J(u_0) < d$, we can get the results similar to the sharp condition, which is to show that the initial data $u_0 \in \{u \in H^1(\Omega) \mid J(u) < d, I(u) > 0\}$ leading to global existence solution and $u_0 \in \{u \in H^1(\Omega) \mid J(u) < d, I(u) < 0\}$ leading to finite time blowup solution. Although in Theorem 17 we extend the restriction $J(u_0) < d$ to $J(u_0) > 0$, we only get a class of sufficient conditions (different from those in Theorem 12) ensuring the finite time blowup of the solution instead of proving anything similar to the sharp condition that can be derived for the sub-critical case, i.e., $J(u_0) < d$. Hence the conclusions for $J(u_0) < d$ and $J(u_0) > 0$ are different, and the conclusions for case $J(u_0) < d$ are better.

Indeed $J(u_0) < d$ and $J(u_0) > 0$ are only the conditions about the initial energy acting as part of the initial conditions, so when we compare the conclusions of these two conditions, we need to pay attention to other conditions also. On one hand, in the frame of $J(u_0) < d$, we have an invariant set V in which $I(u_0) < 0$, and a relation between $I(u) < 0$ and $\|\nabla u\|_2 > r_0 = \left(\frac{1}{C_*^{p+1}}\right)^{\frac{1}{p-1}}$, but we cannot get neither of them when we extend $J(u_0) < d$ to $J(u_0) > 0$. On the other hand, in the frame of $J(u_0) > 0$, (6.56) is an important condition on the initial data, and (6.56) seems interesting here. First we can derive $u \in \mathcal{N}_-$ if u_0 satisfies (6.56), that means the initial datum given by (6.56) leads to $I(u) < 0$ for any $t > 0$, which can be considered as another version of the invariant manifold V in the case $J(u_0) < d$, but the difference is that we replace $u_0 \in V$ by (6.56).

For $J(u_0) > 0$, the condition (6.56) gives

$$0 < J(u_0) < \frac{(p-1)C}{2(p+1)(1+C)} \|u_0\|_{H^1}^2 \leq \frac{p-1}{2(p+1)} \|\nabla u_0\|_2^2. \quad (6.107)$$

Naturally, for sufficiently large initial datum u_0 , we have

$$\frac{p-1}{2(p+1)} \|\nabla u_0\|_2^2 \geq \frac{(p-1)C}{2(p+1)(1+C)} \|u_0\|_{H^1}^2 > d,$$

which says that the following initial condition

$$d < J(u_0) < \frac{(p-1)C}{2(p+1)(1+C)} \|u_0\|_{H^1}^2 \leq \frac{p-1}{2(p+1)} \|\nabla u_0\|_2^2$$

is achievable. Therefore, for $J(u_0) > 0$ we call the initial condition (6.107) high or supercritical initial energy.

Acknowledgement: This work was supported by the National Natural Science Foundation of China (11871017), the Ph.D. Student Research and Innovation Fund of the Fundamental Research Funds for the Central Universities (HEUGIP201808), the China Postdoctoral Science Foundation (2013M540270), the Fundamental Research Funds for the Central Universities.

Professor Xu Runzhang was an Editor of the ANONA although had no involvement in the final decision.

References

- [1] C. Budd, B. Dold and A. Stuart, Blowup in a partial differential equation with conserved first integral, *SIAM J. Appl. Math.* 53 (1993), no. 3, 718-742.
- [2] J. Rubenstein and P. Sternberg, Nonlocal reaction-diffusion equations and nucleation, *IMA J. Appl. Math.* 48 (1992), no. 3, 249-264.
- [3] R. Schaaf, Stationary solutions of chemotaxis systems, *Trans. Amer. Math. Soc.* 292 (1985), no. 2, 531-556.
- [4] S. Childress, Chemotactic collapse in two dimensions, in *Modelling of Patterns in Space and Time*, Lecture Notes in Biomathematics 55, Springer-Verlag, Berlin, 1984.
- [5] L.E. Payne and D.H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, *Israel. J. Math.* 22 (1975), no. 3-4, 273-303.
- [6] P.L. Lions, Structure of the set of steady-state solutions and asymptotic behaviour of semilinear heat equations, *J. Differential Equations* 53 (1984), no. 3, 362-386.
- [7] R. Ikehata and T. Suzuki, Semilinear parabolic equations involving critical Sobolev exponent: local and asymptotic behavior of solutions, *Differential Integral Equations* 13 (2000), no. 7-9, 869-901.
- [8] P. Quittner, Universal bound for global positive solutions of a superlinear parabolic problem, *Math. Ann.* 320 (2001), no. 2, 299-305.
- [9] P. Poláčik and F. Simondon, Nonconvergent bounded solutions of semilinear heat equations on arbitrary domains, *J. Differential Equations* 186 (2002), no. 2, 586-610.
- [10] P. Quittner, P. Souplet and M. Winkler, Initial blow-up rates and universal bounds for nonlinear heat equations, *J. Differential Equations* 196 (2004), no. 2, 316-339.
- [11] F. Gazzola and T. Weth, Finite time blow up and global solutions for semilinear parabolic equations with initial data at high energy level, *Differential Integral Equations* 18 (2005), no. 9, 961-990.
- [12] Y.C. Liu and J.S. Zhao, On potential wells and applications to semilinear hyperbolic equations and parabolic equations, *Nonlinear Anal.* 64 (2006), no. 12, 2665-2687.
- [13] L.E. Payne and P.W. Schaefer, Lower bounds for blow-up time in parabolic problems under Dirichlet conditions, *J. Math. Anal. Appl.* 328 (2007), no. 2, 1196-1205.
- [14] N. Ackermann, T. Bartsch, P. Kaplický and P. Quittner, A priori bounds, nodal equilibria and connecting orbits in indefinite superlinear parabolic problems, *Trans. Amer. Math. Soc.* 360 (2008), no. 7, 3493-3539.
- [15] R.Z. Xu, Initial boundary value problem for semilinear hyperbolic equations and parabolic equations with critical initial data, *Quart. Appl. Math.* 68 (2010), no. 3, 459-468.
- [16] R.Z. Xu, Y.X. Chen, Y.B. Yang, S.H. Chen, J.H. Shen, T. Yu and Z.S. Xu, Global well-posedness of semilinear hyperbolic equations, parabolic equations and Schrödinger equations, *Electron. J. Differential Equations* 55 (2018), 52 pp.
- [17] H. Alsayed, H. Fakhri, A. Miranville and A. Wehbe, Finite difference scheme for 2D parabolic problem modelling electrostatic micro-electromechanical systems, *Electron. Res. Announc. Math. Sci.* 26 (2019), 54-71.

- [18] R.Z. Xu, W. Lian and Y. Niu, Global well-posedness of coupled parabolic systems, *Sci. China Math.* 2019, <https://doi.org/10.1007/s11425-017-9280-x>.
- [19] M.Q. Xiang, V.D. Rădulescu and B.L. Zhang, Nonlocal Kirchhoff diffusion problems: local existence and blow-up of solutions, *Nonlinearity* 31 (2018), no. 7, 3228-3250.
- [20] P. Pucci, M.Q. Xiang and B.L. Zhang, A diffusion problem of Kirchhoff type involving the nonlocal fractional p -Laplacian, *Discrete Contin. Dyn. Syst.* 37 (2017), no. 7, 4035-4051.
- [21] Y.Q. Fu and P. Pucci, On solutions of space-fractional diffusion equations by means of potential wells, *Electron. J. Qual. Theory Differ. Equ.* 70 (2016), 17pp.
- [22] J. Giacomoni, V.D. Rădulescu and G. Warnault, Quasilinear parabolic problem with variable exponent: qualitative analysis and stabilization, *Commun. Contemp. Math.* 20 (2018), no. 8, 38 pp.
- [23] W. Lian, J. Wang and R.Z. Xu, Global existence and blow up of solutions for pseudo-parabolic equation with singular potential, *J. Differential Equations* 269 (2020), no. 6, 4914-4959.
- [24] M. Fila and H.A. Levine, On the boundedness of global solutions of abstract semilinear parabolic equations, *J. Math. Anal. Appl.* 216 (1997), no. 2, 654-666.
- [25] A.El Soufi, M. Jazar and R. Monneau, A Gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24 (2007), no. 1, 17-39.
- [26] M. Jazar and R. Kiwan, Blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 25 (2008), no. 2, 215-218.
- [27] W.J. Gao and Y.Z. Han, Blow-up of a nonlocal semilinear parabolic equation with positive initial energy, *Appl. Math. Lett.* 24 (2011), no. 5, 784-788.
- [28] E. Vitillaro, Global nonexistence theorems for a class of evolution equations with dissipation, *Arch. Ration. Mech. Anal.* 149 (1999), no. 2, 155-182.
- [29] A. Khelghati and K. Baghaei, Blow-up phenomena for a nonlocal semilinear parabolic equation with positive initial energy, *Comput. Math. Appl.* 70 (2015), no. 5, 896-902.
- [30] R.E. Showalter and T.W. Ting, Pseudoparabolic partial differential equations, *SIAM J. Math. Anal.* 1 (1970), 1-26.
- [31] R.Z. Xu and J. Su, Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations, *J. Funct. Anal.* 264 (2013), no. 12, 2732-2763.
- [32] R.Z. Xu and Y. Niu, Addendum to "Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations" [*J. Funct. Anal.* 264 (12) (2013) 2732-2763], *J. Funct. Anal.* 270 (2016), no. 10, 4039-4041.
- [33] W.J. Liu and J.Y. Yu, A note on blow-up of solution for a class of semilinear pseudo-parabolic equations, *J. Funct. Anal.* 274 (2018), no. 5, 1276-1283.
- [34] R.Z. Xu, X.C. Wang and Y.B. Yang, Blowup and blowup time for a class of semilinear pseudo-parabolic equations with high initial energy, *Appl. Math. Lett.* 83 (2018), 176-181.
- [35] V.K. Kalantarov and O.A. Ladyzhenskaya, The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types, *J. Soviet Math.* 10 (1978), no. 1, 53-70.
- [36] M. Feng and J. Zhou, Global existence and blow-up of solutions to a nonlocal parabolic equation with singular potential, *J. Math. Anal. Appl.* 464 (2018), no. 2, 1213-1242.
- [37] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš, *Nonlinear Analysis-Theory and Methods*, Springer Monographs in Mathematics, Springer, Cham, 2019.
- [38] K. Ono, On global existence, asymptotic stability and blowing up of solutions for some degenerate non-linear wave equations of Kirchhoff type with a strong dissipation, *Math. Methods Appl. Sci.* 20 (1997), no. 2, 151-177.
- [39] F. Gazzola and M. Squassina, Global solutions and finite time blow up for damped semilinear wave equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 23 (2006), no. 2, 185-207.